# Weak Convergence of Measures and a Limit Theorem for Correspondences without Closed Graphs

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**Abstract**

A sequence of measurable, closed-valued correspondences between metric spaces converges to a correspondence with the same properties. These correspondences need not have closed graphs. Each element of a convergent sequence of measures gives full measure to the graph of the associated correspondence. Under a uniform absolute continuity condition, the limit measure gives full measure to the limit correspondence. An application to the upper hemicontinuity of the Nash equilibrium correspondence of large games is provided.

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# 1 Introduction

In equilibrium theory, existence and upper hemicontinuity are two issues of paramount importance. The properties of the underlying correspondences play a crucial role in answers to these questions. Correspondences with closed graphs help prove the existence of equilibrium and in addition, if the correspondences satisfy certain limit properties then the equilibrium correspondences are upper hemicontinuous.

We will be concerned with correspondences between metric spaces. The domain can be identified with the payoff functions and the range with actions. In general, if the payoffs are continuous and the space is endowed with the sup-norm topology, then the best reply correspondences have closed graphs. These also have nice limit properties, which ensure upper hemicontinuity of the equilibrium correspondence. On the other hand, if the continuous payoffs are given an alternative topology, or the payoffs are discontinuous then the graphs of the best reply correspondences may not be closed and need not have desired limit properties. The objective is to make some progress in this direction.

In recent years there has been a considerable growth in the literature on finite-player games with discontinuous payoffs; see Reny (2016) and the articles in the same volume. Carmona and Podczeck (2014) examines continuum-player games with discontinuous payoffs. The emphasis in these papers has been the existence of Nash equilibria. Earlier investigations of anonymous games with discontinuous payoffs include Khan (1989) and Rath (1996). There, existence of Nash equilibrium distributions was obtained under alternative topologies on the space of payoff functions but upper hemicontinuity of the equilibrium correspondence was not obtained in all cases.

In this paper, a limit theorem is obtained for measurable correspondences. A sequence of measurable, closed-valued correspondences between metric spaces converges to a correspondence with the same properties. These correspondences need not have closed graphs. Each element of a convergent sequence of measures gives full measure to the graph of the associated correspondence. Under a uniform absolute continuity condition, the limit measure gives full measure to the limit correspondence. An example shows that this result need not obtain if the uniform absolute continuity condition is violated. An application to the upper hemicontinuity of the Nash equilibrium correspondence of large games is provided.

The paper is organized as follows. The preliminaries are given in the next section. Section 3 provides the main results. Section 4 gives an illustration with examples. Section 5 contains the proofs. An application to large games is given in section 6.

## 2 Preliminary Concepts

The set of positive integers is denoted by $\mathbb{N}$. Let $X$ be a metric space. Denote by $B_r(\cdot)$ the open neighborhood of radius $r$ of a point or set in $X$. Let $\{A_n\}$ be a sequence of sets in $X$. A point $x \in X$ belongs to $\mathrm{Ls}(A_n)$ if for every neighborhood $V$ of $x$ there are infinitely many $n$ with $V \cap A_n \neq \emptyset$. A point $x \in X$ belongs to $\mathrm{Li}(A_n)$ if for every neighborhood $V$ of $x$ there is an integer $\bar{n}$ such that $V \cap A_n \neq \emptyset$ for all $n \geq \bar{n}$. If $\mathrm{Li}(A_n) = \mathrm{Ls}(A_n) = A$ then the set $A$ is called closed limit of the sequence $\{A_n\}$.

A metric space, such as $X$, is always endowed with its Borel $\sigma$-algebra and all measures are Borel probability measures. A measure $\mu$ is regular if for any measurable set $A$, $\mu(A) = \sup\{\mu(C) : C \subseteq A, C \text{ is closed}\} = \inf\{\mu(U) : A \subseteq U, U \text{ is open}\}$. A family of measures $\{\mu^n\}$ is tight if for any $\epsilon > 0$, there exists a compact set $K_\epsilon \subseteq X$ such that $\mu^n(K_\epsilon) > 1 - \epsilon$ for all $n$. A sequence of measures $\{\mu^n\}$ converges weakly to $\mu$ if $\{\int f \, \mathrm{d}\mu^n\} \to \int f \, \mathrm{d}\mu$ for every continuous and bounded real valued function $f$. Alternative characterizations of weak convergence can be found in Hildenbrand (1974, p. 48).

Let $Y$ be a metric space. If $\rho$ is a measure on the product space $X \times Y$, the marginals of $\rho$ (denoted by subscripts) are given by $\rho_X(P) = \rho(P \times Y)$ for any measurable subset $P$ of $X$ and $\rho_Y(Q) = \rho(X \times Q)$ for any measurable subset $Q$ of $Y$.

A correspondence $F : X \longrightarrow\!\!\!\!\!\!\to Y$ associates with each $x \in X$ a nonempty subset of $Y$. $F$ is closed-valued if for every $x \in X$, $F(x)$ is a closed subset of $Y$. The graph of $F$ is $\{(x, y) \in X \times Y : y \in F(x)\}$. $F$ has a closed graph if its graph is a closed subset of $X \times Y$ and has a measurable graph if its graph is a measurable subset of $X \times Y$. $F$ is measurable if for every closed subset $C$ of $Y$, $\{x \in X : F(x) \cap C \neq \emptyset\}$ is a measurable set. $F$ is weakly measurable if for every open subset $G$ of $Y$, $\{x \in X : F(x) \cap G \neq \emptyset\}$ is a measurable set. Every measurable correspondence is weakly measurable and a compact-valued, weakly measurable correspondence is measurable. If $Y$ is separable, then a closed-valued, weakly measurable correspondence has a measurable graph. These results can be found in Aliprantis and Border (2006, pp.593−596).

## 3 The Results

**Proposition 1** *Let $M$ be a nonempty Borel subset of a complete separable metric space and $K$ a nonempty compact metric space. Let $\Gamma^n$, $n \in \mathbb{N}$, be correspondences from $M$ to $K$ with closed graphs in $M^* = M \times K$. Let $\rho$ and $\rho^n$, $n \in \mathbb{N}$, be measures on $M^*$ such that $\{\rho^n\} \to \rho$ and for each $n$, $\rho^n(\text{Graph of } \Gamma^n) = 1$. Then $\rho(\text{Ls}(\text{Graph of } \Gamma^n)) = 1$.*

*In particular, if $\Gamma$ is a correspondence from $M$ to $K$ with a measurable graph and* $\text{Ls}(\text{Graph of } \Gamma^n) \subseteq \text{Graph of } \Gamma$ *then* $\rho(\text{Graph of } \Gamma) = 1$.

**Theorem 1** *Let $M$ be a nonempty Borel subset of a complete separable metric space and $K$ a nonempty compact metric space. Let $\Gamma$ and $\Gamma^n$, $n \in \mathbb{N}$, be measurable, closed-valued correspondences from $M$ to $K$ such that* $\text{Ls}\,(\Gamma^n(x)) \subseteq \Gamma(x)$ *for each $x \in M$.*

*Let $\rho$ and $\rho^n$, $n \in \mathbb{N}$, be measures on the product space $M^* = M \times K$. Let $\mu$ and $\mu^n$ respectively denote their marginals on $M$. Suppose that (i) $\{\rho^n\} \to \rho$ (ii) for each $n$,* $\rho^n(\text{Graph of } \Gamma^n) = 1$ *and (iii) the family $\{\mu^n\}$ is uniformly absolutely continuous with respect to some measure $\eta$ on $M$, i.e., given $\epsilon > 0$, $\exists\, \delta > 0$ such that $\eta(B) < \delta \Rightarrow \mu^n(B) < \epsilon$ for every $n$. Then* $\rho(\text{Graph of } \Gamma) = 1$.

Proposition 1 serves as an interesting back drop to Theorem 1. Consider the following two conditions pertaining to limit properties of the sequence $\{\Gamma^n\}$. (a) $\text{Ls}(\text{Graph of } \Gamma^n) \subseteq \text{Graph of } \Gamma$. (*b*) For each $x \in M$, $\text{Ls}\,(\Gamma^n(x)) \subseteq \Gamma(x)$.

If (*a*) holds, then it can be deduced from weak convergence of measures that $\rho(\text{Graph of } \Gamma) = 1$. Lemma 2 in Khan (1989) is relevant here. However, in some contexts, the correspondences $\Gamma^n$s may not be so well-behaved, i.e., (a) may not hold. In that case, $\rho(\text{Graph of } \Gamma)$ may be less than one. (*b*) is, of course, significantly weaker than (*a*), and therefore, more often satisfied. Naturally, under (*b*), additional restrictions on the measures are required to assert that $\rho(\text{Graph of } \Gamma) = 1$.

The uniform absolute continuity condition is sufficient to obtain the result. Denoting by $h^n$ the Radon-Nikodym derivative of $\mu^n$ with respect to $\eta$, this is equivalent to the statement that the collection $\{h^n\}$ is uniformly integrable. It needs to be noted that $\eta$ is any measure on $M$ and need not be the same as $\mu$, the marginal of $\rho$ on $M$.

The strategy of the proof is as follows. Consider any neighborhood $V$ of graph of $\Gamma$. By constructing suitable sequences of sets in $V$ it can be shown that $\rho(V)$ is 1. The regularity of $\rho$ then implies that $\rho$ gives unit measure to graph of $\Gamma$. Example 2 below shows that if the uniform absolute continuity condition is relaxed, then the result may not hold.

# 4 Examples

Two examples are given below. Example 1 is a prelude to the next one, it generates a sequence of measures which are not uniformly absolutely continuous. These measures are used in Example 2 to demonstrate that the uniform absolutely continuity condition in Theorem 1 cannot be relaxed.

**Example 1** Let $M = [0, 1]$, $\nu$ the Dirac measure at 0 and $\lambda$ the Lebesgue measure. Define $\mu$ on $M$ by $\mu = (\nu + \lambda)/2$. For each $n \in \mathbb{N}$, define a sequence of real valued functions $\{h^n\}$ on $M$ as follows.

$$\begin{aligned} h^n(x) &= 0 && \text{if } x = 0 \\ &= n+1 && \text{if } 0 < x < \frac{1}{n} \\ &= 1 && \text{if } \frac{1}{n} \leq x \leq 1. \end{aligned}$$

For any Borel subset $B$ of $M$, let $\mu^n(B) = \int_B h^n \, \mathrm{d}\mu$. For each $n$, $\mu^n$ is absolutely continuous with respect to $\mu$. Moreover, for any $n$, $\mu^n((0, 1]) = 1$. That $\{\mu^n\} \to \mu$ is seen as follows.

Let $\mathrm{bd}(\cdot)$ and $\mathrm{int}(\cdot)$ be the boundary and interior of a set respectively. Consider a set $B$ with $\mu(\mathrm{bd}(B)) = 0$. Then $0 \notin \mathrm{bd}(B)$. Therefore, $0 \notin \mathrm{bd}(B^c)$ as well. Suppose that $0 \in B$. Then $0 \in \mathrm{int}(B)$, i.e., $\exists\, N$ such that $[0, 1/N) \subseteq B$. If $n \geq N$ then $\mu^n(B^c) = \lambda(B^c)/2 = \mu(B^c)$. So, $\mu^n(B) = \mu(B)$. Similarly, if $0 \in B^c$ then $0 \in \mathrm{int}(B^c)$, i.e., $\exists\, N$ such that $[0, 1/N) \subseteq B^c$. If $n \geq N$ then $\mu^n(B) = \lambda(B)/2 = \mu(B)$. This shows that $\{\mu^n\} \to \mu$.

For any integer $k \geq 1$, consider the open interval $A_k = (0, 1/k)$. For any $k$, $\int_{A_k} h^n \, \mathrm{d}\mu \geq (n+1)/(2n) \geq 1/2$ for sufficiently large $n$. Since the $\lambda$-measure of $A_k$, and consequently, the $\mu$-measure of $A_k$, can be chosen to be arbitrarily small, this means that $\{h^n\}$ is not uniformly integrable with respect to $\mu$. Therefore, $\{\mu^n\}$ is not uniformly absolutely continuous with respect to $\mu$.

To show that $\{\mu^n\}$ is not uniformly absolutely continuous with respect to any measure $\eta$ on $M$, consider any measure $\eta$ on $M$ and the sets $A_k$ considered above. Since $A_{k+1} \subseteq A_k$

and $\cap_{k=1}^{\infty} A_k = \emptyset$, $\lim_{k\to\infty} \eta(A_k) = 0$. On the other hand, for any $k$, there exists $n$ such that $\mu^n(A_k) \geq 1/2$.

**Example 2** This example shows that the assumption that the collection of marginals of $\rho^n$ on $M$ being uniformly absolutely continuous with respect to some measure $\eta$ on $M$ in Theorem 1 cannot be relaxed.

Let $M = [0, 1]$ and $K = \{0, 1\}$. Let $f : M \longrightarrow K$ be given by

$$
\begin{aligned}
f(x) &= 0 \quad \text{if } x = 0 \\
&= 1 \quad \text{if } x > 0.
\end{aligned}
$$

Clearly, $f$ does not have a closed graph. Interpret $f = \Gamma = \Gamma^n$ for each $n$. Let $\mu^n$ and $\mu$ be as defined in Example 1. For any Borel set $P$ of $M \times K$, let

$$
\rho^n(P) = \mu^n(\{x \in M : (x, 1) \in P\}), \qquad \rho(P) = \mu(\{x \in M : (x, 1) \in P\}).
$$

[The term $(x, 1)$ appearing in the above expressions means the cartesian product of $x \in M$ and $1 \in K$. It does not denote an open interval.] It follows that $\mu^n$ is the marginal of $\rho^n$ on $M$ and $\mu$ is the marginal of $\rho$ on $M$.

To show that $\{\rho^n\} \to \rho$, consider a closed set $F$ in $M \times K$. Then $F = (F_0 \times \{0\}) \cup (F_1 \times \{1\})$, where $F_0$ and $F_1$ are closed subsets of $M$. $\rho(F) = \rho(F_1 \times \{1\}) = \mu(F_1)$ and for any $n$, $\rho^n(F) = \rho^n(F_1 \times \{1\}) = \mu^n(F_1)$. Since $\{\mu^n\} \to \mu$, $\mu(F_1) \geq \text{limsup}_n\, \mu^n(F_1)$, i.e., $\rho(F) \geq \text{limsup}_n\, \rho^n(F)$. So, $\{\rho^n\} \to \rho$. However, $\rho^n(\text{Graph of } f) = 1$ and $\rho(\text{Graph of } f) = 1/2$.

A variant of this example is worth noting. Keeping $f(x) = \Gamma^n(x) = \Gamma(x) = 1$ if $x > 0$ as above, if we redefine $f(0) = \Gamma^n(0) = 1$ for each $n$, and $\Gamma(0) = 1$, or $\Gamma(0) = \{0, 1\}$; then $\rho^n(\text{Graph of } \Gamma^n) = 1$ for each $n$ and $\rho(\text{Graph of } \Gamma) = 1$. Now $\Gamma^n$ has a closed graph for each $n$ and the result follows from Proposition 1. The uniform absolute continuity condition is not needed.

## 5 Proofs of Proposition 1 and Theorem 1

**Proof of Proposition 1** Let $C = \text{Ls}(\text{Graph of } \Gamma^n)$. Then $C$ is a closed subset of $M^*$. To show that $\rho(C) = 1$, suppose to the contrary that $\rho(C) < 1$. Then for some $\delta > 0$,

$\rho(C) < 1-\delta$. For each $m \in \mathbb{N}$, consider $B_{1/m}(C)$. $\{B_{1/m}(C)\}$ is a decreasing sequence of sets and $\cap_{m\in\mathbb{N}} B_{1/m}(C) = C$. So, $\{\rho(B_{1/m}(C))\} \to \rho(C)$. Since $\rho(C) < 1-\delta$, there exists $\bar{m}$ such that for all $m \geq \bar{m}$, $\rho(B_{1/m}(C)) < 1-\delta$.

Fix $m^* > \bar{m}$ and let $D$ denote the closure of $B_{1/m^*}(C)$. Then $D \subseteq B_{1/\bar{m}}(C))$ and $\rho(D) < 1-\delta$. If $\rho^n(D) \geq 1-\delta$ for infinitely many $n$ then $\limsup_n \rho^n(D) \geq 1-\delta$. The closedness of $D$ implies that $\rho(D) \geq 1-\delta$, a contradiction. So, $\rho^n(D) \geq 1-\delta$ for at most finitely many $n$. Assume that $\rho^n(D) < 1-\delta$ for all $n$. It follows that $\rho^n(B_{1/m^*}(C)) < 1-\delta$ for all $n$.

Since $M$ is a Borel subset of a complete separable metric space and $K$ is a compact metric space, $M^* = M \times K$ is a Borel subset of a complete separable metric space. So, each of the measures $\rho$ and $\rho^n$, $n \in \mathbb{N}$, is tight; Parthasarathy (1967, p. 29). Since $\{\rho^n\} \to \rho$, the family of measures $\{\rho, \rho^1, \rho^2, \ldots\}$ is tight; Hildenbrand (1974, p. 49). Let $0 < \epsilon < \delta$. There is a compact subset $Z_\epsilon$ of $M^*$ such that $\rho(Z_\epsilon) > 1-\epsilon$ and $\rho^n(Z_\epsilon) > 1-\epsilon$ for $n \in \mathbb{N}$.

Let $P = Z_\epsilon \setminus B_{1/m^*}(C)$. Then $P$ is compact and $P \subseteq (B_{1/m^*}(C))^c$. Since $C \subseteq B_{1/m^*}(C)$, $P$ is disjoint from $C$.

For any $n \in \mathbb{N}$, $\rho^n(P) \geq \rho^n(Z_\epsilon) - \rho^n(B_{1/m^*}(C)) > 1-\epsilon-(1-\delta) = \delta-\epsilon > 0$. This shows that $P \cap \text{Graph of } \Gamma^n \neq \emptyset$ for each $n$. Let $z^n \in P \cap \text{Graph of } \Gamma^n$. Since $P$ is compact, some subsequence of $\{z^n\}$ converges, say to $z \in P$. Then $z \in \text{Ls}(\text{Graph of } \Gamma^n)$. This is a contradiction since $P$ is disjoint from $C = \text{Ls}(\text{Graph of } \Gamma^n)$. Therefore, $\rho(C) = 1$. ■

**Proof of Theorem 1** For $x \in M$, denote by $G(x)$ and $G^n(x)$ the sets $\{x\}\times\Gamma(x)$ and $\{x\}\times\Gamma^n(x)$ respectively. $G$ and $G^n$ for each $n$ are compact-valued correspondences from $M$ to $M^*$. That these are measurable correspondences is seen as follows.

Aliprantis and Border (2006, p. 594) has shown that the cartesian product of a countable collection of weakly measurable correspondences from a measurable space to a (common) separable metric space is weakly measurable. The proof there, however, applies to the case where the ranges are distinct separable metric spaces. In the present context, only a two-fold cartesian product is needed. Notice that the identity map $x \longmapsto x$ is measurable on $M$ and by assumption, $\Gamma$ and $\Gamma^n$ from $M$ to $K$ are measurable. Hence, these are weakly measurable. Thus, $G$ and $G^n$ for each $n$ from $M$ to $M^*$ are weakly measurable. Being compact-valued, these are measurable.

Let $V$ be an open subset of $M^*$ such that Graph of $\Gamma$ is a subset of $V$. Below it is shown that $\rho(V) = 1$. If $V = M^*$, then $\rho(V) = 1$. Henceforth, suppose that $V$ is a proper subset of $M^*$.

For each positive integer $m$, let

$$W_m = \{x \in M : \ B_{1/m}(V^c) \cap G(x) \ = \ \emptyset\}.$$

Since $B_{1/m}(V^c)$ is an open set and $G$ is weakly measurable, $\{x \in M : \ B_{1/m}(V^c) \cap G(x) \neq \emptyset\}$ is a Borel set. So, its complement $W_m$ is a Borel set.

Clearly, $W_m \subseteq W_{m+1}$. It is shown below that if $x \in M$ then $x \in W_m$ for some $m$. Therefore, $\cup_{m=1}^{\infty} W_m = M$.

Suppose that for some $x \in M$, $x \notin W_m$ for any $m$. Then for some $y^m \in \Gamma(x)$ and $z^m = (x, y^m)$, $z^m \in B_{1/m}(V^c)$. $\{y^m\}$ is a sequence in $\Gamma(x)$, which is compact. So, some subsequence converges, say to $y$. Without loss of generality assume that the entire sequence converges. Clearly, $y \in \Gamma(x)$ and $z = (x, y) \in G(x)$. Since $z^m \in B_{1/m}(V^c)$ for every $m$ and $V^c$ is closed, $z \in V^c$. This is a contradiction since $G(x) \subseteq V$.

For each positive integer $n$, let

$$W_{mn} \ = \ \{x \in W_m : \text{ for all } k \geq n, \ B_{1/2m}(V^c) \cap G^k(x) \ = \ \emptyset\}.$$

For $k \in \mathbb{N}$, let $U_{mk} \ = \ \{x \in W_m : \ B_{1/2m}(V^c) \cap G^k(x) \ \neq \ \emptyset\}$. Since $B_{1/2m}(V^c)$ is an open set and $G^k$ is weakly measurable, $U_{mk}$ is a Borel set. Fix any $n$ and let $U^*_{mn} = \cup_{k \geq n} U_{mk}$. Then $U^*_{mn}$ is a Borel set, so its complement $W_{mn}$ is Borel set.

Clearly, $W_{mn} \subseteq W_{m,n+1}$. Now we show that if $x \in W_m$ then $x \in W_{mn}$ for some $n$. So, $\cup_{n=1}^{\infty} W_{mn} = W_m$.

Let $x \in W_m$. Suppose that $x \notin W_{mn}$ for any $n$. Then there are sequences $\{n_1, n_2, \ldots\}$ and $\{k_1, k_2, \ldots\}$ such that $k_t \geq n_t$ and $B_{1/2m}(V^c) \cap G^{k_t}(x) \neq \emptyset$. So, $\exists \, y^t$ such that $y^t \in \Gamma^{k_t}(x)$ and $(x, y^t) \in B_{1/2m}(V^c)$. $\{y^t\}$ is a sequence in $K$, so some subsequence converges, say to $y$. Since $\{(x, y^t)\}$ is a sequence in $B_{1/2m}(V^c)$, $(x, y)$ belongs to the closure of $B_{1/2m}(V^c)$. The closure of $B_{1/2m}(V^c)$ is a subset of $B_{1/m}(V^c)$.

Since $\mathrm{Ls}(\Gamma^i(x)) \subseteq \Gamma(x)$, $y \in \Gamma(x)$ and $(x, y) \in G(x)$. Since $B_{1/m}(V^c) \cap G(x) = \emptyset$, this is a contradiction. So, if $x \in W_m$ then $x \in W_{mn}$ for some $n$.

Next it is shown that for any $\epsilon > 0$, there exist $\bar{m}$ and $\bar{n}$ such that, $\mu^k(W_{\bar{m}\bar{n}}) > 1 - \epsilon$ for every $k$. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that $\eta(B) < \delta \Rightarrow \mu^k(B) < \epsilon$ for every $k$. Since $W_m \subseteq W_{m+1}$ and $\cup_{m=1}^{\infty} W_m = M$, there exists $N$ such that for all $m \geq N$, $\eta(W_m) > 1 - \delta$. Fix any $\bar{m} \geq N$. $W_{\bar{m}n} \subseteq W_{\bar{m},n+1}$ and $\cup_{n=1}^{\infty} W_{\bar{m}n} = W_{\bar{m}}$. So, for some $\bar{n}$, $\eta(W_{\bar{m}\bar{n}}) > 1 - \delta$. This implies that $\mu^k(W_{\bar{m}\bar{n}}) > 1 - \epsilon$ for every $k$.

For $k \in \mathbb{N}$, let $F^k =$ Graph of $\Gamma^k \cap (W_{\bar{m}\bar{n}} \times K)$. Since $\rho^k$(Graph of $\Gamma^k$) $= 1$, $\rho^k(F^k) = \rho^k(W_{\bar{m}\bar{n}} \times K) = \mu^k(W_{\bar{m}\bar{n}}) > 1 - \epsilon$.

Observe that if $k \geq \bar{n}$, then $B_{1/2\bar{m}}(V^c) \cap F^k = \emptyset$. Let $(x, y) \in F^k$. Then $(x, y) \in$ Graph of $\Gamma^k$ and $(x, y) \in W_{\bar{m}\bar{n}} \times K$. So, $(x, y) \in G^k(x)$ and $x \in W_{\bar{m}\bar{n}}$. If $x \in W_{\bar{m}\bar{n}}$ then $B_{1/2\bar{m}}(V^c) \cap G^k(x) = \emptyset$ for $k \geq \bar{n}$. So, if $k \geq \bar{n}$ then $B_{1/2\bar{m}}(V^c) \cap F^k = \emptyset$.

Now we show that $\rho(V) = 1$. For each $k \in \mathbb{N}$, $\rho^k(F^k) > 1 - \epsilon$. From above, $B_{1/2\bar{m}}(V^c) \cap F^k = \emptyset$ if $k \geq \bar{n}$. Therefore, for $k \geq \bar{n}$, $\rho^k(B_{1/2\bar{m}}(V^c)) < \epsilon$.

$B_{1/2\bar{m}}(V^c)$ is open and since $\{\rho^k\} \to \rho$, $\rho(B_{1/2\bar{m}}(V^c)) \leq \liminf_k \rho^k(B_{1/2\bar{m}}(V^c)) \leq \epsilon$. This implies that $\rho(V^c) \leq \epsilon$. So, $\rho(V) \geq 1 - \epsilon$. Since this is true for any $\epsilon > 0$, $\rho(V) = 1$.

That $\rho$(Graph of $\Gamma$) $= 1$ is established as follows. Since $M^*$ is a metric space, $\rho$ is regular; Parthasarathy (1967, p. 27). So, $\rho$(Graph of $\Gamma$) $= \inf \{\rho(V) :$ Graph of $\Gamma \subseteq V$, $V$ is open$\}$. Since $\rho(V) = 1$ for any open set $V$ containing the Graph of $\Gamma$, $\rho$(Graph of $\Gamma$) $= 1$. This completes the proof. ∎

# 6 An Application

In this section, some of the ideas in Theorem 1 are illustrated in the context of large games. Let $A$ be a compact metric space and $\mathcal{M}(A)$ the compact metric space of probability measures on $A$ under weak convergence of measures. Let $\mathcal{U}$ be a set of real valued payoff functions on $A \times \mathcal{M}(A)$, suitably metrized. If $(T, \mathcal{T}, \beta)$ is an atomless probability space then a non-anonymous game is a measurable mapping $\mathcal{G}$ from $T$ to $\mathcal{U}$. A Nash equilibrium of $\mathcal{G}$ is a measurable function from $T$ to $A$ such that for almost all $t \in T$, $\mathcal{G}_t(f(t), \beta \circ f^{-1}) \geq \mathcal{G}_t(x, \beta \circ f^{-1})$ for all $x \in A$. An anonymous game $\mu$ is a probability measure $\mu$ on $\mathcal{U}$. A Nash equilibrium distribution (NED) of $\mu$ is a probability measure $\tau$ on $\mathcal{U} \times A$ such that $\tau_{\mathcal{U}} = \mu$ and $\tau(B_\tau) = 1$, where $B_\tau = \{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(x, \tau_A)\}$ for all $x \in A$.

If $\mathcal{G}$ is a non-anonymous game and $\mu = \beta \circ \mathcal{G}^{-1}$ is an anonymous game then the Nash equilibria of $\mathcal{G}$ and the NEDs of $\mu$ are intimately related. This aspect is explored in detail in Khan et al. (2017). In particular, if $f$ is a Nash equilibrium of $\mathcal{G}$ then $\tau = \beta \circ (\mathcal{G}, f)^{-1}$ is an NED of $\mu$.

In terms of Theorem 1, the metric space $\mathcal{U}$ of payoff functions can be identified with $M$ and the space of actions $A$ can be identified with $K$. $\mu$ and $\mu^n$ there are games. The correspondences $\Gamma$ and $\Gamma^n$ are the best reply correspondences and their graphs correspond to the sets $B_\tau$, or $B_{\tau^n}$. The measures $\rho$ and $\rho^n$s there can be interpreted as NEDs.

If $X$ is a metric space and $g : X \longrightarrow \mathbb{R}$ is a function, then its hypograph is $\{(x, \alpha) \in X \times \mathbb{R} : g(x) \geq \alpha\}$. A function is upper semicontinuous (usc) iff its hypograph is a closed set. The hypotopology on the space of usc functions can be defined as: two functions are close in the hypotopology if their hypographs are close in the topology of closed convergence (of closed sets). The topology of closed convergence has been used extensively in the economics literature as a topology on preferences.

If $X$ is a compact metric space then $X \times \mathbb{R}$ is a locally compact separable metric space and the set of all closed subsets it endowed with the topology of closed convergence is a compact metrizable space; Hildenbrand (1974, p. 19). It is well known that every compact metric space is complete and separable and that every subset of a separable metric space is separable.

In what follows, $\mathcal{U}$ will be taken to be the space of bounded usc functions on $A \times \mathcal{M}(A)$, endowed with the hypotopology. Any usc function can be identified with its hypograph. So, it is a subset of a compact metric space. It can be shown that it is a countable union of closed subsets; see Rath (1996, p. 316), and hence a Borel subset and is separable.

Two examples are given below. In the first, the uniform absolute continuity condition is violated and the conclusion of Theorem 1 does not obtain. In the second, this condition is satisfied and the conclusion holds.

**Example 3** This example builds upon Example 1 in Rath (1996). The payoff functions in this example are continuous. Let $A = \{0, 1\}$ and $S = [0, 1]$. The set of probability measures on $A$ can be identified with $S$, where $y \in S$ denotes the probability of 1. Let $n \in \mathbb{N}$.

$$
\begin{aligned}
u^n(0,y) &= 1/2 \quad \text{if } y \in S \\
u^n(1,y) &= ny \quad \text{if } 0 \le y < 1/n \\
&= 1 \quad \text{if } 1/n \le y.
\end{aligned}
$$

Also let

$$
\begin{aligned}
u(0,y) &= 1/2 \quad \text{if } y \in S \\
u(1,y) &= 1 \quad \text{if } y \in S.
\end{aligned}
$$

It is relatively easy to check that $\{u^n\} \to u$ in the hypotopology. Let $\mu^n$ be the Dirac measure on $u^n$ and $\mu$ the Dirac measure on $u$. Then $\{\mu^n\} \to \mu$. The NEDs of $\mu$ and $\mu^n$ are characterized below.

Notice that $u(1,y) > u(0,y)$ for every $y \in [0,1]$. So, under $u$, $\{1\}$ is always the best response. Thus, $\mu$ has a unique NED $\tau$ where $\tau(\{u,1\}) = 1$. In equilibrium $\tau_A(\{1\}) = 1$, i.e., $y = 1$.

If $y = 0$ then $u^n(0,y) = 1/2 > 0 = u^n(1,y)$. If $y = 1$ then $u^n(1,y) = 1 > 1/2 = u^n(0,y)$. If $y = 1/2n$ then $u^n(0,y) = 1/2 = u^n(1,y)$. So, for $n \in \mathbb{N}$, $\mu^n$ has three NEDs, each denoted by $\tau^n$, where: (1) $\tau^n(\{u^n,0\}) = 1$, (2) $\tau^n(\{u^n,1\}) = 1$ and (3) $\tau^n(\{u^n,1\}) = 1/2n$ and $\tau^n(\{u^n,0\}) = (2n-1)/2n$. In equilibrium, $y^n = 0$, $y^n = 1$ and $y^n = 1/2n$ respectively.

Of particular interest is the NED $\tau^n$ given in (1), $\tau^n(\{u^n,0\}) = 1$. As $n \to \infty$, the limit of $\{\tau^n\}$ is the Dirac measure on $(u,0)$. This is clearly different from $\tau$, the only NED of $\mu$. This shows that the conclusion of Theorem 1 does not hold.

The family of games $\{\mu^n\}$ is not uniformly absolutely continuous. Suppose to the contrary that the family is uniformly absolutely continuous with respect to $\eta$. The absolute continuity of each $\mu^n$ implies that $\eta(\{u^n\}) > 0$ for each $n$. Since $\eta$ is finite, $\{\eta(\{u^n\})\} \to 0$ as $n \to \infty$. Let $0 < \epsilon < 1$. Suppose $\exists\, \delta$ such that $\eta(B) < \delta \Rightarrow \mu^n(B) < \epsilon$ for each $n$. For infinitely many $n$, $\eta(\{u^n\}) < \delta$ but $\mu^n(\{u^n\}) = 1 > \epsilon$.

**Example 4** Let $A = \{0,1\}$ and $S = [0,1]$ where $y \in S$ denotes the probability of 1. Let $W = [0,1]$ with Lebesgue measure $\lambda$. For $n \in \mathbb{N}$, define a function $\mathcal{G}^n : W \longrightarrow \mathcal{U}$ as follows.

$$\begin{aligned}\mathcal{G}^n_t(0,y) &= 1/2 \quad \text{if } y \in S\\ \mathcal{G}^n_t(1,y) &= 1 \quad\quad \text{if } y \geq \max\{1/n, t\}\\ &= t/4 \quad \text{if } y < \max\{1/n, t\}.\end{aligned}$$

Also define $\mathcal{G} : W \longrightarrow \mathcal{U}$ as follows.

$$\begin{aligned}\mathcal{G}_t(0,y) &= 1/2 \quad \text{if } y \in S\\ \mathcal{G}_t(1,y) &= 1 \quad\quad \text{if } y \geq t\\ &= t/4 \quad \text{if } y < t.\end{aligned}$$

It is easy to check that for any fixed $t$, $\mathcal{G}_t$ and $\mathcal{G}^n_t$ for $n \in \mathbb{N}$ are usc functions.

Claim 1: For each $n \in \mathbb{N}$, $\mathcal{G}^n$ is a one-to-one, continuous function.

Let $t < t'$. Choose $y$ such that $t < y < t'$. Then $\mathcal{G}^n_{t'}(1,y) = t'/4$ and $\mathcal{G}^n_t(1,y)$ is either $t/4$, or 1. So, $\mathcal{G}^n_{t'} \neq \mathcal{G}^n_t$.

To show that $\mathcal{G}^n$ is continuous, let $\{t^k\} \to t$. $\{\mathcal{G}^n_{t^k}\}$ converges to $\mathcal{G}^n_t$ in the hypotopology, provided that Ls(hypograph of $\mathcal{G}^n_{t^k}$) $\subseteq$ hypograph of $\mathcal{G}^n_t \subseteq$ Li(hypograph of $\mathcal{G}^n_{t^k}$).

Consider a sequence $\{(1, y^k, \alpha_k)\}$ where each term belongs to the hypograph of $\mathcal{G}^n_{t^k}$. Suppose that there is a subsequence $\{k_j\}$ such that $\{(1, y^{k_j}, \alpha_{k_j})\}$ converges to $(1, y, \alpha)$. Then the limit belongs to Ls(hypograph of $\mathcal{G}^n_{t^k}$).

Suppose that $\alpha > t/4$. Then for large $j$, $\alpha_{k_j} > t/4$. Since $(1, y^{k_j}, \alpha_{k_j}) \in$ hypograph of $\mathcal{G}^n_{t^{k_j}}$, $\mathcal{G}^n_{t^{k_j}}(1, y^{k_j}) = 1$. Therefore, $y^{k_j} \geq \max\{1/n, t^{k_j}\}$ which implies that $y \geq \max\{1/n, t\}$. Thus, $\mathcal{G}^n_t(1,y) = 1$ and $(1, y, \alpha) \in$ hypograph of $\mathcal{G}^n_t$. Now suppose that $\alpha \leq t/4$. Since $\mathcal{G}^n_t(1,y)$ is either $t/4$, or 1; $(1, y, \alpha) \in$ hypograph of $\mathcal{G}^n_t$. This shows that Ls(hypograph of $\mathcal{G}^n_{t^k}$) $\subseteq$ hypograph of $\mathcal{G}^n_t$.

Now let $(1, y, \alpha) \in$ hypograph of $\mathcal{G}^n_t$. Suppose that $\alpha > t/4$. Then $\mathcal{G}^n_t(1,y) = 1$ and $y \geq \max\{1/n, t\}$. If $y = t \geq 1/n$, let $y^k = t^k$ if $t^k > t$ and $y^k = y$ if $t^k < t$. In either case, $\mathcal{G}^n_{t^k}(1, y^k) = 1$ and $(1, y^k, \alpha) \in$ hypograph of $\mathcal{G}^n_{t^k}$. Since $\{y^k\} \to y$, $(1, y, \alpha) \in$ Li(hypograph of $\mathcal{G}^n_{t^k}$).

Continue to assume that $\alpha > t/4$ but let $y > t$. Then for large $k$, $y > t^k$. Since $y \geq 1/n$,

$\mathcal{G}^n_{t^k}(1,y) = 1$ and $(1,y,\alpha) \in$ hypograph of $\mathcal{G}^n_{t^k}$. Since this is true for large $k$, $(1,y,\alpha) \in$ Li(hypograph of $\mathcal{G}^n_{t^k}$).

Now suppose that $\alpha \leq t/4$. Let $\beta = (t/4) - \alpha \geq 0$. Since $\mathcal{G}^n_{t^k}(1,y)$ is either $t^k/4$, or 1; $(1,y,(t^k/4)-\beta) \in$ hypograph of $\mathcal{G}^n_{t^k}$. This implies that $(1,y,(t/4)-\beta) = (1,y,\alpha) \in$ Li(hypograph of $\mathcal{G}^n_{t^k}$). Thus, hypograph of $\mathcal{G}^n_t \subseteq$ Li(hypograph of $\mathcal{G}^n_{t^k}$).

<u>Claim 2</u>: $\mathcal{G}$ is a one-to-one, continuous function.

The proof is analogous to that of Claim 1.

<u>Claim 3</u>: If $t \geq 1/n$ then $\mathcal{G}_t = \mathcal{G}^n_t$.

Let $t \geq 1/n$. If $y \geq t$ then $\mathcal{G}_t(1,y) = 1$. Since $y \geq t \geq 1/n$, $\mathcal{G}^n_t(1,y) = 1$, i.e., $\mathcal{G}_t(1,y) = \mathcal{G}^n_t(1,y)$. If $y < t$ then $\mathcal{G}_t(1,y) = t/4$. Since $t \geq 1/n$, $\max\{1/n, t\} = t$. So, $\mathcal{G}^n_t(1,y) = t/4$. This gives $\mathcal{G}_t(1,y) = \mathcal{G}^n_t(1,y)$.

Let $\mu = \lambda \circ \mathcal{G}^{-1}$ and $\mu^n = \lambda \circ (\mathcal{G}^n)^{-1}$, $n \in \mathbb{N}$.

<u>Claim 4</u>: For any Borel subset $B$ of $\mathcal{U}$, $\mu^n(B) \leq \mu(B) + (1/n)$.

Claim 3 is used in the proof.

$$\begin{aligned}
\mu^n(B) &= \lambda(\{t : \mathcal{G}^n_t \in B\}) \\
&= \lambda(\{t \geq 1/n : \mathcal{G}^n_t \in B\}) + \lambda(\{t < 1/n : \mathcal{G}^n_t \in B\}) \\
&= \lambda(\{t \geq 1/n : \mathcal{G}_t \in B\}) + \lambda(\{t < 1/n : \mathcal{G}^n_t \in B\}) \\
&\leq \mu(B) + \frac{1}{n}.
\end{aligned}$$

<u>Claim 5</u>: $\{\mu^n\} \to \mu$. The family $\{\mu, \mu^1, \mu^2, \ldots\}$ is tight.

Let $B$ be a closed subset of $\mathcal{U}$. Then $\mu^n(B) \leq \mu(B) + (1/n)$, which implies that $\limsup_n \mu^n(B) \leq \mu(B)$. So, $\{\mu^n\} \to \mu$.

It follows from Parthasarathy (1967, p. 29) that each of the measures $\mu$ and $\mu^n$, $n \in \mathbb{N}$, is tight. Since $\{\mu^n\} \to \mu$ and each of these measures is tight, the family $\{\mu, \mu^1, \mu^2, \ldots\}$ is tight; Hildenbrand (1974, p. 49).

Let $\eta = (1/2)\mu + (1/2^{n+1})\mu^n$.

<u>Claim 6</u>: The family $\{\mu^n\}$ is uniformly absolutely continuous with respect to $\eta$.

Obviously, $\mu$ and $\mu^n$ for each $n$ are absolutely continuous with respect to $\eta$. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $1/N < \epsilon$. Then there exists $\delta > 0$ such that $\eta(B) < \delta$ implies $\mu(B) < \epsilon - (1/N)$ and $\mu^n(B) < \epsilon - (1/N)$ if $n \leq N$. Consider the same $\delta$ and let $n > N$. Then $\mu^n(B) \leq \mu(B) + (1/n) < \epsilon - (1/N) + (1/n) < \epsilon$. This completes the proof.

For $n \in \mathbb{N}$, define functions $f_0^n$ and $f_y^n$, $y \geq 1/n$, from $W$ to $A$ as follows. $f_0^n(t) = 0$ for all $t \in W$. $f_y^n(t) = 1$ if $y \geq t$ and $f_y^n(t) = 0$ if $y < t$.

Claim 7: For fixed $n$, $f_0^n$ and $f_y^n$, $y \geq 1/n$, are Nash equilibria of $\mathcal{G}^n$. (The proportions of players choosing 1 are 0 and $y$ respectively.)

If $f_0^n(t) = 0$ for all $t$, then $y = \lambda \circ (f_0^n)^{-1}(\{1\}) = 0$. $\mathcal{G}_t^n(0, y) = 1/2$, and, since $y < \max\{1/n, t\}$, $\mathcal{G}_t^n(1, y) = t/4$. Thus, $\{0\}$ is the only best response and $f_0^n$ is a Nash equilibrium.

Consider $f_y^n$, $y \geq 1/n$. If $y \geq t$, then $\mathcal{G}_t^n(1, y) = 1$ and if $y < t$, then $\mathcal{G}_t^n(1, y) = t/4$. Since $\mathcal{G}_t^n(0, y) = 1/2$, $\{1\}$ is the best response if $y \geq t$ and $\{0\}$ is the best response if $y < t$. So, $f_y^n$ is a Nash equilibrium.

For $0 \leq y \leq 1$, define $f_y$ as $f_y(t) = 1$ if $y \geq t$ and $f_y(t) = 0$ if $y < t$.

Claim 8: For any $y \in [0, 1]$, $f_y$ is a Nash equilibrium of $\mathcal{G}$. (The proportion of players choosing 1 is $y$.)

It is immediate that $\lambda \circ (f_y)^{-1}(\{1\}) = y$. $\mathcal{G}_t(1, y) = 1$ if $y \geq t$ and $\mathcal{G}_t(1, y) = t/4$ if $y < t$. On the other hand, $\mathcal{G}_t(0, y) = 1/2$. So, $\{1\}$ is the best response if $y \geq t$ and $\{0\}$ is the best response if $y < t$. Therefore, $f_y$ is a Nash equilibrium.

For $n \in \mathbb{N}$, let $h^n$ be a Nash equilibrium of $\mathcal{G}^n$ and $\rho^n = \lambda \circ (\mathcal{G}^n, h^n)^{-1}$. It was mentioned earlier that $\rho^n$ is an NED of $\mu^n = \lambda \circ (\mathcal{G}^n)^{-1}$.

Claim 9: The family $\{\rho^n\}$ is tight and contains a convergent subsequence.

It follows that $\rho_{\mathcal{U}}^n = \mu^n$ and $\rho_A^n = \lambda \circ (h^n)^{-1}$. It was claimed earlier that the family $\{\mu^n\}$ is tight. The family $\{\rho_A^n\}$ is tight because each of these measures is on the compact set $A$. The family $\{\rho^n\}$ is tight by Hildenbrand (1974, p. 50). That it contains a convergent subsequence follows from Hildenbrand (1974, p. 49). We will assume below without loss of generality that the entire sequence $\{\rho^n\}$ converges.

Let $Z$ be the range of $\mathcal{G}$, $Z_n$ the range of $\mathcal{G}^n$ and $Z^* = Z \cup (\cup_{n \in \mathbb{N}} Z_n)$. Being the continuous images of a compact set (Claims 1 and 2), $Z$ and $Z_n$ for each $n$ is compact. So, $Z^*$ is a Borel subset of $\mathcal{U}$.

Let $0 \leq y^* \leq 1$. To establish the limit property of the correspondence $\Gamma$, three cases are examined below: $(i)$ $y^* = 0$, $(ii)$ $y^* = 1$ and $(iii)$ $0 < y^* < 1$.

<u>Case 1</u>: $y^* = 0$. The correspondences $\Gamma$ and $\Gamma^n$ are defined on $Z^*$. If $u = \mathcal{G}_t$ and $t = 0$ then $\Gamma(u) = \Gamma^n(u) = \{1\}$, $n \in \mathbb{N}$. For any other $u \in Z^*$, $\Gamma(u) = \Gamma^n(u) = \{0\}$, $n \in \mathbb{N}$. Given $y^* = 0$, these are the best responses.

For each $n \in \mathbb{N}$, define $h^n(t) = 0$ for each $t \in W$. From Claim 7, $h^n$ is a Nash equilibrium of $\mathcal{G}^n$. Let $\rho^n = \lambda \circ (\mathcal{G}^n, h^n)^{-1}$. Then $\rho^n$ is an NED of $\mu^n$, i.e., $\rho^n(B_{\rho^n}) = 1$. Clearly, $\rho^n_A(\{1\}) = 0$.

Since Graph of $\Gamma^n$ coincides with $B_{\rho^n}$, $\rho^n(\text{Graph of } \Gamma^n) = 1$. The LS condition for $\Gamma$ is satisfied, so $\rho(\text{Graph of } \Gamma) = 1$ by Theorem 1. The convergence of $\{\rho^n\}$ to $\rho$, implies the convergence of marginals. So, $\rho_{\mathcal{U}} = \mu$ and $\rho_A(\{1\}) = 0$. The Graph of $\Gamma$ coincides with $B_\rho$, so $\rho$ is an NED of $\mu$.

<u>Case 2</u>: $y^* = 1$. The correspondences $\Gamma$ and $\Gamma^n$ are defined on $Z^*$. For every $u \in Z^*$, let $\Gamma(u) = \Gamma^n(u) = \{1\}$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define $h^n(t) = 1$ for each $t \in W$. Arguments similar to above lead to $\rho(\text{Graph of } \Gamma) = 1$, $\rho_A(\{1\}) = 1$ and $\rho$ is an NED of $\mu$.

<u>Case 3</u>: $0 < y^* < 1$. Let $N$ be the smallest integer such that $1/N < y^*$. Let $Z^*_N = Z \cup (\cup_{n \geq N} Z_n)$. The correspondences $\Gamma$ and $\Gamma^n$ for $n \geq N$ are defined on $Z^*_N$. For $u \in Z^*_N$, let $\Gamma(u) = \Gamma^n(u) = \{1\}$ if $u = \mathcal{G}_t$, or $u = \mathcal{G}^k_t$ for $k \geq N$ and $t \leq y^*$. For $u \in Z^*_N$, let $\Gamma(u) = \Gamma^n(u) = \{0\}$ if $u = \mathcal{G}_t$, or $u = \mathcal{G}^k_t$ for $k \geq N$ and $t > y^*$. Notice the discontinuities in the correspondences $\Gamma$ and $\Gamma^n$, $n \geq N$. As a result, these correspondences do not have closed graphs.

For each $n \in \mathbb{N}$, define $h^n(t) = 1$ if $t \leq y^*$ and $h^n(t) = 0$ if $t > y^*$. Arguments similar to above lead to $\rho(\text{Graph of } \Gamma) = 1$, $\rho_A(\{1\}) = y^*$ and $\rho$ is an NED of $\mu$.

There is another force at work as well in this example and that is convergence in distribution. Let $d$ denote the metric on $\mathcal{U} \times A$. Note that in each of the cases considered above, the functions $h^n$ is the same for each $n$. Let $h = h^n$. The function $t \longmapsto d((\mathcal{G}^n_t, h^n(t)), (\mathcal{G}_t, h(t)))$ is measurable. Moreover, for any $\epsilon > 0$, $\{t : d((\mathcal{G}^n_t, h^n(t)), (\mathcal{G}_t, h(t))) > \epsilon\} \subseteq [0, 1/n)$. Since $\{\lambda([0, 1/n))\} \to 0$, $\{(\mathcal{G}^n, h^n)\} \to (\mathcal{G}, h)$ in measure. Therefore, the sequence converges in distribution; Hildenbrand (1974, pp. 51$-$52). Thus, $\{\rho^n\} \to \rho$. Since $\rho^n$ is an NED of $\mu^n$ and $\rho$ is an NED of $\mu$, $\rho^n(\text{Graph of } \Gamma^n) = 1$ for each $n$ and $\rho(\text{Graph of } \Gamma) = 1$.

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