Product differentiation with elastic demand and unit specific transportation cost

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Abstract

The consumers are uniformly distributed in the unit interval and there are two producers. A consumer purchases from the producer with the lower delivered price (product price plus the quadratic transportation cost). The consumer's quantity demanded depends on the product price and the transportation cost which is paid for every unit of the product purchased. The producers first choose locations and then compete in prices. For each fixed pair of locations, there is a unique Nash equilibrium in prices. The equilibrium locations are unique, symmetric and depend upon the ratio of the reservation price and the transportation cost parameter. When this ratio exceeds a certain critical value, the locations are at the extreme endpoints of the market. As the ratio decreases, the two firms gradually move inwards, approximately to the quartiles of the market. The firms never agglomerate at the center.

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1 Introduction

Hotelling, in his 1929 classic, examined a two stage differentiated product model in which duopolists first choose product type and then compete in prices. The linear transportation cost is borne by the consumers and they buy a unit from the cheapest seller. His conclusion was that the firms locate close to each other and the products are minimally differentiated. However, subsequent research has found a flaw in his Hotelling's argument. Due to the incentive to undercut the competitor located closeby the payoff functions exhibit discontinuities resulting in non-existence of pure strategy equilibrium in prices when firms are located sufficiently close to each other.

Osborne and Pitchik (1987) restored equilibrium in the second stage by allowing for mixed strategies over prices, the resulting equilibrium locations are then near the quartiles. Indeed, d'Aspremont et al. (1983) found that minimum differentiation would never obtain in location-price models in the spirit of Hotelling. The reason is that, when firms locate coincidentally, intense price competition drives profits to zero. Anticipating this outcome, firms will never choose to locate together.

The discovery of the flaw in Hotelling's argument [Vickrey (1964, p. 323–334), d'Aspremont et al. (1979)] has led to many variants of this model and a significant body of research. Gabszewicz and Thisse (1992) contains a review of this extensive literature. One important development in this line of research is models with nonlinear transportation cost. The two stage model with quadratic transportation cost exhibits maximal product differentiation [d'Aspremont et al. (1979), Neven (1985)].

A predominant theme in the literature since Hotelling's paper is that the models are characterized by perfectly inelastic demand. The consumers buy either zero or a unit of the product subject to a reservation price. One implication of inelastic demand is that a firm does not make any sacrifice in its own market segment as it moves towards its competitor. However, such a movement intensifies competition. So the net effect depends on the relative strengths of these effects.

In many contexts it is reasonable to suppose that the quantity demanded by consumers is inversely related with price. Smithies (1941), Anderson et al. (1992) and Hamilton et al. (1994) have examined models with this feature.

Elastic demand can act as a check towards moving nearer to one's rival [Smithies (1941, p. 423)]. In a model with linear transportation cost, Smithies concludes that the firms locate apart from each other and each locates inside the quartiles of the market to maximize profits. However, because of subsequent difficulties associated with Hotelling's model due to discontinuities, Smithies' results need to be interpreted with care and caution.

If consumers' demand is inversely related to price then a distinction can be made between two issues: whether the transportation cost is lump sum or proportional to the quantity demanded. This distinction has already been made in Stahl (1987, p. 793–794), Anderson et al. (1992, p. 283) and Rath and Zhao (2001). Anderson et al. (1992) call it a "shopping model" if the transportation cost is a lump sum and a "shipping model" if the transportation cost is paid for every unit of the product.

Anderson et al. (1992, p. 283) examines a model of elastic demand with lump sum transportation cost. The prices are assumed to be identical for both producers. In such a model, both firms locate at the center of the market. The reasoning is that, since the prices are the same each consumer buys from the nearest seller and each consumer buys the same quantity. The model is thus similar to one with inelastic demand and identical prices which exhibits central agglomeration.

In Rath and Zhao (2001) the consumers' demand functions are linear in prices. The quadratic transportation cost is paid only once and doesn't depend on quantity purchased. The interesting result of their model is that the extent of product differentiation depends on the relative magnitudes of (squared) reservation price and transportation cost parameter. In particular, if the relative magnitude is sufficiently high then firms locate at the extremes, as the ratio decreases firms gradually move towards the center of the market.

This paper is concerned with a two stage model with elastic demand. The firms choose locations in the first stage and compete in prices in the second stage. The (quadratic) transportation cost is paid per unit purchased. The demand of each consumer varies inversely with delivered price (price plus the transportation cost) and is assumed to be linear. The firms charge the Nash equilibrium prices in the second stage and choose locations accordingly in the initial stage. A unique Nash equilibrium in prices is shown to exist at each possible pair of locations. The analysis of the location stage turns out to be considerably difficult. While we were not able to show the existence and uniqueness of the optimal location choices, we provide results from numerical simulations. The algorithm converges to a unique location pair irrespective of initial guess of the equilibrium locations and is always symmetric.

If A is the reservation price and b is the transportation cost parameter then the equilibrium locations are determined by the ratio A/b. If A/b exceeds a certain critical value, then the locations are at the extreme endpoints of the market. As A/b decreases the locations gradually move inwards to approximately (1/4, 3/4). The model then implies that duopolists never agglomerate. Thus elastic demand and quadratic costs paid per unit of good bought limit the extent of differentiation.

One explanation for this phenomenon of extreme locations can be given as follows. As the reservation price rises, this model gradually approximates the two stage model with inelastic demand. The unique equilibrium locations of the latter model are at the extremes. So for large reservation price, one still obtains extreme locations in this model as well.

However, for low A/b, the issue is a lot more intricate. To contrast the settings when demand is inelastic and elastic, first consider the case of inelastic demand. When consumers' demand is price inelastic, the profit of firm 1 can be written as $p_1 z$ (price times the market share). Of course, the market share is determined by own price, more importantly it depends on the price differential of the two firms. As a firm moves away from its rival, its price goes up or competition is softened. This also affects the market share (because of the price changes). So, the net effect on profit depends on these two countervailing effects. As it turns out in the inelastic demand case, the increase in price offsets the loss in market share. So, profit does increase because of the price increase as the firm moves away from its rival. This results in extreme locations.

In contrast, when consumers' demand is linear in price, the profit of firm 1 can be written as $p_1(A - p_1)z - p_1 T$, where $p_1 T$ represents the value of aggregate transportation cost in the duopolist's market segment. When demand is price elastic, the contribution of high price to profit is restricted by price-inversed low demand and by the value of total transportation cost. Hence, competition on demand and market share becomes more important than achieving a high price by locating apart.

The parameters A and b determine the relative magnitudes of the effect of higher price and lower demand and market share on profit. Thus, a move away from its rival may not be always profitable for a firm and so location choices can be in the interior of the market segment. Another interesting feature of the model is that customers paying transportation cost per unit of good purchased imposes a lower bound on the degree of differentiation. Since transportation cost now affects demand, this has direct consequence for the profit of the firm through the lower demand of the consumers. This makes agglomeration an unattractive option for the firms since they would lose too much of demand from the periphery of the market segment.

The paper is organized as follows. The next section specifies consumers' preferences and derives the demand functions for differentiated product. The demand functions of duopolists are derived in Section 3. The existence of Nash equilibrium in prices for fixed locations of the firms is proved in section 4. Section 5 provides the results from numerical simulations for equilibrium locations. Section 6 concludes.

2 Demand for Differentiated Products

There are two goods. Good 1 is to be thought of as a (Hicksian) composite good and good 2 is a differentiated product. A consumer's utility from buying q_h and q_d units of the two goods is assumed to be $u(q_h, q_d) = q_h + \rho_1 q_d - \rho_2 q_d^2 - q_d f_d$. The function f_d captures the notion of transportation cost paid per unit of good 2 bought. The budget constraint is $p_h q_h + p_d q_d = m$. p_h is normalized to 1. p_d can vary, and in particular, may be zero.

The utility function is strictly quasiconcave in the two goods. Utility maximization subject to budget constraint yields the demand functions: $q_h = m - p_d[(\rho_1 - p_d - f_d)/2\rho_2]$ and $q_d = (\rho_1 - p_d - f_d)/2\rho_2$ whenever positive and zero otherwise.

Now suppose that the consumers are uniformly distributed over the unit interval [0, 1] and the two producers of differentiated products are located at x_1 and x_2, $x_1 \leq x_2$. Each consumer's problem is to decide which product to buy and how much. Minor modifications in the utility function can take into account the two differentiated products at different locations. The quantity of the differentiated products are now denoted by q_1 and q_2 (corresponding to firm locations at x_1, x_2) and the prices by p_1 and p_2. Let the utilities of consumer t while consuming products 1 and 2 be $u_1(q_h, q_1) = q_h + \rho_1 q_1 - \rho_2 q_1^2 - q_1 f_1(t)$ and $u_2(q_h, q_2) = q_h + \rho_1 q_2 - \rho_2 q_2^2 - q_2 f_2(t)$.

The functions f_1 and f_2 capture the utility loss of consumer t from consuming a product at a different location. From above, the demand of a consumer of the two differentiated products is $q_1 = (\rho_1 - p_1 - f_1(t))/(2\rho_2)$ and $q_2 = (\rho_1 - p_2 - f_2(t))/(2\rho_2)$. The corresponding indirect utility functions are $u_1^* = m + (\rho_1 - p_1 - f_1(t))\, q_1 - \rho_2 q_1^2$ and $u_2^* = m + (\rho_1 - p_2 - f_2(t))\, q_2 - \rho_2 q_2^2$, where q_1 and q_2 are demand functions defined earlier. Consumers for which $u_1^* \geq u_2^*$ buy from firm 1 and others buy from firm 2.

At this stage, it is worthwhile to note two alternative interpretations. In the preceding paragraph it is assumed that the indifferent consumer, for whom $u_1^* = u_2^*$, buys from firm 1. Alternatively, one can postulate that the indifferent consumer chooses randomly between the two firms. The subsequent analysis is unaffected by this assumption. The preceding analysis interprets the transportation cost as a cost in utility. Instead, if the transportation cost is included in the budget constraint, then the utility function becomes $u(q_h, q_d) = q_h + \rho_1 q_d - \rho_2 q_d^2$ and the budget constraint becomes $p_h q_h + p_d q_d + q_d f_d = m$. It is readily checked that, by including f_d in the budget constraint, q_d remains identical whereas q_h is different, depending now on f_d. Of more significance is the fact that u_1^* and u_2^* given above remain the same and the subsequent analysis is obviously not affected.

Typically, ρ_1, ρ_2, f_1 and f_2 would be functions of t, x_1 and x_2. However, the model becomes almost intractable at that level of generality. So, for simplicity let $\rho_1 = A$ and $\rho_2 = 1/2$. Furthermore, let $f_1(t) = b(t - x_1)^2$ and $f_2(t) = b(t - x_2)^2$.

The constant A plays the role of the reservation price of the consumers. The transportation cost parameter is b and the transportation cost is quadratic and paid per unit of differentiated product purchased.

Now $q_1 = A - p_1 - b(t - x_1)^2$ and $q_2 = A - p_2 - b(t - x_2)^2$. Solving $u_1^* = u_2^*$ for t, which now becomes $[A - p_1 - b(t - x_1)^2]q_1 - \frac{1}{2}q_1^2 = [A - p_2 - b(t - x_2)^2]q_2 - \frac{1}{2}q_2^2$, gives the identity of marginal consumer denoted by z. The market shares of the two firms are z and $1 - z$ and the expression for z is given in the next section.

3 The Model

From the preceding discussion the model can be described as follows. The consumers are uniformly distributed over the unit interval [0, 1]. The reservation price of the consumers is

A. The transportation cost parameter is b. The transportation cost is quadratic and paid per unit of good purchased. Throughout it is assumed that $A \geq b$.

The producers are located at x_1 and x_2, $x_1 \leq x_2$. The production costs are zero. The market shares of the two firms are z and $1-z$ respectively. Each consumer $t \in [0, z]$ buys $A - p_1 - b(t - x_1)^2$ units from firm 1 and each consumer $t \in [z, 1]$ buys $A - p_2 - b(t - x_2)^2$ units from firm 2. Notice that demand of a consumer in either segment of the market now depends on own location and location of the supplier in the corresponding markets. Firms set uniform price p_i per unit of quantity sold and collect $p_i \left[A - p_i - b(t - x_i)^2\right]$ amount of profit from a consumer located at t. Therefore, when $x_1 < x_2$,

$$
\begin{aligned}
z &= \frac{p_2 - p_1}{2b(x_2 - x_1)} + \frac{x_1 + x_2}{2} \\
D_1(p_1, p_2) &= \int_0^z \left[A - p_1 - b(t - x_1)^2\right] dt \\
D_2(p_1, p_2) &= \int_z^1 \left[A - p_2 - b(t - x_2)^2\right] dt \\
\pi_1(p_1, p_2) &= p_1 D_1(p_1, p_2) = \int_0^z p_1 \left[A - p_1 - b(t - x_1)^2\right] dt \\
\pi_2(p_1, p_2) &= p_2 D_2(p_1, p_2) = \int_z^1 p_2 \left[A - p_2 - b(t - x_2)^2\right] dt
\end{aligned}
$$

The main difference between this model and models with perfectly inelastic demand is the following. The demand of each consumer is inversely related and varies continuously with the delivered price. An increase in price affects the firm's demand in two ways, the market share diminishes and the quantity sold to each consumer diminishes. Had consumer demand been completely inelastic, firm demand would be affected only through loss in market share.

The difference between this specification and one where the quadratic transportation cost is paid only once is that in the latter case the demand functions of each consumers are not affected by transportation cost parameter and the only effect it induces is through identity of marginal consumer. In the current model the transportation cost parameter effects both individual demand and identity of indifferent consumer since it is paid for each unit purchased.

4 Existence of Nash Equilibrium in Prices

The purpose of this section is to prove the existence of a Nash equilibrium in prices for any given pair of locations. For fixed locations of the firms, a Nash equilibrium is a pair of prices (p_1^*, p_2^*) such that $\pi_1(p_1^*, p_2^*) \geq \pi_1(p_1, p_2^*)$ for all p_1 and $\pi_2(p_1^*, p_2^*) \geq \pi_2(p_1^*, p_2)$ for all p_2. If $x_1 = x_2$ then a zero price for each firm is the unique Nash equilibrium. So, for the remainder of the section it is assumed that $x_1 < x_2$.

We can further describe the tradeoff firms are facing. From the profit function of firm 1, its derivative with respect to p_1 is

$$\int_0^z \left[A - 2p_1 - b(t - x_1)^2\right] dt + p_1 \theta \frac{\partial z}{\partial p_1}$$

The first term measures marginal change in profits by slightly increasing price while holding the market share constant. Notice that the integrand refers to the consumer located at t and integrating over all consumers in the market segment of firm 1 corresponds to the aggregated marginal effect. The second term measures marginal decrease in profits that is due to loss in the market share by setting higher prices. Therefore, the firm tries to set price such that the effect of a loss in the market share is exactly balanced out by the effect of aggregate marginal increase in profits from it's market segment.

The next theorem deals with existence of the price equilibrium in the second stage of the game.

Theorem 1. *Let $x_1 < x_2$ be given. The profit functions $\pi_1(\cdot, \cdot)$ and $\pi_2(\cdot, \cdot)$ are strictly quasi-concave in their own arguments. Consequently for each pair of locations a Nash equilibrium in prices exists. The Nash equilibrium prices are (implicitly) given by the following pair of first order conditions.*

$$(A - 2p_1 - \eta_1)z - \frac{p_1}{2b(x_2 - x_1)}\theta = 0 \qquad (1)$$

$$(A - 2p_2 - \eta_2)(1 - z) - \frac{p_2}{2b(x_2 - x_1)}\theta = 0 \qquad (2)$$

where $\eta_1 = b/3[(z - x_1)^2 - (z - x_1)x_1 + x_1^2]$, $\eta_2 = b/3[(1 - x_2)^2 + (1 - x_1)(z - x_2) + (z - x_2)^2]$ and $\theta = A - p_1 - b(z - x_1)^2$.

Appendix 1 shows that the profit functions are strictly quasiconcave. In addition, using a contraction argument, it can be shown that the Nash equilibrium prices are unique. The details are given in the same appendix.

Unfortunately, it is not possible to solve for Nash equilibrium prices explicitly, however the following can be shown.

Lemma 1. *If* $1 - x_1 - x_2 = 0$, *then* $p_2 = p_1$ *and* $z = 1/2$. *If* $1 - x_1 - x_2 > 0$, *then* $p_2 > p_1$ *and* $(x_1 + x_2)/2 < z < 1/2$. *If* $1 - x_1 - x_2 < 0$, *then* $p_2 < p_1$ *and* $(x_1 + x_2)/2 > z > 1/2$.

The sketch of the proof is given in Appendix 2. The lemma plays a crucial role in characterizing equilibrium locations.

5 Equilibrium Locations

The firms choose locations in the first stage and compete in prices in the second stage. The existence of Nash equilibrium prices for any given pair of locations was proved in the preceding section. In this section, we provide results from the numerical simulations, draw a conjecture and discuss a possible way to prove existence and uniqueness.

Denote by $\tilde{\pi}_1(x_1, x_2)$ and $\tilde{\pi}_2(x_1, x_2)$ the Nash equilibrium profits of the two firms at locations x_1 and x_2. So,

$$\begin{aligned} \tilde{\pi}_1(x_1, x_2) &= \int_0^z p_1 \left[A - p_1 - b(t - x_1)^2\right] dt \\ \tilde{\pi}_2(x_1, x_2) &= \int_z^1 p_1 \left[A - p_2 - b(t - x_2)^2\right] dt \end{aligned}$$

where p_1 and p_2 are the Nash equilibrium prices at the locations (x_1, x_2) and z and $1 - z$ are the corresponding market shares. An equilibrium pair of locations is such that each profit function is maximized given the location choice of the other firm. Using the expression for $\tilde{\pi}_1(x_1, x_2)$ and $\tilde{\pi}_2(x_1, x_2)$ one can analyze the tradeoff firms are facing when deciding where to locate. For example, for firm 1 the derivative of profit with respect to it's own location can be written as:

$$\begin{aligned}\frac{\partial \tilde{\pi}_1}{\partial x_1} &= \frac{\partial \pi_1}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial \pi_1}{\partial x_2} \\ &= p_1 \theta \frac{\partial z}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \int_0^z p_1 \left[2b(t - x_1)\right] dt + p_1 \theta \frac{\partial z}{\partial x_1}\end{aligned}$$

The first term corresponds to the indirect channel when firm 1 changes it's location. This itself has two effects. First, it induces endogenous change in second firm's price choice. Second, the change in competitor's price affects the profit of firm 1 through change in the market share. By moving closer towards its competitor firm 1 induces a more aggressive price competition. Notice that indeed, two effects come into play, when firm 2 changes it's price it induces change in market share of firm 1, this is further dampened/amplified by pricing decision of the competitor.

The second and third terms capture the direct effect of change in location on first stage profits. The second term captures the aggregate marginal effect on demand through transportation cost when market share is held constant. The third term accounts for the direct effect induced from a change in market share as result of the firm changing it's location. Finally, notice that own price effect is irrelevant due to the fact that firm is reoptimizing for each new location decision. See Appendix 3 for derivations of first order conditions and comparative statics expressions for prices.

The issue of existence of equilibrium locations is not quite straightforward. Part of the difficulty is caused by the fact that one does not have explicit solutions for Nash equilibrium prices. Furthermore, it is not clear whether the functions $\tilde{\pi}_1(\cdot,\cdot)$ and $\tilde{\pi}_2(\cdot,\cdot)$ are quasiconcave in their own arguments. Thus one cannot appeal to any well known existence argument and has to resort to indirect means to prove existence. At the end of this section we outline one possible way to show existence of the equilibrium, although we were not able to provide the poof.

The following conjecture summarizes the results from numerical simulations.

Conjecture 1. *There is a unique equilibrium pair of locations* (x_1^*, x_2^*). *These locations are symmetric,* $x_1^* = 1 - x_2^*$. *There is a constant* C *(approx: 7.5170) such that if* $A/b \geq C$ *then the equilibrium locations are at the extreme endpoints of the market,* $x_1^* = 0$. *If* $A/b \in [1, C)$ *then the equilibrium locations are interior,* $x_1^* \in (0,\ 0.2676]$.

The constant C is obtained from numerical simulations. We normalize A to unity and vary b. For each value of transportation cost parameter the algorithm converges to a symmetric location equilibrium irrespective of the initial guess. As b decreases towards zero, x_1^* also decreases towards zero and as b goes to A, x_1^* converges to approximately 0.2676. This is illustrated in the following figure.

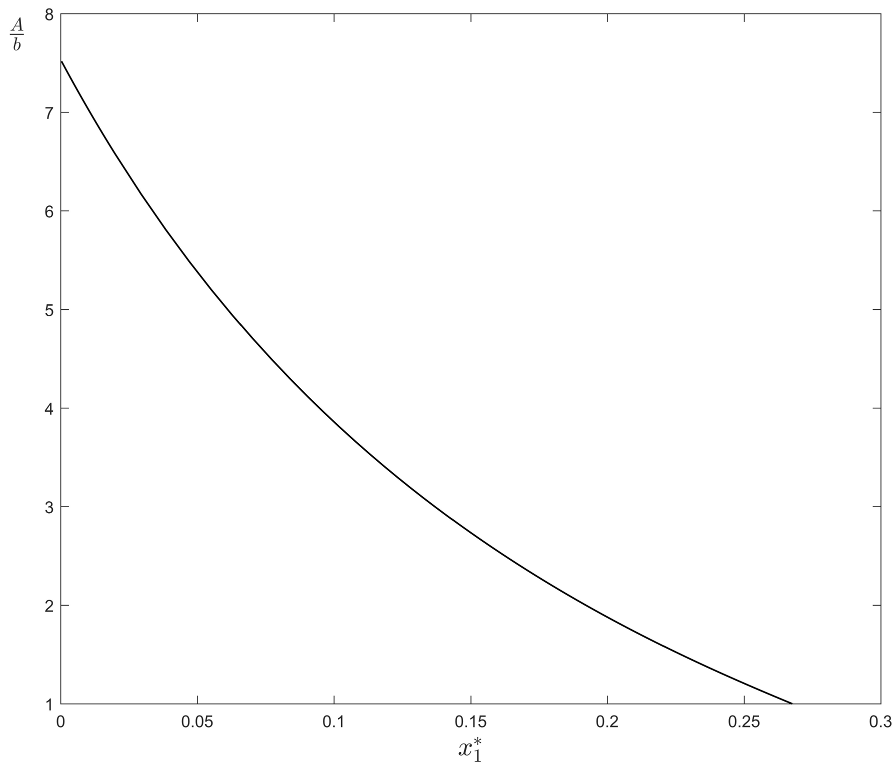

Figure 1: Symmetric equilibrium locations

The issue of nonextreme locations can be briefly examined. Consider the Nash equilibrium profit of firm 1, $\tilde{\pi}_1(x_1, x_2)$, at any given pair of locations x_1 and x_2. Then, as we argued earlier, the marginal effect of change in the own location can be decomposed into two effects: the effect due to price change and the effect due to competitors change in market share. Numerical simulations show that $\partial p_2/\partial x_1 < 0$. Suppose that b is low, so that the second effect is negligible, then first effect dominates (indeed, $\partial z/\partial p_2$ is high in such case) and the profits are decreasing in x_1, therefore firms choose to locate at the extremes of the market. On the other hand, if b is high enough so that second effect is not negligible, then firms choose locations such that they balance these two countervailing forces resulting in an interior location choice. The magnitude of A/b is what determines the relative strength of these two effects, if A/b is high enough then firms locate at the extremes, when A/b decreases firms locate inside the market.

The remainder of this section is devoted to an outline of a possible structure of a proof of the Conjecture 1. It can be shown that $\partial\tilde{\pi}_1/\partial x_1 = -V_1K_1$ and $\partial\tilde{\pi}_2/\partial x_2 = V_2K_2$. All of these K_1, K_2, V_1, V_2 are functions of the locations x_1 and x_2. Furthermore, V_1 and V_2 are positive. The details are given in Appendix 3. So, the signs of the derivatives are determined by K_1 and K_2. At symmetric locations, K_1 and K_2 are identical. Then consider the following two conjectures.

Claim 1. *$K_2 - K_1$ has the same sign as $1 - x_1 - x_2$.*

Conjecture 2. *$\partial(K_1 + K_2)/\partial x_1 > 0$ and $\partial(K_1 + K_2)/\partial x_2 < 0$ whenever $K_1 + K_2$ is zero.*

These are akin to first and second order conditions. Using Conjecture 1 one could show that the model cannot have asymmetric equilibrium locations, i.e., $1 - x_1 - x_2 \neq 0$ cannot hold at an equilibrium. This can be seen as follows.

Suppose there is an equilibrium with $1 - x_1 - x_2 > 0$. Since $x_2 < 1$, $\partial\tilde{\pi}_2/\partial x_2$ must be zero, i.e., $K_2 = 0$. From Claim 1, $K_2 - K_1 > 0$. So, $K_1 < 0$ which implies that $\partial\tilde{\pi}_1/\partial x_1 > 0$. So, firm 1 changes its location. The case $1 - x_1 - x_2 < 0$ is handled in an analogous manner.

From the preceding discussion, we could rule out any asymmetric equilibrium. This leaves only symmetric location pairs as candidates for equilibrium. Conjecture 2 now would provides the sufficient conditions for existence, that is, the symmetric locations are indeed equilibrium locations. Consider any pair of locations (x_1^*, x_2^*) given in Claim 1. If $x_1^* > 0$ then $K_1 + K_2 = 0$, if $x_1^* = 0$ then $K_1 + K_2 \geq 0$. Keep x_2^* fixed. From Conjecture 2, if firm 1 chooses a location x_1 just to the right of x_1^* then $K_1 + K_2 > 0$ at locations x_1 and x_2^*. Conjecture 2 further ensures that $K_1 + K_2 > 0$ at all locations x_1 to the right of x_1^*. For any such x_1, $1 - x_1 - x_2^* < 0$. So, from Claim 1, $K_2 - K_1 < 0$. So, $K_1 > 0$ and $\partial\tilde{\pi}_1/\partial x_1 < 0$ (at all locations x_1 to the right of x_1^*).

Now suppose that $x_1^* > 0$. Then $K_1 + K_2 = 0$. At a location x_1 to the immediate left of x_1^*, $K_1 + K_2 < 0$ by Conjecture 2 and $K_1 + K_2 < 0$ for all locations x_1 to the left of x_1^*. Now $1 - x_1 - x_2^* > 0$, so $K_2 - K_1 > 0$ from Claim 1. Thus $K_1 < 0$ and $\partial\tilde{\pi}_1/\partial x_1 > 0$.

Therefore, this would show that x_1^* is the optimal location of firm 1 given x_2^*. Similarly,

x_2^* is the optimal location of firm 2 given x_1^*. So the pair (x_1^*, x_2^*) given in Conjecture 1 would be the unique equilibrium locations in this model.

6 Conclusion

This paper has dealt with a situation where the transportation cost is quadratic and paid per unit of the quantity purchased. Consumers' demand is linear in delivered price. The results are: there is a Nash equilibrium in prices for each location pair and the unique symmetric equilibrium locations depend on the relative magnitudes of the reservation price, A, and the transportation cost parameter, b. Through numerical simulations we found a constant $C \approx 7.5170$ such that if $A/b \geq C$ the equilibrium locations are at the extremes of the market. As A/b decreases to unity, the locations gradually move from the endpoints towards $(0.2676, 0.7324)$, slightly more than a quarter of a market length. Interestingly, these locations are close to socially optimal locations.

This phenomenon is quite different from the existing results in the literature. If demand is assumed to be completely inelastic, then the firms locate at the extreme endpoints of the market [d'Aspremont et al. (1979), Neven (1985)]. If prices are assumed to be identical irrespective of locations, then the firms agglomerate at the center [Anderson et al. (1992)].

Hamilton et al. (1994) have examined a model of quantity competition. They assume elastic demand, linear transportation cost and that the transportation cost is paid for every unit of the product. Barring the issue of existence of pure strategy equilibrium in the second stage (due to linear transportation cost), they claim that the firms nearly agglomerate for low transportation costs and move away from the center of the market as transportation costs rise. This is in direct contrast with the result obtained in this paper. In the model examined above, for lower transportation costs (higher A/b) the firms locate at the extreme endpoints of the market and for higher transportation costs (lower A/b) the firms gradually move towards location slightly more than a quarter distance of their market periphery.

We also contrast these results with Rath and Zhao (2001). Their setup is similar to ours, except that in their specification, the transportation cost is paid only once. In their model the ratio A^2/b determines the equilibrium locations. In both models when the transportation cost is sufficiently low, firms locate at the extremes. However, in contrast to their result, in our model firms never agglomerate at the center. When transportation cost is sufficiently high, firms locate approximately at $(0.2676, 0.7324)$. We interpret this difference as follows. In our model the transportation cost is paid on every unit bought. This further lowers the quantity demanded through the quadratic transportation cost paid per product. By agglomerating at the center firms forgo suboptimally high amount of profits from the edge of the market.

Appendix 1.

It is shown below that the profit functions π_1 and π_2 are strictly quasiconcave in their own arguments.

From the definition of z one obtains: $\partial z/\partial p_1 = -1/[2b(x_2 - x_1)]$ and $\partial z/\partial p_2 = 1/[2b(x_2 - x_1)]$. Therefore, the first and second order derivatives can be written as:

$$\begin{aligned}
\frac{\partial \pi_1}{\partial p_1} &= (A - 2p_1 - \eta_1)z - \frac{p_1}{2b(x_2 - x_1)}\theta \\
\frac{\partial \pi_2}{\partial p_2} &= (A - 2p_2 - \eta_2)(1 - z) - \frac{p_2}{2b(x_2 - x_1)}\theta \\
\frac{\partial^2 \pi_1}{\partial p_1^2} &= -2\left(\frac{\theta - p_1}{2b(x_2 - x_1)} + z\right) - \frac{p_1}{2b(x_2 - x_1)} \times \frac{z - x_1}{x_2 - x_1} \\
\frac{\partial^2 \pi_2}{\partial p_2^2} &= -2\left(\frac{\theta - p_2}{2b(x_2 - x_1)} + 1 - z\right) + \frac{p_2}{2b(x_2 - x_1)} \times \frac{z - x_2}{x_2 - x_1}
\end{aligned}$$

where η_1, η_2 and θ are defined in the statement of Theorem 1.

We show that $\partial^2\pi_1/\partial p_1^2$ is negative when $\partial\pi_1/\partial p_1$ is zero. The second-order condition can be rewritten as

$$z + \frac{A - 2p_1}{2b(x_2 - x_1)} - \frac{b(z - x_1)^2}{2b(x_2 - x_1)} + \frac{bp_1(z - x_1)}{[2b(x_2 - x_1)]^2} > 0.$$

We will show that this is positive when $\partial\pi_1/\partial p_1$ is zero.

First suppose that $z - x_1 \geq 0$. Suppose further that $z > p_1/[2b(x_2 - x_1)]$. Then the expression is positive since $A - p_1 - b(z - x_1)^2$ is nonnegative.

Next suppose that $z \leq p_1/[2b(x_2 - x_1)]$. Then $-(z - x_1)^2 + [p_1(z - x_1)/[2b(x_2 - x_1)]] \geq -(z - x_1)^2 + z(z - x_1) = (z - x_1)(z - z + x_1) \geq 0$.

Therefore, suppose that $z - x_1 < 0$. The fact that $\partial\pi_1/\partial p_1 = 0$ is actually needed here.

First note that when $\partial\pi_1/\partial p_1 = 0$, $A - 2p_1 - (b/3)[(z - x_1)^2 - (z - x_1)x_1 + x_1^2] \geq 0$. Moreover, $(z - x_1)^2 - (z - x_1)z + (1/3)z^2 = (1/3)[(z - x_1)^2 - (z - x_1)x_1 + x_1^2]$. When $z - x_1 < 0$, $(z - x_1)^2 < (1/3)[(z - x_1)^2 - (z - x_1)x_1 + x_1^2]$. Therefore, $A - p_1 - b(z - x_1)^2 \geq A - 2p_1 - b(z - x_1)^2 \geq A - 2p_1 - (b/3)[(z - x_1)^2 - (z - x_1)x_1 + x_1^2]$. This means, $z \geq p_1/[2b(x_2 - x_1)]$.

Since $z - x_1 < 0$, $p_1(z-x_1)/[2b(x_2-x_1)] \geq z(z-x_1)$. Therefore, it suffices to show that

$$z + \frac{A - 2p_1}{2b(x_2 - x_1)} - \frac{b(z-x_1)^2}{2b(x_2-x_1)} + \frac{bz(z-x_1)}{2b(x_2-x_1)}$$

is positive. From above, $A - 2p_1 - b(z-x_1)^2 + bz(z-x_1) = A - 2p_1 - (b/3)[(z-x_1)^2 - (z - x_1)x_1 + x_1^2] + (b/3)z^2 \geq 0$.

Analogous arguments applies to firm 2. This shows that Nash equilibrium in prices exist.

The contraction argument. The remainder of the Appendix is devoted to the uniqueness issue. From (1) it can be shown that

$$\frac{\partial p_1}{\partial p_2} = \frac{\theta + p_1\left[(z-x_2)/(x_2-x_1)\right]}{2\left[\theta + 2b(x_2-x_1)z\right] + p_1\left[(z-x_2)/(x_2-x_1)\right] - p_1}.$$

The denominator is positive by the second order condition of firm 1. It can be shown that the numerator is positive and the ratio is less than 1. A similar argument applies to firm 2. Therefore, the best response mapping is a contraction, hence the price equilibrium is unique.

Appendix 2.

This appendix provides a sketch of the proof for Lemma 1.

Taking the difference of (2) and (1), one can show that

$$\frac{p_2 - p_1}{b(x_2 - x_1)} = \frac{\beta_1 + \beta_2 - (b/3) + (b/2)(x_2 - x_1)}{\theta + \beta_1 + \beta_2 + (b/6) + (3b/2)(x_2 - x_1)}(1 - x_1 - x_2)$$

where, $\beta_i = A - 2p_i - \eta_i$, $i = 1, 2$. Clearly, the denominator positive. It can be shown that the numerator is positive as well.

Appendix 3.

This Appendix provides the expressions for $\partial\tilde{\pi}_1/\partial x_1$ and $\partial\tilde{\pi}_2/\partial x_2$.

The Nash equilibrium prices p_1 and p_2 are functions of x_1 and x_2. Denote $\partial p_i/\partial x_j$ by w_{ij}; $i, j = 1, 2$.

Applying implicit function theorem to first-order conditions (1) and (2) gives the comparative statics expressions w_{ij}:

$$w_{ij} = \left[-\frac{\partial^2 \pi_j}{\partial p_j^2} \frac{\partial^2 \pi_i}{\partial x_j \partial p_i} + \frac{\partial^2 \pi_i}{\partial p_j \partial p_i} \frac{\partial^2 \pi_j}{\partial x_j \partial p_j} \right] / \Gamma$$

where

$$\Gamma = \frac{\partial^2 \pi_1}{\partial p_1^2} \frac{\partial^2 \pi_2}{\partial p_2^2} - \frac{\partial^2 \pi_1}{\partial p_2 \partial p_1} \frac{\partial^2 \pi_2}{\partial p_1 \partial p_2}$$

is the determinant of Jacobian matrix of best response mapping and is positive by contraction.

The first order conditions for the maximization are:

$$\begin{aligned}
\frac{\partial \tilde{\pi}_1}{\partial x_1} &= \frac{\partial \pi_1}{\partial p_2} \frac{\partial p_2}{\partial x_1} + \frac{\partial \pi_1}{\partial x_1} \\
&= p_1 \left[\theta \left(\frac{w_{21}}{2b(x_2 - x_1)} + \frac{z - x_1}{x_2 - x_1} \right) + b\,(z - x_1)^2 - b x_1^2 \right] \\
\frac{\partial \tilde{\pi}_2}{\partial x_2} &= \frac{\partial \pi_2}{\partial p_1} \frac{\partial p_1}{\partial x_2} + \frac{\partial \pi_2}{\partial x_2} \\
&= p_2 \left[\theta \left(\frac{w_{12}}{2b(x_2 - x_1)} + \frac{x_2 - z}{x_2 - x_1} \right) - b\,(z - x_2)^2 + b\,(1 - x_2)^2 \right]
\end{aligned}$$

and K_1 and K_2 in section 5 are defined from the above expressions.

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