The Weak α-Core of Large Games

Kali P. Rath
University of Notre Dame

Haomiao Yu
Ryerson University

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Literature

- Aumann and Peleg (1960) introduced the notions of α and β cores for finite player games. Aumann (1961) explores the issues further.
- General existence theorems are proved in Scarf (1967, 1971). (The notion of balancedness is important.)
- Notable contributions since have been many; e.g., Shapley (1973), Border (1982), Ichiishi (1982) and Konishi et al. (1997).
- Shapley and Vohra (1991) provides a proof of Scarf's theorem using Kakutani's fixed point theorem.
- The existence of core for infinite player games are proved in Ichiishi and Weber (1978) and Weber (1979, 1981).
- Recent important contributions for games with a non-atomic space of players and finite player games with nonatomic spaces of incomplete information are Askoura (2011), Askoura et al. (2013) and Noguchi (2014).

Literature, contd.

- Askoura (2011). The payoff to a player depends only on the societal distribution. It does not depend on the choice of own action.
  - The existence of weak α-core is shown.
  - For each coalition, the set of strongly unblocked distributions is a nonempty compact set.
  - The set of strongly unblocked distributions of coalitions have the finite intersection property. (This step requires characteristic function form construction and Scarf's theorem.)
- Ichiishi and Weber (1978). The game is in characteristic function form.
  - The connection between strategies and payoffs is not specified.
  - The notion of core may not correspond to either α or β cores.
  - The convexity of the set of feasible payoffs of the grand coalition is assumed. No other notion of balancedness is needed.
  - The proof uses Fan's theorem on linear inequalities.

This Talk

- We consider games over an atomless probability space of players with finite actions. The set of randomized strategy profiles is endowed with its weak topology.
- The payoff to a player depends on the choice of own action and the average action of all others.
- A coalition is a subset of the players of nonzero measure.
- A coalition E strongly blocks a strategy profile f if the coalition has a strategy h_E such that for any strategy of the complement of the coalition h_{E^c} and $h = (h_E, h_{E^c})$, the payoff to each member of the coalition under h exceeds by ϵ the payoff from f for some $\epsilon > 0$.
- The weak α-core is the set of strategy profiles which is not strongly blocked by any coalition.
- We show that under some conditions, the weak α-core is nonempty.
- The relationship between Nash equilibria and the weak α-core is explored.

Large Games

- Let $E = \{e^1, \dots, e^L\}$ be the set of unit vectors in $\mathbb{R}^L$ and
  $S = \{s \in \mathbb{R}^L_+ : \sum_{k=1}^L s_k = 1\}$ the unit simplex in $\mathbb{R}^L$.
- Let $\mathcal{U}$ be the set of real valued continuous functions defined on $S \times S$, endowed with sup norm.
  (We can restrict attention to $u \in \mathcal{U}$ where u is linear in the first coordinate.)
- Let $(T, \mathcal{T}, \mu)$ be an atomless, probability space.
- A *game* is a measurable function $\mathcal{G} : T \longrightarrow \mathcal{U}$.
- A *pure strategy profile* is a measurable function $f : T \longrightarrow E$.

  A *randomized strategy profile* is a measurable function $f : T \longrightarrow S$.
- A $f : T \longrightarrow E$ is a (pure strategy) *Nash equilibrium* of $\mathcal{G}$ if for almost all t, $\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(a, \int_T f \, \mathrm{d}\mu\right)$ for all $a \in E$.

  A $f : T \longrightarrow S$ is a (randomized strategy) *Nash equilibrium* of $\mathcal{G}$ if for almost all t, $\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(y, \int_T f \, \mathrm{d}\mu\right)$ for all $y \in S$.

The Notion of α-Core

- Let $\mathcal{F}$ denote the set of measurable mappings from T to S with the weak topology. $\mathcal{F}$ corresponds to $L_1(T \times \{1, \ldots, L\})$.
  Under the weak topology, $\mathcal{F}$ is a compact, convex subset of a locally convex linear topological space.
- A *coalition* is a measurable subset of T with positive measure.
- Given a coalition E, $B(E, S)$ denotes the set of measurable functions from E to S.
- A coalition E *blocks* a strategy profile f if there is a measurable function $h_E \in B(E, S)$, such that for every $h_{E^c} \in B(E^c, S)$ and $h = (h_E, h_{E^c})$,

$$u_t(h(t), \textstyle\int_T h \,\mathrm{d}\mu) > u_t(f(t), \textstyle\int_T f \,\mathrm{d}\mu) \text{ for almost all } t \in E.$$

- The *α-core* of the game is the set of profiles that are not blocked by any coalition E.

The Notion of Weak α-Core

- A coalition E *blocks* a strategy profile f if there is a measurable function $h_E \in B(E, S)$, such that for every $h_{E^c} \in B(E^c, S)$ and $h = (h_E, h_{E^c})$,

$$u_t(h(t), \int_T h \, d\mu) > u_t(f(t), \int_T f \, d\mu) \text{ for almost all } t \in E.$$

- The *α-core* of the game is the set of profiles that are not blocked by any coalition E.
- A coalition E *strongly blocks* a strategy profile f if there is $\epsilon > 0$ and a measurable function $h_E \in B(E, S)$, such that for every $h_{E^c} \in B(E^c, S)$ and $h = (h_E, h_{E^c})$,

$$u_t(h(t), \int_T h \, d\mu) > u_t(f(t), \int_T f \, d\mu) + \epsilon \text{ for almost all } t \in E.$$

- The *weak α-core* of the game is the set of profiles that are not strongly blocked by any coalition E.

Assumptions

The following three assumptions are respectively; integrably boundedness, equicontinuity and quasiconcavity. (The utility function $\mathcal{G}(t)$ is denoted by u_t.)

Assumption 1

The family of functions $\{u_t(f(t), \int_T f \, \mathrm{d}\mu) : f \in \mathcal{F}\}$ is integrably bounded.

Assumption 2

Let $f \in \mathcal{F}$. If $\epsilon > 0$ then there is an open neighborhood $U(f, \epsilon)$ such that

$$|u_t(f(t), \int_T f \, \mathrm{d}\mu) - u_t(g(t), \int_T g \, \mathrm{d}\mu)| < \epsilon$$

for all $g \in U(f, \epsilon)$ and $t \in T$.

For a coalition E and $f \in \mathcal{F}$, let $z(E, f) = \int_E u_t(f(t), \int_T f \, \mathrm{d}\mu) \, \mathrm{d}\mu$.

Assumption 3

For every coalition E, $z(E, \cdot)$ is continuous and quasiconcave on $\mathcal{F}$.

Existence

Theorem

Under assumptions 1-3, the weak α-core of a game is nonempty.

The proof consists of two lemmas.

For a coalition E, let $\mathcal{H}(E) = \{f \in \mathcal{F} : f \text{ is not strongly blocked by } E\}$.

Lemma 1

For every coalition E, $\mathcal{H}(E)$ is a nonempty, closed (and hence compact) subset of $\mathcal{F}$.

Lemma 2

Let E_i, $i \in I$ be a finite collection of coalitions. Then $\cap_{i \in I} \mathcal{H}(E_i)$ is nonempty.

Proof of Lemma 1

$\mathcal{H}(E) = \{f \in \mathcal{F} : f$ is not strongly blocked by $E\}$.

- $\mathcal{H}(E) \neq \emptyset$. The function $z(E, f) = \int_E u_t(f(t), \int_T f \,\mathrm{d}\mu)\,\mathrm{d}\mu$ is continuous on $\mathcal{F}$. Since $\mathcal{F}$ is compact, $z(E, \cdot)$ attains its maximum, say at f^*. The coalition E cannot strongly block the strategy profile f^* and $f^* \in \mathcal{H}(E)$.
- If E strongly blocks f then there exist $\epsilon > 0$ and $h_E \in B(E, S)$, such that for every $h_{E^c} \in B(E^c, S)$ and $h = (h_E, h_E^c)$,

$$u_t(h(t), \int_T h \,\mathrm{d}\mu) > u_t(f(t), \int_T f \,\mathrm{d}\mu) + \epsilon \text{ for almost all } t \in E.$$

By assumption 2, given $\epsilon/2 > 0$, there is an open neighborhood $V(f, \epsilon/2)$ of f such that if $g \in V(f, \epsilon/2)$ then

$$|u_t(f(t), \int_T f)\,\mathrm{d}\mu - u_t(g(t), \int_T g \,\mathrm{d}\mu)| < \epsilon/2 \text{ for all } t \in T.$$

For almost all $t \in E$,

$$u_t(g(t), \int_T g \,\mathrm{d}\mu) + (\epsilon/2) < u_t(f(t), \int_T f \,\mathrm{d}\mu) + \epsilon < u_t(h(t), \int_T h \,\mathrm{d}\mu).$$

This means the coalition E strongly blocks every profile $g \in V(f, \epsilon/2)$. Thus, the complement of $\mathcal{H}(E)$ is open and $\mathcal{H}(E)$ is closed. ∎

Outline of Proof of Lemma 2

If I is a finite set then $\cap_{i \in I} \mathcal{H}(E_i) \neq \emptyset$.

- Let $\{E_i\}_{i \in I}$ be a finite family of coalitions such that $\cup_{i \in I} E_i = T$.
- Let $\{K_j\}_{j \in J}$ be a finite family of pairwise disjoint elements of $\mathcal{T}$ such that $\mu(K_j) > 0$ for all j and each E_i is a union of some of the K_js.
- For $B \subseteq J$, define $K_B = \cup_{j \in B} K_j$. If $B \subset J$ then K_{B^c} is nonempty and automatically defined as $T \setminus (\cup_{j \in B} K_j)$.
- For $B \subseteq J$, define a subset $V(B)$ of $\mathbb{R}^J$ as follows.

$$V(B) = \{v \in \mathbb{R}^J : \exists\, h_{K_B} \text{ such that } \forall\, h_{K_{B^c}} \text{ and } h = (h_{K_B}, h_{K_{B^c}}), \\ z(K_j, h) \geq v_j,\ \forall\, j \in B\}.$$

  Note that if $j \notin B$ then $v_j \in V(B)$ can be any number in $\mathbb{R}$.
- The following properties hold:
  (1) For every $B \subseteq J$, $V(B)$ is nonempty and closed.
  (2) For every $B \subseteq J$, if $v \in V(B)$ and $v' \leq v$ then $v' \in V(B)$.
  (3) $V(J)$ is bounded from above.
  (4) J is balanced.

Proof of Lemma 2, contd.

- Scarf' theorem: The core of $G = (J, V)$ is nonempty.

  (If v is in the core then v is not in the interior of $V(B)$ for any $B \subseteq J$.)

- If the core of $G = (J, V)$ is not empty, then $\cap_{i \in I} \mathcal{H}(E_i) \neq \emptyset$.
- Let v be in the core of $G = (J, V)$. Let $f : T \longrightarrow S$ such that $z(K_j, f) \geq v_j$ for all $j \in J$.
- Fix an arbitrary index $i \in I$. E_i is a finite union of some sets K_j, $j \in J$. Let $E_i = \cup_{j \in J_i} K_j$ where $J_i \subseteq J$.
- Since v is not in the interior of $V(J_i)$, for every h_{E_i}, there exists $h_{E_i^c}$ and an index $j \in J_i$ such that for $h = (h_{E_i}, h_{E_i^c})$,

$$z(K_j, h) \leq v_j \leq z(K_j, f).$$

- Thus, for any h_{E_i}, there exists $h_{E_i^c}$ and a subset D_i of E_i of positive measure such that $u_t(h(t), \int_T h \, d\mu) \leq u_t(f(t), \int_T f \, d\mu)$ for all $t \in D_i$.
- This shows that $f \in \cap_{i \in I} \mathcal{H}(E_i)$ and completes the proof. ■

Purification

- We have proved the existence of a randomized strategy profile in the core. Does the core contain a pure strategy profile?
- Three possible ways of proving a pure strategy profile in the core.
  1. Use the DWW theorem. (A partitioning of the player set and a consequent refinement of the DWW theorem may be needed.)
  2. Extreme point argument. Consider the closed convex hull of the set of core profiles. It has an extreme point.
     Is the extreme point a pure strategy profile?
  3. The set of pure strategies are dense in the set of randomized strategies. Does this imply that there is a pure strategy profile in the core?

Example 1

- The player space is $T = [0, 1]$ and λ denotes Lebesgue measure.
- The set of Nash equilibria is a proper subset of the core.
- Let $A = \{a_1, a_2\}$. For any $\eta \in \mathcal{M}^1_+(A)$, let

$$u(a_1, \eta) = \frac{1}{2}, \qquad u(a_2, \eta) = 1 - \eta(a_2).$$

  For each $t \in T$, let $u_t = u$.

- f is a Nash equilibrium of this game iff $\lambda \circ f^{-1}(a_2) = 1/2$.
- Since the payoff function is the same for all the players, the weak α-core and the α-core are the same.
- We will show that the α-core of this game is any f such that $\lambda \circ f^{-1}(a_2) \leq 1/2$.
  (Thus, the set of Nash equilibria is contained in the core.)

Example 1: Blocked Profiles

- If $\lambda \circ f^{-1}(a_2) > 1/2$ then f is not in the core.
- Let $E \subseteq \{t \in T : f(t) = a_2\}$ such that $\lambda(E) > 0$.
- For any $t \in E$,

$$u_t(f(t), \lambda \circ f^{-1}) = 1 - \lambda \circ f^{-1}(a_2) < \frac{1}{2}.$$

- Let $h_E(t) = a_1$ for any $t \in E$. Then for any h_{E^c} and $h = (h_E, h_{E^c})$,

$$u_t(h(t), \lambda \circ h^{-1}) = \frac{1}{2} \text{ for } t \in E.$$

- So, the coalition E blocks f.

Example 1: Unblocked Profiles

- Now consider any f such that $\lambda \circ f^{-1}(a_2) \leq 1/2$.
  We will show that it is in the core.
- Suppose there is a coalition E which blocks f.
  Let h_E be the function on E such that for any function h_{E^c} on E^c and $h = (h_E, h_{E^c})$,

$$u_t(h(t), \lambda \circ h^{-1}) > u_t(f(t), \lambda \circ f^{-1}).$$

- Consider

$$S_{ij} = \{t \in E : f(t) = a_i \text{ and } h(t) = a_j,\ i, j = 1, 2\}.$$

  - If $t \in S_{11}$ then $u_t(h(t), \lambda \circ h^{-1}) = u_t(f(t), \lambda \circ f^{-1}) = 1/2$, a contradiction. So, $\lambda(S_{11}) = 0$.
  - If $t \in S_{21}$ then $u_t(f(t), \lambda \circ f^{-1}) = 1 - \lambda \circ f^{-1}(a_2) \geq 1/2$ and $u_t(h(t), \lambda \circ h^{-1}) = 1/2$, again a contradiction. So, $\lambda(S_{21}) = 0$.
  - Thus, $E = S_{12} \cup S_{22}$.

Example 1: Unblocked Profiles, contd.

- We have

$$S_{ij} = \{t \in E : f(t) = a_i \text{ and } h(t) = a_j,\ i, j = 1, 2\}, \qquad E = S_{12} \cup S_{22}.$$

  - If $t \in S_{12}$ then $u_t(f(t), \lambda \circ f^{-1}) = 1/2$.
    If $t \in S_{22}$ then $u_t(f(t), \lambda \circ f^{-1}) = 1 - \lambda \circ f^{-1}(a_2) \geq 1/2$.
  - Let $h_{E^c}(t) = a_2$. Then $\lambda \circ h^{-1}(a_2) = 1$.
  - For any $t \in E$, $u_t(h(t), \lambda \circ h^{-1}) = 1 - \lambda \circ h^{-1}(a_2) = 0$.
    This is a contradiction.

- So, no coalition can block f and any f with $\lambda \circ f^{-1}(a_2) \leq 1/2$ is in the core.

Example 2

- In this example the weak core does not contain any Nash equilibrium.
- Let $A = \{a_1, a_2, a_3\}$, $M_t = \max\{1/10, t\}$ and $m_t = \min\{9/10, t\}$. For $t \in T$ define

$$\begin{aligned} u_t(a_1, \eta) &= 2[1 - \eta(a_2)]M_t \\ u_t(a_2, \eta) &= 1 - \eta(a_2) \\ u_t(a_3, \eta) &= 3[\eta(a_1) - \eta(a_2)](1 - m_t) \end{aligned}$$

- This game has two Nash equilibria f_1 and f_2 where:
  - (1) $f_1(t) = a_1$ if $t > 1/2$ and $f_1(t) = a_2$ if $t \leq 1/2$ and
  - (2) $f_2(t) = a_2$ for all t.
- None of the Nash equilibrium is in the weak core.

Example 2: Nash Equilibria

Payoff Functions:

$$u_t(a_1,\eta) = 2[1-\eta(a_2)]M_t$$
$$u_t(a_2,\eta) = 1-\eta(a_2)$$
$$u_t(a_3,\eta) = 3[\eta(a_1)-\eta(a_2)](1-m_t)$$

Nash Equilibria:

(1) $f_1(t) = a_1$ if $t > 1/2$
$f_1(t) = a_2$ if $t \leq 1/2$.

(2) $f_2(t) = a_2$ for all t.

- Observation: If $\eta(a_2) < 1$ then for any $t > 1/2$, $u_t(a_1,\eta) > u_t(a_2,\eta)$ and for $t < 1/2$, $u_t(a_2,\eta) > u_t(a_1,\eta)$.
- (1) If $\eta = \lambda \circ (f_1)^{-1}$ then $\eta(a_1) = \eta(a_2) = 1/2$.
  The payoffs from a_3 is zero and from a_1 and a_2 are positive for all t.
  a_1 is the BR for $t > 1/2$ and a_2 is the BR for $t < 1/2$. So, f_1 is an NE.
- (2) If $f_2(t) = a_2$ and $\eta = \lambda \circ (f_2)^{-1}$ then $\eta(a_2) = 1$.
  For all t, the payoffs from a_1 and a_2 are zero and from a_3 is negative.
  So, a_2 is a BR for $t \in [0,1]$ and f_2 is an NE.
- The arguments to show that these are the only NE are omitted.

Example 2: No Nash Equilibrium in the Weak Core

Payoff Functions:

$$u_t(a_1,\eta) = 2[1-\eta(a_2)]M_t$$
$$u_t(a_2,\eta) = 1-\eta(a_2)$$
$$u_t(a_3,\eta) = 3[\eta(a_1)-\eta(a_2)](1-m_t)$$

Nash Equilibria:

(1) $f_1(t) = a_1$ if $t > 1/2$
$f_1(t) = a_2$ if $t \leq 1/2$.

(2) $f_2(t) = a_2$ for all t.

- At f_2 the payoff to each player is zero.
  At f_1, the payoff is t if $t > 1/2$ and the payoff is $1/2$ if $t \leq 1/2$.
  So, $u_t(f_1(t), \lambda \circ (f_1)^{-1}) \geq u_t(f_2(t), \lambda \circ (f_2)^{-1}) + (1/2)$ for all t.
  So, f_2 is not in the weak core.
- At f_1 the payoff is t if $t > 1/2$ and the payoff is $1/2$ if $t \leq 1/2$.
  - Let $h(t) = a_1 = f_1(t)$ if $t > 1/2$ and $h(t) = a_3$ if $t \leq 1/2$.
  - If $\rho = \lambda \circ h^{-1}$ then $\rho(a_1) = 1/2$ and $\rho(a_2) = 0$.
  - The payoff at h is $2t$ if $t > 1/2$ and $(3/2)(1-t) \geq 3/4$ if $t \leq 1/2$.
  - $u_t(h(t), \lambda \circ h^{-1}) \geq u_t(f_1(t), \lambda \circ (f_1)^{-1}) + (1/4)$ for almost all t.

  So, f_1 is not in the weak core.

Example 2: A Core Profile

Payoff Functions:

$$u_t(a_1,\eta) = 2[1-\eta(a_2)]M_t$$
$$u_t(a_2,\eta) = 1-\eta(a_2)$$
$$u_t(a_3,\eta) = 3[\eta(a_1)-\eta(a_2)](1-m_t)$$

A Core Profile:

$f(t) = a_1$ if $t > 1/2$
$f(t) = a_3$ if $t \le 1/2$.

- If $\eta = \lambda \circ f^{-1}$ then $\eta(a_1) = \eta(a_3) = 1/2$ and $\eta(a_2) = 0$.
  $t > 1/2$: $u_t(a_1,\eta) = 2t > 1$. $t \le 1/2$: $u_t(a_3,\eta) = (3/2)(1-t) \ge 3/4$.
- f is not an NE because at $t = 1/2$, $u_t(a_3,\eta) = 3/4 < 1 = u_t(a_2,\eta)$.
- Suppose a coalition E blocks f. Let $h = (h_E, h_{E^c})$ and $\rho = \lambda \circ h^{-1}$.
- Let $t > 1/2$. Then $u_t(a_2,\rho) \le u_t(a_1,\rho) \le u_t(a_1,\eta)$.
  - If $t \ge 2/3$ then $1-m_t \le 1/3$ and $u_t(a_3,\rho) \le 1$. $\lambda(E \cap [2/3,1]) = 0$.
  - Let $h(t) = a_2$ on $[2/3,1]$. Then $\rho(a_1) - \rho(a_2) \le 1/3$ and $u_t(a_3,\rho) \le 1$ if $t \in (1/2, 2/3)$. $\lambda(E \cap (1/2,2/3)) = 0$.
- Let $t \le 1/2$. Assume that $h(t) = a_2$ if $t > 1/2$.
  Then $u_t(a_1,\rho) \le u_t(a_2,\rho) \le 1/2$ and $u_t(a_3,\rho) \le 0$. $\lambda(E \cap [0,1/2]) = 0$.

Example 3

Payoff Functions:

$$u_t(a_1, \eta) = \eta(a_1) - \eta(a_3)$$
$$u_t(a_2, \eta) = 0$$
$$u_t(a_3, \eta) = -2$$

Nash Equilibria:

(1) $f_1(t) = a_1$ for all t.

(2) $f_2(t) = a_2$ for all t.

f_1 is in the core but not f_2.

- (1) If $\eta = \lambda \circ (f_1)^{-1}$ then $\eta(a_1) = 1$ and $\eta(a_2) = \eta(a_3) = 0$. a_1 is the unique BR for $t \in [0, 1]$. So, f_1 is an NE.
- (2) If $\eta = \lambda \circ (f_2)^{-1}$ then $\eta(a_2) = 1$ and $\eta(a_1) = \eta(a_3) = 0$. So, a_2 is a best response for $t \in [0, 1]$ and f_2 is an NE.
- Conversely suppose that f is an NE and $\eta = \lambda \circ (f_1)^{-1}$.
  - If $\eta(a_1) > \eta(a_3)$ then $u_t(a_1, \eta) > u_t(a_i, \eta)$ for $i = 2, 3$. So, $f = f_1$.
  - If $\eta(a_1) \leq \eta(a_3)$ then $u_t(a_2, \eta) = u_t(a_1, \eta) > u_t(a_3, \eta)$. So, $\eta(a_3) = 0$ which implies that $\eta(a_1) = 0$. Thus, $f = f_2$.
- The payoff to every player from f_1 is 1, which is the highest payoff in the game. So, no coalition can block it and f_1 is in the core.
- The payoff is zero to every player from f_2. So, the all member coalition can strongly block f_2 (via f_1) and f_2 is not in the weak core.

Example 4

- The core is a proper subset of the set of NE.
- Let $A = \{a_1, a_2\}$ and $u(a_i, \eta) = \eta(a_1)$ for $i = 1, 2$.
  For all $t \in [0, 1]$, let $u_t = u$.
- Each player has the same payoff function and the payoff depends only on the measure.
  So, every measure (or the corresponding strategy profile) is an NE.
- We will show that $f(t) = a_1$ for all t is the only core profile.
- Let $\eta = \lambda \circ f^{-1}$. Then $\eta(a_1) = 1$ and the payoff is 1 to each. This is the highest payoff in the game. So, no coalition can block it and f_1 is in the core.
- Let h be any strategy profile, $\rho = \lambda \circ h^{-1}$ and $\rho(a_1) < 1$. Then the payoff to each player is $\rho(a_1) < 1$. The all member coalition strongly blocks h.
- So, f is the unique core allocation and the core is a proper subset of the set of NE.

Example 5

- The core and set of NE are identical.
- Let $A = \{a_1, a_2\}$ and $\quad u_t(a_1, \eta) = \eta(a_1), \quad u_t(a_2, \eta) = \eta(a_1) - 1.$
- Let $f^*(t) = a_1$ for each t and $\eta^* = \lambda \circ (f^*)^{-1}$. Then $\eta^*(a_1) = 1$ and $\eta^*(a_2) = 0$. $u_t(a_1, \eta^*) = 1$ and $u_t(a_2, \eta^*) = 0$. So, f^* is an NE.
- Conversely, suppose that f is an NE. Then

$$u_t(a_1, \lambda \circ f^{-1}) = \lambda \circ f^{-1}(a_1), \qquad u_t(a_2, \lambda \circ f^{-1}) = \lambda \circ f^{-1}(a_1) - 1.$$

  So, $f(t) = a_1$ for almost all t. Thus f^* is the unique NE.
- f^* is in the core. The payoff to t at f^* is 1 and a player never gets more than 1. So, no coalition can block f^*.
- Let f be any profile such that $\lambda \circ f^{-1}(a_2) > 0$. The payoffs are:

$$u_t(a_1, \lambda \circ f^{-1}) = \lambda \circ f^{-1}(a_1) < 1,$$
$$u_t(a_2, \lambda \circ f^{-1}) = \lambda \circ f^{-1}(a_1) - 1 < 0.$$

  The all member coalition strongly blocks f (via f^*).
- This shows that the unique NE f^* is in the unique element of the core.