Weak Convergence of Measures and a Limit Theorem for Correspondences without Closed Graphs

Kali P. Rath

Department of Economics, University of Notre Dame, Notre Dame, IN 46556, USA

(email: rath.1@nd.edu)

Abstract

A sequence of measurable, closed-valued correspondences between metric spaces converges to a correspondence with the same properties. These correspondences need not have closed graphs. Each element of a convergent sequence of measures gives full measure to the graph of the associated correspondence. Under a uniform absolute continuity condition, the limit measure gives full measure to the limit correspondence. An application to the upper hemicontinuity of the Nash equilibrium correspondence of large games is provided.

JEL Classification: C62; C79; D50

Keywords: Absolute continuity; Closed graph; Correspondence; Upper hemicontinuity; Weak convergence

1 Introduction

Correspondences and their properties have played an important role in the development of general equilibrium theory and game theory over the last several decades. Correspondences with closed graphs help prove the existence of equilibrium and in addition, if the correspondences satisfy certain limit properties then the equilibrium correspondences are upper hemicontinuous.

We will be concerned with correspondences from a measure space to the space of payoff functions. In general, if the payoffs are continuous and the space is endowed with the sup-norm topology, then the best reply correspondences have closed graphs. These also have nice limit properties, which ensure upper hemicontinuity. On the other hand, if the continuous payoffs are given an alternative topology, or the payoffs are discontinuous then the graphs of the best reply correspondences may not be closed and need not have desired limit properties. The objective is to make some progress in this direction.

In recent years there has been a considerable growth in the literature on finite-player games with discontinuous payoffs; see Reny (2016). Carmona and Podczeck (2014) examines continuum-player games with discontinuous payoffs and obtains existence of Nash equilibria. Earlier investigations of anonymous games with discontinuous payoffs include Khan (1989) and Rath (1996). There, existence of Nash equilibrium distributions was obtained under alternative topologies on the space of payoff functions but upper hemicontinuity of the equilibrium correspondence was not obtained in all cases.

In this paper, a limit theorem is obtained for measurable correspondences. A sequence of measurable, closed-valued correspondences between metric spaces converges to a correspondence with the same properties. These correspondences need not have closed graphs. Each element of a convergent sequence of measures gives full measure to the graph of the associated correspondence. Under a uniform absolute continuity condition, the limit measure gives full measure to the limit correspondence. An example shows that this result need not obtain if the uniform absolute continuity condition is violated. An application to the upper hemicontinuity of the Nash equilibrium correspondence of large games is provided.

The paper is organized as follows. The preliminaries are given in the next section. Section 3 provides the main results. Section 4 gives an illustration with examples. Section 5 contains the proofs. An application to large games is given in section 6.

2 Preliminary Concepts

The set of positive integers is denoted by $\mathbb{N}$. Let X be a metric space. Denote by $B_r(\cdot)$ the open neighborhood of radius r of a point or set in X. Let $\{A_n\}$ be a sequence of sets in X. A point $x \in X$ belongs to $\text{Ls}(A_n)$ if every neighborhood V of x there are infinitely many n with $V \cap A_n \neq \emptyset$. A point $x \in X$ belongs to $\text{Li}(A_n)$ if every neighborhood V of x there is an integer $\bar{n}$ such that $V \cap A_n \neq \emptyset$ for all $n \geq \bar{n}$. If $\text{Li}(A_n) = \text{Ls}(A_n) = A$ then the set A is called closed limit of the sequence $\{A_n\}$.

A metric space (such as X) is always endowed with its Borel σ-algebra and all measures are Borel probability measures. A measure μ is regular if for any measurable set A, $\mu(A) = \sup\{\mu(C) : C \subseteq A, C \text{ is closed}\} = \inf\{\mu(U) : A \subseteq U, U \text{ is open}\}$. A family of measures $\{\mu^n\}$ is tight if for any $\epsilon > 0$, there exists a compact set $K_\epsilon \subseteq X$ such that $\mu^n(K_\epsilon) > 1 - \epsilon$ for all n. A measure μ is absolutely continuous with respect to a measure η if given $\epsilon > 0$ there exists $\delta > 0$ such that for any measurable set A, $\eta(A) < \delta$ implies that $\mu(A) < \epsilon$. A sequence of measures $\{\mu^n\}$ is uniformly absolutely continuous with respect to η if given $\epsilon > 0$ there exists $\delta > 0$ such that for any measurable set A, $\eta(A) < \delta$ implies that $\mu^n(A) < \epsilon$ for all n. A sequence of measures $\{\mu^n\}$ converges weakly to μ if $\{\int f \, \mathrm{d}\mu^n\} \to \int f \, \mathrm{d}\mu$ for every continuous and bounded real valued function f. Alternative characterizations of weak convergence can be found in Hildenbrand (1974, p. 48).

Let Y be a metric space. If ρ is a measure on the product space $X \times Y$, the *marginals* of ρ (denoted by subscripts) are given by $\rho_X(P) = \rho(P \times Y)$ for any measurable subset P of X and $\rho_Y(Q) = \rho(X \times Q)$ for any measurable subset Q of Y.

A correspondence $F : X \longrightarrow\!\!\!\!\to Y$ associates with each $x \in X$ a nonempty subset of Y. F is closed-valued if for every $x \in X$, $F(x)$ is a closed subset of Y. The graph of F is $\{(x, y) \in X \times Y : y \in F(x)\}$. F has a closed graph if its graph is a closed subset of $X \times Y$ and has a measurable graph if its graph is a measurable subset of $X \times Y$. F is measurable if for every closed subset C of Y, $\{x \in X : F(x) \cap C \neq \emptyset\}$ is a measurable set. F is weakly measurable if for every open subset G of Y, $\{x \in X : F(x) \cap G \neq \emptyset\}$ is a measurable set. Every measurable correspondence is weakly measurable and a compact valued, weakly measurable correspondence is measurable; Lemma 18.2 in Aliprantis and Border (2006). If Y is separable, then a closed-valued, weakly measurable correspondence has a measurable graph; Theorem 18.6 in Aliprantis and Border (2006).

3 The Results

Proposition 1 *Let M be a nonempty Borel subset of a complete separable metric space and K a nonempty compact metric space. Let Γ^n, $n \in \mathbb{N}$, be correspondences from M to K with closed graphs in $M^* = M \times K$. Let ρ and ρ^n, $n \in \mathbb{N}$, be measures on M^* such that $\{\rho^n\} \to \rho$ and for each n, $\rho^n(\text{Graph of } \Gamma^n) = 1$. Then $\rho(\text{Ls}(\text{Graph of } \Gamma^n)) = 1$.*

In particular, if Γ is a correspondence from M to K with a measurable graph and $\text{Ls}(\text{Graph of } \Gamma^n) \subseteq \text{Graph of } \Gamma$ *then* $\rho(\text{Graph of } \Gamma) = 1$.

Theorem 1 *Let M be a nonempty Borel subset of a complete separable metric space and K a nonempty compact metric space. Let Γ and Γ^n, $n \in \mathbb{N}$, be measurable, closed-valued correspondences from M to K such that* $\text{Ls}\,(\Gamma^n(x)) \subseteq \Gamma(x)$ *for each $x \in M$.*

Let ρ and ρ^n, $n \in \mathbb{N}$, be measures on the product space $M^* = M \times K$. Let μ and μ^n respectively denote their marginals on M. Suppose that (i) $\{\rho^n\} \to \rho$ (ii) for each n, $\rho^n(\text{Graph of } \Gamma^n) = 1$ and (iii) the family $\{\mu^n\}$ is uniformly absolutely continuous with respect to some measure η on M, i.e., given $\epsilon > 0$, $\exists\, \delta > 0$ such that $\eta(B) < \delta \Rightarrow \mu^n(B) < \epsilon$ for every n. Then $\rho(\text{Graph of } \Gamma) = 1$.

Proposition 1 serves as an interesting back drop to Theorem 1. Consider the following two conditions pertaining to limit properties of the sequence $\{\Gamma^n\}$. (a) $\text{Ls}(\text{Graph of } \Gamma^n) \subseteq \text{Graph of } \Gamma$. (*b*) For each $x \in M$, $\text{Ls}\,(\Gamma^n(x)) \subseteq \Gamma(x)$.

If (*a*) holds, then it can be deduced from weak convergence of measures that $\rho(\text{Graph of } \Gamma) = 1$. Lemma 2 in Khan (1989) is relevant here. However, in some contexts, the correspondences Γ^ns may not be so well-behaved, i.e., (*a*) may not hold. In that case, $\rho(\text{Graph of } \Gamma)$ may be less than one. (*b*) is, of course, significantly weaker than (*a*), and therefore, more often satisfied. Naturally, under (*b*), additional restrictions on the measures are required to assert that $\rho(\text{Graph of } \Gamma) = 1$.

The uniform absolute continuity condition is sufficient to obtain the result. Denoting by h^n the Radon-Nikodym derivative of μ^n with respect to η, this is equivalent to the statement that the collection $\{h^n\}$ is uniformly integrable. It needs to be noted that η is any measure on M and need not be the same as μ, the marginal of ρ on M.

The strategy of the proof is as follows. Consider any neighborhood V of graph of Γ. By constructing suitable sequences of sets in V it can be shown that $\rho(V)$ is 1. The regularity of ρ then implies that ρ gives unit measure to graph of Γ.

4 Examples

Two examples are given below. Example 1 is a prelude to the next one, it generates a sequence of measures which are not uniformly absolutely continuous. These measures are used in Example 2 to demonstrate that the uniform absolutely continuity condition in Theorem 1 cannot be relaxed.

Example 1 Let $M = [0,\, 1]$, ν the Dirac measure at 0 and λ the Lebesgue measure. Define μ on M by $\mu = (\nu + \lambda)/2$. For each $n \in \mathbb{N}$, define a real valued functions h^n on M as follows.

$$\begin{aligned} h^n(x) &= 0 && \text{if } x = 0 \\ &= n+1 && \text{if } 0 < x < \frac{1}{n} \\ &= 1 && \text{if } \frac{1}{n} \le x \le 1. \end{aligned}$$

For any Borel subset B of M, let $\mu^n(B) = \int_B h^n \, \mathrm{d}\mu$. For each n, μ^n is absolutely continuous with respect to μ. Moreover, for any n, $\mu^n((0,1]) = 1$. That $\{\mu^n\} \to \mu$ is seen as follows.

Let $\mathrm{bd}(\cdot)$ and $\mathrm{int}(\cdot)$ be the boundary and interior of a set respectively. Consider a set B with $\mu(\mathrm{bd}(B)) = 0$. Then $0 \notin \mathrm{bd}(B)$. Therefore, $0 \notin \mathrm{bd}(B^c)$ as well. Suppose that $0 \in B$. Then $0 \in \mathrm{int}(B)$, i.e., $\exists\, N$ such that $[0, 1/N) \subseteq B$. If $n \ge N$ then $\mu^n(B^c) = \lambda(B^c)/2 = \mu(B^c)$. So, $\mu^n(B) = \mu(B)$. Similarly, if $0 \in B^c$ then $0 \in \mathrm{int}(B^c)$, i.e., $\exists\, N$ such that $[0, 1/N) \subseteq B^c$. If $n \ge N$ then $\mu^n(B) = \lambda(B)/2 = \mu(B)$. This shows that $\{\mu^n\} \to \mu$.

For any integer $k \ge 1$, consider the open interval $A_k = (0, 1/k)$. For any k, $\int_{A_k} h^n \, \mathrm{d}\mu \ge (n+1)/(2n) \ge 1/2$ for sufficiently large n. Since the λ-measure of A_k, and consequently, the μ-measure of A_k, can be chosen to be arbitrarily small, this means that $\{h^n\}$ is not uniformly integrable with respect to μ. Therefore, $\{\mu^n\}$ is not uniformly absolutely continuous with respect to μ.

To show that $\{\mu^n\}$ is not uniformly absolutely continuous with respect to any measure η on M, consider any measure η on M and the sets A_k defined above. Since $A_{k+1} \subseteq A_k$

and $\cap_{k=1}^{\infty} A_k = \emptyset$, $\lim_{k\to\infty} \eta(A_k) = 0$. On the other hand, for any k, there exists n such that $\mu^n(A_k) \geq 1/2$.

Example 2 This example shows that the assumption that the collection of marginals of ρ^n on M being uniformly absolutely continuous with respect to some measure η on M in Theorem 1 cannot be relaxed.

Let $M = [0, 1]$ and $K = \{0, 1\}$. Let $f : M \to K$ be given by

$$\begin{aligned} f(x) &= 0 \quad \text{if } x = 0 \\ &= 1 \quad \text{if } x > 0. \end{aligned}$$

Clearly, f does not have a closed graph. Interpret $f = \Gamma = \Gamma^n$ for each n. Let μ^n and μ be as defined in Example 1. For any Borel set P of $M \times K$, let

$$\rho^n(P) = \mu^n(\{x \in M : (x, 1) \in P\}), \qquad \rho(P) = \mu(\{x \in M : (x, 1) \in P\}).$$

[The term $(x, 1)$ appearing in the above expressions means the cartesian product of $x \in M$ and $1 \in K$. It does not denote an open interval.] It follows that μ^n is the marginal of ρ^n on M and μ is the marginal of ρ on M.

To show that $\{\rho^n\} \to \rho$, consider a closed set F in $M \times K$. Then $F = (F_0 \times \{0\}) \cup (F_1 \times \{1\})$, where F_0 and F_1 are closed subsets of M. $\rho(F) = \rho(F_1 \times \{1\}) = \mu(F_1)$ and for any n, $\rho^n(F) = \rho^n(F_1 \times \{1\}) = \mu^n(F_1)$. Since $\{\mu^n\} \to \mu$, $\mu(F_1) \geq \limsup_n \mu^n(F_1)$, i.e., $\rho(F) \geq \limsup_n \rho^n(F)$. So, $\{\rho^n\} \to \rho$. However, $\rho^n(\text{Graph of } f^n) = 1$ and $\rho(\text{Graph of } f) = 1/2$.

A variant of this example is worth noting. Keeping everything as above, if we redefine $f(0) = \Gamma^n(0) = 1$ for each n, and $\Gamma(0) = 1$, or $\Gamma(0) = \{0, 1\}$; then $\rho^n(\text{Graph of } \Gamma^n) = 1$ for each n and $\rho(\text{Graph of } \Gamma) = 1$. Now Γ^n has a closed graph for each n and the result follows from Proposition 1. The uniform absolute continuity condition is not needed.

5 Proofs of Proposition 1 and Theorem 1

Proof of Proposition 1 Let $C = \text{Ls}(\text{Graph of } \Gamma^n)$. Then C is a closed subset of M^*. We will show that $\rho(C) = 1$. Suppose that $\rho(C) < 1$. Then for some $\delta > 0$, $\rho(C) < 1 - \delta$. For each

$m \in \mathbb{N}$, consider $B_{1/m}(C)$. $\{B_{1/m}(C)\}$ is a decreasing sequence of sets and $\cap_{m\in\mathbb{N}} B_{1/m}(C) = C$. So, $\{\rho(B_{1/m}(C))\} \to \rho(C)$. Since $\rho(C) < 1 - \delta$, there exists $\bar{m}$ such that for all $m \geq \bar{m}$, $\rho(B_{1/m}(C)) < 1 - \delta$.

Fix $m^* > \bar{m}$ and let D denote the closure of $B_{1/m^*}(C)$. Since $D \subseteq B_{1/\bar{m}}(C))$, $\rho(D) < 1-\delta$. Suppose that $\rho^n(D) \geq 1-\delta$ for infinitely many n. Then $\limsup_n \rho^n(D) \geq 1-\delta$. The closedness of D implies that $\rho(D) \geq 1-\delta$, a contradiction. So, $\rho^n(D) \geq 1-\delta$ for at most finitely many n. We can assume that $\rho^n(D) < 1-\delta$ for all n. It follows that $\rho^n(B_{1/m^*}(C)) < 1-\delta$ for all n.

Since M is a Borel subset of a complete separable metric space and K is a compact metric space, $M^* = M \times K$ is a Borel subset of a complete separable metric space. So, each of the measures ρ and ρ^n, $n \in \mathbb{N}$, is tight, Parthasarathy (1967, p. 29). Since $\{\rho^n\} \to \rho$, the family of measures $\{\rho, \rho^1, \rho^2, \ldots\}$ is tight, Hildenbrand (1974, p. 49). Let $0 < \epsilon < \delta$. There is a compact subset Z_ϵ of M^* such that $\rho(Z_\epsilon) > 1-\epsilon$ and $\rho^n(Z_\epsilon) > 1-\epsilon$ for $n \in \mathbb{N}$.

Let $P = Z_\epsilon \setminus B_{1/m^*}(C)$. Then P is compact and for any $n \in \mathbb{N}$, $\rho^n(P) = \rho^n(Z_\epsilon) - \rho^n(B_{1/m^*}(C)) > 1-\epsilon-(1-\delta) = \delta-\epsilon > 0$. This shows that $P \cap \text{Graph of } \Gamma^n \neq \emptyset$ for each n. Let $z^n \in P \cap \text{Graph of } \Gamma^n$. Since P is compact, some subsequence of $\{z^n\}$ converges, say to $z \in P$. Then $z \in \text{Ls}(\text{Graph of } \Gamma^n)$. This is a contradiction since P is disjoint from $C = \text{Ls}(\text{Graph of } \Gamma^n)$. Therefore, $\rho(C) = 1$. ∎

Proof of Theorem 1 Denote by $G(x)$ and $G^n(x)$ the sets $\{x\} \times \Gamma(x)$ and $\{x\} \times \Gamma^n(x)$ respectively. Since K is compact, $G(x)$ and $G^n(x)$ are compact for any x. G and G^n for each n are correspondences from M to M^*.

Let V be an open subset of M^* such that Graph of Γ is a subset of V. We will show that $\rho(V) = 1$. If $V = M^*$, then $\rho(V) = 1$. Henceforth, suppose that V is a proper subset of M^*.

For each positive integer m, let

$$W_m = \{x \in M :\ B_{1/m}(V^c) \cap G(x) \ = \ \emptyset\}.$$

Since $B_{1/m}(V^c)$ is an open set and G is weakly measurable, $\{x \in M :\ B_{1/m}(V^c) \cap G(x) \neq \emptyset\}$ is a Borel set. So, its complement W_m is a Borel set.

Clearly, $W_m \subseteq W_{m+1}$. It is shown below that if $x \in M$ then $x \in W_m$ for some m. Therefore, $\cup_{m=1}^{\infty} W_m = M$.

Suppose that for some $x \in M$, $x \notin W_m$ for any m. Then for some $y^m \in \Gamma(x)$ and $z^m = (x, y^m)$, $z^m \in B_{1/m}(V^c)$. $\{y^m\}$ is a sequence in $\Gamma(x)$, which is compact. So, some subsequence converges, say to y. Without loss of generality we assume that the entire sequence converges. Clearly, $y \in \Gamma(x)$ and $z = (x, y) \in G(x)$. Since $z^m \in B_{1/m}(V^c)$ for every m and V^c is closed, $z \in V^c$. This is a contradiction since $G(x) \subseteq V$.

For each positive integer n, let

$$W_{mn} \;=\; \{x \in W_m : \text{ for all } k \geq n,\ B_{1/2m}(V^c) \cap G^k(x) \;=\; \emptyset\}.$$

For $k \in \mathbb{N}$, let $U_{mk} \;=\; \{x \in W_m :\ B_{1/2m}(V^c) \cap G^k(x) \;\neq\; \emptyset\}$. Since $B_{1/2m}(V^c)$ is an open set and G^k is weakly measurable, U_{mk} is a Borel set. Fix any n and let $U^*_{mn} = \cup_{k \geq n} U_{mk}$. Then U^*_{mn} is a Borel set, so its complement W_{mn} is Borel set.

Clearly, $W_{mn} \subseteq W_{m,n+1}$. Now we show that if $x \in W_m$ then $x \in W_{mn}$ for some n. So, $\cup_{n=1}^{\infty} W_{mn} = W_m$.

Let $x \in W_m$. Suppose that $x \notin W_{mn}$ for any n. Then there are sequences $\{n_1, n_2, \ldots\}$ and $\{k_1, k_2, \ldots\}$ such that $k_t \geq n_t$ and $B_{1/2m}(V^c) \cap G^{k_t}(x) \neq \emptyset$. So, $\exists\, y^t$ such that $y^t \in \Gamma^{k_t}(x)$ and $(x, y^t) \in B_{1/2m}(V^c)$. $\{y^t\}$ is a sequence in K, so some subsequence converges, say to y. Since $\{(x, y^t)\}$ is a sequence in $B_{1/2m}(V^c)$, (x, y) belongs to the closure of $B_{1/2m}(V^c)$. The closure of $B_{1/2m}(V^c)$ is a subset of $B_{1/m}(V^c)$.

Since $\mathrm{Ls}(\Gamma^i(x)) \subseteq \Gamma(x)$, $y \in \Gamma(x)$ and $(x, y) \in G(x)$. Since $B_{1/m}(V^c) \cap G(x) = \emptyset$, we get a contradiction. So, if $x \in W_m$ then $x \in W_{mn}$ for some n.

Next we show that for any $\epsilon > 0$, there exist m and n such that, $\mu^k(W_{mn}) > 1 - \epsilon$ for every k. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$, such that $\eta(B) < \delta \Rightarrow \mu^k(B) < \epsilon$ for every k. Since $W_m \subseteq W_{m+1}$ and $\cup_{m=1}^{\infty} W_m = M$, there exists N such that for all $m \geq N$, $\eta(W_m) > 1 - \delta$. Fix any $m \geq N$. $W_{mn} \subseteq W_{m,n+1}$ and $\cup_{n=1}^{\infty} W_{mn} = W_m$. So, for some n, $\eta(W_{mn}) > 1 - \delta$. This implies that $\mu^k(W_{mn}) > 1 - \epsilon$ for every k.

Let $0 < \epsilon < 1$. Fix any m and n such that $\mu^k(W_{mn}) > 1 - \epsilon > 0$ for every k. Let $F_k \;=\;$ Graph of $\Gamma^k \cap (W_{mn} \times K)$. Since ρ^k(Graph of $\Gamma^k) \;=\; 1$, $\rho^k(F_k) \;=\; \rho^k(W_{mn} \times K) \;=\; \mu^k(W_{mn}) > 1 - \epsilon$.

We will show that if $k \geq n$, then $B_{1/2m}(V^c) \cap F_k = \emptyset$. Let $(x, y) \in F_k$. Then $(x, y) \in$ Graph of Γ^k and $(x, y) \in W_{mn} \times K$. So, $(x, y) \in G^k(x)$ and $x \in W_{mn}$. If $x \in W_{mn}$ then

$B_{1/2m}(V^c) \cap G^k(x) = \emptyset$ for $k \geq n$. So, if $k \geq n$ then $B_{1/2m}(V^c) \cap F_k = \emptyset$.

Now we show that $\rho(V) = 1$. For each k, $\rho^k(F_k) > 1 - \epsilon$. From above, $B_{1/2m}(V^c) \cap F_k = \emptyset$ for $k \geq n$. Therefore, for $k \geq n$, $\rho^k(B_{1/2m}(V^c)) < \epsilon$.

$B_{1/2m}(V^c)$ is open and since $\{\rho^k\} \to \rho$, $\rho(B_{1/2m}(V^c)) \leq \liminf_k \rho^k(B_{1/2m}(V^c)) \leq \epsilon$. This implies that $\rho(V^c) \leq \epsilon$. So, $\rho(V) \geq 1 - \epsilon$. Since this is true for any $\epsilon > 0$, $\rho(V) = 1$.

That $\rho(\text{Graph of } \Gamma) = 1$ is established as follows. Since M^* is a metric space, ρ is regular, Parthasarathy (1967, p. 27). So, $\rho(\text{Graph of } \Gamma) = \inf \{\rho(V) : \text{ Graph of } \Gamma \subseteq V,\ V \text{ is open}\}$. Since $\rho(V) = 1$ for any open set V containing the Graph of Γ, $\rho(\text{Graph of } \Gamma) = 1$. This completes the proof. ∎

6 An Application

First provide a general description of large games and its Nash equilibria. Then provide two examples. In the first example, the payoff functions are continuous, the uniform absolute continuity condition in Theorem 1 does not hold and the result does not obtain. In the second example, the payoff functions are upper semicontinuous, all the conditions of Theorem 1 hold and its conclusion follows.

References

C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, 3rd edition, Springer-Verlag, Berlin, 2006.

G. Carmona and K. Podczeck, Existence of Nash Equilibrium in Games with a Measure Space of Players and Discontinuous Payoff Functions, *Journal of Economic Theory*, 152 (2014), 130–178.

W. Hildenbrand, *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, 1974.

M. Ali Khan, On Cournot-Nash Equilibrium Distributions for Games with a Nonmetrizable Action Space and Upper Semicontinuous Payoffs, *Transactions of the American Mathematical Society* 315 (1989), 127–146.

M. Ali Khan, K. P. Rath, H. Yu and Y. Zhang, On the Equivalence of Large Individualized and Distributionalized Games, *Theoretical Economics*, 12 (2017), 533–554.

K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.

K. P. Rath, Existence and Upper Hemicontinuity of Equilibrium Distributions of Anonymous Games with Discontinuous Payoffs, *Journal of Mathematical Economics* 26 (1996), 305–324.

P. J. Reny, Introduction to the Symposium on Discontinuous Games, *Econ Theory* 61 (2016), 423–429.