The Perron-Frobenius Theorem under Weak Indecomposability and Weak Monotonicity

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Background and Motivation

- The Perron-Frobenius theorem for nonnegative square matrices originated with a series of writings by Perron and Frobenius in early 1900's.
  - It asserts that such a matrix has a nonnegative eigenvalue and the corresponding eigenvector is also nonnegative.
  - The eigenvalue dominates any other eigenvalue in modulus and is often referred to as the dominant eigenvalue or the Frobenius root.
  - In addition, if the matrix is indecomposable then both the eigenvalue and the eigenvector are positive.
- The Perron-Frobenius theorem is the principal tool in the analysis of linear input-output models.
  - The dominant eigenvalue of the input matrix determines the rate of growth and the eigenvector corresponds to a balanced growth path.
  - Indecomposability in this context means each good is *directly or indirectly needed for the production of every other good*.
  - Monotonicity is a natural assumption in this setting.

Background and Motivation, contd.

- Solow and Samuelson (1953) considered a more general production system by dropping the additivity assumption for matrices (but retaining the homogeneity condition) and obtained a balanced growth path for all the sectors in the economy.
- In this nonlinear setting, Morishima (1964) and Morishima and Fujimoto (1974) proved the existence of a positive eigenvalue and a positive eigenvector under indecomposability.
- Kohlberg (1982) showed that if the mapping is primitive, repeated iterations take any semipositive vector to the positive eigenvector.
- Bounds for the dominant eigenvalue were obtained in Rath (1986).
- Further developments and applications are contained in the monograph by Lemmens and Nussbaum (2012) and in Chang (2014).

- Neither indecomposability nor monotonicity are as attractive in the nonlinear setting. Furthermore, non-monotonicity and indecomposability may be incompatible some times. These issues are explored below.

Mathematical Preliminaries

- For a positive integer L, $\mathbb{R}^L$ is the L-dimensional Euclidean space and $\mathbb{R}^L_+$ its nonnegative orthant.
- If $x \in \mathbb{R}^L$ then $\| x \| = \sum_{i=1}^{L} |x_i|$ and $|x| = (|x_1|, \ldots, |x_L|) \in \mathbb{R}^L_+$.
- If $x, y \in \mathbb{R}^L$,

$$\begin{aligned} x \leqq y &\quad \text{means} \quad x_i \leq y_i \text{ for every } i, \\ x \leq y &\quad \text{means} \quad x \leqq y \text{ but } x \neq y \\ x < y &\quad \text{means} \quad x_i < y_i \text{ for every } i. \end{aligned}$$

- For any two vectors x and y, $E_{x,y} = \{i : x_i = y_i\}$.
- A matrix A is *decomposable* if there is a nonempty proper subset J of $\{1, \ldots, L\}$ such that $a_{ij} = 0$ for $i \notin J$ and $j \in J$.
- A matrix is *indecomposable* if it is not decomposable and is not the zero matrix of order 1.
- A matrix A is *primitive* if for some positive integer p, $A^p > 0$.

  (Every *primitive* matrix is *indecomposable* but not the converse.)

Properties of Nonlinear Mappings

If A is a matrix then it has two properties:

$$A(\alpha x) = \alpha Ax \quad \text{(homogeneity)} \quad \text{and} \quad A(x+y) = Ax + Ay \quad \text{(additivity)}.$$

Nonlinear mappings typically relax the additivity assumption.

Let $H : \mathbb{R}^L \longrightarrow \mathbb{R}^L$ be a continuous mapping. The following are some of the common assumptions made in the literature.

(A1) Homogeneity. $H(\alpha x) = \alpha H(x)$ for any $\alpha \in \mathbb{R}$.

(A2) Nonnegativity. $H(x) \geqq 0$ for all $x \geqq 0$.

(A3) Monotonicity. If $x \leqq y$ then $H(x) \leqq H(y)$.

(A4) Indecomposability. If $L = 1$ then $H(1) > 0$.
For $L \geq 2$, if $x \leq y$ and $E_{x,y}$ is a nonempty proper subset of $\{1, \ldots, L\}$ then $H_i(x) \neq H_i(y)$ for some $i \in E_{x,y}$.

(A5) Primitivity. For an integer $\ell \geq 1$, $x \leq y \Rightarrow H^{\ell}(x) < H^{\ell}(y)$.

The Linear Input-Output Model & the Nonlinear Model

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

- a_{ij} is the amount of the i-th good needed to produce *one* unit of the j-th good.
- Each row stands for a good and each column stands for a production process.

- If $x \geq 0$ is the output vector then Ax is the input requirement. (Unique)
- If A_{11} is square and $A_{21} = 0$ then A is decomposable, otherwise indecomposable. It is a very reasonable assumption in this context.

The Nonlinear Model. *Joint production is possible.*

- For each $x \geq 0$, the input requirement $H(x)$ is a priori given.
  The same output vector can be produced in many different ways.
- The overall production structure may have a high level of dependence among all the goods but the dependence may be less strong among specific individual production processes.
- Monotonicity implies free disposal of inputs, but really means more. More output requires more of *each* input, which need not be true.

An Illustration with Non-monotonicity

- Define $H : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ as $H(x) = (|x_1 - x_2|, x_1 + x_2)$.

- Let $x = (2, 1)$ and $y = (2, 2)$. Then $H(x) = (1, 3)$ and $H(y) = (0, 4)$. So, H is not monotone.

- H is not indecomposable either.

  Let $0 < \epsilon < x_1 = y_1$, $x_2 = x_1 - \epsilon$ and $y_2 = y_1 + \epsilon$.

  Then $x \leq y$, $H(x) = (\epsilon, 2x_1 - \epsilon)$ and $H(y) = (\epsilon, 2y_1 + \epsilon)$.

  Since $E_{x,y} = \{1\}$ and $H_1(x) = H_1(y)$, indecomposability is violated.

- Non-monotonicity and indecomposability will often be in conflict.

Weak indecomposability: Definition

(A4) Indecomposability. If $L = 1$ then $H(1) > 0$.

For $L \geq 2$, if $x \leq y$ and $E_{x,y}$ is a nonempty proper subset of $\{1, \ldots, L\}$ then $H_i(x) \neq H_i(y)$ for some $i \in E_{x,y}$.

(A6) Weak indecomposability. If $L = 1$ then $H(1) > 0$.

For $L \geq 2$, if $x \leq y$ and $E_{x,y}$ is a nonempty proper subset of $\{1, \ldots, L\}$, then for some integer $k \geq 1$,

$$H_i^k(x) \neq H_i^k(y) \quad \text{for some} \quad i \in E_{x,y}.$$

In general, the k in the above definition will depend upon x and y.

Two inter-connections are immediate.

(1) Every indecomposable mapping is weakly indecomposable.

(2) A primitive mapping is weakly indecomposable.

Example 1

Consider the following regions in $\mathbb{R}^2$:

$$W_1 = \{x \in \mathbb{R}^2_+ : x_1 \geq x_2\},$$

$$W_2 = \{x \in \mathbb{R}^2_+ : x_2 > x_1\},$$

$$W_3 = \{x \in \mathbb{R}^2 : x_1 < 0,\ x_2 > 0\},$$

$$W = W_1 \cup W_2 \cup W_3.$$

Define $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as follows.

$$\begin{aligned} H(x) &= (2x_2, x_1) && \text{if } x \in W_1 \\ &= (x_1 + x_2, x_2) && \text{if } x \in W_2 \\ &= (x_2, x_1 + x_2) && \text{if } x \in W_3 \\ &= -H(-x) && \text{if } x \notin W. \end{aligned}$$

H is continuous, homogenous, nonnegative.

- Monotonicity: if $x \leq y$ then $H(x) \leq H(y)$.
- H is neither indecomposable nor primitive.
- H is weakly indecomposable.
- H has two eigenvalues: $\lambda^* = \sqrt{2}$ and $\bar{\lambda} = (1 - \sqrt{5})/2$.
- Eigenvectors: $x^* = \left(\sqrt{2}, 1\right)$ and $\bar{x} = \left(-(\sqrt{5}+1)/2, 1\right)$.

$$\lambda^* > 0,\ x^* > 0,\ \lambda^* \geq |\bar{\lambda}|,\ \bar{x} \not\geq 0.$$

These are consequences of Theorem 1.

Some Auxiliary Results (Weak ind and Mon)

Proposition 1

A nonnegative, square matrix is indecomposable iff it is weakly indecomposable.

It is well known that if a mapping is monotonic and indecomposable, then

$\star\star$ the image of a semipositive vector is semipositive and

$\star\star$ the image of a positive vector is positive.

These conclusions are valid under weak indecomposability as well.

Lemma 1

Suppose that (A1)–(A3) and (A6) hold. Then $H(x) \geq 0$ if $x \geq 0$ and $H(x) > 0$ if $x > 0$.

Weak indecomposability and Monotonicity: Existence

Theorem 1

Suppose that (A1)–(A3) and (A6) hold.

(i) There exist $\lambda^* > 0$ *and* $x^* > 0$ *such that* $H(x^*) = \lambda^* x^*$.

(ii) x^* *is unique up to scalar multiplication.*

(iii) If $\lambda \neq \lambda^*$ *then there is no* $x \geq 0$ *such that* $H(x) = \lambda x$.

(iv) $\lambda^* \geq |\bar{\lambda}|$ *for any eigenvalue* $\bar{\lambda}$ *of* H.

(iii) shows that x^* is the only nonnegative eigenvector of H.

λ^* is called the dominant eigenvalue because of (iv).

Weak indecomposability and Monotonicity: Bounds

Theorem 2

Suppose that (A1)–(A3) and (A6) hold. Then for any $x > 0$,

$$\min_i \frac{H_i(x)}{x_i} \leq \lambda^* \leq \max_i \frac{H_i(x)}{x_i}.$$

If $x = x^*$ then both inequalities become equalities.

If $x \neq x^*$ then both the inequalities are strict.

Corollary 1 (Viability condition): *$x > H(x)$ for some $x > 0$ iff $\lambda^* < 1$.*

$\lambda^* < 1 \Rightarrow x^* > H(x^*)$. $\quad x > H(x) \Rightarrow \lambda^* \leq \max_i(H_i(x)/x_i) < 1$.

In an economic system, the gross output of each commodity exceeds its total input requirement, so the net output vector is positive. The system is *viable*, i.e., capable of economic growth. If x^* is the output vector then this is the balanced growth path and the rate of balanced growth is $(1/\lambda^*) - 1$.

Weak monotonicity

(A3) Monotonicity. If $x \leqq y$ then $H(x) \leqq H(y)$.

Let $\mathbb{R}^2_- = -\mathbb{R}^2+$.

(A7) Weak monotonicity. (I) If $x, y \in \mathbb{R}^L_+$, $x \leq y$ and $x \not< y$ then for some $p \geq 1$, $\quad H^p(x) \leqq H^p(y)$.

(II) If $x \notin \mathbb{R}^L_+ \cup \mathbb{R}^L_-$ then for every $t \geq 1$, $\quad H^t(|x|) \geqq H^t(x) \geqq H^t(-|x|)$.

Notice that (I) and (II) refer to distinct regions of $\mathbb{R}^L$.

If monotonicity holds then weak monotonicity holds. The converse is not true.

Iterative semipositivity

(A8) Iterative semipositivity. For some $\bar{z} \in \mathbb{R}^L_+$, $H^t(\bar{z}) \geq 0$ for every $t \geq 1$.

This is not a very restrictive condition.

If weak indecomposability and monotonicity hold then this condition is satisfied for every $x \geq 0$, by Lemma 1.

Lemma 2

Suppose that (A1), (A2), (A6), (I) of (A7) and (A8) hold. Then $H(x) \geq 0$ if $x \geq 0$.

Unlike Lemma 1, one cannot claim that if $x > 0$ then $H(x) > 0$.

Weak indecomposability & Weak monotonicity: Existence

Theorem 3

Suppose that (A1), (A2) and (A6)–(A8) hold.

(i) There exist $\lambda^* > 0$ *and* $x^* > 0$ *such that* $H(x^*) = \lambda^* x^*$.

(ii) x^* *is unique up to scalar multiplication.*

(iii) If $\lambda \neq \lambda^*$ *then there is no* $x \geq 0$ *such that* $H(x) = \lambda x$.

(iv) $\lambda^* \geq |\bar{\lambda}|$ *for any eigenvalue* $\bar{\lambda}$ *of* H.

In the proof, weak monotonicity (II) is needed only in part (iv).

If weak monotonicity (II) is violated then (iv) need not hold.

G-property

Given a mapping H, define G as $G(x) = x + H(x)$ for every $x \in \mathbb{R}^L_+$.

(A9) G-property. (I) If $x, y \in \mathbb{R}^L_+$, $x \leq y$ and $x \nless y$ then for some $k \geq 1$ and

$$i \in E_{x,y}, \quad G_i^k(x) < G_i^k(y).$$

(II) Let $x > 0$ and α and β be positive constants.

If $\alpha x \geqq G(x)$ then $\alpha G(x) \geqq G^2(x)$. If $\beta x \leqq G(x)$ then $\beta G(x) \leqq G^2(x)$.

(I) is somewhat stronger than weak indecomposability of G on the nonnegative orthant. Here, the direction of inequality is specifically given.

If G is primitive on the nonnegative orthant then (I) automatically holds.

Primitivity of G does not imply G-property.

Weak indecomposability & Weak monotonicity: Bounds

Theorem 4

Suppose that (A1), (A2) and (A6)–(A8) hold. Suppose that
(a) either G is primitive on $\mathbb{R}^L_+$, or (b) (A9) holds.
Then for any $x > 0$,

$$\min_i \frac{H_i(x)}{x_i} \leq \lambda^* \leq \max_i \frac{H_i(x)}{x_i}.$$

If $x = x^*$ then both inequalities become equalities.
If $x \neq x^*$ then both the inequalities are strict.

Corollary 2 (Viability condition): *$x > H(x)$ for some $x > 0$ iff $\lambda^* < 1$.*

The proof and the interpretations are the same as of Corollary 1.

Summary of Results

- This paper has relaxed the notion of indecomposability to weak indecomposability and the notion of monotonicity to weak monotonicity.
- The Perron-Frobenius theorem holds under weak indecomposability and monotonicity. (Theorems 1 and 2)
- The Perron-Frobenius theorem holds under weak indecomposability and weak monotonicity. (Theorem 3)
- Under weak indecomposability and weak monotonicity, additional restrictions are needed to obtain bounds for the dominant eigenvalue. (Theorem 4)