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Duality between Harmonic and Bergman spaces

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Abstract. In this paper we study the duality of the harmonic spaces on the annulus $\Omega = \Omega_1 \setminus \overline{\Omega}^-$ between two pseudoconvex domains with $\Omega^- \subset\subset \Omega_1$ in $\mathbb{C}^n$ and the Bergman spaces on Ω^-. We show that on the annulus Ω, the space of harmonic forms for the critical case on $(0, n-1)$-forms is infinite dimensional and it is dual to the the Bergman space on the pseudoconvex domain Ω^-. The duality is further identified explicitly by the Bochner-Martinelli transform, generalizing a result of Hörmander.

Introduction

Let Ω^- and Ω_1 be two bounded pseudoconvex domains in $\mathbb{C}^n$ with $\Omega^- \subset\subset \Omega_1$. In this paper we study the duality of the harmonic spaces on the annulus $\Omega = \Omega_1 \setminus \overline{\Omega}^-$ and the Bergman spaces on Ω^-. This paper is inspired by a recent paper of Hörmander [**Hö 2**] where the null space of the $\bar{\partial}$-Neumann operator on a spherical shell as well as on an ellipsoid in $\mathbb{C}^n$ has been computed by explicit formula for the critical case for $(0, n-1)$-forms.

The $\bar{\partial}$-Neumann problem on the annulus has been studied in [**Sh 1**] on an annulus between two pseudoconvex domains in $\mathbb{C}^n$ or in a hermitian Stein manifold. When the boundary is smooth, the closed range property and boundary regularity for $\bar{\partial}$ were established in the earlier work (see [**BS**] or [**Sh1**]) for $0 < q \leq n-1$ and $n \geq 2$. In the case when $0 < q < n-1$, the space of harmonic forms is trivial. In this paper, we will study the critical case when $q = n-1$ on the annulus Ω. In this case the space of harmonic forms is infinite dimensional. Our goal is to establish the duality between the harmonic forms in the critical degree with the Bergman spaces on the domain Ω^-.

In the first section, we recall the Hodge decomposition theorem on the annulus between two pseudoconvex domains. In the second section we establish the duality between the harmonic forms with coefficients in the Sobolev $W^1(\Omega)$ spaces with the Bergman spaces on Ω^-. We then refine the duality to duality between L^2 spaces in Section 3.

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1. L^2 theory for $\bar\partial$ on the annulus between two weakly pseudoconvex domains in $\mathbb{C}^n$

We recall the following L^2 existence and estimates for $\bar\partial$ in the annulus between two pseudoconvex domains (see Theorems 3.2 and 3.3 in Shaw [**Sh4**]).

THEOREM 1.1. *Let $\Omega \subset\subset \mathbb{C}^n$, $n \geq 3$, be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}^-$ between two pseudoconvex domains Ω_1 and Ω^-. We assume that $\Omega^- \subset\subset \Omega_1$ and Ω^- has C^2 boundary. For any $f \in L^2_{(p,q)}(\Omega)$, where $0 \leq p \leq n$ and $0 \leq q < n-1$, such that $\bar\partial f = 0$ in Ω, the following hold:*

(1) *there exists $F \in W^{-1}_{(p,q)}(\Omega_1)$ such that $F|_\Omega = f$ and $\bar\partial F = 0$ in Ω_1 in the distribution sense.*

(2) *there exists $u \in L^2_{(p,q-1)}(\Omega)$ satisfying $\bar\partial u = f$ in Ω.*

For $q = n-1$, there is an additional compatibility condition for the $\bar\partial$-closed extension of $(p, n-1)$-forms.

THEOREM 1.2. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}^-$ between two pseudoconvex domains Ω_1 and Ω^-. We assume that $\Omega^- \subset\subset \Omega_1$ and Ω^- has C^2 boundary. For any $\bar\partial$-closed $f \in L^2_{(p,n-1)}(\Omega)$, where $0 \leq p \leq n$, the following hold:*

(1) *There exists $F \in W^{-1}_{(p,n-1)}(\Omega_1)$ such that $F|_\Omega = f$ and $\bar\partial F = 0$ in Ω_1 in the distribution sense.*

(2) *The restriction of f to $b\Omega^-$ satisfies the compatibility condition*

$$\int_{b\Omega^-} f \wedge \phi = 0, \quad \phi \in W^1_{(n-p,0)}(\Omega^-) \cap Ker(\bar\partial). \tag{1.1}$$

(3) *There exists $u \in L^2_{(p,n-2)}(\Omega)$ satisfying $\bar\partial u = f$ in Ω.*

Corollary 1.3. *Let Ω be the same as in Theorem 1.2. Then $\bar\partial$ has closed range in $L^2_{(p,n-1)}(\Omega)$ and the $\bar\partial$-Neumann operator $N_{(p,n-1)}$ exists on $L^2_{(p,n-1)}(\Omega)$.*

Theorem 1.4 (Hodge Decomposition Theorem). *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}^-$ between two pseudoconvex domains Ω_1 and Ω^-. We assume that $\Omega^- \subset\subset \Omega_1$ and Ω^- has C^2 boundary. Then the $\bar\partial$-Neumann operator $N_{(p,q)}$ exists on $L^2_{(p,q)}(\Omega)$ for $0 \leq p \leq n$ and $0 \leq q \leq n$. For any $f \in L^2_{(p,q)}(\Omega)$, we have*

$$\begin{aligned}
&f = \bar\partial^*\bar\partial N_{(p,0)} f + H_{(p,0)} f, \qquad q = 0.\\
&f = \bar\partial\bar\partial^* N_{(p,q)} f + \bar\partial^*\bar\partial N_{(p,q)} f, \qquad 1 \leq q \leq n-2.\\
&f = \bar\partial\bar\partial^* N_{(p,n-1)} f + \bar\partial^*\bar\partial N_{(p,n-1)} f + H_{(p,n-1)} f, \qquad q = n-1.\\
&f = \bar\partial\bar\partial^* N_{(p,n)} f, \qquad q = n.
\end{aligned}$$

We have used the notation $H_{(p,q)}$ to denote the projection operator from $L^2_{(p,q)}(\Omega)$ onto the harmonic space $\mathcal{H}_{(p,q)}(\Omega) = ker(\Box_{(p,q)})$.

For a proof of Theorem 1.4, see Theorem 3.5 in [**Sh4**].

Remark: All the results can be extended to annulus between pseudoconvex domains in a Stein manifold with trivial modification.

2. The duality between $H^{(p,n-1)}_{W^1}(\Omega)$ and $H^{(n-p,0)}(\Omega^-)$

For $k \geq 0$, we define the Dolbeault cohomology $H^{(p,q)}_{W^k}(\Omega)$ with $W^k(\Omega)$-coefficients by

$$H^{(p,q)}_{W^k}(\Omega) = \frac{\{f \in W^k_{(p,q)}(\Omega) \mid \bar\partial f = 0\}}{\{f \in W^k_{(p,q)}(\Omega) \mid f = \bar\partial u, u \in W^k_{(p,q-1)}(\Omega)\}}.$$

For $k \in \mathbb{R}$, we define $H^{(p,0)}_{W^k}(\Omega)$ to be the space of $(p,0)$-forms with holomorphic coefficients in $W^k(\Omega)$.

If Ω is the annulus between two pseudoconvex domains as in Theorem 1.2, we have that the space $\{f \in W^k_{(p,q)}(\Omega) \mid f = \bar\partial u, u \in W^k_{(p,q-1)}(\Omega)\}$ is closed. Furthermore, we have from Theorem 1.4:

$$\mathcal{H}_{(p,n-1)}(\Omega) \simeq H^{(p,n-1)}_{L^2}(\Omega).$$

We will use the notation $H^{(p,n-1)}(\Omega)$ for $H^{(p,n-1)}_{L^2}(\Omega)$ and $H_{W^1}(\Omega) = H^{(0,0)}_{W^1}(\Omega)$.

Theorem 2.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega^-}$ between two pseudoconvex domains Ω_1 and Ω^- with smooth boundary and $\Omega^- \subset\subset \Omega_1$, $n \geq 2$. For each $k \geq 0$ and $0 \leq p \leq n$, the space $H^{(p,q)}_{W^k}(\Omega) = \{0\}$, when $0 < q < n-1$ and the space $H^{(p,n-1)}_{W^k}(\Omega)$ is of infinite dimension. Furthermore, we have the isomorphism:*

$$H^{(p,n-1)}_{W^k}(\Omega) \simeq (H^{(n-p,0)}_{W^{-k+1}}(\Omega^-))'$$

where the right-hand side is the space of all bounded linear functionals on the space $H^{(n-p,0)}_{W^{-k+1}}(\Omega^-)$.

Proof. First we assume that $k = 0$. Suppose that $f \in L^2_{(p,n-1)}(\Omega)$ and $\bar\partial f = 0$. We define a pairing between $H^{(p,n-1)}(\Omega)$ and $H^{(n-p,0)}_{W^1}(\Omega^-)$

$$l : H^{(p,n-1)}(\Omega) \times H^{(n-p,0)}_{W^1}(\Omega^-) \to \mathbb{C}$$

by

$$l([f], h) = \int_{b\Omega^-} f \wedge h, \quad h \in H^{(n-p,0)}_{W^1}(\Omega^-). \tag{2.1}$$

First we note that the pairing (2.1) is well-defined. It is well-known any holomorphic function or forms with $L^2(\Omega)$ coefficients has a well-defined trace in $W^{-\frac{1}{2}}(b\Omega)$ (see e.g. [**LM**]). For any f in $L^2_{(p,n-1)}(\Omega)$ with $\bar\partial f = 0$ and $\bar\partial^* f = 0$, we also have a well-defined trace in $W^{-\frac{1}{2}}(b\Omega)$ (see [**Sh3**] for details). Any function or form with $W^1(\Omega^-)$ coefficients has trace in $W^{\frac{1}{2}}(b\Omega^-)$ from the Sobolev Trace Theorem. Thus the pairing between f and ϕ in (2.1) is well-defined since

$$|\int_{b\Omega^-} f \wedge h| \leq \|f\|_{W^{-\frac{1}{2}}(b\Omega^-)} \|h\|_{W^{\frac{1}{2}}(b\Omega^-)} \leq \|f\|_{L^2(\Omega)} \|h\|_{W^1(\Omega^-)}.$$

We also note that the pairing in (2.1) is independent of the choice of the representation function $[f]$. Let $\tilde f$ be another representation of $[f]$, then $\tilde f = f + \bar\partial u$ for some element of the form $\bar\partial u \in L^2_{(p,n-1)}(\Omega)$ with $u \in L^2_{(p,n-2)}(\Omega)$. Using Friedrichs' lemma, there exists a sequence $\{u_\nu\}$ such that $u_\nu \in C^\infty_{(p,n-2)}(\overline{\Omega})$ such that $u_\nu \to u$ in $L^2_{(p,n-2)}(\Omega)$ and $\bar\partial u_\nu \to \bar\partial u$ in $L^2_{(p,n-1)}(\Omega)$. It follows from Stokes' Theorem that

$$\begin{aligned}\int_{b\Omega^-} \bar\partial u \wedge h &= \lim_{\nu\to\infty} \int_{b\Omega^-} \bar\partial u_\nu \wedge h \\ &= \lim_{\nu\to\infty} (-1)^{p+n} \int_{b\Omega^-} u_\nu \wedge \bar\partial h = 0, \quad h \in H^{(n-p,0)}_{W^1}(\Omega^-).\end{aligned}$$

Thus the pairing (2.1) is well-defined.

If we assume that f satisfies the condition

$$\int_{b\Omega^-} f \wedge \phi = 0, \quad \phi \in W^1_{(n-p,0)}(\Omega^-) \cap \mathrm{Ker}(\bar\partial),$$

from Theorem 1.2, there exists a $\bar\partial$-closed form $F \in W^{-1}_{(p,n-2)}(\Omega_1)$ which is equal to f on Ω and one can find a solution $u \in L^2_{(p,n-2)}(\Omega)$ satisfying $\bar\partial u = f$. This implies that $[f] = 0$. Thus there is a 1-1 map from $H^{(p,n-1)}(\Omega)$ to $H^{(n-p,0)}_{W^1}(\Omega^-)'$.

On the other hand, suppose that f is a bounded linear functional on $H^{(n-p,0)}_{W^1}(\Omega^-)$. We will show that l can be represented by some $[f]$ in (2.1). Since we assume that Ω^- is pseudoconvex and has smooth boundary, one has the duality for holomorphic space $H^1(\Omega^-) = L^2(\Omega^-) \cap \mathrm{Ker}(\bar\partial)$ and $H^{-1}(\Omega^-)$ (see [**BB**]). If the $\bar\partial$-Neumann operator is exact regular on $W^1(\Omega^-)$, we can use the duality between the usual L^2 spaces. Otherwise, one can use the exact regularity for the weighted $\bar\partial$-Neumann operator with weights $t|z|^2$ for sufficiently large $t > 0$. The weight function can be viewed as the bundle metric $e^{-t|z|^2}$ for the trivial line bundle $\mathbb{C}$ and the dual space will be equipped with the dual metric $e^{t|z|^2}$ for $\mathbb{C}$. In particular the pairing (2.1) is well-defined. For simplicity, we assume that the $\bar\partial$-Neumann operator is exact regular. But all the arguments remain the same if we use weighted spaces with the dual weighted norms.

Thus l can be represented by $(n-p, 0)$-form g with distribution coefficients in $H^{-1}(\Omega^-) = \ker(\bar\partial) \cap W^{-1}(\Omega^-)$. Extending g to be zero outside Ω^-, then $\star g$ is a (p, n)-form on Ω_1, a top degree form which is always $\bar\partial$-exact. The extension by zero of g results in a form which is in $W^{-1}(\Omega_1)$. This is due to the fact that holomorphic functions in $W^{-1}(\Omega^-)$ is also in the dual of $W^1(\Omega^-)$. We remark that for a general function or forms, this is not true. But when the functions or forms are harmonic, then the dual space of W^1_0, denoted by W^{-1}, coincides with the dual space of W^1 for domains with smooth boundary. For detailed explanation of this subtle point, we refer the reader to the paper by Boas (see Appendix B in [**Boa**] where the dual space of W^1 is denoted by W^{-1}_*.).

Thus we have that $\star g = \bar\partial U$ on Ω_1 for some $U \in L^2_{(p,n-1)}(\Omega_1)$. Let $f = U$ on Ω. It follows that $\bar\partial f = 0$ on Ω and the linear functional

$$l(h) = \int_{\Omega^-} \star g \wedge h = \int_{\Omega^-} \bar\partial U \wedge h = \int_{b\Omega^-} f \wedge h, \quad h \in H^{(n-p,0)}_{W^1}(\Omega^-).$$

Since $f \in L^2_{(p,n-1)}(\Omega)$, we have that the bounded linear functional l is represented by $[f] \in H^{(p,n-1)}(\Omega)$. This proves the theorem for $k = 0$.

Suppose that $k \geq 1$ and $f \in W^k_{(p,n-1)}(\Omega)$ and $\bar\partial f = 0$. We define a pairing between $H^{(p,n-1)}_{W^k}(\Omega)$ and $H^{(n-p,0)}_{W^{-k+1}}(\Omega^-)$ by

$$l : H^{(p,n-1)}_{W^k}(\Omega) \times H^{(n-p,0)}_{W^{-k+1}}(\Omega^-)$$

$$l([f], h) = \int_{b\Omega^-} f \wedge h. \tag{2.2}$$

It is easy to see that the pairing (2.2) is well-defined as before. If f satisfies the condition

$$\int_{b\Omega^-} f \wedge \phi = 0, \quad \phi \in W^{-k+1}_{(n-p,0)}(\Omega^-) \cap \mathrm{Ker}(\bar\partial),$$

there exists a $\bar\partial$-closed form $F \in W^{k-1}_{(p,n-1)}(\Omega_1)$ which is equal to f on Ω and one can find a solution $u \in W^k_{(p,n-2)}(\Omega)$ satisfying $\bar\partial u = f$. For a proof, see Corollary 2 in the recent paper by Chakrabarti-Shaw [**CS2**]. This implies that $[f] = 0$.

Thus there is a one to one map from $H^{(p,n-1)}_{W^k}(\Omega)$ to $H^{(n-p,0)}_{W^{-k+1}}(\Omega^-)'$. Any element in $H^{(n-p,0)}_{W^{-k+1}}(\Omega^-)'$ can be identified as a (p,n)-form $\star g$ with $H_{W^{k-1}}(\Omega^-)$-coefficients. Thus repeating the same arguments as before, there exists $f \in W^k(\Omega)$ with $\bar\partial f = 0$ such that any bounded linear functional can be given by f in the equation (2.1). $\square$

We remark that in Theorem 2.1, the boundary is of Ω is assumed to be C^∞ smooth in order to have the duality for all $k \geq 0$. For each fixed k, the duality result holds for sufficiently smooth (depending on k) domains Ω^- and Ω_1. In particular, Theorem 2.1 holds for $k = 1$ for much less smooth domains Ω_1 and Ω^-. In the following, we will only assume that the boundary for Ω^- be Lipschitz, i.e., locally it is the graph of a Lipschitz function.

Let $\bar\partial_c : L^2_{(p,n-1)}(\Omega^-) \to L^2_{(p,n)}(\Omega^-)$ be the minimal closure of $\bar\partial$. By this we mean that $f \in \mathrm{Dom}(\bar\partial_c)$ if and only if that there exists a sequence of smooth forms f_ν in $C^\infty_{(p,n-1)}(\Omega)$ compactly supported in Ω such that $f_\nu \to f$ and $\bar\partial f_\nu \to \bar\partial f$ in L^2. Let ϑ be the dual of $\bar\partial_c$. Then ϑ is equal to the maximal closure of the operator

$$\vartheta : L^2_{(p,n)}(\Omega^-) \to L^2_{(p,n-1)}(\Omega^-).$$

We set

$$\Box^c_{(p,n)}(\Omega^-) = \bar\partial_c \vartheta : L^2_{(p,n)}(\Omega^-) \to L^2_{(p,n)}(\Omega^-).$$

The kernel of $\Box^c_{(p,n)}(\Omega^-)$ is denoted by $\mathcal{H}^{(p,n)}_c(\Omega^-)$, the space of harmonic forms of degree (p,n) with compact support.

Theorem 2.2. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}^-$ between two bounded pseudoconvex domains Ω_1 and Ω^- with $\Omega^- \subset\subset \Omega_1$, $n \geq 2$. We assume that Ω_1 has C^2 boundary and Ω^- has Lipschitz boundary. The space $H^{(p,n-1)}_{W^1}(\Omega)$ is of infinite dimension and we have the isomorphism:*

$$H^{(p,n-1)}_{W^1}(\Omega) \simeq \mathcal{H}^{(p,n)}_c(\Omega^-) \simeq H^{(n-p,0)}(\Omega^-).$$

Proof. From the closed range property for $\bar\partial$ on Ω^- and its L^2 dual for all degrees, it follows (see [**CS2**]) that range of $\bar\partial_c$ is also closed for all degrees. In particular, we have

$$L^2_{(p,n)}(\Omega^-) = \mathrm{Range}(\bar\partial_c) \oplus \mathrm{Ker}(\vartheta).$$

Here we only need the boundary Ω^- to be Lipschitz smooth (see [**CS2**] for details).

This gives that

$$\mathcal{H}_c^{(p,n)}(\Omega^-) = \mathrm{Ker}(\vartheta). \tag{2.3}$$

Using star operator, one has that

$$\mathrm{Ker}(\vartheta) \simeq H^{(n-p,0)}(\Omega^-). \tag{2.4}$$

From the extension of the $\bar\partial$-closed forms from the annulus to Ω_1 as in the proof of Theorem 1.2, we will show that $H^{(p,n-1)}_{W^1}(\Omega)$ is isomorphic to the perp of $\mathrm{Range}(\bar\partial_c)$. To see this, let $f \in W^1_{(p,n-1)}(\Omega)$ and let $\bar\partial f = 0$ in Ω. We extend f to be a form $\tilde f \in W^1_{(p,n-1)}(\Omega_1)$. The equation

$$\bar\partial_c u = \bar\partial \tilde f \tag{2.5}$$

for some $u \in L^2_{(p,n-1)}(\Omega^-)$ if and only if

$$\int_{b\Omega^-} f \wedge \phi = 0, \quad \phi \in L^2_{(n-p,0)}(\Omega^-) \cap \mathrm{Ker}(\bar\partial). \tag{2.6}$$

For a proof of the equivalence of (2.5) and (2.6), see the proof Proposition 5 in [**CS2**].

In this case, f can be extended to be $\bar\partial$-closed form F where

$$F = \begin{cases} f, & z \in \Omega, \\ \tilde f - u, & z \in \Omega^-. \end{cases}$$

It follows that $\bar\partial F = 0$ in Ω_1 and $F = f$ on Ω. The form F is in $L^2_{(p,n-1)}(\Omega_1)$ but F is in $W^1(\Omega)$ since $F = f$ on Ω. Since we assume that the boundary Ω_1 is C^2, we can find a solution (see [**Ha**]) $F = \bar\partial U$ for some $U \in W^1_{(p,n-2)}(\Omega_1)$. In fact we can use the solution $U = F + \bar\partial_t^* N_t F$ by the weighted $\bar\partial$-Neumann operator N_t on Ω_1. Then U is in W^1 near the boundary Ω_1 from the boundary regularity for the weighted $\bar\partial$-Neumann operator. Since the weighted $\bar\partial$-Neumann operator N_t is elliptic in the interior of Ω_1, $N_t F$ is in $W^2(\Omega, \mathrm{loc})$. Thus the solution U is in $W^1(\Omega, \mathrm{loc})$. Thus U is in $W^1_{(p,n-2)}(\Omega_1)$. This shows that for any $[f] \in H^{(p,n-1)}_{W^1}(\Omega)$, $[f] = 0$ if and only if (2.5) or (2.6) is satisfied for any representation $f \in W^1_{(p,n-1)}(\Omega)$. Repeating the arguments as in Theorem 2.1 and using (2.3) and (2.4), we have proved the theorem. □

3. The isomorphism between $H^{(p,n-1)}(\Omega)$ and $H^{(n-p,0)}(\Omega^-)$

In this section we will further establish the isomorphism between the spaces $H^{(p,n-1)}(\Omega)$ and $H^{(n-p,0)}(\Omega^-)$.

Theorem 3.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega^-}$ between two pseudoconvex domains Ω_1 and Ω^- and $\Omega^- \subset\subset \Omega_1$, $n \geq 2$. We assume that the boundary of Ω^- is C^2 smooth. Then we have the isomorphism:*

$$H^{(p,n-1)}(\Omega) \simeq H^{(n-p,0)}(\Omega^-). \tag{3.1}$$

Furthermore, if we assume that Ω has C^2 smooth boundary, then we have the isomorphism:

$$H^{(p,n-1)}_{W^1}(\Omega) \simeq H^{(p,n-1)}(\Omega). \tag{3.2}$$

Proof. It follows from Theorem 2.2 that

$$H^{(p,n-1)}_{W^1}(\Omega) \simeq H^{(n-p,0)}(\Omega^-).$$

Thus it suffices to prove (3.1).

Let $h \in H^{(n-p,0)}(\Omega^-)$. We will associate h with a $\bar\partial$-closed form h^+ in Ω as follows:

Let ρ be a normalized C^2 defining function for Ω^-. Since h has holomorphic coefficients in $L^2(\Omega^-)$, it is well known that h has $W^{-\frac{1}{2}}$ boundary values on $b\Omega^-$. Let $h_1 = h \wedge \bar\partial\rho$. The form $\star h_1$ is a $(p, n-1)$-form on Ω^- and it has boundary value in $W^{-\frac{1}{2}}(b\Omega^-)$. We denote the restriction of $\star h_1$ to $b\Omega^-$ by

$$h_b = \star h_1 \mid_{b\Omega} . \tag{3.3}$$

Let $h^+ = B^+ h_b$ and $h^- = B^- h_b$ be the Bochner transform of h_b defined by

$$h^+ = B^+ h_b = \int_{b\Omega^-} B(\zeta, z) \wedge h_b, \quad z \in \Omega, \tag{3.4}$$

$$h^- = B^- h_b = \int_{b\Omega^-} B(\zeta, z) \wedge h_b, \quad z \in \Omega^-. \tag{3.5}$$

We have the jump formula:

$$h_b = B^+ h_b - B^- h_b = h^+ - h^-$$

in terms of distributions. Also each h^+ and h^- are $\bar\partial$-closed and L^2 on Ω and Ω^- respectively.

We define a map $l^+ : H^{(n-p,0)}(\Omega^-) \to H^{(p,n-1)}(\Omega)$ by

$$l^+ h = [h^+], \qquad h \in H^{(n-p,0)}(\Omega^-)$$

where h^+ is defined by (3.4). Since h^+ is $\bar\partial$-closed on Ω and has L^2 coefficients, the map l^+ is well-defined.

We next show that l^+ is one to one. If $l^+(h) = [h^+] = 0$ for some $h \in H^{(n-p,0)}(\Omega^-)$, we will show that $h = 0$. Since $[h^+] = 0$, this implies that h^+ can be represented by a $\bar\partial$-exact form and there exists $u^+ \in L^2_{(p,n-2)}(\Omega)$ such that

$$h^+ = \bar\partial u^+. \tag{3.6}$$

Let h^- be defined by by (3.5). Since Ω^- is pseudoconvex, we have

$$h^- = \bar\partial u^- \tag{3.7}$$

for some $u^- \in L^2_{(p,n-2)}(\Omega^-)$. It follows from (3.6) and (3.7) that that for each $g \in H^{(n-p,0)}_{W^1}(\Omega^-)$,

$$\int_{b\Omega^-} h_b \wedge g = \int_{b\Omega^-} (h^+ - h^-) \wedge g = \int_{b\Omega^-} (\bar\partial u^+ - \bar\partial u^-) \wedge g = 0.$$

This implies that h_b is a linear functional vanishing on $H^{(n-p,0)}_{W^1}(\Omega^-)$. But from the regularity for the weighted $\bar\partial$-Neumann operator for $W^1_{(n-p,1)}(\Omega)$ (since we assume that Ω^- is C^2), we have that the space $H^{(n-p,0)}_{W^1}(\Omega^-)$ is dense in $H^{(n-p,0)}_{L^2}(\Omega^-)$. Since the functional vanishes on a dense subspace, h_b must be zero. This proves that $h = 0$ if $l^+h = [h^+] = 0$. Thus l^+ is one to one.

To show that l^+ is onto, take an element $F \in L^2_{(p,n-1)}(\Omega)$ such that $\bar\partial F = 0$. For simplicity, we assume that $p = n$. We will construct a holomorphic function h in $L^2(\Omega)$ such that $l^+h = F$. Note that from Theorem 1.4, any element $[F]$ can be represented by a harmonic form and we may assume that F is in $\mathcal{H}_{(n,n-1)}(\Omega)$. This implies that $\bar\partial F = \bar\partial^* F = 0$. It follows that F has boundary value with $W^{-\frac12}$-coefficients. Choose a special orthonormal frame field basis $w_1, \cdots, w_n = \sqrt{2}\partial\rho$ for $(1,0)$-forms. Then near the boundary F written in the special orthonormal frame fields as

$$F = \sum_{i=1}^n F_i (dV \lrcorner \bar w_i)$$

where $dV = w_1 \wedge \bar w_1 \ldots w_n \wedge \bar w_n$ is the volume element. Using

$$\star \bar w_n = \star w_n = \frac12 \star d\rho = \frac12 d\sigma$$

on $b\Omega^-$, we have

$$F\ |_{b\Omega^-} = F_n dV \lrcorner \bar w_n\ |_{b\Omega^-} = \frac12 F_n d\sigma,$$

where $d\sigma$ is the surface element on $b\Omega^-$.

We claim that $\bar F_n$ is a CR distribtution on $b\Omega^-$. To see this, note that $\bar\partial \star F = 0$ since $\vartheta F = 0$. Restricted to the boundary, this implies that $\bar\partial \bar F_n \wedge \bar w_n = 0$ on $b\Omega^-$. Thus $\bar F_n$ is a CR distribution in $W^{-\frac12}(b\Omega^-)$. Let $h = \bar F_n$ be the holomorphic extension of $\bar F_n$ from the boundary to Ω^-. Then h is an L^2 holomorphic function on Ω^-. Let $h^+ = l^+h$ be the $\bar\partial$-closed form in $L^2_{(n,n-1)}(\Omega)$. It remains to show that $[h^+] = [F]$. This follows from $h^+ - h^- = h_b = \star(\bar F_n \wedge \bar\partial\rho) = F_n dV \lrcorner \bar w_n$ on $b\Omega^-$. If we define

$$G = \begin{cases} F - h^+, & z \in \Omega \\ h^-, & z \in \Omega^-. \end{cases}$$

Then G is an L^2 $\bar\partial$-closed form in Ω_1. Thus we have $G = \bar\partial U$ for some $U \in L^2_{(n,n-2)}(\Omega_1)$ is $\bar\partial$-exact on Ω_1. Thus $[F] = [h^+]$ in $H^{(n,n-1)}(\Omega)$. This proves that l^+ is onto. The theorem is proved.

Corollary 3.2. *Let Ω be the same as Theorem 3.1. Each element f in the harmonic space $\mathcal{H}_{(p,n-1)}(\Omega)$ can be represented by some h^+, where h is a harmonic form in $L^2_{(n-p,0)}(\Omega^-)$. We have the following representation for the harmonic space*

$$\mathcal{H}_{(p,n-1)}(\Omega) = \{h^+ \mid h \in L^2_{(n-p,0)}(\Omega^-),\ \bar\partial h = 0.\}$$

where h^+ is defined by (3.4).

Proof. We have proved that for every $f \in \mathcal{H}_{(p,n-1)}(\Omega)$, we can write $f = \star h \partial\rho = \bar{h} \star \bar{\partial}\rho$ for some holomorphic h in $L^2_{(n-p,0)}(\Omega^-)$. On the other hand, any $h \in \mathcal{H}_{(n-p,0)}(\Omega^-)$, the associated $B^+(h_b) = h^+$ is in $L^2_{(p,n-1)}(\Omega)$. The form h^+ is automatically $\bar{\partial}$-closed. To see that it is in the domain of $\bar{\partial}^*$ and $\bar{\partial}^*\bar{h}^+ = 0$, we repeat the arguments before and the corollary is proved.

Remarks:

(1) All the results can be extended to any annulus between two pseudoconvex domains in a Stein manifold with trivial modification. It can also be applied to an annulus between two pseudoconvex domains in complex manifolds if one has the existence and the W^1 regularity of the $\bar{\partial}$-Neumann operator on the pseudoconvex domain Ω^-. We refer the reader to some related results in [**HI**] (see also [**CaS**] and [**CS1**] and [**CS2**]).

(2) If we assume that the boundary is C^∞ smooth, we can also have the isomorphism between $H^{(p,n-1)}(\Omega)$ and $H^{(p,n-1)}_{W^k}(\Omega)$ for all k following the same proof.

4. The null space for the $\bar{\partial}$-Neumann operator between balls

When the domain $\Omega = \{z \in \mathbb{C}^n \mid 0 < R_0 < |z| < R_1\}$ is the annulus between two balls centered at 0, the harmonic space $\mathcal{H}_{(0,n-1)}$ has been computed explicitly in Hörmander (see equation (2.3) in [**Hör2**]). He proved that any $(0, n-1)$-form f is in the null space of the $\bar{\partial}$-Neumann operator if and only if

$$f = \sum_1^n (-1)^j \frac{\bar{z}_j}{|z|^{2n}} h\left(\frac{\bar{z}}{|z|^2}\right) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{j-1} \wedge d\bar{z}_{j+1} \wedge \cdots \wedge d\bar{z}_n, \tag{4.1}$$

where h is a holomorphic function in $L^2(\Omega^*)$, $\Omega^* = \{z \in \mathbb{C}^n \mid |z| < \frac{1}{R_0}\}$. It is easy to check that $\bar{\partial} f = \vartheta f = 0$.

To see that f is in the domain of $\bar{\partial}^*$, we note that $f = \frac{1}{|z|^{2n}} h\left(\frac{\bar{z}}{|z|^2}\right) \star \bar{\partial}\rho$ where $\rho = |z|^2$. Thus $f \in \mathrm{Dom}(\bar{\partial}^*)$. We will show that the representation by the Bochner transform stated the harmonic forms from Corollary 3.2 agrees with the formula (4.1). For simplicity, we will assume that the inner ball is the unit ball, i.e., $R_0 = 1$ and h is a holomorphic $(n, 0)$ form in $L^2_{(n,0)}(B_1)$. Let $\rho(z) = |z|^2 - 1$. Then $\overline{h(\bar{z})}$ is holomorphic with L^2 coefficients. The Bochner transform described in Section 3 is given by

$$\begin{aligned} B^+ h_b &= \int_{\zeta \in bB_1} B_0(\zeta, z) \wedge h(\bar{\zeta}) \wedge \star\bar{\partial}\rho(\zeta) \\ &= \int_{\zeta \in bB_1} B_0(\zeta, z) \wedge h\left(\frac{\bar{\zeta}}{|\zeta|^2}\right) \wedge \sum_j \frac{\bar{\zeta}_j}{|\zeta|^{2n}} [d\bar{\zeta}_j] \end{aligned} \tag{4.2}$$

where $[d\bar{\zeta}_j]$ denote the $(n, n-1)$-form $dV \lrcorner d\bar{\zeta}_j$. It is easy to see that

$$\bar{\partial}\left(\sum_j \frac{\bar{\zeta}_j}{|\zeta|^{2n}} [d\bar{\zeta}_j]\right) = 0, \qquad \zeta \neq 0.$$

We also have

$$\bar\partial\left(h\left(\frac{\bar\zeta}{|\zeta|^2}\right)\wedge\sum_j\frac{\bar\zeta_j}{|\zeta|^{2n}}[d\bar\zeta_j]\right)=\sum_k\sum_j\frac{\partial h}{\partial w_k}\left(\frac{\delta_{jk}}{|\zeta|^2}-\frac{\bar\zeta_k\zeta_j}{|\zeta|^4}\right)\frac{\bar\zeta_j}{|\zeta|^{2n}}dV$$
$$=\sum_k\frac{\partial h}{\partial w_k}\left(\frac{\bar\zeta_k}{|\zeta|^2}-\frac{\bar\zeta_k}{|\zeta|^2}\right)dV=0.$$

Applying Stokes's Theroem to (4.2), we see that

$$B^+h_b=\sum\frac{\bar z_j}{|z|^{2n}}h\left(\frac{\bar z}{|z|^2}\right)dV\lrcorner d\bar z_j.$$

We mention that in [**Hör2**], it is also proved that for any $f\in\mathrm{Dom}(\bar\partial)\cap\mathrm{Dom}(\bar\partial^*)\cap\mathcal{H}_{(n,n-1)}(\Omega)^\perp$,

$$\max(n-2,1)\|f\|^2\le R_1^2\|\bar\partial f\|^2+\|\bar\partial^* f\|^2. \tag{4.3}$$

Notice that the constant in (4.3) is independent of the inner diameter R_0. It is not known if one can have such estimates on the more general annulus between two pseudoconvex domains.

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