Approximate Equilibria in Games and Economies over Finitely Additive Measure Spaces

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Background and Motivation

- Finite agent economies and games: Arrow-Debreu (1954), McKenzie (1954), Nash (1951).
- Economies and games with a continuum of agents: Aumann (1964, 1966), Vind (1964), Milnor-Shapley (1961), Schmeidler (1973).
- Many macro economics papers assume infinite agents with mass 1.
- Modeling many agents:
  - Replication/Large finite approximations: Edgeworth (1881), Debreu-Scarf (1963), Anderson (1978).
  - Continuum models with an atomless measure: Milnor-Shapley (1961), Aumann (1964), Schmeidler (1973), Hildenbrand (1974), Khan-Sun (2002).
  - Infinitesimals, Loeb spaces: Brown-Robinson (1972, 1975), Khan (1974), Brown-Loeb (1976), Khan-Sun (1996, 1999).
  - Finitely additive economies: Armstrong-Richter (1984, 1986), Weiss (1981), Feldman-Gilles (1985), Basile (1993).

Mathematical Preliminaries

- Let T be a nonempty set and $\mathcal{T}$ a σ-algebra of subsets of T.
- Let μ be a set function from $\mathcal{T}$ to $[0, 1]$ with $\mu(T) = 1$.
  - μ is a finitely additive measure on $\mathcal{T}$ if for any $A, B \in \mathcal{T}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$.
  - μ is a countably additive measure on $\mathcal{T}$ if for any sequence $\{A_n\}$ of pairwise disjoint sets in $\mathcal{T}$, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.
- The triple $(T, \mathcal{T}, \mu)$ will be called a (finitely additive/countably additive) measure space.

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- The triple $(T, \mathcal{T}, \mu)$ will be called a (finitely additive/countably additive) measure space.

- A measure μ is atomless if for every $\epsilon > 0$, there exists a $\mathcal{T}$-measurable partition $\{F_1, \ldots, F_n\}$ of T such that $\mu(F_i) < \epsilon$ for every i.

- Let $\mathbb{N}$ be the set of positive integers and $\mathcal{P}(\mathbb{N})$ its power set. There are finitely additive, atomless measures on $\mathcal{P}(\mathbb{N})$ (such as a density charge).

Preview of the Results

- Negative results on finitely additive spaces.
  - An economy may not have a competitive equilibrium.
    (Two examples)
  - A game may not have a Nash equilibrium.
    (Two examples)
  - An economy may not have the idealized limit property.
  - A game may not have the idealized limit property.
- Consequences.
  - Necessity of countably additivity for economies:
    both existence and idealized limit property hold.
  - Necessity of countably additivity for games:
    both existence and idealized limit property hold.
- Approximate equilibria on finitely additive spaces.
  - An economy may not have an approximate competitive equilibrium.
    A tightness assumption is sufficient for existence.
  - A game may not have an approximate Nash equilibrium.
    A tightness assumption is sufficient for existence.

Economies and Competitive Equilibria

- There are L goods and the commodity space is $\mathbb{R}^L_+$.
- $u : \mathbb{R}^L_+ \longrightarrow \mathbb{R}$ is *strongly monotone* if $x \geq y$, $x \neq y \Rightarrow u(x) > u(y)$.
- Let $\mathcal{U}$ denote the class of real valued, continuous and strongly monotone functions on $\mathbb{R}^L_+$. (endowed with the compact open topology)
- Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space. (space of agents)

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- Let $(T, \mathcal{T}, \mu)$ be a finitely additive measure space. (space of agents)
- An *economy* is a measurable mapping $\mathcal{E} = (u, \omega) : T \longrightarrow \mathcal{U} \times \mathbb{R}^L_+$ such that ω is integrable and $\bar{\omega} = \int_T \omega \, \mathrm{d}\mu \gg 0$.
- An *allocation* of $\mathcal{E}$ is an integrable mapping f from T to $\mathbb{R}^L_+$. An allocation is *feasible* if $\int_T f \, \mathrm{d}\mu = \int_T \omega \, \mathrm{d}\mu$.
- Given a price vector $p \in \mathbb{R}^L_+$, the *budget set* of consumer t is
$$B_t(p) = \{x \in \mathbb{R}^L_+ : p \cdot x \leq p \cdot \omega_t\}.$$
- A *competitive equilibrium* of $\mathcal{E}$ is a pair (p, f), where $p \in \mathbb{R}^L_+ \setminus \{0\}$, f is a feasible allocation and μ-a.e.;
  (a) $f(t) \in B_t(p)$ and (b) $u_t(f(t)) \geq u_t(x)$ for all $x \in B_t(p)$.
- An allocation f of $\mathcal{E}$ is a *competitive allocation* if for some p, (p, f) is a competitive equilibrium.

Nonexistence of a CE: An Example on Integers

- The measure space is $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$. Fix $\theta \in [1/2, 1)$.
  Economy $\mathcal{E}$: for $t \in \mathbb{N}$,

$$u_t(x_1, x_2) = \frac{t+1}{t} x_1^{\frac{t}{t+1}} + x_2, \qquad \omega_t = (\theta, \theta).$$

- Equilibrium prices: $p \gg 0$, $p_1 + p_2 = 1$.
- For any $t \in \mathbb{N}$, the demand functions are:

$$D_{t1} = \min\left\{\frac{p_2^{t+1}}{p_1^{t+1}}, \frac{\theta}{p_1}\right\}, \qquad D_{t2} = \frac{\theta}{p_2} - \frac{p_1 D_{t1}}{p_2}.$$

- Case 1. $p_2/p_1 < 1$. $\lim_{t\to\infty} D_{t1} = 0$. $\int_{\mathbb{N}} D_{t1}\, \mathrm{d}\mu = 0$.

$$\int_{\mathbb{N}} D_{t2}\, \mathrm{d}\mu = \frac{\theta}{p_2} > \theta = \int_{\mathbb{N}} \omega_{t2}\, \mathrm{d}\mu. \qquad \text{(contradiction)}$$

- Case 2. $p_2/p_1 \geq 1$. $D_{t1} \geq \min\{1, 2\theta\} = 1$,

$$\int_{\mathbb{N}} D_{t1}\, \mathrm{d}\mu \geq 1 > \theta = \int_{\mathbb{N}} \omega_{t1}\, \mathrm{d}\mu. \qquad \text{(contradiction)}$$

Nonexistence of a CE on General Measure Spaces

Claim

Let $(T, \mathcal{T}, \mu)$ be an atomless finitely additive measure space. Assume that μ is not countably additive. Then there is an economy on $(T, \mathcal{T}, \mu)$ which has no competitive equilibrium.

Games and Nash Equilibria

- Let $E = \{e^1, \ldots, e^L\}$ be the set of unit vectors in $\mathbb{R}^L$ and
  $S = \{s \in \mathbb{R}_+^L : \sum_{k=1}^L s_k = 1\}$ the unit simplex in $\mathbb{R}^L$.
- Let $\mathcal{V}$ be the set of real valued continuous functions defined on $E \times S$, endowed with sup norm.
- $(T, \mathcal{T}, \mu)$ is an atomless, countably/finitely additive probability space.
- A *game* is a measurable function $\mathcal{G} : T \longrightarrow \mathcal{V}$.
- A *pure strategy profile* is a measurable function $f : T \longrightarrow E$.
- A $f : T \longrightarrow E$ is a *pure strategy Nash equilibrium* of $\mathcal{G}$ if μ-a.e.;

$$\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(a, \int_T f \, \mathrm{d}\mu\right) \text{ for all } a \in E.$$

Games and Nash Equilibria, contd.

- Pure strategy profile: $f : T \longrightarrow E$.
- Mixed strategy profile: $g : T \longrightarrow S$.
- Given a mixed strategy profile g and $y \in S$, the *payoff to player t* is

$$\mathcal{G}(t)\left(y, \int_T g \, \mathrm{d}\mu\right) = \sum_{k=1}^{L} y_k \mathcal{G}(t)\left(e^k, \int_T g \, \mathrm{d}\mu\right).$$

- A $g : T \longrightarrow S$ is a *mixed strategy Nash equilibrium* of $\mathcal{G}$ if μ-a.e.;

$$\mathcal{G}(t)\left(g(t), \int_T g \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(a, \int_T g \, \mathrm{d}\mu\right) \text{ for all } a \in E.$$

Nonexistence of an NE: An Example on Integers

- Let $A = \{0, 1\}$ and $K = [0, 1]$. Any $x \in K$ is the weight on action 1.
- The measure space is $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$. For each $t \in \mathbb{N}$,

$$\mathcal{G}(t)(a, x) = a\left(\frac{1}{t} - x\right), \ a \in A.$$

- Best responses:

$$\text{argmax}_{a \in A}\mathcal{G}(t)(a, x) = \begin{cases} \{0, 1\} & \text{if } x = 1/t \\ 1 & \text{if } x < 1/t \\ 0 & \text{if } x > 1/t. \end{cases}$$

- Suppose that g from $\mathbb{N}$ to K is a (mixed) NE. Let $x = \int_{\mathbb{N}} g \, d\mu$.
  - If $x = 0$ then $x < 1/t$ for all $t \in \mathbb{N}$ which implies that $g(t) = 1$ for all t and $\int_{\mathbb{N}} g \, d\mu = 1$. (contradiction)
  - If $x > 0$ then $x > 1/t$ for almost all t (since the measure of a finite set is zero), which implies that $g(t) = 0$ for almost all t and $\int_{\mathbb{N}} g \, d\mu = 0$. (contradiction)

Nonexistence of an NE on General Measure Spaces

Claim

Let $(T, \mathcal{T}, \mu)$ be an atomless finitely additive measure space. Assume that μ is not countably additive. Then there is a game on $(T, \mathcal{T}, \mu)$ which has no Nash equilibrium.

Idealized Limits: Economies

Definition

A measurable mapping $\alpha^m : T \longrightarrow \{1, \ldots, m\}$ is a *replication function* if $\mu(\alpha^m)^{-1}(\{i\}) = 1/m$ for $i = 1, \ldots, m$.

Definition

An economy $\mathcal{E} = (u, \omega)$ on an atomless, finitely additive measure space $(T, \mathcal{T}, \mu)$ is said to have the *idealized limit property* if

(1) for any sequence $\{\mathbb{E}^n = (u^n, \omega^n)\}$ of finite-agent economies with $\{f^n\}$ as competitive allocations, where the number of agents in $\mathbb{E}^n$ is k_n and $\lim_{n\to\infty} k_n = \infty$,

(2) for any sequence of replication functions $\{\alpha^{k_n}\}$ such that $\{\mathbb{E}^n \circ \alpha^{k_n}\}$ converges to $\mathcal{E}$ pointwise on T, $\{f^n \circ \alpha^{k_n}\}$ converges to some allocation f pointwise on T, and $\lim_{n\to\infty} \int_T \omega^n \circ \alpha^{k_n} \, d\mu = \int_T \omega \, d\mu$,

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There is an economy $\mathcal{E}$ on $\mathbb{N}$ which does not have the idealized limit property.

Idealized Limits: Games

Definition

A game $\mathcal{G}$ on an atomless, finitely additive measure space $(T, \mathcal{T}, \mu)$ is said to have the *idealized limit property* if

(1) for any sequence $\{\mathbb{G}^n\}$ of finite-player games with $\{g^n\}$ as mixed strategy Nash equilibria, where the number of players in $\mathbb{G}^n$ is k_n and $\lim_{n\to\infty} k_n = \infty$,

(2) for any sequence of replication functions $\{\alpha^{k_n}\}_{n=1}^{\infty}$ such that $\{\mathbb{G}^n \circ \alpha^{k_n}\}$ converges to $\mathcal{G}$ pointwise on T, and $\{g^n \circ \alpha^{k_n}\}$ converges to some mixed strategy profile g pointwise on T,

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then g is a mixed strategy Nash equilibrium of $\mathcal{G}$.

There is a game $\mathcal{G}$ on $\mathbb{N}$ which does not have the idealized limit property.

Necessity of Countable Additivity: Economies

We have seen failures of both *existence* and the *idealized limit property* for competitive equilibria in economies over a finitely additive measure space. The next theorem shows the equivalence of countable additivity of the agent space with the validity of each of the properties.

Theorem

Let $(T, \mathcal{T}, \mu)$ be an atomless, finitely additive measure space. Then the following are equivalent.

(i) *Every economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ has a competitive equilibrium.*

(ii) *Every economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ has the idealized limit property.*

(iii) *$(T, \mathcal{T}, \mu)$ is a countably additive measure space.*

CA $\Rightarrow$ Existence: Aumann (1966). Existence $\Rightarrow$ CA: Earlier example.

CA $\Rightarrow$ ILP: Proof in the paper.
ILP $\Rightarrow$ CA: Earlier example on $\mathbb{N}$ can be modified to any T.

Necessity of Countable Additivity: Games

We have seen failures of both *existence* and the *idealized limit property* for Nash equilibria in games over a finitely additive measure space.

The next theorem shows the equivalence of countable additivity of the player space with the validity of each of the properties.

Theorem

Let $(T, \mathcal{T}, \mu)$ be an atomless, finitely additive measure space. Then the following are equivalent.

(i) *Every game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ has a mixed strategy Nash equilibrium.*

(ii) *Every game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ has the idealized limit property.*

(iii) *$(T, \mathcal{T}, \mu)$ is a countably additive measure space.*

CA $\Rightarrow$ Existence: Schmeidler (1973).

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ILP $\Rightarrow$ CA: Earlier example on $\mathbb{N}$ can be modified to any T.

Approximate Competitive Equilibria

Earlier, we have seen examples that an economy may not have a competitive equilibrium. It is natural to ask if approximate competitive equilibria exist.

Definition

Let $\mathcal{E}$ be an economy on $(T, \mathcal{T}, \mu)$ and $\epsilon > 0$. (p, f) is an *ϵ-competitive equilibrium* of $\mathcal{E}$ if $p \in \mathbb{R}^L_+ \setminus \{0\}$, f is a feasible allocation, $f(t) \in B_t(p)$ for almost all t and there exists $T_\epsilon \in \mathcal{T}$ such that:

(a) $\mu(T_\epsilon) \leq \epsilon$ and

(b) for almost all $t \in T^c_\epsilon$, $u_t(f(t)) \geq u_t(y) - \epsilon$ for any $y \in B_t(p)$.

In general, an ϵ-competitive equilibrium may not exist. There is an example.

Existence of Approximate Competitive Equilibria

Definition

An economy $\mathcal{E}$ on $(T, \mathcal{T}, \mu)$ is *tight* if for any $\epsilon > 0$, there exists $\bar{T} \subseteq T$ such that

(a) $\mu(\bar{T}) < \epsilon$ and

(b) $\mathcal{E}(T \setminus \bar{T})$ is a relatively compact subset of $\mathcal{U} \times \mathbb{R}^L_+$.

Proposition

If an economy is $\mathcal{E}$ is tight, then it has an ϵ-competitive equilibrium for every $\epsilon > 0$.

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Proposition

If an economy is $\mathcal{E}$ is tight, then it has an ϵ-competitive equilibrium for every $\epsilon > 0$.

- The existence of an ϵ-competitive equilibrium for every $\epsilon > 0$ does not imply that there is a competitive equilibrium.
  An earlier example demonstrates this.

Approximate Nash Equilibria

Earlier, we have seen examples that a game may not have a Nash equilibrium. It is natural to ask whether approximate Nash equilibria exist.

Definition

Let $\mathcal{G}$ be a game on $(T, \mathcal{T}, \mu)$ and $\epsilon > 0$. A strategy profile $g : T \longrightarrow S$ is an *ϵ-Nash equilibrium* of $\mathcal{G}$ if there exists $T_\epsilon \in \mathcal{T}$ such that

(a) $\mu(T_\epsilon) \leq \epsilon$ and

(b) for almost all $t \in T_\epsilon^c$, $\mathcal{G}(t)\left(g(t), \int_T g \, \mathrm{d}\mu\right) \geq \mathcal{G}(t)\left(a, \int_T g \, \mathrm{d}\mu\right) - \epsilon$ for all $a \in E$.

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Proposition

If a game is $\mathcal{G}$ is tight, then it has a pure strategy ϵ-Nash equilibrium for every $\epsilon > 0$.

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A game $\mathcal{G}$ on $(T, \mathcal{T}, \mu)$ is *tight* if for any $\epsilon > 0$, there exists $\bar{T} \subseteq T$ such that

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Proof of Existence of Approximate Nash Equilibria

- Fix $0 < \epsilon < 1$. Since $\mathcal{G}$ is tight, there is a subset $\bar{T} \subseteq T$ such that $0 < \mu(\bar{T}) < \epsilon$ and $\mathcal{G}(T \setminus \bar{T})$ is relatively compact.
- There are m disjoint sets $T_1, \ldots, T_m$ such that $\cup_{k=1}^{m} T_k = T \setminus \bar{T}$, and for any $k \in \{1, 2, \ldots, m\}$, $\| \mathcal{G}(t) - \mathcal{G}(t') \| < \epsilon/2$ if $t, t' \in T_k$.
- Assume that $\mu(T_k) > 0$ for $1 \leq k \leq m$. Denote $\bar{T}$ by T_{m+1}.
- For $1 \leq k \leq m+1$, fix a player $i_k \in T_k$. Construct a game $\mathcal{H}$ on T.

  If $t \in T_k$ then $\mathcal{H}(t) = \mathcal{G}(i_k)$. The range of $\mathcal{H}$ is finite. It has a PSNE f.

  If $t \in T_k$ then
  $$\mathcal{G}(i_k)\left(f(t), \int_T f \, \mathrm{d}\mu\right) = \mathcal{H}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) \geq \mathcal{H}(t)\left(a, \int_T f \, \mathrm{d}\mu\right) = \mathcal{G}(i_k)\left(a, \int_T f \, \mathrm{d}\mu\right) \quad \text{for any} \quad a \in E.$$
- We will show that f is an ϵ-NE of $\mathcal{G}$. Recall that $\mu(T_{m+1}) < \epsilon$. Fix any $1 \leq k \leq m$ and let $t \in T_k$. Then for any $a \in E$,

$$\begin{aligned}
\mathcal{G}(t)\left(f(t), \int_T f \, \mathrm{d}\mu\right) &\geq \mathcal{G}(i_k)\left(f(t), \int_T f \, \mathrm{d}\mu\right) - \frac{\epsilon}{2} \\
&\geq \mathcal{G}(i_k)\left(a, \int_T f \, \mathrm{d}\mu\right) - \frac{\epsilon}{2} \geq \mathcal{G}(t)\left(a, \int_T f \, \mathrm{d}\mu\right) - \epsilon.
\end{aligned}$$

Summary of Results

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    (Two examples)
  - A game may not have a Nash equilibrium.
    (Two examples)
  - An economy may not have the idealized limit property.
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