Modeling Infinitely Many Agents: Why Countable Additivity Is Necessary

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Background and Motivation

- Finite agent economies and games: Arrow-Debreu (1954), McKenzie (1954), Nash (1951).
- Economies and games with a continuum of agents: Aumann (1964, 1966), Vind (1964), Milnor-Shapley (1961), Schmeidler (1973).
- Many macro economics papers assume infinite agents with mass 1.
- Modeling many agents:
  - Replication/Large finite approximations: Edgeworth (1881), Debreu-Scarf (1963), Anderson (1978).
  - Continuum models with an atomless measure: Milnor-Shapley (1961), Aumann (1964), Schmeidler (1973), Hildenbrand (1974), Khan-Sun (2002).
  - Infinitesimals, Loeb spaces: Brown-Robinson (1972, 1975), Khan (1974), Brown-Loeb (1976), Khan-Sun (1996, 1999).
  - Finitely additive economies: Armstrong-Richter (1984, 1986), Weiss (1981), Feldman-Gilles (1985), Basile (1993).

Mathematical Preliminaries

- Let T be a nonempty set and $\mathcal{T}$ a σ-algebra of subsets of T,
  (*i*) $T \in \mathcal{T}$, (*ii*) $A \in \mathcal{T}$ implies $A^c \in \mathcal{T}$,
  (*iii*) $A_n \in \mathcal{T}$ $(n = 1, 2 \ldots)$ implies $\cup_{n=1}^{\infty} A_n \in \mathcal{T}$.
- Let μ be a set function from $\mathcal{T}$ to $[0, 1]$ with $\mu(T) = 1$.
  - μ is a finitely additive measure on $\mathcal{T}$ if for any $A, B \in \mathcal{T}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$.
  - μ is a countably additive measure on $\mathcal{T}$ if for any sequence $\{A_n\}$ of pairwise disjoint sets in $\mathcal{T}$, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.
- The triple $(T, \mathcal{T}, \mu)$ will be called a (finitely additive/countably additive) measure space.

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- The triple $(T, \mathcal{T}, \mu)$ will be called a (finitely additive/countably additive) measure space.

- A measure μ is atomless if for every $\epsilon > 0$, there exists a $\mathcal{T}$-measurable partition $\{F_1, \ldots, F_n\}$ of T such that $\mu(F_i) < \epsilon$ for every i.
- Let $\mathbb{N}$ be the set of positive integers and $\mathcal{P}(\mathbb{N})$ its power set. There are finitely additive, atomless measures on $\mathcal{P}(\mathbb{N})$ (such as a density charge).

Preview of the Results

- Negative results on finitely additive spaces.
  - An economy may not have a competitive equilibrium.
    (Two examples)
  - A game may not have a Nash equilibrium.
    (Two examples)
  - An economy may not have the idealized limit property.
  - A game may not have the idealized limit property.
- Consequences.
  - Necessity of countably additivity for economies:
    both existence and idealized limit property hold.
  - Necessity of countably additivity for games:
    both existence and idealized limit property hold.
- Approximate equilibria on finitely additive spaces.
  - An economy may not have an approximate competitive equilibrium.
    A tightness assumption is sufficient for existence.
  - A game may not have an approximate Nash equilibrium.
    A tightness assumption is sufficient for existence.

Economies and Competitive Equilibria

- There are L goods and the commodity space is $\mathbb{R}_+^L$.
- Let $\mathcal{U}$ denote the class of real valued, continuous utility functions on $\mathbb{R}_+^L$ (endowed with the compact open topology).
- A $u \in \mathcal{U}$ is *strongly monotone* if $x \geq y$, $x \neq y$ implies that $u(x) > u(y)$.
- Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space. (space of agents)

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- Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space. (space of agents)
- An *economy* is a measurable mapping $\mathcal{E} = (u, \omega) : T \longrightarrow \mathcal{U} \times \mathbb{R}^L_+$ such that ω is integrable and $\bar{\omega} = \int_T \omega \, \mathrm{d}\mu \gg 0$.
- An *allocation* of $\mathcal{E}$ is an integrable mapping f from T to $\mathbb{R}^L_+$. An allocation is *feasible* if $\int_T f \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$.
- Given a price vector $p \in \mathbb{R}^L_+$, the *budget set* of consumer t is
$$B_t(p) = \{x \in \mathbb{R}^L_+ : p \cdot x \leq p \cdot \omega_t\}.$$
- A *competitive equilibrium* of $\mathcal{E}$ is a pair (p, f), where $p \in \mathbb{R}^L_+ \setminus \{0\}$, f is a feasible allocation and μ-a.e.;
$$(a)\ f(t) \in B_t(p) \quad \text{and} \quad (b)\ u_t(f(t)) \geq u_t(x) \text{ for all } x \in B_t(p).$$
- An allocation f of $\mathcal{E}$ is a *competitive allocation* if for some p, (p, f) is a competitive equilibrium.

Nonexistence of a CE: An Example on Integers

- The measure space is $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$. $0 \le c < 1$. Economy $\mathcal{E}$: for $t \in \mathbb{N}$,

$$u_t(x_1, x_2) = \frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2, \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2} \right),$$

- Equilibrium prices: $p_1 + p_2 = 1$, $p \gg 0$.
- For any $t \in \mathbb{N}$, the demand function are:

$$D_{t1} = \min \left\{ \frac{p_2^{t+1}}{p_1^{t+1}}, \frac{c+1}{2p_1} \right\}, \qquad D_{t2} = \frac{c+1}{2p_2} - \frac{p_1 D_{t1}}{p_2}.$$

- Case 1. $p_2/p_1 < 1$. $\lim_{t\to\infty} D_{t1} = 0$. $\int_{\mathbb{N}} D_{t1}\, d\mu = 0$.

$$\int_{\mathbb{N}} D_{t2}\, d\mu = \frac{c+1}{2p_2} > \frac{c+1}{2} = \int_{\mathbb{N}} \omega_{t2}\, d\mu. \qquad \text{(contradiction)}$$

- Case 2. $p_2/p_1 \ge 1$. $p_2^{t+1}/p_1^{t+1} \ge 1$. $(c+1)/(2p_1) \ge c+1$.

Therefore, $D_{t1} \ge \min\{1, c+1\} = 1$.

$$\int_{\mathbb{N}} D_{t1}\, d\mu \ge 1 > \frac{c+1}{2} = \int_{\mathbb{N}} \omega_{t1}\, d\mu. \qquad \text{(contradiction)}$$

Nonexistence of a CE on General Measure Spaces

Claim

Let $(T, \mathcal{T}, \mu)$ be an atomless finitely additive measure space. Assume that μ is not countably additive. Then there is an economy on $(T, \mathcal{T}, \mu)$ which has no competitive equilibrium.

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- Fact: Let $(T, \mathcal{T}, \mu)$ be a finitely additive probability space. Then the following are equivalent.
  - (i) μ is not countably additive.
  - (ii) There is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that $\cup_{n=1}^{\infty} B_n = T$ and $\lim_{n \to \infty} \mu(B_n) = c < 1$.

- Since μ is not countably additive, there is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that $\cup_{n=1}^{\infty} B_n = T$ and $\lim_{n \to \infty} \mu(B_n) = c < 1$.
- For $n \in \mathbb{N}$, let $C_1 = B_1$ and for $n \geq 2$, $C_n = B_n \setminus B_{n-1}$.
- $\{C_n\}$ is a sequence of pairwise disjoint sets and $\cup_{n=1}^{\infty} C_n = T$.

Nonexistence on General Measure Spaces, contd.

- Preferences and endowments: Let $t \in C_n$.

$$u_t(x_1, x_2) = \frac{n+1}{n} x_1^{\frac{n}{n+1}} + x_2, \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2} \right).$$

- Assume that $p_1, p_2 \in \mathbb{R}^2_+$ is a pair of competitive equilibrium prices. We must have $p_1 > 0$ and $p_2 > 0$ since u_t is strongly monotone for each t.
- Let $p_2 = 1$ and $p_1 > 0$. If $t \in C_n$, then

$$D_{t1} = \min \left\{ \frac{1}{p_1^{n+1}}, \frac{c+1}{2} \left(1 + \frac{1}{p_1} \right) \right\}, \quad D_{t2} = \frac{c+1}{2}(1 + p_1) - p_1 D_{t1}.$$

- To show that there is *no* competitive equilibrium, consider two cases: (i) $1 \geq p_1$ and (ii) $1 < p_1$. Obtain contradictions.

Games and Nash Equilibria

- Let $E = \{e^1, \ldots, e^L\}$ be the set of unit vectors in $\mathbb{R}^L$ and
  $S = \{s \in \mathbb{R}^L_+ : \sum_{k=1}^L s_k = 1\}$ the unit simplex in $\mathbb{R}^L$.
- Let $\mathcal{V}$ be the set of real valued continuous functions defined on $E \times S$, endowed with sup norm.
- $(T, \mathcal{T}, \mu)$ is an atomless, countably/finitely additive probability space.
- A *game* is a measurable function $\mathcal{G} : T \longrightarrow \mathcal{V}$.
- A *pure strategy profile* is a measurable function $f : T \longrightarrow E$.
- A $f : T \longrightarrow E$ is a *pure strategy Nash equilibrium* of $\mathcal{G}$ if μ-a.e.;

$$\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(a, \int_T f \, \mathrm{d}\mu\right) \text{ for all } a \in E.$$

Games and Nash Equilibria, contd.

- Pure strategy profile: $f : T \longrightarrow E$.
- Mixed strategy profile: $g : T \longrightarrow S$.
- Given a mixed strategy profile g and $y \in S$, the *payoff to player t* is

$$\mathcal{G}(t)\left(y, \int_T g \, \mathrm{d}\mu\right) = \sum_{k=1}^{L} y_k \mathcal{G}(t)\left(e^k, \int_T g \, \mathrm{d}\mu\right).$$

- A $g : T \longrightarrow S$ is a *mixed strategy Nash equilibrium* of $\mathcal{G}$ if μ-a.e.;

$$\mathcal{G}(t)\left(g(t), \int_T g \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(y, \int_T g \, \mathrm{d}\mu\right) \text{ for all } y \in S.$$

Nonexistence of an NE: An Example on Integers

- Let $A = \{0, 1\}$ and $K = [0, 1]$. Any $x \in K$ is the weight on action 1.
- The measure space is $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$. For each $t \in \mathbb{N}$,

$$\mathcal{G}(t)(a, x) = a\left(\frac{1}{t} - x\right), \quad a \in A.$$

- Best responses:

$$\text{argmax}_{a \in A} \mathcal{G}(t)(a, x) = \begin{cases} \{0, 1\} & \text{if } x = 1/t \\ 1 & \text{if } x < 1/t \\ 0 & \text{if } x > 1/t. \end{cases}$$

- Suppose that g from $\mathbb{N}$ to K is a (mixed) Nash equilibrium. Let $x = \int_{\mathbb{N}} g \, d\mu$.
  - If $x = 0$ then $x < 1/t$ for all $t \in \mathbb{N}$ which implies that $g(t) = 1$ for all t and $\int_{\mathbb{N}} g \, d\mu = 1$. (contradiction)
  - If $x > 0$ then $x > 1/t$ for almost all t (since the measure of a finite set is zero), which implies that $g(t) = 0$ for almost all t and $\int_{\mathbb{N}} g \, d\mu = 0$. (contradiction)

Nonexistence of an NE on General Measure Spaces

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Let $(T, \mathcal{T}, \mu)$ be an atomless finitely additive measure space. Assume that μ is not countably additive. Then there is a game on $(T, \mathcal{T}, \mu)$ which has no Nash equilibrium.

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Claim

Let $(T, \mathcal{T}, \mu)$ be an atomless finitely additive measure space. Assume that μ is not countably additive. Then there is a game on $(T, \mathcal{T}, \mu)$ which has no Nash equilibrium.

- Since μ is not countably additive, there is an increasing sequence of sets $\{B_n\}$ in $\mathcal{T}$ such that

$$\cup_{n=1}^{\infty} B_n = T \text{ and } \lim_{n\to\infty} \mu(B_n) = c < 1.$$

- For $n \in \mathbb{N}$, let $C_1 = B_1$ and for $n \geq 2$, $C_n = B_n \setminus B_{n-1}$.
- $\{C_n\}$ is a sequence of pairwise disjoint sets and $\cup_{n=1}^{\infty} C_n = T$.
- $A = \{0, 1\}$, $K = [0, 1]$. For each $t \in C_n$, let

$$\mathcal{G}(t)(a, x) = a(\ell_n - x), \qquad \text{where} \qquad \ell_n = c + \frac{1-c}{n}.$$

- Note that $\ell_1 = 1$, $\ell_n > c$ for each n and $\{\ell_n\} \downarrow c$.
- To show that there is no mixed strategy NE, consider two cases: (i) $x \leq c < 1$ and (ii) $x > c$. Obtain contradictions.

Idealized Limits: Economies

Definition

A measurable mapping $\alpha^m : T \longrightarrow \{1, \ldots, m\}$ is a *replication function* if $\mu(\alpha^m)^{-1}(\{i\}) = 1/m$ for $i = 1, \ldots, m$.

Definition

An economy $\mathcal{E}$ on an atomless finitely additive measure space $(T, \mathcal{T}, \mu)$ is said to have the *idealized limit property* if

(1) for any sequence $\{\mathcal{E}^n\}_{n=1}^{\infty}$ of finite-agent economies with $\{f_n\}_{n=1}^{\infty}$ as competitive allocations, where the number of agents in $\mathcal{E}^n$ is k_n and $\lim_{n\to\infty} k_n = \infty$,

(2) for any sequence of replication functions $\{\alpha^{k_n}\}_{n=1}^{\infty}$ such that $\mathcal{E}^n \circ \alpha^{k_n}$ converges to $\mathcal{E}$ pointwise on T, $f_n \circ \alpha^{k_n}$ converges to some allocation f pointwise on T, and $\lim_{n\to\infty} \int_T \omega^n \circ \alpha^{k_n} \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$,

then f is a competitive allocation of $\mathcal{E}$.

Example: No Idealized Limit (Economies)

- Consider the economy $\mathcal{E} = (u, \omega)$. $0 \leq c < 1$. For each $t \in \mathbb{N}$,

$$u_t(x_1, x_2) = \frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2, \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2} \right),$$

- Fix any $n \in \mathbb{N}$. Let $\mathcal{E}^n$ be the restriction of $\mathcal{E}$ on $\{1, \ldots, n\}$.
- Since $\mathcal{E}^n$ is a finite economy with concave and strictly increasing utility functions, there exists a competitive equilibrium f_n.

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- Fix any $n \in \mathbb{N}$. Let $\mathcal{E}^n$ be the restriction of $\mathcal{E}$ on $\{1, \ldots, n\}$.
- Since $\mathcal{E}^n$ is a finite economy with concave and strictly increasing utility functions, there exists a competitive equilibrium f_n.
- Let $\{A_k^n\}_{k=1}^n$ be a partition of $\mathbb{N}$ such that $A_k^n = \{mn + k : m = 0, 1, \ldots\}$.
- Let $\alpha^n(t) = k$ for any $t \in A_k^n$, where $k = 1, \ldots, n$.
- Note that for any $n \ge t$, $t \in A_t^n$. Then $u_{\alpha^n(t)} = u_t$ for any $n \ge t$, which implies $\mathcal{E}^n \circ \alpha^n$ converges to $\mathcal{E}$ pointwise.
- Moreover, $f_n \circ \alpha^n$ converges pointwise and $\lim_{n \to \infty} \int_T \omega^n \circ \alpha^n \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$.
- However, the limit economy $\mathcal{E}$ has no competitive equilibrium, which implies $\mathcal{E}$ does not have the idealized limit property.

Idealized Limits: Games

Definition

A game $\mathcal{G}$ on an atomless finitely additive measure space $(T, \mathcal{T}, \mu)$ is said to have the *idealized limit property* if

(1) for any sequence $\{\mathcal{G}^n\}_{n=1}^{\infty}$ of finite-agent games with $\{g_n\}_{n=1}^{\infty}$ as mixed strategy Nash equilibria, where the number of agents in $\mathcal{G}^n$ is k_n and $\lim_{n\to\infty} k_n = \infty$,

(2) for any sequence of replication functions $\{\alpha^{k_n}\}_{n=1}^{\infty}$ such that $\mathcal{G}^n \circ \alpha^{k_n}$ converges to $\mathcal{G}$ pointwise on T, and $g_n \circ \alpha^{k_n}$ converges to some mixed strategy profile g pointwise on T,

then g is a mixed strategy Nash equilibrium of $\mathcal{G}$.

The next example shows that the idealized limit property may fail for a game with countably many agents.

Example: No Idealized Limit (Games)

- Consider the game $\mathcal{G}$, for $t \in \mathbb{N}$, $\mathcal{G}(t)(a,x) = a\left[(1/t) - x\right]$
- Fix any $n \in \mathbb{N}$. Let $\mathcal{G}^n$ be the restriction of $\mathcal{G}$ on $\{1, \ldots, n^2\}$.
- Let $\{A_i^n\}_{i=1}^{n^2}$ be a partition of $\mathbb{N}$ such that
$$A_i^n = \{mn^2 + i : m = 0, 1, \ldots\}.$$
- Let $\alpha^{n^2}(t) = k$ for any $t \in A_k^n$, where $k = 1, \ldots, n^2$.
Note that for any $n \geq \sqrt{t}$, $\alpha^{n^2}(t) = t$.
- Then $\mathcal{G}^n \circ \alpha^{n^2}(t) = \mathcal{G}(t)$ for any $n \geq \sqrt{t}$, which implies $\mathcal{G}^n \circ \alpha^{n^2}$ converges to $\mathcal{G}$ pointwise on T.

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Note that for any $n \geq \sqrt{t}$, $\alpha^{n^2}(t) = t$.
- Then $\mathcal{G}^n \circ \alpha^{n^2}(t) = \mathcal{G}(t)$ for any $n \geq \sqrt{t}$, which implies $\mathcal{G}^n \circ \alpha^{n^2}$ converges to $\mathcal{G}$ pointwise on T.
- Fix any $n \geq 2$. Let
$$g_n(i) = \begin{cases} 1 & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases}$$
Then g_n is a Nash equilibrium of $\mathcal{G}^n$.
- Fix any $t \in \mathbb{N}$. For any $n \geq t$, $\alpha^{n^2}(t) = t$, which implies $g_n \circ \alpha^{n^2}(t) = 1$.
Then $g_n \circ \alpha^{n^2}(t) \to 1$ as $n \to \infty$.
- However, the limit game $\mathcal{G}$ has no mixed strategy Nash equilibrium.

Necessity of Countable Additivity: Economies

We have seen failures of both existence and the idealized limit property for competitive equilibria in economies over a finitely additive measure space.
The next theorem shows the equivalence of countable additivity of the agent space with the validity of each of the properties.

Theorem

Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space. Assume that all the preferences are strongly monotone. Then the following statements hold.

(i) *Every economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ has a competitive equilibrium if and only if μ is countably additive.*

(ii) *Every economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ has the idealized limit property if and only if μ is countably additive.*

CA $\Rightarrow$ Existence: Aumann (1966). Existence $\Rightarrow$ CA: Earlier example.

CA $\Rightarrow$ ILP: Proof in the paper. (Follows Hildenbrand (1974))
ILP $\Rightarrow$ CA: Earlier example on $\mathbb{N}$ can be modified to any T.

Necessity of Countable Additivity: Games

We have seen failures of both existence and the idealized limit property for Nash equilibria in games over a finitely additive measure space.

The next theorem shows the equivalence of countable additivity of the agent space with the validity of each of the properties.

Theorem

Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space. Then the following statements hold.

(i) *Every game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ has a pure strategy Nash equilibrium if and only if μ is countably additive.*

(ii) *Every game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ has idealized limit property if and only if μ is countably additive.*

CA $\Rightarrow$ Existence: Schmeidler (1973).

Existence $\Rightarrow$ CA: Earlier example.

CA $\Rightarrow$ ILP: Proof in the paper.

ILP $\Rightarrow$ CA: Earlier example on $\mathbb{N}$ can be modified to any T.

Approximate Competitive Equilibria

Earlier, we have seen examples that an economy may not have a competitive equilibrium. It is natural to ask if approximate competitive equilibria exist.

Definition

Let $\mathcal{E}$ be an economy on $(T, \mathcal{T}, \mu)$ and $\epsilon > 0$. (p, f) is an *ϵ-competitive equilibrium* of $\mathcal{E}$ if $p \in \mathbb{R}^L_+ \setminus \{0\}$, f is a feasible allocation, $f(t) \in B_t(p)$ for almost all t and there exists $T_\epsilon \in \mathcal{T}$ such that:

(*a*) $\mu(T_\epsilon) \leq \epsilon$ and

(*b*) for almost all $t \in T^c_\epsilon$, $u_t(f(t)) \geq u_t(y) - \epsilon$ for any $y \in B_t(p)$.

In general, an ϵ-competitive equilibrium may not exist, as shown by the next Example.

Nonexistence of Approximate Competitive Equilibria

- The economy is on $\mathbb{N}$.
- The utility function and endowment of $t \in \mathbb{N}$ is,

$$u_t(x_1, x_2) = e^t \left[\frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2 \right], \qquad \omega_t = \left(\frac{c+1}{2}, \frac{c+1}{2} \right),$$

  where $0 \leq c < 1/3$.

- This economy does not have an ϵ-competitive equilibrium if $0 < \epsilon \leq 1/3$.

Existence of Approximate Competitive Equilibria

Definition

An economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ is *tight* if for any $\epsilon > 0$, there exists $\bar{T} \subseteq T$ such that

(a) $\mu(\bar{T}) < \epsilon$ and

(b) $\mathcal{E}(T \backslash \bar{T})$ is a relatively compact subset of $\mathcal{U} \times \mathbb{R}_+^L$.

Proposition

If an economy is $\mathcal{E}$ is tight, then it has an ϵ-competitive equilibrium for every $\epsilon > 0$.

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Proposition

If an economy is $\mathcal{E}$ is tight, then it has an ϵ-competitive equilibrium for every $\epsilon > 0$.

- The existence of an ϵ-competitive equilibrium for every $\epsilon > 0$ does not imply that there is a competitive equilibrium. We demonstrate this by means of an earlier example.

Approximate Competitive Equilibria in an Example

- Take $c = 0$ in the first example. The (tight) economy is

$$u_t(x_1, x_2) = \frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2, \qquad \omega_t = \left(\frac{1}{2}, \frac{1}{2}\right).$$

- If $p \gg 0$ and $p_1 + p_2 = 1$, then the demand functions are

$$D_{t1} = \min\left\{\frac{p_2^{t+1}}{p_1^{t+1}}, \frac{1}{2p_1}\right\}, \qquad D_{t2} = \frac{1}{2p_2} - \frac{p_1 D_{t1}}{p_2}.$$

- Let $p = (1/2, 1/2)$ and $f(t) = (1/2, 1/2) = \omega_t$.
  For any $\epsilon > 0$, (p, f) is an ϵ-competitive equilibrium.
- $D_{t1} = 1$ and $D_{t2} = 0$. The maximized utility is $(t+1)/t$.
  For each t, $f(t)$ is in the budget set and f is a feasible allocation.
- We will show that for any $\epsilon > 0$, and for almost all t,

$$\frac{t+1}{t}\left(\frac{1}{2}\right)^{\frac{t}{t+1}} + \frac{1}{2} > \frac{t+1}{t} - \epsilon, \qquad \epsilon > \frac{t+1}{t} - \frac{t+1}{t}\left(\frac{1}{2}\right)^{\frac{t}{t+1}} - \frac{1}{2}.$$

- As t tends to infinity, the RHS tends to zero. So, given $\epsilon > 0$, there exists $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$, the above inequality holds.

Approximate Nash Equilibria

Earlier, we have seen examples that a game may not have a Nash equilibrium. It is natural to ask whether approximate Nash equilibria exist.

Definition

Let $\mathcal{G}$ be a game on $(T, \mathcal{T}, \mu)$ and $\epsilon > 0$. A strategy profile $g : T \longrightarrow S$ is an *ϵ-Nash equilibrium* of $\mathcal{G}$ if there exists $T_\epsilon \in \mathcal{T}$ such that

(a) $\mu(T_\epsilon) \leq \epsilon$ and

(b) for almost all $t \in T_\epsilon^c$, $\mathcal{G}(t)\left(g(t), \int_T g \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(y, \int_T g \, \mathrm{d}\mu\right) - \epsilon$ for any $y \in S$.

In general, an ϵ-Nash equilibrium may not exist, as shown by the next Example.

Nonexistence of Approximate Nash Equilibria

- The game is on $\mathbb{N}$, with $A = \{0, 1\}$ and $K = [0, 1]$.
- For each player $t \in \mathbb{N}$, the payoff function is $\mathcal{G}(t)(0, x) = 0$ and

$$\mathcal{G}(t)(1, x) = \begin{cases} 1 + 2^{t-1}(1 - 2x) & \text{if } -1 \leq 1 + 2^{t-1}(1 - 2x) \leq 1 \\ 1 & \text{if } 1 + 2^{t-1}(1 - 2x) > 1 \\ -1 & \text{if } 1 + 2^{t-1}(1 - 2x) < -1. \end{cases}$$

- The best responses are:

$$\text{argmax}_{a \in A} \mathcal{G}(t)(a, x) = \begin{cases} \{0, 1\} & \text{if } x = (1/2) + 2^{-t} \\ 1 & \text{if } x < (1/2) + 2^{-t} \\ 0 & \text{if } x > (1/2) + 2^{-t}. \end{cases}$$

- This game does not have an ϵ-Nash equilibrium if $0 < \epsilon \leq 1/4$.

Existence of Approximate Nash Equilibria

Definition

A game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ is *tight* if for any $\epsilon > 0$, there exists $\bar{T} \subseteq T$ such that

(a) $\mu(\bar{T}) < \epsilon$ and

(b) $\mathcal{G}(T \backslash \bar{T})$ is a relatively compact subset of $\mathcal{V}$.

Proposition

If a game is $\mathcal{G}$ is tight, then it has a pure strategy ϵ-Nash equilibrium for every $\epsilon > 0$.

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Proposition

If a game is $\mathcal{G}$ is tight, then it has a pure strategy ϵ-Nash equilibrium for every $\epsilon > 0$.

- The existence of an ϵ-Nash equilibrium for every $\epsilon > 0$ does not ensure the existence of an NE. Example: $\mathcal{G}(t)(a, x) = a[(1/t) - x]$ on $\mathbb{N}$.
- The game is tight. It has an ϵ-Nash equilibrium for every $\epsilon > 0$.
- Explicitly, $f(t) = 0$ for all $t \in \mathbb{N}$ is an ϵ-Nash equilibrium.
  $\mathcal{G}(t)(0, 0) = 0$, $\mathcal{G}(t)(1, 0) = 1/t$, $0 \geq (1/t) - \epsilon$ for almost all t.
- However, as has been shown, the game does not have a Nash equilibrium.

Summary of Results

- Negative results on finitely additive spaces.
  - An economy may not have a competitive equilibrium.
    (Two examples)
  - A game may not have a Nash equilibrium.
    (Two examples)
  - An economy may not have the idealized limit property.
  - A game may not have the idealized limit property.
- Consequences.
  - Necessity of countably additivity for economies:
    both existence and idealized limit property hold.
  - Necessity of countably additivity for games:
    both existence and idealized limit property hold.
- Approximate equilibria on finitely additive spaces.
  - An economy may not have an approximate competitive equilibrium.
    A tightness assumption is sufficient for existence.
  - A game may not have an approximate Nash equilibrium.
    A tightness assumption is sufficient for existence.