The Perron-Frobenius Theorem under Weak Indecomposability and Weak Monotonicity

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Abstract

This paper examines continuous and homogenous mappings from an Euclidean space to itself. A new concept weak indecomposability is introduced. A mapping is called weakly indecomposable if whenever one vector dominates another in some but not all the components then the images of these two vectors are not equal in one of the components where equality held originally for some iterate of the mapping. For nonnegative square matrices, the two concepts weak indecomposability and indecomposability are equivalent. For nonlinear mappings, weak indecomposability is strictly weaker than indecomposability. The existence of a positive eigenvalue is proved under monotonicity and weak indecomposability. The corresponding eigenvector is also positive. This eigenvalue dominates all other eigenvalues in modulus. Bounds for this dominant eigenvalue are given. Subsequently monotonicity is weakened to weak monotonicity. Analogous results are obtained for weakly monotone and weakly indecomposable mappings.

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1 Introduction

The Perron-Frobenius theorem for nonnegative square matrices originated with a series of writings by Perron (1907a, 1907b) and Frobenius (1908, 1909, 1912). It asserts that such a matrix has a nonnegative eigenvalue and the corresponding eigenvector is also nonnegative. The eigenvalue dominates any other eigenvalue in modulus and is often referred to as the dominant eigenvalue or the Frobenius root. In addition, if the matrix is indecomposable then both the eigenvalue and the eigenvector are positive. This result has been of considerable interest in various fields.

In economics, the Perron-Frobenius theorem is the principal tool in the analysis of linear input-output models. It has led to further studies in matrix theory such as Debreu and Herstein (1953) and McKenzie (1960). In the context of input-output models, the dominant eigenvalue of the input matrix determines the rate of growth and the eigenvector corresponds to a balanced growth path.

Solow and Samuelson (1953) considered a more general production system by dropping the additivity assumption for matrices and obtained a balanced growth path for all the sectors in the economy. In this nonlinear setting, Morishima (1964) and Morishima and Fujimoto (1974) proved the existence of a positive eigenvalue and a positive eigenvector under indecomposability. Kohlberg (1982) showed that if the mapping is primitive, in addition to existence, repeated iterations take any semipositive vector to the positive eigenvector. Bounds for the dominant eigenvalue were obtained in Rath (1986). The monograph by Lemmens and Nussbaum (2012) contains an excellent account of the developments in nonlinear mappings and various applications. The Krein-Rutman theorem is an important result related to the Perron-Frobenius theorem. Chang (2014) presents a unifying approach to both the results.

In the context of production, indecomposability means a strong linkage among the different production sectors. In the linear case, joint production is not possible and inputs are combined in fixed proportions to produce a unit of a single good. The inputs are combined linearly to produce a positive output vector. Indecomposability in this context, i.e., each good needs every other good directly or indirectly for its production, is a compelling assumption.

In nonlinear systems, however, joint production is possible and the input requirement for each configuration of outputs is specified. A specific output vector may be produced using its own input requirements, but can also be produced by combining the outputs of several other production processes. Suppose that from a prespecified output vector, the outputs of some but not all the goods are increased. To require an increase in the input requirement of at least one of the goods whose output has been held fixed seems to be too strong in this context. Some production processes may exhibit this property but other processes might not. The overall production structure may have a high level of dependence among all the goods but the dependence may be less strong among specific individual production processes. The production processes might be interrelated in a notion weaker than indecomposability.

Another important notion in production is monotonicity. Higher output levels means higher input levels of all the inputs. This implies free disposal of inputs, but really means more. Since the production structure is quite general, higher output levels might be attained by increasing the inputs of some of the goods but reducing the inputs of the remaining goods. In other words, the production structure need not be monotonic.

Unfortunately, non-monotonicity and indecomposability may be incompatible sometimes. This strengthens the case for a notion weaker than indecomposability. In this paper, we relax both the notions indecomposability and monotonicity. Indecomposability is weakened to weak indecomposability and monotonicity is weakened to weak monotonicity. Consider a mapping from an Euclidean space to itself. The mapping is weakly indecomposable if whenever one vector dominates another in some but not all the components then the images of these two vectors are not equal in one of the components where equality held originally for some iterate of the mapping. Weak monotonicity has two parts. (I) If one vector dominates another, then the same relationship holds between the images for some iterate of the mapping. (II) In certain regions of the Euclidean space and for every iterate of the mapping, there is dominance among the images of the absolute value of the vector, the vector itself and the negative of the absolute value of the vector.

In terms of results we obtain the following. If the mapping is monotonic and weakly indecomposable then it has a positive eigenvalue and the corresponding eigenvector is also positive. This eigenvalue dominates all other eigenvalues in modulus. Bounds for this dominant eigenvalue are given. Thus, in the presence of monotonicity, indecomposability can be relaxed to weak indecomposability, yet all the results obtain with full generality. Subsequently monotonicity is weakened to weak monotonicity. Analogous results are obtained for weakly monotone and weakly indecomposable mappings.

The paper is organized as follows. In the next section the basic notations and some common assumptions in the literature are listed. Section 3 points out the conflict between non-monotonicity and indecomposability. It also contains the notion of weak indecomposability and a motivating example. Section 4 explores the implications of monotonicity and weak indecomposability. Theorem 1 there is the Perron-Frobenius theorem under these conditions. Theorem 2 provides bounds for the dominant eigenvalue. Section 5 introduces the notion of weak monotonicity and discusses the implications of weak monotonicity and weak indecomposability. Theorems 3 and 5 there are the counterparts of Theorems 1 and 2 under weak monotonicity and weak indecomposability. Section 6 concludes the paper. Details of some examples and proofs of the results are relegated to Appendices A and B.

2 Notations and some common assumptions

For a positive integer L, $\mathbb{R}^L$ is the L-dimensional Euclidean space and $\mathbb{R}^L_+$ its nonnegative orthant. If $x \in \mathbb{R}^L$ then $\| x \| = \sum_{i=1}^L |x_i|$ and $|x| = (|x_1|, \ldots, |x_L|) \in \mathbb{R}^L_+$. The unit simplex is $S = \{x \in \mathbb{R}^L_+ : \| x \| = 1\}$. If $x, y \in \mathbb{R}^L$, $x \leqq y$ means $x_i \leq y_i$ for every i, $x \leq y$ means $x \leqq y$ but $x \neq y$ and $x < y$ means $x_i < y_i$ for every i. For any two vectors x and y, $E_{x,y} = \{i : x_i = y_i\}$.

All matrices considered in this paper are nonnegative and square. A matrix A is *decomposable* if there is a nonempty proper subset J of $\{1, \ldots, L\}$ such that $a_{ij} = 0$ for $i \notin J$ and $j \in J$. A matrix is *indecomposable* if it is not decomposable and is not the zero matrix of order 1. Sometimes the word reducible (irreducible) is used for decomposable (indecomposable). A matrix A is *primitive* if for some positive integer p, $A^p > 0$. Every primitive matrix is indecomposable but not the converse. Proofs of the Perron-Frobenius theorem with or without indecomposability can be found in Nikaido (1968).

If A is a matrix then it has two properties: $A(\alpha x) = \alpha A x$ (homogeneity) and $A(x + y) = Ax + Ay$ (additivity). Nonlinear mappings typically relax the additivity assumption.

Let $H : \mathbb{R}^L \longrightarrow \mathbb{R}^L$ be a continuous mapping. The following are some of the common assumptions made in the literature.

(A1) Homogeneity. $H(\alpha x) = \alpha H(x)$ for any $\alpha \in \mathbb{R}$.

(A2) Nonnegativity. $H(x) \geqq 0$ for all $x \geqq 0$.

(A3) Monotonicity. If $x \leqq y$ then $H(x) \leqq H(y)$.

(A4) Indecomposability. If $L = 1$ then $H(1) > 0$. For $L \geq 2$, if $x \leq y$ and $E_{x,y}$ is a nonempty proper subset of $\{1, \ldots, L\}$ then $H_i(x) \neq H_i(y)$ for some $i \in E_{x,y}$.

(A5) Primitivity. There is an integer $\ell \geq 1$ such that for any $x \leq y$, $H^{\ell}(x) < H^{\ell}(y)$.

Monotone mappings are called order-preserving by some authors. Proofs of the Perron-Frobenius theorem with or without indecomposability can be found in Morishima (1964). If (A1) and A(2) hold, then a nonnegative eigenvalue and a semipositive eigenvector exist. If (A3) holds in addition then there are only a finite number of nonnegative eigenvalues. Under (A1)$-$(A4), both the eigenvalue and the eigenvector are positive.

3 The notion of weak indecomposability and an example

A nonnegative matrix mapping is necessarily monotonic. Even though monotonicity is commonly assumed in the literature in the nonlinear setting, a nonlinear mapping need not be monotonic. Non-monotonicity has significant implications for indecomposability, as the following illustration shows.

Define $H : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ as $H(x) = (|x_1 - x_2|, x_1 + x_2)$.[1] Let $x = (2, 1)$ and $y = (2, 2)$. Then $H(x) = (1, 3)$ and $H(y) = (0, 4)$, so H is not monotone. H is not indecomposable either. Let $0 < \epsilon < x_1 = y_1$, $x_2 = x_1 - \epsilon$ and $y_2 = y_1 + \epsilon$. Then $x \leq y$, $H(x) = (\epsilon, 2x_1 - \epsilon)$ and $H(y) = (\epsilon, 2y_1 + \epsilon)$. Since $E_{x,y} = \{1\}$ and $H_1(x) = H_1(y)$, indecomposability is violated.

This suggests that non-monotonicity and indecomposability will often be in conflict. This makes the case for a relaxed notion of indecomposability to encompass a wider class of mappings. Moreover, in the presence of monotonicity, such a notion should preserve the important results of indecomposable mappings.

(A6) Weak indecomposability. If $L = 1$ then $H(1) > 0$. For $L \geq 2$, if $x \leq y$ and $E_{x,y}$ is a

[1] This mapping is extended to $\mathbb{R}^2$ in Example 5. Many properties of the extended mapping are studied there.

nonempty proper subset of $\{1, \ldots, L\}$, then for some integer $k \geq 1$, $H_i^k(x) \neq H_i^k(y)$ for some $i \in E_{x,y}$. (In general, the integer k above will depend upon x and y.)

Two inter-connections are immediate. (1) Every indecomposable mapping is weakly indecomposable. This follows by taking $k = 1$ for every $x \leq y$ in (A6). (2) A primitive mapping is weakly indecomposable. This follows by taking $k = \ell$ for every $x \leq y$ in (A6).

The mapping in the example below is monotonic and weakly indecomposable but neither indecomposable nor primitive. It demonstrates that weak indecomposability is weaker than both indecomposability and primitivity for nonlinear mappings.

Example 1 Consider the following regions in $\mathbb{R}^2$: $W_1 = \{x \in \mathbb{R}^2_+ : x_1 \geq x_2\}$, $W_2 = \{x \in \mathbb{R}^2_+ : x_2 > x_1\}$, $W_3 = \{x \in \mathbb{R}^2 : x_1 < 0,\ x_2 > 0\}$ and $W = W_1 \cup W_2 \cup W_3$. Define $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as follows.

$$\begin{aligned} H(x) &= (2x_2, x_1) && \text{if } x \in W_1 \\ &= (x_1 + x_2, x_2) && \text{if } x \in W_2 \\ &= (x_2, x_1 + x_2) && \text{if } x \in W_3 \\ &= -H(-x) && \text{if } x \notin W. \end{aligned}$$

The nonnegativity of H is obvious. Continuity and homogeneity are relatively easy to verify and we skip the details. For monotonicity, if $x \leq y$ then $H(x) \leq H(y)$. Moreover, for any pair of vectors x and y in $\mathbb{R}^L$, either $k = 1$, or $k = 2$ in the definition of weak indecomposability. Monotonicity and weak indecomposability are verified in subsection A.1.

We will show that H is neither indecomposable nor primitive. Consider x and y such that $0 < x_1 < y_1 < y_2 = x_2$. Then $x \leq y$ and both are in W_2. $H(x) = (x_1 + x_2, x_2)$ and $H(y) = (y_1 + y_2, y_2)$. Since $x_2 = y_2$ and $H_2(x) = H_2(y)$, H is not indecomposable.

To show that H is not primitive, consider x and y such that $0 < x_1 = x_2 = y_2 = y_1/2$. Then $x \leq y$ and both are in W_1. $H(x) = (2x_2, x_1) \in W_1$ and $H^2(x) = (2x_1, 2x_2) \in W_1$. Inductively it can be shown that $H^t(x) \in W_1$ for every t, $H^t(x) = \left(2^{(t+1)/2}x_2, 2^{(t-1)/2}x_1\right)$ if t is odd and $H^t(x) = \left(2^{t/2}x_1, 2^{t/2}x_2\right)$ if t is even. Similar properties hold for y: $H^t(y) \in W_1$ for every t, $H^t(y) = \left(2^{(t+1)/2}y_2, 2^{(t-1)/2}y_1\right)$ if t is odd and $H^t(y) = \left(2^{t/2}y_1, 2^{t/2}y_2\right)$ if t is even.

Notice that if t is odd then $H_1^t(x) = H_1^t(y)$ and if t is even then $H_2^t(x) = H_2^t(y)$. Thus $H^t(x) < H^t(y)$ never holds, which shows that H is not primitive.

We now address the eigenvalue problem: $H(x) = \lambda x$, $x \neq 0$. H has two eigenvalues $\lambda^* = \sqrt{2}$ and $\bar{\lambda} = (1-\sqrt{5})/2$. The corresponding eigenvectors are $x^* = \left(\sqrt{2}, 1\right)$ and $\bar{x} = \left(-(\sqrt{5}+1)/2, 1\right)$. The reasoning is as follows.

Since eigenvectors are sign independent (by homogeneity), we will assume that an eigenvector of H belongs to W. (1) If $x \in W_2$ then $H(x) \in W_1$. So, H cannot have an eigenvector in W_2. (2) Suppose that $x \in W_1$. Then $(\lambda x_1, \lambda x_2) = (2x_2, x_1)$. If $x_2 = 0$ then $x_1 = 0$, a contradiction. Let $x_2 = 1$. Then $x_1 = \lambda$ and $\lambda^2 = 2$. This yields λ^* and x^*. (3) Suppose that $x \in W_3$. Then $(\lambda x_1, \lambda x_2) = (x_2, x_1 + x_2)$. Since $x_2 > 0$ and $x_1 < 0$, $\lambda < 0$. Let $x_2 = 1$. Then $\lambda = x_1 + 1$, or $x_1 = \lambda - 1$. From $\lambda x_1 = 1$ we get $\lambda(\lambda - 1) = 1$. This gives $\bar{\lambda}$ and $\bar{x}$.

Notice that $\lambda^* > 0$, $x^* > 0$, $\lambda^* \geq |\bar{\lambda}|$ and $\bar{x} \not\geq 0$. These are consequences of Theorem 1 below.

4 Implications of monotonicity and weak indecomposability

Throughout this section, monotonicity and weak indecomposability are assumed. The main results are Theorems 1 and 2. These are direct generalizations of results under indecomposability. First we prove some auxiliary results.

It was mentioned in the preceding section that indecomposability implies weak indecomposability. Example 1 showed that the converse does not hold for nonlinear, monotonic mappings. However, in the linear case the two concepts are equivalent.

Proposition 1 *A matrix is indecomposable iff it is weakly indecomposable.*

It is well known that if a mapping is monotonic and indecomposable, then the image of a semipositive (resp. positive) vector is semipositive (resp. positive). These conclusions are valid under weak indecomposability as well. The proofs, however, are more intricate.

Lemma 1 *Suppose that (A1)–(A3) and (A6) hold. Then $H(x) \geq 0$ if $x \geq 0$ and $H(x) > 0$ if $x > 0$.*

Theorem 1 *Suppose that (A1)–(A3) and (A6) hold.*

(i) There exist $\lambda^* > 0$ *and* $x^* > 0$ *such that* $H(x^*) = \lambda^* x^*$.

(ii) x^* *is unique up to scalar multiplication.*

(iii) If $\lambda \neq \lambda^*$ *then there is no* $x \geq 0$ *such that* $H(x) = \lambda x$.

(iv) $\lambda^* \geq |\bar{\lambda}|$ *for any eigenvalue* $\bar{\lambda}$ *of* H.

In the proof, Brouwer's fixed point theorem is used to establish *(i)*. *(iii)* shows that x^* is the only nonnegative eigenvector of H. λ^* is called the dominant eigenvalue because of *(iv)*.

Theorem 2 *Suppose that (A1)–(A3) and (A6) hold. Then for any* $x > 0$,

$$\min_i \frac{H_i(x)}{x_i} \;\leq\; \lambda^* \;\leq\; \max_i \frac{H_i(x)}{x_i}.$$

If $x = x^*$ *then both inequalities become equalities. If* $x \neq x^*$ *then both the inequalities are strict.*

For monotone and indecomposable mappings these bounds were obtained in Rath (1986). Let e denote the vector all of whose components are 1. It is obvious from the theorem that $\min_i H_i(e) \leq \lambda^* \leq \max_i H_i(e)$. In the matrix case this is equivalent to the statement that the dominant eigenvalue of an indecomposable matrix lies between the minimal and maximal column (row) sums.

Corollary 1 $x > H(x)$ *for some* $x > 0$ *iff* $\lambda^* < 1$.

If $\lambda^* < 1$ then $x^* > H(x^*)$. If $x > H(x)$ for some $x > 0$ then $\lambda^* \leq \max_i(H_i(x)/x_i) < 1$.

This is a very important result and is commonly known as the viability condition. In an economic system, the gross output of each commodity exceeds its total input requirement, so the net output vector is positive. The system is viable, i.e., capable of economic growth. If x^* is the output vector then this is the balanced growth path and the rate of balanced growth is $(1/\lambda^*) - 1$.

Recall that a primitive matrix is always indecomposable. In the following example, the mapping is monotonic and primitive (and hence weakly indecomposable), but not indecomposable.

Thus, unlike the linear case, monotonicity and primitivity do not imply indecomposability.[2]

Example 2 This example is a simple variant of Example 1. Define $\bar{H} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $\bar{H}(x) = x + H(x)$, where H is as in Example 1.

For ease of reference note that, $\bar{H}(z) = (z_1 + 2z_2, z_1 + z_2)$ if $z \in W_1$, $\bar{H}(z) = (2z_1 + z_2, 2z_2)$ if $z \in W_2$ and $\bar{H}(z) = (z_1 + z_2, z_1 + 2z_2)$ if $z \in W_3$. Also note that if $z \notin W$ then $\bar{H}(z) = z + H(z) = -(-z) - H(-z) = -(-z + H(-z)) = -\bar{H}(-z)$.

Since H is continuous, nonnegative, homogenous and monotonic, $\bar{H}$ inherits these properties. Furthermore, since $x \leq y \Rightarrow H(x) \leq H(y)$, it follows that $\bar{H}(x) \leq \bar{H}(y)$ if $x \leq y$. $\bar{H}$ is not indecomposable. Consider x and y such that $0 < x_1 < y_1 < y_2 = x_2$. Then $x \leq y$ and both are in W_2. $\bar{H}(x) = (2x_1 + x_2, 2x_2)$ and $\bar{H}(y) = (2y_1 + y_2, 2y_2)$. Since $x_2 = y_2$ and $\bar{H}_2(x) = \bar{H}_2(y)$, $\bar{H}$ is not indecomposable. $\bar{H}$ is primitive (and hence weakly indecomposable) with $\ell = 3$. Some further details are given in subsection A.6.

Here we show that ℓ cannot be reduced to 2 in this example. Let $x = (1, 6)$ and $y = (2, 6)$. Then both x and y are in W_2, $\bar{H}(x) = (8, 12)$ and $\bar{H}(y) = (10, 12)$. Both $\bar{H}(x)$ and $\bar{H}(y)$ are in W_2. So, $\bar{H}^2(x) = (28, 24)$ and $\bar{H}^2(y) = (32, 24)$. Both $\bar{H}^2(x)$ and $\bar{H}^2(y)$ are in W_1, $\bar{H}^3(x) = (76, 52)$ and $\bar{H}^3(y) = (80, 56)$. It is important to observe that $\bar{H}_2(x) = \bar{H}_2(y)$ and $\bar{H}_2^2(x) = \bar{H}_2^2(y)$.

In terms of the eigenvalues of $\bar{H}$, clearly H and $\bar{H}$ have the same eigenvectors and their eigenvalues differ by 1. So, the eigenvalues of $\bar{H}$ are $\sqrt{2} + 1$ and $(3 - \sqrt{5})/2$. The corresponding eigenvectors are $\left(\sqrt{2}, 1\right)$ and $\left(-(\sqrt{5} + 1)/2, 1\right)$.

5 Implications of weak monotonicity and weak indecomposability

In the previous section, monotonicity played an important role in the proof of Lemma 1 and consequently in establishing Theorems 1 and 2. In this section we relax monotonicity to weak monotonicity. However, this relaxed notion is not adequate to obtain the desired results and we will supplement it with other conditions. Let $\mathbb{R}^2_- = -\mathbb{R}^2_+$.

[2] It was noted in Rath (1986) that a primitive mapping need not be indecomposable. That mapping was non-monotonic, however.

(A7) Weak monotonicity. (I) If $x, y \in \mathbb{R}^L_+$, $x \leq y$ and $x \not< y$ then for some integer $p \geq 1$, $H^p(x) \leqq H^p(y)$. (The integer p, in general will depend upon x and y).

(II) If $x \notin \mathbb{R}^L_+ \cup \mathbb{R}^L_-$ then for every integer $t \geq 1$, $H^t(|x|) \geqq H^t(x) \geqq H^t(-|x|)$.

We will refer these two parts as WM I and WM II respectively. If monotonicity holds then weak monotonicity holds. The converse is not true, as any of the examples below show. A primitive mapping satisfies WM I, the integer ℓ in the definition of primitivity can serve the role of p. Notice that WM I and WM II refer to distinct regions of $\mathbb{R}^L$. In the latter, the restriction to $x \notin \mathbb{R}^L_+ \cup \mathbb{R}^L_-$ is meaningful. If $x \in \mathbb{R}^L_+ \cup \mathbb{R}^L_-$ then the condition is automatically fulfilled. If $x \in \mathbb{R}^L_+$, then $|x| = x$. So, $H^t(|x|) = H^t(x)$ and each belongs to $\mathbb{R}^L_+$. Moreover, $H^t(-|x|) \leqq 0$ for every t. Similarly, for $x \in \mathbb{R}^L_-$.

Next we introduce anther condition.

(A8) Iterative semipositivity. For some $\bar{z} \in \mathbb{R}^L_+$, $H^t(\bar{z}) \neq 0$ for every integer $t \geq 1$.

How restrictive is this condition? Possibly not very. It definitely rules out the mapping which is identically zero on the nonnegative orthant. Otherwise, it is fairly general. If the mapping is weakly indecomposable and monotone then it holds for every $\bar{z} \geq 0$ by Lemma 1. A primitive mapping satisfies this condition as well. Let $\bar{z} \geq 0$ be given. Choose x such that $0 \leq x \leq \bar{z}$. Then $H^\ell(x) < H^\ell(\bar{z})$ and for every $m \geq 1$, $H^{m\ell}(x) < H^{m\ell}(\bar{z})$. So, $H^t(\bar{z}) \neq 0$ for every integer $t \geq 1$. If this condition does not hold then the mapping eventually becomes zero for some integer at any x. If x is a semipositive eigenvector then zero is the only corresponding eigenvalue. This condition plays a key role in establishing Lemma 2.

5.1 Existence of a positive eigenvalue

Lemma 2 *Suppose that (A1), (A2), (A6), WM I and (A8) hold. Then $H(x) \geq 0$ if $x \geq 0$.*

Unlike Lemma 1, one cannot claim that if $x > 0$ then $H(x) > 0$. The examples below demonstrate this.

Theorem 3 *Suppose that (A1), (A2) and (A6)–(A8) hold.*

(i) There exist $\lambda^* > 0$ and $x^* > 0$ such that $H(x^*) = \lambda^* x^*$.

(ii) x^* *is unique up to scalar multiplication.*

(iii) If $\lambda \neq \lambda^*$ *then there is no* $x \geq 0$ *such that* $H(x) = \lambda x$.

(iv) $\lambda^* \geq |\bar{\lambda}|$ *for any eigenvalue* $\bar{\lambda}$ *of* H.

In the proof, Brouwer's fixed point theorem is used to establish *(i)*. Also in the proof, WM II is needed only in part *(iv)*.

For primitive mappings part *(iv)* can be strengthened.

Theorem 4 *Suppose that (A1), (A2) and (A5) hold. Then conclusions (i)−(iii) of Theorem 3 hold. Part (iv) of Theorem 3 holds with strict inequality for any eigenvalue* $\bar{\lambda}$ *distinct from* λ^*.

Primitivity implies weak indecomposability, WM I and iterative semipositivity. So, *(i)−(iii)* follow immediately. Only *(iv)* requires a different proof and is given in subsection B.3. It asserts that under primitivity, the dominant eigenvalue strictly dominates every other eigenvalue in modulus.[3]

5.2 Bounds for the eigenvalue

Theorem 2 in the preceding section provided certain bounds for the dominant eigenvalue of weakly indecomposable and monotone mappings. One would like to prove a similar result when monotonicity is relaxed to weak monotonicity. An extra condition is needed. Given H, define G as $G(x) = x + H(x)$.

(A9) G-property. (I) If $x, y \in \mathbb{R}^L_+$, $x \leq y$ and $x \nless y$ then for some integer $k \geq 1$ and $i \in E_{x,y}$, $G^k_i(x) < G^k_i(y)$.

(II) Let $x > 0$ and α and β be positive constants. If $\alpha x \geqq G(x)$ then $\alpha G(x) \geqq G^2(x)$. If $\beta x \leqq G(x)$ then $\beta G(x) \leqq G^2(x)$.

(I) is somewhat stronger than weak indecomposability of G on the nonnegative orthant. Here, the direction of inequality is specifically given. Notice that if G is primitive on the nonnegative orthant then (I) automatically holds. The main reasons that we impose conditions

[3] For primitive mappings from $\mathbb{R}^L_+$ to itself, Kohlberg (1982) has proved the uniqueness of λ^* and x^* and that repeated iterations of the mapping take any semipositive vector to the positive eigenvector.

on G and not on H are as follows. Even if $x > 0$, then $H(x)$ need not be positive. On the other hand, $G^t(x) > 0$ for any $t \geq 1$. In the absence of monotonicity, the weak indecomposability condition (on H) states that certain components be unequal. It does not give an inequality in any direction. On the other hand, G may have some additional property and the direction of inequality may be more specific.

Theorem 5 *Suppose that (A1), (A2) and (A6)−(A8) hold. Suppose that (a) either G is primitive on $\mathbb{R}^L_+$, or (b) (A9) holds. Then for any $x > 0$,*

$$\min_i \frac{H_i(x)}{x_i} \leq \lambda^* \leq \max_i \frac{H_i(x)}{x_i}.$$

If $x = x^*$ then both inequalities become equalities. If $x \neq x^*$ then both the inequalities are strict.

Corollary 2 *$x > H(x)$ for some $x > 0$ iff $\lambda^* < 1$.*

The proof and the interpretations are the same as Corollary 1

5.3 Examples with weak monotonicity and weak indecomposability

Three examples are given below. In each case G is primitive on $\mathbb{R}^L_+$. In Examples 3 and 4 the G-property (A9) holds but it is violated in Example 5. So, primitivity of G does not imply the G-property. The mappings are weakly indecomposable in all these examples. In Examples 3 and 5 weak monotonicity is satisfied and all the conclusions of Theorems 3 and 5 hold in these two examples. Example 4 satisfies only WM I but violates WM II. Part *(iv)* of Theorem 3 does not hold in this case. This underscores the importance of WM II in ensuring that the positive eigenvalue is dominant.

Example 3 Let $W_1 = \{x \in \mathbb{R}^2_+ : x_1 \geq x_2\}$, $W_2 = \{x \in \mathbb{R}^2_+ : x_2 > x_1\}$, $W_3 = \{x \in \mathbb{R}^2 : x_1 < 0,\ x_2 > 0,\ x_1 + x_2 > 0\}$, $W_4 = \{x \in \mathbb{R}^2 : x_1 < 0,\ x_2 > 0,\ x_1 + x_2 \leq 0\}$ and $W = \cup_{i=1}^4 W_i$. Define $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as follows.

$$
\begin{aligned}
H(x) &= (0, x_1 + x_2) && \text{if } x \in W_1 \cup W_4 \\
&= (x_2 - x_1, x_1 + x_2) && \text{if } x \in W_2 \\
&= (x_1 + x_2, x_1 + x_2) && \text{if } x \in W_3 \\
&= -H(-x) && \text{if } x \notin W.
\end{aligned}
$$

This mapping is not monotone. Let $0 < x_1 < y_1 < x_2 = y_2$. Then $H(x) = (x_2 - x_1, x_1 + x_2)$ and $H(y) = (y_2 - y_1, y_1 + y_2)$. Since $x \leq y$ and $H_1(x) > H_1(y)$, monotonicity is violated. Let $0 < x_2 < y_2 < x_1 = y_1$. Then $H(x) = (0, x_1 + x_2)$ and $H(y) = (0, y_1 + y_2)$. Since $E_{x,y} = \{1\}$ and $H_1(x) = H_1(y)$, indecomposability is violated.

To show that H is not primitive, consider x and y such that $0 < x_1 < y_1 < x_2 = y_2$. Then $x \leq y$ and both are in W_2. $H(x) = (x_2 - x_1, x_1 + x_2) \in W_2$ and $H^2(x) = (2x_1, 2x_2) \in W_2$. Inductively it can be shown that $H^t(x) \in W_2$ for every t, $H^t(x) = \big(2^{(t-1)/2}(x_2 - x_1), 2^{(t-1)/2}(x_1 + x_2)\big)$ if t is odd and $H^t(x) = \big(2^{t/2}x_1, 2^{t/2}x_2\big)$ if t is even. Similar properties hold for y: $H^t(y) \in W_2$ for every t, $H^t(y) = \big(2^{(t-1)/2}(y_2 - y_1), 2^{(t-1)/2}(y_1 + y_2)\big)$ if t is odd and $H^t(y) = \big(2^{t/2}y_1, 2^{t/2}y_2\big)$ if t is even. Notice that if t is odd then $H_1^t(x) > H_1^t(y)$ and if t is even then $H_2^t(x) = H_2^t(y)$. Thus $H^t(x) < H^t(y)$ never holds, which shows that H is not primitive.

(A8) is verified as follows. Take $0 < x_1 < x_2$. Then $x \in W_2$, $H(x) > 0$ and $H_1(x) < H_2(x)$, i.e., $H(x) \in W_2$. Subsequent iterations have this property, i.e., for any $t \geq 1$, $H^t(x) > 0$ and $H_1^t(x) < H_2^t(x)$, which means $H^t(x) \in W_2$. So, any such x can serve as $\bar{z}$ in (A8).

This mapping satisfies weak monotonicity (with p either 1, or 2) and weak indecomposability (with k either 1, or 2). Furthermore, G is primitive (with $\ell = 2$) on $\mathbb{R}_+^L$ and G-property holds. Therefore, the conclusions of Theorems 3 and 5 hold.

H has two eigenvalues, $\lambda^* = \sqrt{2}$ and $\bar{\lambda} = 0$. The corresponding eigenvectors are $x^* = (\sqrt{2} - 1, 1)$ and $\bar{x} = (-1, 1)$.

Example 4 This example is a variant of Example 3. WM II is violated in this example. Let the sets W_i's and W be as in Example 3. Define $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as follows.

$$
\begin{aligned}
H(x) &= (0, x_1 + x_2) && \text{if } x \in W_1 \\
&= (x_2 - x_1, x_1 + x_2) && \text{if } x \in W_2 \\
&= (x_2 - 2x_1, 4x_1 + x_2) && \text{if } x \in W_3 \\
&= (3x_2, x_1 - 2x_2) && \text{if } x \in W_4 \\
&= -H(-x) && \text{if } x \notin W.
\end{aligned}
$$

Notice that on $\mathbb{R}^2_+$, this mapping and the one in Example 3 coincide. So, we can conclude that none of monotonicity, indecomposability and primitivity hold in this example. Furthermore, G is primitive on $\mathbb{R}^L_+$ and G-property holds. WM I and (A8) also follow. This mapping satisfies weak indecomposability, but violates WM II.

As in Example 3, $\lambda^* = \sqrt{2}$ and $x^* = (\sqrt{2} - 1, 1)$ satisfies $\lambda^* x^* = H(x^*)$. Notice that if $\bar{x} = (-1, 1)$ then $H(\bar{x}) = (3, -3)$. So, -3 is an eigenvalue of H with the associated eigenvector $\bar{x}$. Since $|-3| > \sqrt{2}$, (iv) of Theorem 3 does not hold.

To see the failure of WM II, consider $\bar{x}$ again. Then $H(\bar{x}) = (3, -3)$, $|\bar{x}| = (1, 1)$ and $H(|\bar{x}|) = (0, 2)$. Neither $H(|\bar{x}|) \geqq H(\bar{x})$, nor $H(\bar{x}) \geqq H(-|\bar{x}|)$ holds.

Example 5 Let $W_1 = \{x \in \mathbb{R}^2 : x_1 < 0, x_2 > 0\}$ and $W = \mathbb{R}^2_+ \cup W_1$. Define $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as follows.

$$
\begin{aligned}
H(x) &= (|x_1 - x_2|, x_1 + x_2) && \text{if } x \in \mathbb{R}^2_+ \\
&= (x_1 + x_2, x_1 + x_2) && \text{if } x \in W_1 \\
&= -H(-x) && \text{if } x \notin W.
\end{aligned}
$$

As mentioned in section 3, H is neither monotone nor indecomposable. To show that H is not primitive, consider $x = (1, 2)$ and $y = (2, 2)$. Then $H(x) = (1, 3)$ and $H^2(x) = (2, 4)$. Inductively it can be shown that $H^t(x) = \left(2^{(t-1)/2}, 2^{(t-1)/2} \times 3\right)$ if t is odd and $H^t(x) = \left(2^{t/2}, 2^{(t+2)/2}\right)$ if t is even. Similarly, $H(y) = (0, 4)$ and $H^2(y) = (4, 4)$. Inductively it can be shown that $H^t(y) = \left(0, 2^{(t+3)/2}\right)$ if t is odd and $H^t(y) = \left(2^{(t+2)/2}, 2^{(t+2)/2}\right)$ if t is even. If t is odd then $H^t_1(x) > H^t_1(y)$ and if t is even then $H^t_2(x) = H^t_2(y)$. So, $H^t(x) < H^t(y)$ never holds.

This mapping satisfies weak monotonicity and weak indecomposability. (A8) follows form Example 3. Furthermore, G is primitive on $\mathbb{R}^2_+$. Thus, all the conclusions of Theorems 3 and 5 hold. This example does not satisfy (A9).

There are two eigenvalues, $\lambda^* = \sqrt{2}$ and $\bar{\lambda} = 0$. The corresponding eigenvectors are $x^* = (\sqrt{2} - 1, 1)$ and $\bar{x} = (-1, 1)$.

6 Conclusion

In this paper two weaker notions are proposed, indecomposability has been weakened to weak indecomposability and monotonicity has been weakened to monotonicity. Two versions of the Perron-Frobenius theorem have been proved, with monotonicity and weak indecomposability and with weak monotonicity and weak indecomposability. This has widened the applicability of Perron-Frobenius theory.

Often the mapping H is defined on a cone of a vector space to itself, depending on emphasis. In our context that would have meant that $H : \mathbb{R}^L_+ \longrightarrow \mathbb{R}^L_+$. In that case, from *(i)*–*(iii)* of Theorems 1, 3 and 4, λ^* and x^* are unique. Part *(iv)* of the theorems become irrelevant. The task would have been a lot easier in the sense that much fewer cases would have been examined in the examples pertaining to the property of the mappings.

However, $\lambda^* \geq |\bar{\lambda}|$ is an interesting and important result in matrix theory. It is this result that gives λ^* its name dominant eigenvalue. That is the main reason we wanted to explore Perron-Frobenius theory in its entirety. Some key insights have emerged. Theorem 1 shows that in the presence of monotonicity indecomposability can be weakened to weak indecomposability without sacrificing any of the results. If monotonicity is relaxed, then weak monotonicity (II) is the relevant condition to obtain this result.

In terms of the bounds for the dominant eigenvalue, the analysis is of course confined to $\mathbb{R}^n_+$. No difficulty was encountered in the presence of monotonicity. Without it, further conditions are needed to obtain the bounds.

There is an open problem pertaining to primitivity which is worthy of attention. It is well known that if A is an indecomposable matrix then $(I + A)^{L-1}$ is a positive matrix. Therefore, $I + A$ is a primitive matrix with $\ell \leq L - 1$. Given a nonlinear mapping H, define G as

$G(x) = x + H(x)$. It is also known that if H is indecomposable and monotonic then G is primitive with $\ell \leq L-1$. This mapping has played an important role in our analysis. However, not much is known about its properties when H is weakly indecomposable. Example 2 shows that if H is weakly indecomposable then G need not be indecomposable. In the examples, when needed, we have found that G is primitive. Any general result pertaining to primitivity of G when H is only weakly indecomposable will be useful in future.

Appendix A

A.1 Monotonicity and weak indecomposability in Example 1

We will verify monotonicity and weak indecomposability of H on $\mathbb{R}^2_+$ together. We will assume that $x \leq y$ and show that $H(x) \leq H(y)$. On some occasions, $H(x) < H(y)$. In that case, weak indecomposablity is automatically satisfied. On other occasions, we will show that either H, or H^2 satisfies the relevant inequalities for weak indecomposability. Cases (1)$-$(5) below examine situations when x and y are in W.

(1) $x, y \in W_1$, or $x, y \in W_2$, or $x, y \in W_3$. Suppose that $x, y \in W_1$. Then $H(x) = (2x_2, x_1)$ and $H(y) = (2y_2, y_1)$. Since $x \leq y$, $H(x) \leq H(y)$. If $E_{x,y} = \{1\}$ then $x_2 < y_2$ and $H_1(x) = 2x_2 < 2y_2 = H_1(y)$. If $E_{x,y} = \{2\}$ then $x_1 < y_1$ and $H_2(x) = x_1 < y_1 = H_2(y)$.

Suppose that $x, y \in W_3$. Then $H(x) = (x_2, x_1 + x_2)$ and $H(y) = (y_2, y_1 + y_2)$. Clearly, $H(x) \leq H(y)$. If $E_{x,y} = \{1\}$ then $H_1(x) = x_2 < y_2 = H_1(y)$ and if $E_{x,y} = \{2\}$ then $H_2(x) = x_1 + x_2 < y_1 + y_2 = H_2(y)$.

Suppose that $x, y \in W_2$. Then $H(x) = (x_1 + x_2, x_2) \in W_1$ and $H(y) = (y_1 + y_2, y_2) \in W_1$. Since $x \leq y$, $H(x) \leq H(y)$. $H^2(x) = (2x_2, x_1 + x_2)$ and $H^2(y) = (2y_2, y_1 + y_2)$. If $E_{x,y} = \{1\}$ then $H_1(x) = x_1 + x_2 < y_1 + y_2 = H_1(y)$ and if $E_{x,y} = \{2\}$ then $H^2_2(x) = x_1 + x_2 < y_1 + y_2 = H^2_2(y)$.

(2) $x \in W_1$, $y \in W_2$. Then $x_2 \leq x_1 \leq y_1 < y_2$, $H(x) = (2x_2, x_1)$ and $H(y) = (y_1 + y_2, y_2)$. $H_1(x) = 2x_2 \leq 2x_1 \leq 2y_1 < y_1 + y_2 = H_1(y)$ and $H_2(x) = x_1 \leq y_1 < y_2 = H_2(y)$. Thus, $H(x) < H(y)$. This implies weak indecomposablity.

(3) $x \in W_2$, $y \in W_1$. Then $x_1 < x_2 \leq y_2 \leq y_1$, $H(x) = (x_1 + x_2, x_2)$ and $H(y) = (2y_2, y_1)$.

$H_1(x) = x_1 + x_2 < 2x_2 \leq 2y_2 = H_1(y)$ and $H_2(x) = x_2 \leq y_2 \leq y_1 = H_2(y)$. So, $H(x) \leq H(y)$.

Observe that $H(x) \in W_1$, so $H^2(x) = (2x_2, x_1 + x_2)$. If $2y_2 \geq y_1$ then $H(y) \in W_1$ and $H^2(y) = (2y_1, 2y_2)$, but if $2y_2 < y_1$ then $H(y) \in W_2$ and $H^2(y) = (y_1 + 2y_2, y_1)$.

Since $x_1 < y_1$, $E_{x,y} = \{1\}$ cannot hold. Assume that $E_{x,y} = \{2\}$. If $2y_2 \geq y_1$ then $H_2^2(x) = x_1 + x_2 < 2x_2 = 2y_2 = H_2^2(y)$. If $2y_2 < y_1$ then $H_2^2(x) = x_1 + x_2 < 2x_2 = 2y_2 < y_1 = H_2^2(y)$.

(4) $x \in W_3$, $y \in W_1 \cup W_2$. Since $x \in W_3$, $x_1 < 0 < x_2$ and $H(x) = (x_2, x_1 + x_2)$.

Let $y \in W_1$. Then $H(y) = (2y_2, y_1)$. $H_1(x) = x_2 \leq y_2 < 2y_2 = H_1(y)$ and $H_2(x) = x_1 + x_2 < x_2 \leq y_2 \leq y_1 = H_2(y)$. Thus, $H(x) < H(y)$. Weak indecomposability follows.

Now suppose that $y \in W_2$. Then $H(y) = (y_1 + y_2, y_2)$. $H_1(x) = x_2 \leq y_2 \leq y_1 + y_2 = H_1(y)$ and $H_2(x) = x_1 + x_2 < x_2 \leq y_2 = H_2(y)$. Since $x_1 < 0 \leq y_1$, $E_{x,y} = \{1\}$ cannot hold. If $E_{x,y} = \{2\}$ then we have shown that $H_2(x) < H_2(y)$.

(5) $x \in W_1 \cup W_2$, $y \in W_3$. Note that $x_1 \geq 0$ and $y_1 < 0$. So, $x \leq y$ cannot hold, i.e., this possibility does not arise.

(6) $x \notin W$, $y \notin W$. $x \leq y$ implies that $-x \geq -y$. Moreover, $-x \in W$ and $-y \in W$. Above we have shown that $H(-x) \geq H(-y)$. So, $H(x) = -H(-x) \leq -H(-y) = H(y)$.

Note that $E_{-x,-y} = E_{x,y}$. We have shown that if $E_{-x,-y} = \{i\}$, $i = 1, 2$; then either $H_i(-x) > H_i(-y)$, or $H_i^2(-x) > H_i^2(-y)$. Note that $H(x) = -H(-x)$ and $H^2(x) = H(H(x)) = H(-H(-x)) = -H(H(-x)) = -H^2(-x))$. Similarly, $H(y) = -H(-y)$ and $H^2(y) = -H^2(-y))$. Therefore, either $H_i(x) = -H_i(-x) < -H_i(-y) = H_i(y)$, or $H_i^2(x) = -H_i^2(-x) < -H_i^2(-y) = H_i^2(y)$ for $E_{x,y} = \{i\}$.

(7) $x \in W$, $y \notin W$. If $x \in W_1 \cup W_2$ then $0 \leqq x \leq y$. So, $y \in W_1 \cup W_2$, a contradiction. If $x \in W_3$ then $0 < x_2 \leq y_2$ means $y \in W$, a contradiction. Thus, this possibility does not arise.

(8) $x \notin W$, $y \in W$. Suppose that $-x \in W_3$. Then $-x_1 < 0$ and $-x_2 > 0$, i.e., $x_1 > 0$ and $x_2 < 0$. $H(-x) = (-x_2, -x_1 - x_2)$ and $H(x) = (x_2, x_1 + x_2)$. Since $x_1 > 0$, $y \in W_1 \cup W_2$.

If $y \in W_1$ then $H(y) = (2y_2, y_1)$. $H_1(x) = x_2 < 0 \leq 2y_2 = H_1(y)$ and $H_2(x) = x_1 + x_2 < x_1 \leq y_1 = H_2(y)$. If $y \in W_2$ then $H(y) = (y_1 + y_2, y_2)$. $H_1(x) = x_2 < 0 \leq y_1 + y_2 = H_1(y)$ and $H_2(x) = x_1 + x_2 < x_1 \leq y_1 < y_2 = H_2(y)$. Thus, in any case, $H(x) < H(y)$. Weak indecomposability follows.

Suppose that $-x \in W_2$. Then $-x_2 > -x_1 \geq 0$, i.e., $x_2 < x_1 \leq 0$. $H(-x) = (-x_1 - x_2, -x_2)$ and $H(x) = (x_1 + x_2, x_2) < 0$. If $y \in W_1 \cup W_2$ then $H(y) \geqq 0$. So, $H(x) < H(y)$. If $y \in W_3$ then $H(y) = (y_2, y_1 + y_2)$. $H_1(x) = x_1 + x_2 < 0 < y_2 = H_1(y)$ and $H_2(x) = x_2 < x_1 \leq y_1 < y_1 + y_2 = H_2(y)$. So, $H(x) < H(y)$.

Lastly suppose that $-x \in W_1$. Then $-x_1 \geq -x_2 \geq 0$, i.e., $x_1 \leq x_2 \leq 0$. Since $x \neq 0$, $x_1 < 0$. $H(-x) = (-2x_2, -x_1)$ and $H(x) = (2x_2, x_1) \leq 0$. If $y \in W_1 \cup W_2$ then $H(y) \geqq 0$. So, $H(x) \leq H(y)$. Since $x_1 < 0$, $E_{x,y} \neq \{1\}$. If $E_{x,y} = \{2\}$ then $H_2(x) = x_1 < 0 \leq H_2(y)$.

If $y \in W_3$ then $H(y) = (y_2, y_1 + y_2)$. $H_1(x) = 2x_2 \leq 0 < y_2 = H_1(y)$ and $H_2(x) = x_1 \leq y_1 < y_1 + y_2 = H_2(y)$. So, $H(x) < H(y)$.

We have shown that H is monotonic and weakly indecomposable on $\mathbb{R}^2$.

A.2 Proof of Proposition 1

It suffices to show that weak indecomposability implies indecomposability. Assume that a matrix A is decomposable. Then there is a nonempty proper subset J of $\{1, \ldots, L\}$ such that $a_{ij} = 0$ for $i \notin J$ and $j \in J$. Let x and y be such that $0 < x_j < y_j$ if $j \in J$ and $0 \leq x_j = y_j$ if $j \notin J$. Then $x \leq y$ and $E_{x,y} = J^c$.

Let $i \in E_{x,y} = J^c$. Then

$$(Ay)_i - (Ax)_i \;=\; \sum_{j \in J} a_{ij}(y_j - x_j) + \sum_{j \notin J} a_{ij}(y_j - x_j) = \sum_{j \in J} a_{ij}(y_j - x_j).$$

The last equality follows from the fact that $x_j = y_j$ for every $j \notin J$. Since $a_{ij} = 0$ for $i \in J^c$ and $j \in J$, $(Ay)_i - (Ax)_i = 0$.

Now suppose that for some integer $m \geq 1$, $(A^m x)_j = (A^m y)_j$ for every $j \notin J$. Consider $A^{m+1}x$ and $A^{m+1}y$. If $i \in J^c$ then

$$\begin{aligned}(A^{m+1}y)_i - (A^{m+1}x)_i \;&=\; \sum_{j \in J} a_{ij}[(A^m y)_j - (A^m x)_j] + \sum_{j \notin J} a_{ij}[(A^m y)_j - (A^m x)_j] \\ &=\; \sum_{j \in J} a_{ij}[(A^m y)_j - (A^m x)_j].\end{aligned}$$

The last equality follows from the fact that $(A^m x)_j = (A^m y)_j$ for every $j \notin J$. Since $a_{ij} = 0$

for $i \in J^c$ and $j \in J$, $(A^{m+1}y)_i - (A^{m+1}x)_i = 0$.

This shows that A is not weakly indecomposable. ■

A.3 Proof of Lemma 1

First suppose that $x \geq 0$ and for some j, $x_j = 0$. Then $E_{0,x}$ is a nonempty proper subset of $\{1, \ldots, L\}$. Weak indecomposability implies that for some $k \geq 1$, $0 = H_i^k(0) \neq H_i^k(x)$ for some $i \in E_{0,x}$. Since $H^k(x) \neq 0$, $H(x) \neq 0$ and by nonnegativity $H(x) \geq 0$. Now suppose that $x > 0$. Then there exists $z \geq 0$ such that $z_j = 0$ for some j and $z \leq x$. From above, $H(z) \geq 0$ and by (A3), $H(x) \geqq H(z) \geq 0$. We have shown that $H(x) \geq 0$ if $x \geq 0$.

To show that $H(x) > 0$ when $x > 0$, we first construct a vector $w \geq 0$ such that $H(w) > 0$. Fix a component $i \in \{1, \ldots, L\}$. Define $z^i \in \mathbb{R}_+^L$ as: $z_i^i = 0$ and $z_j^i > 0$ if $j \neq i$. Then $E_{0,z^i} = \{i\}$. By nonnegativity and weak indecomposability, there is an integer $k_i \geq 1$ such that $0 < H_i^{k_i}(z^i)$. Let $w^i = z^i$ if $k_i = 1$ and $w^i = H^{k_i-1}(z^i)$ if $k_i > 1$. In either case, $H_i(w^i) > 0$. Such a $w^i \in \mathbb{R}_+^L$ can be constructed for each $i \in \{1, \ldots, L\}$. Let $w = \sum_{i=1}^L w^i$. Then $w \geqq w^i$ for each i and by monotonicity, $H(w) \geqq H(w^i)$. Since $H_i(w^i) > 0$ for each i, $H(w) > 0$.

Now let $x > 0$ be given. Then there is $\theta > 0$ such that $\theta w < x$. By homogeneity, $H(\theta w) > 0$ and by monotonicity, $H(x) \geqq H(\theta w) > 0$. ■

A.4 Proof of Theorem 1

(i) If $x \geq 0$ then $\| H(x) \| > 0$ by Lemma 1. Define a mapping F from S to itself by $F(x) = H(x)/ \| H(x) \|$. Then F is continuous. By Brouwer's fixed point theorem it has a fixed point x^*. Let $\lambda^* = \| F(x^*) \| > 0$. Then $\lambda^* x^* = H(x^*)$.

Suppose that $x_j^* = 0$ for some j. Then E_{0,x^*} is a nonempty proper subset of $\{1, \ldots, L\}$. By weak indecomposability, there exists $k \geq 1$ such that $H_i^k(0) \neq H_i^k(x^*)$ for some $i \in E_{0,x^*}$, i.e., $0 \neq (\lambda^*)^k x_i^*$. This is a contradiction since $x_i^* = 0$. Therefore, $x^* > 0$.

(ii) Suppose that there is $\bar{x} \neq 0$ such that $\lambda^* \bar{x} = H(\bar{x})$ and $\bar{x}$ is not a scalar multiple of x^*. Since eigenvectors are sign independent, without loss of generality we can assume that $\bar{x}_j > 0$ for some j. Let $J = \{j : \bar{x}_j > 0\}$ and $\theta = \max\{\bar{x}_j / x_j^* : j \in J\} > 0$. Then $\theta x^* \geqq \bar{x}$ with equality holding for some $j \in J$. If $\theta x^* = \bar{x}$ then $\bar{x}$ is a scalar multiple of x^*, a contradiction. Therefore,

$\theta x^* \geq \bar{x}$ and $E_{\bar{x},\theta x^*}$ is a nonempty proper subset of $\{1,\ldots,L\}$.

By weak indecomposability, there exists $k \geq 1$ such that $H_i^k(\bar{x}) \neq H_i^k(\theta x^*)$ for some $i \in E_{\bar{x},\theta x^*}$, i.e., $(\lambda^*)^k \bar{x}_i \neq (\lambda^*)^k \theta x_i^*$. This is a contradiction since $\bar{x}_i = \theta x_i^* > 0$. So, x^* is unique up to scalar multiplication.

Henceforth, to facilitate presentation, we will take x^* to be scalar independent.

(*iii*) Suppose for some $\lambda \neq \lambda^*$ and $x \geq 0$, $H(x) = \lambda x$. By Lemma 1, $H(x) \geq 0$, i.e., $\lambda > 0$. Suppose that $x \not> 0$. Then $E_{0,x}$ is a nonempty proper subset of $\{1,\ldots,L\}$. Weak indecomposability implies that for some $k \geq 1$ and $i \in E_{0,x}$, $0 \neq H_i^k(x) = \lambda^k x_i = 0$, a contradiction. So, $x > 0$.

Since $\lambda \neq \lambda^*$, x is not a scalar multiple of x^*. Take $x \leq x^*$, $x \not< x^*$. By monotonicity, $\lambda x = H(x) \leqq H(x^*) = \lambda^* x^*$. Since $x_i = x_i^* > 0$ for some i, $\lambda \leq \lambda^*$. Similarly, take $x^* \leq x$, $x^* \not< x$. Then $\lambda^* x^* = H(x^*) \leqq H(x) = \lambda x$ and $\lambda^* \leq \lambda$. Thus, $\lambda = \lambda^*$. This contradiction proves the claim.

(*iv*) Suppose that for some $\bar{x} \neq 0$, $H(\bar{x}) = \bar{\lambda}\bar{x}$. We have $|\bar{x}| \geqq \bar{x} \geqq -|\bar{x}|$. Choose x^* such that $x^* \geqq |\bar{x}|$ with equality holding in at least one component.

By (A1)$-$(A3), $H(x^*) \geqq H(|\bar{x}|) \geqq H(\bar{x}) \geqq H(-|\bar{x}|) = -H(|\bar{x}|)$. This implies $H(x^*) \geqq H(|\bar{x}|) \geqq |H(\bar{x})|$. Since $H(x^*) = \lambda^* x^*$ and $|H(\bar{x})| = |\bar{\lambda}\bar{x}| = |\bar{\lambda}||\bar{x}|$, $\lambda^* x^* \geq |\bar{\lambda}||\bar{x}|$. For some i, $x_i^* = |\bar{x}|_i > 0$. So, $\lambda^* \geq |\bar{\lambda}|$.

This completes the proof. ∎

A.5 Proof of Theorem 2

If $x = x^*$ then $\lambda^* = H_i(x^*)/x_i^*$ for every i and there is nothing to prove. Suppose that $x > 0$ and $x \neq x^*$. By Lemma 1, $H(x) > 0$ and repeated applications give $H^m(x) > 0$ for $m > 1$.

First we will show that $\lambda^* < \max_i[H_i(x)/x_i]$. Let $\theta_1 = \max_i[H_i(x)/x_i]$ and for $m > 1$, $\theta_m = \max_i[H_i^m(x)/H_i^{m-1}(x)]$. That $\{\theta_m\}$ is a non-increasing sequence is seen as follows. Since $\theta_1 x \geqq H(x)$, $\theta_1 H(x) \geqq H^2(x)$ by homogeneity and monotonicity. Therefore, $\theta_1 \geq H_i^2(x)/H_i(x)$ for every i, i.e., $\theta_1 \geq \theta_2$. For any $m > 1$, $\theta_m H^{m-1}(x) \geqq H^m(x)$ implies that $\theta_m H^m(x) \geqq H^{m+1}(x)$, so $\theta_m \geq \theta_{m+1}$.

Let x^* be such that $x^* \leq x$, $x^* \not< x$. Then $E_{x^*,x}$ is a nonempty proper subset of $\{1, \ldots, L\}$. By weak indecomposability, there is $k \geq 1$ such that $H_j^k(x^*) < H_j^k(x)$ for some $j \in E_{x^*,x}$. We can suppose without loss of generality that $H_j^m(x^*) = H_j^m(x)$ for every $m < k$. Then, $H_j^k(x^*)/H_j^{k-1}(x^*) < H_j^k(x)/H_j^{k-1}(x) \leq \theta_k \leq \theta_1$. Since $H_j^m(x^*) = (\lambda^*)^m x_j^*$ for every m, $H_j^k(x^*)/H_j^{k-1}(x^*) = \lambda^*$. Therefore, $\lambda^* < \theta_1 = \max_i[H_i(x)/x_i]$.

The arguments to establish $\min_i[H_i(x)/x_i] < \lambda^*$ are analogous. Let $\sigma_1 = \min_i[H_i(x)/x_i]$ and for $m > 1$, $\sigma_m = \min_i[H_i^m(x)/H_i^{m-1}(x)]$. It is easy to show that $\{\sigma_m\}$ is a non-decreasing sequence.

Choose x^* such that $x \leq x^*$, $x \not< x^*$. Then E_{x,x^*} is a nonempty proper subset of $\{1, \ldots, L\}$. By weak indecomposability, there is $k' \geq 1$ such that $H_j^{k'}(x^*) > H_j^{k'}(x))$ for some $j \in E_{x,x^*}$. We can suppose without loss of generality that $H_j^m(x^*) = H_j^m(x)$ for every $m < k'$. Therefore, $H_j^{k'}(x^*)/H_j^{k'-1}(x^*) > H_j^{k'}(x)/H_j^{k'-1}(x) \geq \sigma_{k'} \geq \sigma_1$. Since $H_j^{k'}(x^*)/H_j^{k'-1}(x^*) = \lambda^*$, $\lambda^* > \sigma_1 = \min_i[H_i(x)/x_i]$. ■

A.6 Primitivity in Example 2

Since $\bar{H}(z) = z + H(z)$, $\bar{H}(z) = (z_1 + 2z_2, z_1 + z_2)$ if $z \in W_1$, $\bar{H}(z) = (2z_1 + z_2, 2z_2)$ if $z \in W_2$ and $\bar{H}(z) = (z_1 + z_2, z_1 + 2z_2)$ if $z \in W_3$. Also note that if $z \notin W$ then $\bar{H}(z) = z + H(z) = -(-z) - H(-z) = -(-z + H(-z)) = -\bar{H}(-z)$.

Before examining primitivity, we prove a claim pertaining to this example.

Claim 1 *(i) $\bar{H}$ maps W_1 to itself. If $z \in W_2$ then $\bar{H}(z)$ can either be in W_2 but can be in W_1. In either case, $\bar{H}^2(z) \in W_1$.*

(ii) If $u < v$ then $\bar{H}(u) < \bar{H}(v)$.

(iii) If for some $u, v \in \mathbb{R}^2$ and $t \geq 1$, $\bar{H}^t(u) < \bar{H}^t(v)$ then $\bar{H}^m(u) < \bar{H}^m(v)$ for every $m \geq t$.

Proof *(i)* If $z \in W_1$, then $\bar{H}(z) = (z_1 + 2z_2, z_1 + z_2)$. Since $z_2 \geq 0$, $\bar{H}_1(z) \geq \bar{H}_2(z)$ and $\bar{H}(z) \in W_1$. If $z \in W_2$ then $\bar{H}(z) = (2z_1 + z_2, 2z_2)$. If $2z_1 \geq z_2$ then $\bar{H}(z) \in W_1$ and $\bar{H}^2(z) \in W_1$. If $2z_1 < z_2$ then $\bar{H}(z) \in W_2$. Since $H\left(\bar{H}(z)\right) = (2z_1 + 3z_2, 2z_2)$ and $\bar{H}^2(z) = \bar{H}(z) + H\left(\bar{H}(z)\right) = (4z_1 + 4z_2, 4z_2) \in W_1$.

(ii) Since $u < v$, by the monotonicity of H, $H(u) \leqq H(v)$. Therefore, $\bar{H}(u) = u + H(u) < v + H(v) = \bar{H}(v)$.

(iii) Suppose that $\bar{H}^t(u) < \bar{H}^t(v)$. By the monotonicity of H, $H(\bar{H}^t(u)) \leqq H(\bar{H}^t(v))$. So, $\bar{H}^{t+1}(u) = \bar{H}\left(\bar{H}^t(u)\right) = \bar{H}^t(u) + H\left(\bar{H}^t(u)\right) < \bar{H}^t(v) + H\left(\bar{H}^t(v)\right) = \bar{H}^{t+1}(v)$.

This completes the proof. ∎

To show that $\bar{H}$ is primitive, one needs to show that if $x \leq y$ then $\bar{H}^3(x) < \bar{H}^3(y)$. Because of Claim 1 ($iii$), it suffices to show that $\bar{H}^m(x) < \bar{H}^m(y)$ for some $1 \leq m \leq 3$. If $x < y$ then Claim 1 (ii) shows that $\bar{H}(x) < \bar{H}(y)$ and there is nothing more to prove. Henceforth, we will suppose that $x \leq y$ and $x \not< y$.

Several cases were examined to show the weak indecomposability of H in Example 1. Primitivity of $\bar{H}$ can be established by examining those cases. We will illustrate only one.

Suppose that $x \in W_1$ and $y \in W_2$. Then $x_2 \leq x_1 \leq y_1 < y_2$, $\bar{H}(x) = (x_1 + 2x_2, x_1 + x_2)$ and $\bar{H}(y) = (2y_1 + y_2, 2y_2)$. $\bar{H}_1(x) = x_1 + 2x_2 \leq 3x_1 \leq 3y_1 < 2y_1 + y_2 = \bar{H}_1(y)$ and $\bar{H}_2(x) = x_1 + x_2 \leq 2x_1 \leq 2y_1 < 2y_2 = \bar{H}_2(y)$. So, $\bar{H}(x) < \bar{H}(y)$. Claim 1 implies that $\bar{H}^3(x) < \bar{H}^3(y)$.

Appendix B

B.1 Proof of Lemma 2

First suppose that $x \geq 0$ and for some j, $x_j = 0$. Then $E_{0,x}$ is a nonempty proper subset of $\{1, \ldots, L\}$. By nonnegativity, $H(x) \geqq 0$. Weak indecomposability implies that for some $k \geq 1$, $0 = H_i^k(0) \neq H_i^k(x)$ for some $i \in E_{0,x}$. Since $H^k(x) \neq 0$, $H(x) \neq 0$ and $H(x) \geq 0$.

Now let $x > 0$. Consider $\bar{z}$ in (A8). If $\bar{z}$ is a scalar multiple of x then $H(x) \geq 0$. If $\bar{z}$ is not a scalar multiple of x then for some $\theta > 0$, $\theta\bar{z} \leq x$ and $\theta\bar{z} \not< x$. By WM I, $H^p(\theta\bar{z}) \leqq H^p(x)$ for some $p \geq 1$. Since $H^p(\theta\bar{z}) = \theta H^p(\bar{z}) \geq 0$, $H^p(x) \geq 0$. So, $H(x) \neq 0$ and by nonnegativity $H(x) \geq 0$. ∎

B.2 Proof of Theorem 3

(i) If $x \geq 0$ then $\| H(x) \| > 0$ by Lemma 2. Define a mapping F from S to itself by $F(x) = H(x)/ \| H(x) \|$. Then F is continuous. By Brouwer's fixed point theorem it has a fixed point x^*. Let $\lambda^* = \| H(x^*) \| > 0$. Then $\lambda^* x^* = H(x^*)$.

Suppose that $x_j^* = 0$ for some j. Then E_{0,x^*} is a nonempty proper subset of $\{1, \ldots, L\}$. By weak indecomposability, there exists $k \geq 1$ such that $H_i^k(0) \neq H_i^k(x^*)$ for some $i \in E_{0,x^*}$, i.e., $0 \neq (\lambda^*)^k x_i^*$. This is a contradiction since $x_i^* = 0$. Therefore, $x^* > 0$.

(ii) Suppose that there is $z \neq 0$ such that $\lambda^* z = H(z)$ and z is not a scalar multiple of x^*. Without loss of generality we can assume that $z_j > 0$ for some j. Let $J = \{j : z_j > 0\}$ and $\theta = \max\{z_j/x_j^* : j \in J\} > 0$. Then $\theta x^* \geqq z$ with equality holding for some $j \in J$. If $\theta x^* = z$ then z is a scalar multiple of x^*, a contradiction. Therefore, $\theta x^* \geq z$ and $E_{z,\theta x^*}$ is a nonempty proper subset of $\{1, \ldots, L\}$.

By weak indecomposability, there exists $k \geq 1$ such that $H_i^k(z) \neq H_i^k(\theta x^*)$ for some $i \in E_{z,\theta x^*}$, i.e., $(\lambda^*)^k z_i \neq (\lambda^*)^k \theta x_i^*$. This is a contradiction since $z_i = \theta x_i^* > 0$. This shows that x^* is unique up to scalar multiplication.

(iii) Suppose for some $\lambda \neq \lambda^*$ and $x \geq 0$, $H(x) = \lambda x$. By Lemma 2, $H(x) \geq 0$, i.e., $\lambda > 0$. Suppose that $x \not> 0$. Then $E_{0,x}$ is a nonempty proper subset of $\{1, \ldots, L\}$. Weak indecomposability implies that for some $k \geq 1$ and $i \in E_{0,x}$, $0 \neq H_i^k(x) = \lambda^k x_i = 0$, a contradiction. So, $x > 0$.

Since $\lambda \neq \lambda^*$, x is not a scalar multiple of x^*. Take $x \leq x^*$, $x \not< x^*$. By weak monotonicity, for some $p \geq 1$, $H^p(x) \leqq H^p(x^*)$, i.e, $\lambda^p x \leqq (\lambda^*)^p x^*$. Since $x_i = x_i^* > 0$ for some i, $\lambda \leq \lambda^*$. Similarly, take $x^* \leq x$, $x^* \not< x$. By weak monotonicity, for some $p' \geq 1$, $H^{p'}(x^*) \leqq H^{p'}(x)$, i.e, $(\lambda^*)^{p'} x^* \leqq \lambda^{p'} x$. Since $x_i^* = x_i > 0$ for some i, $\lambda^* \leq \lambda$. Thus, $\lambda = \lambda^*$. This contradiction proves the claim.

(iv) Suppose that for some $\bar{x} \neq 0$, $H(\bar{x}) = \bar{\lambda}\bar{x}$. Since eigenvectors are sign independent, we can assume that $\bar{x}_i > 0$ for some i. So, $\bar{x} \notin \mathbb{R}_-^L$. If $\bar{x} \geq 0$ then (iii) of the theorem ensures that $\bar{\lambda} = \lambda^*$ and there is nothing to prove. So, assume that $\bar{x} \notin \mathbb{R}_+^L \cup \mathbb{R}_-^L$.

Choose x^* such that $x^* \geqq |\bar{x}|$, $x^* \not> |\bar{x}|$. Suppose that $x^* = |\bar{x}|$. Then $H(x^*) = H(|\bar{x}|)$ and by WM II, $H(|\bar{x}|) \geqq H(\bar{x}) \geqq H(-|\bar{x}|) = -H(|\bar{x}|)$. So, $H(x^*) \geqq |H(\bar{x})|$. Since $H(x^*) = \lambda^* x^*$,

$|H(\bar{x})| = |\bar{\lambda}\bar{x}| = |\bar{\lambda}||\bar{x}|$ and $x_i^* = |\bar{x}|_i > 0$ for some i, $\lambda^* \geq |\bar{\lambda}|$.

Now suppose that $x^* \geq |\bar{x}|$. By WM I, $H^p(x^*) \geqq H^p(|\bar{x}|)$ for some $p \geq 1$. By WM II, $H^p(|\bar{x}|) \geqq H^p(\bar{x}) \geqq H^p(-|\bar{x}|) = -H^p(|\bar{x}|)$. These yield, $H^p(x^*) \geqq |H^p(\bar{x})|$, i.e., $(\lambda^*)^p x^* \geqq |\bar{\lambda}|^p|\bar{x}|$. Since $x_i^* = |\bar{x}|_i > 0$ for some i, $\lambda^* \geq |\bar{\lambda}|$.

This completes the proof. ■

B.3 Proof of Theorem 4

Suppose that for some $\bar{x} \neq 0$ and $\bar{\lambda} \neq \lambda^*$, $H(\bar{x}) = \bar{\lambda}\bar{x}$. Since eigenvectors are sign independent, we can assume that $\bar{x}_i > 0$ for some i. If $\bar{x} \geq 0$ then (iii) of the theorem ensures that $\bar{\lambda} = \lambda^*$, a contradiction. So, $|\bar{x}| \geq \bar{x}$. Since $\bar{x}_i > 0$ for some i, $\bar{x} \geq -|\bar{x}|$.

Choose x^* such that $x^* \geqq |\bar{x}|$, $x^* \not> |\bar{x}|$. Since H is primitive, there is $\ell \geq 1$ such that $H^\ell(x^*) \geqq H^\ell(|\bar{x}|) > H^\ell(\bar{x}) > H^\ell(-|\bar{x}|) = -H^\ell(|\bar{x}|)$. So, $H^\ell(|\bar{x}|) > |H^\ell(\bar{x})|$ and $H^\ell(x^*) > |H^\ell(\bar{x})|$. The last inequality gives $(\lambda^*)^\ell x^* > |\bar{\lambda}|^\ell|\bar{x}|$. Since $x_i^* = |\bar{x}|_i > 0$ for some i, $\lambda^* > |\bar{\lambda}|$.

B.4 Proof of Theorem 5

If $x = x^*$ then $\lambda^* = H_i(x^*)/x_i^*$ for every i and there is nothing to prove. Henceforth, we will work with the mapping G. Since $G(x) = x + H(x)$, G and H have the same eigenvectors and their eigenvalues differ by 1. So, x^* is an eigenvector of G associated with the eigenvalue $\lambda^* + 1$. Moreover, if $x > 0$ then for any i, $G_i(x)/x_i = (x_i + H_i(x)/x_i = 1 + (H_i(x)/x_i)$. So, we need show that if $x > 0$ and $x \neq x^*$ $\min_i(G_i(x)/x_i) < \lambda^* + 1 < \max_i(G_i(x)/x_i)$.

If G is primitive on $\mathbb{R}_+^L$ then the result has been proved in Theorem 2 of Rath (1986). Below we work with (A9). If $x > 0$ then $G(x) > 0$ and repeated applications give $G^m(x) > 0$ for $m > 1$.

First we will show that $\lambda^* + 1 < \max_i(G_i(x)/x_i)$. Let $\theta_1 = \max_i(G_i(x)/x_i)$ and for $m > 1$, $\theta_m = \max_i(G_i^m(x)/G_i^{m-1}(x))$. That $\{\theta_m\}$ is a non-increasing sequence is seen as follows. Since $\theta_1 x \geqq G(x)$, $\theta_1 G(x) \geqq G^2(x)$ by (A9). Therefore, $\theta_1 \geq G_i^2(x)/G_i(x)$ for every i, i.e., $\theta_1 \geq \theta_2$. For any $m > 1$, $\theta_m G^{m-1}(x) \geqq G^m(x)$ implies that $\theta_m G^m(x) \geqq G^{m+1}(x)$, so $\theta_m \geq \theta_{m+1}$.

Let x^* be such that $x^* \leq x$, $x^* \not< x$. Then $E_{x^*,x}$ is a nonempty proper subset of $\{1, \ldots, L\}$.

By (A9), there is $k \geq 1$ such that $G_j^k(x^*) < G_j^k(x)$ for some $j \in E_{x^*,x}$. We can suppose without loss of generality that $G_j^m(x^*) \geq G_j^m(x)$ for every $m < k$.

Therefore, $G_i^k(x^*)/G_i^{k-1}(x^*) < G_i^k(x)/G_i^{k-1}(x) \leq \theta_k \leq \theta_1$. Since $G_j^m(x^*) = (\lambda^* + 1)^m x_j^*$ for every m, $G_j^k(x^*)/G_j^{k-1}(x^*) = \lambda^* + 1$. Therefore, $\lambda^* + 1 < \theta_1 = \max_i(G_i(x)/x_i)$.

The arguments to establish $\min_i(G_i(x)/x_i) < \lambda^*+1$ are analogous. Let $\sigma_1 = \min_i(G_i(x)/x_i)$ and for $m > 1$, $\sigma_m = \min_i(G_i^m(x)/G_i^{m-1}(x))$. Using (A9), it is easy to show that $\{\sigma_m\}$ is a non-decreasing sequence.

Choose x^* such that $x \leq x^*$, $x \not< x^*$. Then E_{x,x^*} is a nonempty proper subset of $\{1, \ldots, L\}$. By (A9), there is $k' \geq 1$ such that $G_j^{k'}(x^*) > G_j^{k'}(x))$ for some $j \in E_{x,x^*}$. We can suppose without loss of generality that $G_j^m(x^*) \leq G_j^m(x)$ for every $m < k'$. Therefore, $G_j^{k'}(x^*)/G_j^{k'-1}(x^*) > G_j^{k'}(x)/G_j^{k'-1}(x) \geq \sigma_{k'} \geq \sigma_1$. Since $G_j^{k'}(x^*)/G_j^{k'-1}(x^*) = \lambda^* + 1$, $\lambda^* + 1 > \sigma_1 = \min_i(G_i(x)/x_i)$. ■

B.5 Weak monotonicity and weak indecomposability in Example 3

B.6 Weak monotonicity and weak indecomposability in Example 4

B.7 Weak monotonicity and weak indecomposability in Example 5

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