Modeling Infinitely Many Agents: Why Countable Additivity Is Necessary

M. Ali Khan
Johns Hopkins University

Lei Qiao
Shanghai University of Finance and Economics

Kali P. Rath
University of Notre Dame

Yeneng Sun
National University of Singapore

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Background and Motivation

- Finite agent economies and games: Arrow-Debreu (1954), McKenzie (1954), Nash (1951).
- Economies and games with a continuum of agents: Aumann (1964, 1966), Vind (1964), Schmeidler (1969, 1973).
- Modeling individual negligibility:
  - Replication/Large finite approximations: Edgeworth (1881), Debreu-Scarf (1963), Anderson (1978).
  - Continuum models with an atomless measure: Milnor-Shapley (1961), Aumann (1964), Schmeidler (1973), Hildenbrand (1974), Khan-Sun (2002).
  - Infinitesimals, Loeb spaces: Brown-Robinson (1972, 1975), Khan (1974), Brown-Loeb (1976), Khan-Sun (1996, 1999).
  - Finitely additive economies: Armstrong-Richter (1984, 1986), Weiss (1981), Feldman-Gilles (1985), Basile (1993).

Mathematical Preliminaries

- Let T be a nonempty set and $\mathcal{T}$ a σ-algebra of subsets of T,
  (i) $T \in \mathcal{T}$, (ii) $A \in \mathcal{T}$ implies $A^c \in \mathcal{T}$,
  (iii) $A_n \in \mathcal{T}$ $(n = 1, 2 \ldots)$ implies $\cup_{n=1}^{\infty} A_n \in \mathcal{T}$.
- Let μ be a set function from $\mathcal{T}$ to $[0, 1]$ with $\mu(T) = 1$.
  - μ is a finitely additive measure on $\mathcal{T}$ if for any $A, B \in \mathcal{T}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$.
  - μ is a countably additive measure on $\mathcal{T}$ if for any sequence $\{A_n\}$ of pairwise disjoint sets in $\mathcal{T}$, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.
- The triple $(T, \mathcal{T}, \mu)$ will be called a (finitely additive/countably additive) measure space.

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- The triple $(T, \mathcal{T}, \mu)$ will be called a (finitely additive/countably additive) measure space.
- A measure μ is atomless if for every $\epsilon > 0$, there exists a $\mathcal{T}$-measurable partition $\{F_1, \ldots, F_n\}$ of T such that $\mu(F_i) < \epsilon$ for every i.
- Let $\mathbb{N}$ be the set of positive integers and $\mathcal{P}(\mathbb{N})$ its power set. There are finitely additive, atomless measures on $\mathcal{P}(\mathbb{N})$ (such as a density charge).

Preview of the Results

- Negative results on finitely additive spaces.
  - An economy may not have a competitive equilibrium.
    (Two examples)
  - A game may not have a Nash equilibrium.
    (Two examples)
  - An economy may not have the idealized limit property.
  - A game may not have the idealized limit property.
- Consequences.
  - Necessity of countably additivity for economies:
    both existence and idealized limit property hold.
  - Necessity of countably additivity for games:
    both existence and idealized limit property hold.
- Approximate equilibria on finitely additive spaces.
  - An economy may not have an approximate competitive equilibrium.
    A tightness assumption is sufficient for existence.
  - A game may not have an approximate Nash equilibrium.
    A tightness assumption is sufficient for existence.

Upper Hemicontinuity of the Integral

- Let $(T, \mathcal{T}, \mu)$ be an atomless, countably additive measure space and X a metric space.
- Let $F : T \times X \longmapsto\!\!\!\!\to \mathbb{R}^n$ be a correspondence.
- If $F(\cdot, x)$ is measurable and $F(t, \cdot)$ is upper hemicontinuous then

$$\int_T F(\cdot, x)\, \mathrm{d}\mu$$

  is upper hemicontinuous (in x).

- This results fails if μ is a finitely additive measure.

Lack of UHC under Integration

- Let $A = \{0,\ 1\}$ and $K = [0, 1]$.
  Let μ be an atomless, finitely additive measure on $\mathcal{P}(\mathbb{N})$.
- Define a correspondence $F: \mathbb{N} \times K \longmapsto\!\!\rightarrow A$ as:

$$F(t, x) = \begin{cases} \{0, 1\} & \text{if } x = 1/t \\ 1 & \text{if } x < 1/t \\ 0 & \text{if } x > 1/t. \end{cases}$$

- Then

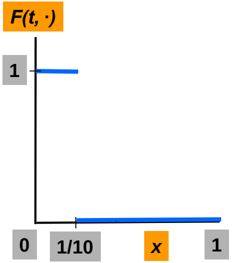
$$\int_{\mathbb{N}} F(\cdot, x)\, \mathrm{d}\mu = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0. \end{cases}$$

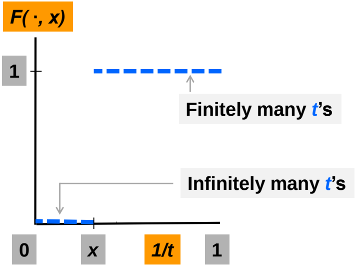
- Clearly, $\int_{\mathbb{N}} F(\cdot, x)\, \mathrm{d}\mu$ is not uhc at $x = 0$.

Graphs of the Correspondence

$$F(t,x) = \begin{cases} \{0,1\} & \text{if } x = 1/t \\ 1 & \text{if } x < 1/t \\ 0 & \text{if } x > 1/t. \end{cases}$$

Let $t = 10$.





Example, contd.

- $F : \mathbb{N} \times K \longmapsto\!\!\!\rightarrow A$.

$$F(t,x) = \begin{cases} \{0,1\} & \text{if } x = 1/t \\ 1 & \text{if } x < 1/t \\ 0 & \text{if } x > 1/t. \end{cases}$$

- Then

$$\int_{\mathbb{N}} F(\cdot,x)\,\mathrm{d}\mu = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0. \end{cases}$$

- Clearly, $\int_{\mathbb{N}} F(\cdot,x)\,\mathrm{d}\mu$ is not uhc at $x = 0$.
- Let f be a measurable selection of $F(\cdot,x)$.
  - If $x = 0$ then $x < 1/t$ for all $t \in \mathbb{N}$, which implies that $f(t) = 1$ for all $t \in \mathbb{N}$ and $\int_{\mathbb{N}} f\,\mathrm{d}\mu = 1$.
  - If $x > 0$ then $x > 1/t$ for almost all t, i.e., $f(t) = 0$ for almost all t and $\int_{\mathbb{N}} f\,\mathrm{d}\mu = 0$.

Economies and Competitive Equilibria

- There are L goods and the commodity space is $\mathbb{R}_+^L$.
- Let $\mathcal{U}$ denote the class of real valued, continuous utility functions on $\mathbb{R}_+^L$ (endowed with the compact open topology).
- A $u \in \mathcal{U}$ is *strongly monotone* if $x \geq y$, $x \neq y$ implies that $u(x) > u(y)$.
- Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space. (space of agents)

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- Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space. (space of agents)
- An *economy* is a measurable mapping $\mathcal{E} = (u, \omega) : T \longrightarrow \mathcal{U} \times \mathbb{R}^L_+$ such that ω is integrable and $\bar{\omega} = \int_T \omega \, \mathrm{d}\mu \gg 0$.
- An *allocation* of $\mathcal{E}$ is an integrable mapping f from T to $\mathbb{R}^L_+$. An allocation is *feasible* if $\int_T f \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$.
- Given a price vector $p \in \mathbb{R}^L_+$, the *budget set* of consumer t is
$$B_t(p) = \{x \in \mathbb{R}^L_+ : p \cdot x \leq p \cdot \omega_t\}.$$
- A *competitive equilibrium* of $\mathcal{E}$ is a pair (p, f), where $p \in \mathbb{R}^L_+ \setminus \{0\}$, f is a feasible allocation and μ-a.e.;
$$(a)\ f(t) \in B_t(p) \quad \text{and} \quad (b)\ u_t(f(t)) \geq u_t(x) \text{ for all } x \in B_t(p).$$
- An allocation f of $\mathcal{E}$ is a *competitive allocation* if for some p, (p, f) is a competitive equilibrium.

Nonexistence of a CE: An Example on Integers

- The measure space is $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$.
- The economy $\mathcal{E}$ is defined as follows. For each $t \in \mathbb{N}$,

$$u_t(x_1, x_2) = \frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2, \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2} \right),$$

where $0 \leq c < 1$.

- Assume that $(p_1, p_2) \in \mathbb{R}^2_+$ is a pair of equilibrium prices. We must have $p_1 > 0$ and $p_2 > 0$ since u_t is strongly monotone for each $t \in \mathbb{N}$.
- Without loss of generality suppose that $p_1 + p_2 = 1$.
- For any $t \in \mathbb{N}$, the unique solution of agent t's problem;

  maximize $u_t(x_1, x_2)$ subject to $p_1 x_1 + p_2 x_2 = (c+1)/2$ is

$$D_{t1} = \min \left\{ \frac{p_2^{t+1}}{p_1^{t+1}}, \frac{c+1}{2p_1} \right\}, \qquad D_{t2} = \frac{c+1}{2p_2} - \frac{p_1 D_{t1}}{p_2}.$$

Nonexistence of a CE, continued

- $\omega_t = ((c+1)/2, (c+1)/2)$, $0 \leq c < 1$ for $t \in \mathbb{N}$.
- $p = (p_1, p_2) \gg 0, \qquad p_1 + p_2 = 1. \qquad$ Income: $(c+1)/2$.
- Demand functions:

$$D_{t1} = \min\left\{\frac{p_2^{t+1}}{p_1^{t+1}}, \frac{c+1}{2p_1}\right\}, \qquad D_{t2} = \frac{c+1}{2p_2} - \frac{p_1 D_{t1}}{p_2}.$$

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- Case 1. $p_2/p_1 < 1$. $\quad \lim_{t\to\infty} D_{t1} = 0$. $\quad \int_{\mathbb{N}} D_{t1}\, \mathrm{d}\mu = 0$.

$$\int_{\mathbb{N}} D_{t2}\, \mathrm{d}\mu = \frac{c+1}{2p_2} > \frac{c+1}{2} = \int_{\mathbb{N}} \omega_{t2}\, \mathrm{d}\mu. \qquad \text{(contradiction)}$$

- Case 2. $p_2/p_1 \geq 1$. $\quad p_2^{t+1}/p_1^{t+1} \geq 1$. $\quad (c+1)/(2p_1) \geq c+1$.

  Therefore, $D_{t1} \geq \min\{1, c+1\} = 1$.

$$\int_{\mathbb{N}} D_{t1}\, \mathrm{d}\mu \geq 1 > \frac{c+1}{2} = \int_{\mathbb{N}} \omega_{t1}\, \mathrm{d}\mu. \qquad \text{(contradiction)}$$

Nonexistence of a CE on General Measure Spaces

Claim

Let $(T, \mathcal{T}, \mu)$ be an atomless finitely additive measure space. Assume that μ is not countably additive. Then there is an economy on $(T, \mathcal{T}, \mu)$ which has no competitive equilibrium.

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- Fact: Let $(T, \mathcal{T}, \mu)$ be a finitely additive probability space. Then the following are equivalent.
  - (i) μ is not countably additive.
  - (ii) There is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that $\cup_{n=1}^{\infty} B_n = T$ and $\lim_{n\to\infty} \mu(B_n) = c < 1$.

- Since μ is not countably additive, there is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that $\cup_{n=1}^{\infty} B_n = T$ and $\lim_{n\to\infty} \mu(B_n) = c < 1$.
- For $n \in \mathbb{N}$, let $C_1 = B_1$ and for $n \geq 2$, $C_n = B_n \setminus B_{n-1}$.
- $\{C_n\}$ is a sequence of pairwise disjoint sets and $\cup_{n=1}^{\infty} C_n = T$.

Nonexistence on General Measure Spaces, contd.

- Preferences and endowments: Let $t \in C_n$.

$$u_t(x_1, x_2) = \frac{n+1}{n} x_1^{\frac{n}{n+1}} + x_2, \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2} \right).$$

- Assume that $p_1, p_2 \in \mathbb{R}^2_+$ is a pair of competitive equilibrium prices. We must have $p_1 > 0$ and $p_2 > 0$ since u_t is strongly monotone for each t.
- Let $p_2 = 1$ and $p_1 > 0$. If $t \in C_n$, then

$$D_{t1} = \min \left\{ \frac{1}{p_1^{n+1}}, \frac{c+1}{2} \left(1 + \frac{1}{p_1} \right) \right\}, \quad D_{t2} = \frac{c+1}{2}(1 + p_1) - p_1 D_{t1}.$$

- To show that there is *no* competitive equilibrium, we will consider two cases: (*i*) $1 \geq p_1$ and (*ii*) $1 < p_1$.

Nonexistence of Equilibrium (Case 1)

Case 1: $1 \geq p_1$. Let $t \in C_n$.

$$D_{t1} = \min\left\{\frac{1}{p_1^{n+1}}, \frac{c+1}{2}\left(1+\frac{1}{p_1}\right)\right\} \geq \min\left\{1, \frac{c+1}{2}\left(1+\frac{1}{p_1}\right)\right\}$$
$$= \frac{c+1}{2} + \min\left\{1-\frac{c+1}{2}, \frac{c+1}{2p_1}\right\}.$$

Let $\theta = \min\left\{1-\frac{c+1}{2}, \frac{c+1}{2p_1}\right\} > 0$.

Then $D_{t1} \geq \frac{c+1}{2} + \theta$ for any $t \in T$. Therefore,

$$\int_T D_{t1}\,\mathrm{d}\mu \geq \frac{c+1}{2} + \theta > \frac{c+1}{2} = \int_T \omega_{t1}\,\mathrm{d}\mu,$$

a contradiction.

Nonexistence of Equilibrium (Case 2)

Case 2: $1 < p_1$. Note that $D_{t1} \leq 1/p_1^{n+1}$ for any $t \in C_n$. Then
(i) $D_{t1} \leq 1$ for any $t \in T$ and
(ii) if $t \in C_{n+1}$ and $n > m$ then $D_{t1} \leq 1/p_1^{n+2} \leq 1/p_1^{m+2}$.

Fix a positive integer m.

$$\begin{aligned} \int_T D_{t1}\, \mathrm{d}\mu &= \int_{B_m} D_{t1}\, \mathrm{d}\mu + \int_{T\setminus B_m} D_{t1}\, \mathrm{d}\mu \\ &\leq \int_{B_m} 1\, \mathrm{d}\mu + \int_{T\setminus B_m} \frac{1}{p_1^{m+2}}\, \mathrm{d}\mu = \mu(B_m) + \frac{1}{p_1^{m+2}}\mu(T\setminus B_m). \end{aligned}$$

Observe that $\mu(T\setminus B_m) \geq 1 - c$ for any m. Let m tend to infinity. $\int_T D_{t1}\, \mathrm{d}\mu \leq \lim_{m\to\infty} \mu(B_m) = c$.

$$\begin{aligned} \int_T D_{t2}\, \mathrm{d}\mu &= \frac{c+1}{2}(1+p_1) - p_1 \int_T D_{t1}\, \mathrm{d}\mu \\ &\geq \frac{c+1}{2}(1+p_1) - p_1 c > \frac{c+1}{2} = \int_T \omega_{t2}\, \mathrm{d}\mu, \end{aligned}$$

a contradiction.

Games and Nash Equilibria

- Let $E = \{e^1, \ldots, e^L\}$ be the set of unit vectors in $\mathbb{R}^L$ and $S = \{s \in \mathbb{R}^L_+ : \sum_{k=1}^L s_k = 1\}$ the unit simplex in $\mathbb{R}^L$.
- Let $\mathcal{V}$ be the set of real valued continuous functions defined on $E \times S$, endowed with sup norm.
- $(T, \mathcal{T}, \mu)$ is an atomless, countably/finitely additive probability space.
- A *game* is a measurable function $\mathcal{G} : T \longrightarrow \mathcal{V}$.
- A *pure strategy profile* is a measurable function $f : T \longrightarrow E$.
- A $f : T \longrightarrow E$ is a *pure strategy Nash equilibrium* of $\mathcal{G}$ if μ-a.e.;

$$\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(a, \int_T f \, \mathrm{d}\mu\right) \text{ for all } a \in E.$$

Games and Nash Equilibria, contd.

- Pure strategy profile: $f : T \longrightarrow E$.
- Mixed strategy profile: $g : T \longrightarrow S$.
- Given a mixed strategy profile g and $y \in S$, the *payoff to player t* is

$$\mathcal{G}(t)\left(y, \int_T g \, \mathrm{d}\mu\right) = \sum_{k=1}^{L} y_k \mathcal{G}(t)\left(e^k, \int_T g \, \mathrm{d}\mu\right).$$

- A $g : T \longrightarrow S$ is a *mixed strategy Nash equilibrium* of $\mathcal{G}$ if μ-a.e.;

$$\mathcal{G}(t)\left(g(t), \int_T g \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(y, \int_T g \, \mathrm{d}\mu\right) \text{ for all } y \in S.$$

Existence of Nash Equilibrium

Theorem (Schmeidler)

Every finite action game on a countably additive measure space has a pure strategy Nash equilibrium.

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Every finite action game on a countably additive measure space has a pure strategy Nash equilibrium.

- Define a correspondence $B : T \times S \longmapsto\!\!\!\!\to E$ by

$$B(t,s) = \{e^k \in E : \ \mathcal{G}(t)(e^k, s) \geq \mathcal{G}(t)(a,s) \ \text{ for all } a \in E\}.$$

- $B(t,s)$ is nonempty, $B(\cdot, s)$ is measurable and $B(t, \cdot)$ is uhc.
- Let $\Gamma(s) = \int_T B(\cdot, s) \, \mathrm{d}\mu$.
  - $\Gamma(s)$ is nonempty for each $s \in S$.
  - $\Gamma(\cdot)$ is uhc (integration preserves uhc).
  - $\Gamma(\cdot)$ is convex valued (by Lyapunov's theorem).
- Γ has a fixed point s^* (by Kakutani's fixed point theorem).
- There is $f : T \longrightarrow E$ such that $\int_T f \, \mathrm{d}\mu = s^*$ and μ-a.e., $f(t) \in B(t, s^*)$.
- This f is a pure strategy Nash equilibrium of $\mathcal{G}$. ■

Nonexistence of an NE: An Example on Integers

- Let $A = \{0, 1\}$ and $K = [0, 1]$.
  Any $x \in K$ can be interpreted as the weight on action 1.
- For each $t \in \mathbb{N}$, let the payoff function on $A \times K$ is

$$\mathcal{G}(t)(a, x) = a\left(\frac{1}{t} - x\right), \ a \in A.$$

- We will show that this game has no Nash equilibrium.
- Best responses:

$$\text{argmax}_{a \in A}\mathcal{G}(t)(a, x) = \begin{cases} \{0, 1\} & \text{if } x = 1/t \\ 1 & \text{if } x < 1/t \\ 0 & \text{if } x > 1/t. \end{cases}$$

  - $x = 1/t$: $\mathcal{G}(t)(0, x) = \mathcal{G}(t)(1, x) = 0$.
  - $x < 1/t$: $\mathcal{G}(t)(0, x) = 0 < \mathcal{G}(t)(1, x)$.
  - $x > 1/t$: $\mathcal{G}(t)(0, x) = 0 > \mathcal{G}(t)(1, x)$.

Example, contd.

- Best responses:

$$\text{argmax}_{a \in A} \mathcal{G}(t)(a,x) = \begin{cases} \{0,1\} & \text{if } x = 1/t \\ 1 & \text{if } x < 1/t \\ 0 & \text{if } x > 1/t. \end{cases}$$

- Suppose that f from $\mathbb{N}$ to K is a (mixed) Nash equilibrium. Let $x = \int_{\mathbb{N}} f \, \mathrm{d}\mu$.
  - If $x = 0$ then $x < 1/t$ for all $t \in \mathbb{N}$ which implies that $f(t) = 1$ for all t and $\int_{\mathbb{N}} f \, \mathrm{d}\mu = 1$. (contradiction)
  - If $x > 0$ then $x > 1/t$ for almost all t (since the measure of a finite set is zero), which implies that $f(t) = 0$ for almost all t and $\int_{\mathbb{N}} f \, \mathrm{d}\mu = 0$. (contradiction)

- The game does not have a Nash equilibrium in pure or mixed strategies.

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- Since μ is not countably additive, there is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that

$$\cup_{n=1}^{\infty} B_n = T \text{ and } \lim_{n\to\infty} \mu(B_n) = c < 1.$$

- For $n \in \mathbb{N}$, let $C_1 = B_1$ and for $n \geq 2$, $C_n = B_n \setminus B_{n-1}$.
- $\{C_n\}$ is a sequence of pairwise disjoint sets and $\cup_{n=1}^{\infty} C_n = T$.
- $A = \{0, 1\}$, $K = [0, 1]$. For each $t \in C_n$, let

$$\mathcal{G}(t)(a, x) = a(\ell_n - x), \qquad \text{where} \qquad \ell_n = c + \frac{1-c}{n}.$$

- Note that $\ell_1 = 1$, $\ell_n > c$ for each n and $\{\ell_n\}$ is a monotonically decreasing sequence converging to c.

The Example, contd.

- $\mathcal{G}(t)(a,x) = a(\ell_n - x)$. $\ell_n = c + [(1-c)/n]$.
- Best responses:

$$\text{argmax}_{a \in A} \mathcal{G}(t)(a,x) = \begin{cases} \{0,1\} & \text{if } x = \ell_n \\ 1 & \text{if } x < \ell_n \\ 0 & \text{if } x > \ell_n. \end{cases}$$

- Let $f : T \longrightarrow [0,1]$ be a mixed strategy Nash equilibrium and $x = \int_T f \, d\mu$.
  - Suppose that $x \leq c < 1$. Then $x < \ell_n$ for all n.
    For all $t \in T$, $f(t) = 1$, i.e., $x = 1$. (contradiction)
  - Now suppose that $x > c$.
    There is a unique $n_0 \in \mathbb{N}$ such that $\ell_{n_0+1} < x \leq \ell_{n_0}$.
    If $n \geq n_0 + 1$ and $t \in C_n$ then $f(t) = 0$.
    So, $x = \int_T f \, d\mu \leq \sum_{i=1}^{n_0} \mu(C_i) = \mu(B_{n_0}) \leq c$. (contradiction)
- The game does not have a Nash equilibrium in pure or mixed strategies.

Idealized Limits: Economies

Definition

A measurable mapping $\alpha^m : T \longrightarrow \{1, \dots, m\}$ is a *replication function* if $\mu(\alpha^m)^{-1}(\{i\}) = 1/m$ for $i = 1, \dots, m$.

Definition

An economy $\mathcal{E}$ on an atomless finitely additive measure space $(T, \mathcal{T}, \mu)$ is said to have the *idealized limit property* if

(1) for any sequence $\{\mathcal{E}^n\}_{n=1}^{\infty}$ of finite-agent economies with $\{f_n\}_{n=1}^{\infty}$ as competitive allocations, where the number of agents in $\mathcal{E}^n$ is k_n and $\lim_{n\to\infty} k_n = \infty$,

(2) for any sequence of replication functions $\{\alpha^{k_n}\}_{n=1}^{\infty}$ such that $\mathcal{E}^n \circ \alpha^{k_n}$ converges to $\mathcal{E}$ pointwise on T, $f^n \circ \alpha^{k_n}$ converges to some allocation f pointwise on T, and $\lim_{n\to\infty} \int_T \omega^n \circ \alpha^{k_n} \, d\mu = \int_T \omega \, d\mu$,

then f is a competitive allocation of $\mathcal{E}$.

Example: No Idealized Limit

- Consider the economy $\mathcal{E} = (u, \omega)$, for each $t \in \mathbb{N}$,

$$u_t(x_1, x_2) = \frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2, \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2} \right),$$

  where $0 \leq c < 1$.
- Fix any $n \in \mathbb{N}$. Let $\mathcal{E}^n$ be the restriction of $\mathcal{E}$ on $\{1, \ldots, n\}$.
- Since $\mathcal{E}^n$ is a finite economy with concave and strictly increasing utility functions, there exists a competitive equilibrium f^n.

Example: No Idealized Limit

- Consider the economy $\mathcal{E} = (u, \omega)$, for each $t \in \mathbb{N}$,

$$u_t(x_1, x_2) = \frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2, \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2}\right),$$

  where $0 \leq c < 1$.
- Fix any $n \in \mathbb{N}$. Let $\mathcal{E}^n$ be the restriction of $\mathcal{E}$ on $\{1, \ldots, n\}$.
- Since $\mathcal{E}^n$ is a finite economy with concave and strictly increasing utility functions, there exists a competitive equilibrium f^n.
- Let $\{A_k^n\}_{k=1}^n$ be a partition of $\mathbb{N}$ such that $A_k^n = \{mn + k : m = 0, 1, \ldots\}$.
- Let $\alpha^n(t) = k$ for any $t \in A_k^n$, where $k = 1, \ldots, n$.
- Note that for any $n \geq t$, $t \in A_t^n$. Then $u_{\alpha^n(t)} = u_t$ for any $n \geq t$, which implies $\mathcal{E}^n \circ \alpha^n$ converges to $\mathcal{E}$ pointwise.
- Moreover, $f^n \circ \alpha^n$ converges pointwise and $\lim_{n\to\infty} \int_T \omega^n \circ \alpha^n \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$.
- However, the limit economy $\mathcal{E} = (u, \omega)$ has no competitive equilibrium, which implies $\mathcal{E} = (u, \omega)$ does not have the idealized limit property.

Idealized Limits: Games

Definition

A game $\mathcal{G}$ on an atomless finitely additive measure space $(T, \mathcal{T}, \mu)$ is said to have the *idealized limit property* if

(1) for any sequence $\{\mathcal{G}^n\}_{n=1}^{\infty}$ of finite-agent games with $\{f_n\}_{n=1}^{\infty}$ as pure strategy Nash equilibria, where the number of agents in $\mathcal{G}^n$ is k_n and $\lim_{n\to\infty} k_n = \infty$,

(2) for any sequence of replication functions $\{\alpha^{k_n}\}_{n=1}^{\infty}$ such that $\mathcal{G}^n \circ \alpha^{k_n}$ converges to $\mathcal{G}$ pointwise on T, and $f^n \circ \alpha^{k_n}$ converges to some pure strategy profile f pointwise on T,

then f is a pure strategy Nash equilibrium of $\mathcal{G}$.

The next example shows that the idealized limit property may fail for a game with countably many agents.

Example: No Idealized Limit

- Consider the game $\mathcal{G}$, for $t \in \mathbb{N}$, $\mathcal{G}(t)(a,x) = a\left[(1/t) - x\right]$
- Fix any $n \in \mathbb{N}$. Let $\mathcal{G}^n$ be the restriction of $\mathcal{G}$ on $\{1, \ldots, n^2\}$.
- Let $\{A_i^n\}_{i=1}^{n^2}$ be an partition of $\mathbb{N}$ such that
$$A_i^n = \{mn^2 + i : m = 0, 1, \ldots\}.$$
- Let $\alpha^{n^2}(t) = k$ for any $t \in A_k^n$, where $k = 1, \ldots, n^2$.
Note that for any $n \geq \sqrt{t}$, $\alpha^{n^2}(t) = t$.
- Then $\mathcal{G}^n \circ \alpha^{n^2}(t) = \mathcal{G}(t)$ for any $n \geq \sqrt{t}$, which implies $\mathcal{G}^n \circ \alpha^{n^2}$ converges to $\mathcal{G}$ pointwise on T.

Example: No Idealized Limit

- Consider the game $\mathcal{G}$, for $t \in \mathbb{N}$, $\mathcal{G}(t)(a,x) = a\,[(1/t) - x]$
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$$A_i^n = \{mn^2 + i : m = 0, 1, \ldots\}.$$
- Let $\alpha^{n^2}(t) = k$ for any $t \in A_k^n$, where $k = 1, \ldots, n^2$.
Note that for any $n \geq \sqrt{t}$, $\alpha^{n^2}(t) = t$.
- Then $\mathcal{G}^n \circ \alpha^{n^2}(t) = \mathcal{G}(t)$ for any $n \geq \sqrt{t}$, which implies $\mathcal{G}^n \circ \alpha^{n^2}$ converges to $\mathcal{G}$ pointwise on T.
- Fix any $n \geq 2$. Let
$$f^n(i) = \begin{cases} 1 & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases}$$
Then f^n is a Nash equilibrium of $\mathcal{G}^n$.
- Fix any $t \in \mathbb{N}$. For any $n \geq t$, $\alpha^{n^2}(t) = t$, which implies $f^n \circ \alpha^{n^2}(t) = 1$.
Then $f^n \circ \alpha^{n^2}(t) \to 1$ as $n \to \infty$.
- However, the limit game $\mathcal{G}$ has no mixed strategy Nash equilibrium.

Necessity of Countable Additivity: Economies

We have seen failures of both existence and the idealized limit property for competitive equilibria in economies over a finitely additive measure space. The next theorem shows the equivalence of countable additivity of the agent space with the validity of each of the properties.

Theorem

Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space. Assume that all the preferences are strongly monotone. Then the following statements hold.

(i) *Every economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ has a competitive equilibrium if and only if μ is countably additive.*

(ii) *Every economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ has the idealized limit property if and only if μ is countably additive.*

CA $\Rightarrow$ Existence: Aumann (1966). Existence $\Rightarrow$ CA: Earlier example.

CA $\Rightarrow$ ILP: Proof in the paper. (Follows Hildenbrand (1974))
ILP $\Rightarrow$ CA: Earlier example on $\mathbb{N}$ can be modified to any T.

Necessity of Countable Additivity: Games

We have seen failures of both existence and the idealized limit property for Nash equilibria in games over a finitely additive measure space.

The next theorem shows the equivalence of countable additivity of the agent space with the validity of each of the properties.

Theorem

Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space. Then the following statements hold.

(i) *Every game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ has a pure strategy Nash equilibrium if and only if μ is countably additive.*

(ii) *Every game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ has idealized limit property if and only if μ is countably additive.*

CA $\Rightarrow$ Existence: Schmeidler (1973). Existence $\Rightarrow$ CA: Earlier example.

CA $\Rightarrow$ ILP: Proof in the paper.

ILP $\Rightarrow$ CA: Earlier example on $\mathbb{N}$ can be modified to any T.

Approximate Competitive Equilibria

Earlier, we have seen examples that an economy may not have a competitive equilibrium. It is natural to ask if approximate competitive equilibria exist.

Definition

Let $\mathcal{E}$ be an economy on $(T, \mathcal{T}, \mu)$ and $\epsilon > 0$. (p, f) is an *ϵ-competitive equilibrium* of $\mathcal{E}$ if $p \in \mathbb{R}^L_+ \setminus \{0\}$, f is a feasible allocation, $f(t) \in B_t(p)$ for almost all t and there exists $T_\epsilon \in \mathcal{T}$ such that:

(a) $\mu(T_\epsilon) \leq \epsilon$ and

(b) for almost all $t \in T^c_\epsilon$, $u_t(f(t)) \geq u_t(y) - \epsilon$ for any $y \in B_t(p)$.

In general, an ϵ-competitive equilibrium may not exist, as shown by the next Example.

Nonexistence of Approximate Competitive Equilibria

- The economy is on $\mathbb{N}$.
- The utility function and endowment of $t \in \mathbb{N}$ is,

$$u_t(x_1, x_2) = e^t \left[\frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2 \right], \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2} \right),$$

where $0 \leq c < 1/3$.

- This economy does not have an ϵ-competitive equilibrium if $0 < \epsilon \leq 1/3$.

Existence of Approximate Competitive Equilibria

Definition

An economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ is *tight* if for any $\epsilon > 0$, there exists $\bar{T} \subseteq T$ such that

(a) $\mu(\bar{T}) < \epsilon$ and

(b) $\mathcal{E}(T \setminus \bar{T})$ is a relatively compact subset of $\mathcal{U} \times \mathbb{R}^L +$.

Proposition

If an economy is $\mathcal{E}$ is tight, then it has an ϵ-competitive equilibrium for every $\epsilon > 0$.

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Proposition

If an economy is $\mathcal{E}$ is tight, then it has an ϵ-competitive equilibrium for every $\epsilon > 0$.

- The existence of an ϵ-competitive equilibrium for every $\epsilon > 0$ does not imply that there is a competitive equilibrium. We demonstrate this by means of an earlier example.

Approximate Competitive Equilibria in an Example

- Take $c = 0$ in the first example. The (tight) economy is

$$u_t(x_1, x_2) = \frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2, \qquad \omega_t = \left(\frac{1}{2}, \frac{1}{2}\right).$$

- If $p \gg 0$ and $p_1 + p_2 = 1$, then the demand functions are

$$D_{t1} = \min\left\{\frac{p_2^{t+1}}{p_1^{t+1}}, \frac{1}{2p_1}\right\}, \qquad D_{t2} = \frac{1}{2p_2} - \frac{p_1 D_{t1}}{p_2}.$$

- Let $p = (1/2, 1/2)$ and $f(t) = (1/2, 1/2) = \omega_t$.
  For any $\epsilon > 0$, (p, f) is an ϵ-competitive equilibrium.
- $D_{t1} = 1$ and $D_{t2} = 0$. The maximized utility is $(t+1)/t$.
  For each t, $f(t)$ is in the budget set and f is a feasible allocation.
- We will show that for any $\epsilon > 0$, and for almost all t,

$$\frac{t+1}{t}\left(\frac{1}{2}\right)^{\frac{t}{t+1}} + \frac{1}{2} > \frac{t+1}{t} - \epsilon, \qquad \epsilon > \frac{t+1}{t} - \frac{t+1}{t}\left(\frac{1}{2}\right)^{\frac{t}{t+1}} - \frac{1}{2}.$$

- As t tends to infinity, the RHS tends to zero. So, given $\epsilon > 0$,
  there exists $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$, the above inequality holds.

Approximate Nash Equilibria

Earlier, we have seen examples that a game may not have a Nash equilibrium. It is natural to ask whether approximate Nash equilibria exist.

Definition

Let $\mathcal{G}$ be a game on $(T, \mathcal{T}, \mu)$ and $\epsilon > 0$. A strategy profile $f : T \longrightarrow S$ is an *ϵ-Nash equilibrium* of $\mathcal{G}$ if there exists $T_\epsilon \in \mathcal{T}$ such that

(a) $\mu(T_\epsilon) \leq \epsilon$ and

(b) for almost all $t \in T_\epsilon^c$, $\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(y, \int_T f \, \mathrm{d}\mu\right) - \epsilon$ for any $y \in S$.

In general, an ϵ-Nash equilibrium may not exist, as shown by the next Example.

Nonexistence of Approximate Nash Equilibria

- The game is on $\mathbb{N}$, with $A = \{0,1\}$ and $K = [0,1]$.
- For each player $t \in \mathbb{N}$, the payoff function is $\mathcal{G}(t)(0,x) = 0$ and

$$\mathcal{G}(t)(1,x) = \begin{cases} 1 + 2^{t-1}(1-2x) & \text{if } -1 \leq 1 + 2^{t-1}(1-2x) \leq 1 \\ 1 & \text{if } 1 + 2^{t-1}(1-2x) > 1 \\ -1 & \text{if } 1 + 2^{t-1}(1-2x) < -1. \end{cases}$$

- The best responses are:

$$\text{argmax}_{a \in A} \mathcal{G}(t)(a,x) = \begin{cases} \{0,1\} & \text{if } x = (1/2) + 2^{-t} \\ 1 & \text{if } x < (1/2) + 2^{-t} \\ 0 & \text{if } x > (1/2) + 2^{-t}. \end{cases}$$

- This game does not have an ϵ-Nash equilibrium if $0 < \epsilon \leq 1/4$.

Nonexistence of Approximate NE: Case 1

- Best responses:

$$\text{argmax}_{a \in A} \mathcal{G}(t)(a, x) = \begin{cases} \{0, 1\} & \text{if } x = (1/2) + 2^{-t} \\ 1 & \text{if } x < (1/2) + 2^{-t} \\ 0 & \text{if } x > (1/2) + 2^{-t}. \end{cases}$$

- Let $0 < \epsilon \leq 1/4$ and suppose that f from $\mathbb{N}$ to $[0, 1]$ is an ϵ-equilibrium. Then there exists $I_\epsilon \subseteq \mathbb{N}$ such that, $\mu(I_\epsilon) \leq \epsilon$ and for any $t \in I_\epsilon^c$,

$$\mathcal{G}(t)(f(t), x) \geq \max\{\mathcal{G}(t)(0, x), \mathcal{G}(t)(1, x)\} - \epsilon,$$

where $x = \int_T f \, d\mu$.

Nonexistence of Approximate NE: Case 1

- Best responses:

$$\operatorname{argmax}_{a\in A}\mathcal{G}(t)(a,x)=\begin{cases}\{0,1\} & \text{if } x=(1/2)+2^{-t}\\ 1 & \text{if } x<(1/2)+2^{-t}\\ 0 & \text{if } x>(1/2)+2^{-t}.\end{cases}$$

- Let $0<\epsilon\le 1/4$ and suppose that f from $\mathbb{N}$ to $[0,1]$ is an ϵ-equilibrium. Then there exists $I_\epsilon \subseteq \mathbb{N}$ such that, $\mu(I_\epsilon)\le\epsilon$ and for any $t\in I_\epsilon^c$,

$$\mathcal{G}(t)(f(t),x)\ge \max\{\mathcal{G}(t)(0,x),\mathcal{G}(t)(1,x)\}-\epsilon,$$

where $x=\int_T f\,\mathrm{d}\mu$.

- Case 1. $x\le 1/2$. For all $t\in\mathbb{N}$, $1+2^{t-1}(1-2x)\ge 1$. $\mathcal{G}(t)(1,x)=1>\mathcal{G}(t)(0,x)$. Therefore, for any $t\in I_\epsilon^c$, $\mathcal{G}(t)(f(t),x)\ge 1-\epsilon$, which means $f(t)\ge 1-\epsilon$.

$$x=\int_{I_\epsilon} f\,\mathrm{d}\mu+\int_{I_\epsilon^c} f\,\mathrm{d}\mu\ge\int_{I_\epsilon^c} f\,\mathrm{d}\mu\ge(1-\epsilon)^2>\frac{1}{2}.$$ contradiction

Nonexistence of Approximate NE: Case 2

- Best responses:

$$\text{argmax}_{a\in A}\mathcal{G}(t)(a,x) = \begin{cases} \{0,1\} & \text{if } x = (1/2) + 2^{-t} \\ 1 & \text{if } x < (1/2) + 2^{-t} \\ 0 & \text{if } x > (1/2) + 2^{-t}. \end{cases}$$

- $0 < \epsilon \leq 1/4$. There is $I_\epsilon \subseteq \mathbb{N}$ such that, $\mu(I_\epsilon) \leq \epsilon$ and for any $t \in I_\epsilon^c$,

$$\mathcal{G}(t)(f(t),x) \geq \max\{\mathcal{G}(t)(0,x), \mathcal{G}(t)(1,x)\} - \epsilon,$$

where $x = \int_T f \, \mathrm{d}\mu$.

Nonexistence of Approximate NE: Case 2

- Best responses:

$$\operatorname{argmax}_{a\in A}\mathcal{G}(t)(a,x)=\begin{cases}\{0,1\} & \text{if } x=(1/2)+2^{-t}\\ 1 & \text{if } x<(1/2)+2^{-t}\\ 0 & \text{if } x>(1/2)+2^{-t}.\end{cases}$$

- $0<\epsilon\le 1/4$. There is $I_\epsilon\subseteq\mathbb{N}$ such that, $\mu(I_\epsilon)\le\epsilon$ and for any $t\in I_\epsilon^c$,

$$\mathcal{G}(t)(f(t),x)\ge\max\{\mathcal{G}(t)(0,x),\mathcal{G}(t)(1,x)\}-\epsilon,$$

where $x=\int_T f\,\mathrm{d}\mu$.

- Case 2. $x>1/2$. For almost all $t\in I_\epsilon^c$, $1+2^{t-1}(1-2x)<-1$. $\mathcal{G}(t)(0,x)>\mathcal{G}(t)(1,x)=-1$. Therefore, for all $t\in I_\epsilon^c$, $\mathcal{G}(t)(f(t),x)\ge-\epsilon$, which means $f(t)\le\epsilon$.

$$x=\int_{I_\epsilon} f\,\mathrm{d}\mu+\int_{I_\epsilon^c} f\,\mathrm{d}\mu\le\mu(I_\epsilon)+\epsilon(1-\epsilon)\le\epsilon+\epsilon\le\frac{1}{2}.$$ contradiction

Existence of Approximate Nash Equilibria

Definition

A game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ is *tight* if for any $\epsilon > 0$, there exists $\bar{T} \subseteq T$ such that

(a) $\mu(\bar{T}) < \epsilon$ and

(b) $\mathcal{G}(T \setminus \bar{T})$ is a relatively compact subset of $\mathcal{V}$.

Proposition

If a game is $\mathcal{G}$ is tight, then it has a pure strategy ϵ-Nash equilibrium for every $\epsilon > 0$.

Existence of Approximate Nash Equilibria

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A game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ is *tight* if for any $\epsilon > 0$, there exists $\bar{T} \subseteq T$ such that

(a) $\mu(\bar{T}) < \epsilon$ and

(b) $\mathcal{G}(T \backslash \bar{T})$ is a relatively compact subset of $\mathcal{V}$.

Proposition

If a game is $\mathcal{G}$ is tight, then it has a pure strategy ϵ-Nash equilibrium for every $\epsilon > 0$.

- The existence of an ϵ-Nash equilibrium for every $\epsilon > 0$ does not ensure the existence of an NE. Example: $\mathcal{G}(t)(a, x) = a[(1/t) - x]$ on $\mathbb{N}$.
- The game is tight. It has an ϵ-Nash equilibrium for every $\epsilon > 0$.
- Explicitly, $f(t) = 0$ for all $t \in \mathbb{N}$ is an ϵ-Nash equilibrium.

  $\mathcal{G}(t)(0, 0) = 0$, $\mathcal{G}(t)(1, 0) = 1/t$, $0 \geq (1/t) - \epsilon$ for almost all t.
- However, as has been shown, the game does not have a Nash equilibrium.

Summary of Results

- Negative results on finitely additive spaces.
  - An economy may not have a competitive equilibrium.
    (Two examples)
  - A game may not have a Nash equilibrium.
    (Two examples)
  - An economy may not have the idealized limit property.
  - A game may not have the idealized limit property.
- Consequences.
  - Necessity of countably additivity for economies:
    both existence and idealized limit property hold.
  - Necessity of countably additivity for games:
    both existence and idealized limit property hold.
- Approximate equilibria on finitely additive spaces.
  - An economy may not have an approximate competitive equilibrium.
    A tightness assumption is sufficient for existence.
  - A game may not have an approximate Nash equilibrium.
    A tightness assumption is sufficient for existence.