Check Your Understanding

Definitions and Proofs

Definition: A topological space X with topology T_X is *Hausdorff* if for every $x, y \in X$ with $x \neq y$, there are $A, B \in T_X$ such that $x \in A$, $y \in B$, and $A \cap B = \emptyset$.

Problem 1: *Is* $\mathbb{N}$ *with* $T_{\mathbb{N}} = \mathcal{P}(\mathbb{N})$ *Hausdorff? Prove you are correct.*

Solution: Yes: Let $x, y \in \mathbb{N}$ with $x \neq y$ as in the definition of Hausdorff. Then $\{x\} \in \mathcal{P}(\mathbb{N})$ and $\{y\} \in \mathcal{P}(\mathbb{N})$ by the definition of the power set. Clearly $\{x\} \cap \{y\} = \emptyset$ since $x \neq y$. Thus x and y witness that this topological space is not Hausdorff.

Problem 2: *Prove* $\{\emptyset, \mathbb{N}\}$ *is a topology on* $\mathbb{N}$.

Solution: By the definition of a topological space, we need to verify three things:

- $\emptyset \in \{\emptyset, \mathbb{N}\}$ and $\mathbb{N} \in \{\emptyset, \mathbb{N}\}$ trivially.
- We need to show that the intersection of any two elements of this set is also an element. But $\emptyset \cap \emptyset = \emptyset$, $\emptyset \cap \mathbb{N} = \mathbb{N} \cap \emptyset = \emptyset$, and $\mathbb{N} \cap \mathbb{N} = \mathbb{N}$ so this is clear.
- Finally, we need to show that the union over any subset is an element of this set. But the only subsets are $\emptyset$, $\{\emptyset\}$, $\{\mathbb{N}\}$, and $\{\emptyset, \mathbb{N}\}$. Their unions respectively are $\emptyset$, $\emptyset$, $\mathbb{N}$, and $\mathbb{N}$, so this holds.

Problem 3: *Is* $\mathbb{N}$ *with* $T_{\mathbb{N}}$ *Hausdorff? Prove you are correct!*

Solution: No: Consider 0 and 1 in $\mathbb{N}$. $0 \notin \emptyset$ and $1 \notin \emptyset$ by definition. Therefore if we have $X \in T_{\mathbb{N}}$ with $0 \in X$ and $Y \in T_{\mathbb{N}}$ with $1 \in Y$, then $X = Y = \mathbb{N}$, so $X \cap Y \neq \emptyset$. Thus it is not Hausdorff by definition.

Definition: A subset $X \subset \mathbb{N}$ is cofinite if $\{n \in \mathbb{N} : n \notin X\}$ is finite.

Problem 4: *Is the set of cofinite subsets of* $\mathbb{N}$ *a topology on* $\mathbb{N}$*? Prove you are correct!*

Solution: No, as it does not contain the empty set: $\{n \in \mathbb{N} : n \notin \emptyset\} = \mathbb{N}$, and therefore is not finite. Thus $\emptyset$ is not cofinite, and the set of all cofinite sets is not a topology by definition.