CHAPTER 1

REAL AND COMPLEX MANIFOLDS

We shall begin by defining holomorphic functions and the Cauchy-Riemann equations in $\mathbb{C}^n$. In Sections 1.2-1.4 of this chapter we will review the definitions and various properties of a smooth real or complex manifold. In Section 1.5, the Cauchy-Riemann complex is introduced on complex manifolds. Section 1.6 is devoted to the Frobenius theorem. In the last section, in contrast to the Riemann mapping theorem in one complex variable, we prove the inequivalence between the ball and the polydisc in several variables.

1.1 Holomorphic Functions in Complex Euclidean Spaces

Let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the n-dimensional complex Euclidean space with product topology. The coordinates of $\mathbb{C}^n$ will be denoted by $z = (z_1, \cdots, z_n)$ with $z_j = x_j + iy_j$, $1 \le j \le n$. Thus, $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n}$ in a natural manner, $z \mapsto (x_1, y_1, \cdots, x_n, y_n)$.

Definition 1.1.1. *A complex-valued C^1 function $f(z)$ defined on an open subset D of $\mathbb{C}^n$ is called holomorphic, denoted by $f \in \mathcal{O}(D)$, if $f(z)$ is holomorphic in each variable z_j when the other variables are fixed. In other words, $f(z)$ satisfies*

$$\frac{\partial f}{\partial \overline{z}_j} = 0, \tag{1.1.1}$$

for each $j = 1, \cdots, n$, where

$$\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \tag{1.1.2}$$

is the so-called Cauchy-Riemann operator.

The objective of this book is to study the behavior of holomorphic functions. It is closely related to the solvability and regularity of the inhomogeneous Cauchy-Riemann equations

$$\frac{\partial u}{\partial \overline{z}_j} = f_j, \qquad \text{for } j = 1, \cdots, n, \tag{1.1.3}$$

where f_j's are given functions.

Some of the properties of holomorphic functions, like power series expansion, do extend from one variable to several variables. They differ, however, in many important aspects. It is therefore, not correct to consider the theory of several complex variables as a straightforward generalization of that of one complex variable. For example, in one variable the zero set of a holomorphic function is a discrete set. The zero set of a holomorphic function in $\mathbb{C}^n$, $n \geq 2$, has a real $2n-2$ dimension. In $\mathbb{C}$, it is trivial to construct a holomorphic function in a domain D which is singular at one boundary point $p \in bD$. In contrast, in $\mathbb{C}^n$ when $n \geq 2$, it is not always possible to construct a holomorphic function in a given domain $D \subset \mathbb{C}^n$ which is singular at one boundary point. This leads to the existence of a domain in several variables such that any holomorphic function defined on this domain can be extended holomorphically to a fixed larger set, a feature that does not exist in one variable. In Chapter 3 we will discuss this phenomenon in detail. Another main difference is that there is no analog to the Riemann mapping theorem of one complex variable in higher dimensional spaces. This phenomenon is analyzed in Section 1.7. Many of these important differences will be further investigated in Chapters 4-6 using solutions of the inhomogeneous Cauchy-Riemann equations (1.1.3).

There is yet another major difference in solving (1.1.3) in one and several variables. When $n \geq 2$, a compatibility condition must be satisfied in order for Equations (1.1.3) to be solvable:

$$\frac{\partial f_i}{\partial \bar{z}_j} = \frac{\partial f_j}{\partial \bar{z}_i}, \qquad \text{for } 1 \leq i < j \leq n. \tag{1.1.4}$$

This will be discussed in the next few chapters on bounded domains in $\mathbb{C}^n$.

We recall here the definition concerning the differentiability of the boundary of a domain.

Definition 1.1.2. *A domain D in $\mathbb{R}^n$, $n \geq 2$, is said to have C^k $(1 \leq k \leq \infty)$ boundary at the boundary point p if there exists a real-valued C^k function r defined in some open neighborhood U of p such that $D \cap U = \{x \in U |\ r(x) < 0\}$, $bD \cap U = \{x \in U |\ r(x) = 0\}$ and $dr(x) \neq 0$ on $bD \cap U$. The function r is called a C^k local defining function for D near p. If U is an open neighborhood of $\overline{D}$, then r is called a global defining function for D, or simply a defining function for D.*

The relationship between two defining functions is clarified in the next lemma.

Lemma 1.1.3. *Let r_1 and r_2 be two local defining functions for D of class C^k $(1 \leq k \leq \infty)$ in a neighborhood U of $p \in bD$. Then there exists a positive C^{k-1} function h on U such that*

(1) *$r_1 = h r_2$ on U,*
(2) *$dr_1(x) = h(x) dr_2(x)$ for $x \in U \cap bD$.*

Proof. Since $dr_2 \neq 0$ on the boundary near p, after a C^k change of coordinates, we may assume that $p = 0$, $x_n = r_2(x)$ and $bD \cap U = \{x \in U |\ x_n = 0\}$. Let $x' = (x_1, \cdots, x_{n-1})$. Then $r_1(x', 0) = 0$. By the fundamental theorem of calculus,

$$r_1(x', x_n) = r_1(x', x_n) - r_1(x', 0) = x_n \int_0^1 \frac{\partial r_1}{\partial x_n}(x', t x_n) dt.$$

This shows $r_1 = hr_2$ for some C^{k-1} function h on U. For $k \geq 2$, we clearly have (2) and $h > 0$ on U. When $k = 1$, (2) also follows directly from the definition of differentiation at 0. This proves the lemma.

1.2 Real and Complex Manifolds

Let M be a Hausdorff space. M is called a topological manifold of dimension n if each point p of M has a neighborhood U_p homeomorphic to an open subset V_p in $\mathbb{R}^n$. Let the homeomorphism be given by $\varphi_p : U_p \to V_p$. We call the pair (U_p, φ_p) a coordinate neighborhood of M near p. Since, for any $q \in U_p$, $\varphi_p(q)$ is a point in $\mathbb{R}^n$, we have the usual Euclidean coordinates $(x_1(\varphi_p(q)), \cdots, x_n(\varphi_p(q)))$ for $\varphi_p(q)$. We shall call the set $(x_1(\varphi_p(q)), \cdots, x_n(\varphi_p(q)))$ the local coordinates for the points q in U_p with respect to the coordinate neighborhood (U_p, φ_p), and it will be abbreviated by $(x_1(q), \cdots, x_n(q))$, and the n-tuple $(x_1, \cdots, x_n)$ of functions on U_p will be called the local coordinate system on (U_p, φ_p).

Let M be a topological manifold, then M is covered by a family of such coordinate neighborhoods $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$, where Λ is an index set. If for some α, β in Λ we have $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, then there is a well-defined homeomorphism

$$f_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta}).$$

These will be called the transition functions with respect to the coordinate neighborhood system $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$. Obviously, we have $f_{\beta\alpha} = f_{\alpha\beta}^{-1}$. Now we give the definition of a differentiable manifold.

Definition 1.2.1. *Let M be a topological manifold together with a coordinate neighborhood system $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$. We call M an n-dimensional differentiable manifold of class C^r, or a C^r manifold, $1 \leq r \leq \infty$, if all of the corresponding transition functions are of class C^r. If $r = \infty$, we call M a smooth manifold. If all of the corresponding transition functions are real analytic, M will be called a real analytic manifold, or a C^ω manifold.*

Next we define complex manifolds.

Definition 1.2.2. *Let M be a topological manifold together with a coordinate neighborhood system $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$, where $\varphi_\alpha(U_\alpha) = V_\alpha$ are open sets in $\mathbb{C}^n$. M is called a complex manifold of complex dimension n if the transition function $f_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U_{\alpha\beta}) \subset \mathbb{C}^n$, whenever $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$ for all α, β.*

It follows that a complex manifold is automatically a real analytic manifold. Here are some important examples of real and complex manifolds.

Example 1.2.3. Any connected open subset M of $\mathbb{R}^n$ is a real analytic manifold. The local chart (M, ι) is simply the induced one given by the identity mapping ι from M into $\mathbb{R}^n$. Similarly, any connected open subset M of $\mathbb{C}^n$ is a complex manifold of complex dimension n.

Example 1.2.4 (Real projective space, $\mathbb{RP}^n$). Define an equivalence relation on the set $\mathbb{R}^{n+1} \setminus \{0\}$. Two points x and y in $\mathbb{R}^{n+1} \setminus \{0\}$ are said to be equivalent if there is a nonzero real number $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$. The set of equivalence classes given by this equivalence relation is called the real projective space $\mathbb{RP}^n$ of dimension n. In other words, $\mathbb{RP}^n$ can be identified with the space of all lines passing through the origin in $\mathbb{R}^{n+1}$. The mapping π from $\mathbb{R}^{n+1} \setminus \{0\}$ onto $\mathbb{RP}^n$ so that $\pi(x)$ is the equivalence class containing the point x is continuous, provided that $\mathbb{RP}^n$ is equipped with the quotient topology, namely, a subset U of $\mathbb{RP}^n$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$. Since π also maps the compact set S^n onto $\mathbb{RP}^n$, we see that $\mathbb{RP}^n$ is compact.

The coordinate neighborhood system $\{(U_j, \varphi_j)\}_{j=1}^{n+1}$ is constructed as follows: for each $p \in \mathbb{RP}^n$ pick an element $x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$ such that $\pi(x) = p$. The point p can be represented by the corresponding homogeneous coordinates $[x_1 : x_2 : \cdots : x_{n+1}]$. This representation is clearly independent of the choice of x. Let $U_j = \{[x_1 : \cdots : x_j : \cdots : x_{n+1}] | x_j \neq 0\}$ be an open subset of $\mathbb{RP}^n$, and the homeomorphism φ_j from U_j onto $\mathbb{R}^n$ is given by

$$\begin{aligned}&\varphi_j([x_1 : \cdots : x_j : \cdots : x_{n+1}])\\ &\quad = (x_1/x_j, \cdots, x_{j-1}/x_j, x_{j+1}/x_j, \cdots, x_{n+1}/x_j).\end{aligned}$$

Hence, if $U_i \cap U_j = U_{ij} \neq \emptyset$, say, $i < j$, then the transition function f_{ij} is

$$\begin{aligned}f_{ij}(y) &= \varphi_i \circ \varphi_j^{-1}(y)\\ &= \varphi_i([y_1 : \cdots : y_{j-1} : 1 : y_j : \cdots : y_n])\\ &= \left(\frac{y_1}{y_i}, \cdots, \frac{y_{i-1}}{y_i}, \frac{y_{i+1}}{y_i}, \cdots, \frac{y_{j-1}}{y_i}, \frac{1}{y_i}, \frac{y_j}{y_i}, \cdots, \frac{y_n}{y_i}\right).\end{aligned}$$

It follows that the real projective space $\mathbb{RP}^n$ is a real analytic compact manifold.

Example 1.2.5 (Complex projective space, $\mathbb{CP}^n$). If $\mathbb{C}$ is substituted for $\mathbb{R}$ in the definition of the real projective space $\mathbb{RP}^n$, we will end up with a compact complex manifold of complex dimension n which we call the complex projective space and denote by $\mathbb{CP}^n$.

Example 1.2.6 (Riemann surface). A Riemann surface M is by definition a complex manifold of complex dimension one. Hence, any open subset U of $\mathbb{C}$ is a Riemann surface. Complex projective space $\mathbb{CP}^1$ is a compact Riemann surface, also known as the Riemann sphere.

From now on we shall concentrate on complex manifolds, unless the contrary is stated explicitly in the text. Let f be a continuous complex-valued function defined on an open subset U of a complex manifold M, and let p be a point in U. We say that f is *holomorphic* at p if there exists a small open neighborhood V of p, contained in $U \cap U_\alpha$ for some local coordinate neighborhood U_α, such that $f \circ {\varphi_\alpha}^{-1}$ is holomorphic on the open subset $\varphi_\alpha(V)$ in $\mathbb{C}^n$. Clearly, the above definition of holomorphic functions at a point $p \in U$ is independent of the choice of the local coordinate system $(U_\alpha, \varphi_\alpha)$. The function f is said to be holomorphic on U if f is holomorphic at every point $p \in U$. In particular, the local coordinate functions $z_i, 1 \le i \le n$, on U_α of a complex manifold are holomorphic.

Let M and N be two complex manifolds of complex dimensions m and n with local coordinate systems $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ and $\{(V_\beta, \psi_\beta)\}_{\beta \in \Gamma}$ respectively, and let f be a continuous mapping from M into N. We shall say that f defines a holomorphic mapping at $p \in M$, if there exists an open neighborhood U_p of p, contained in a local coordinate neighborhood U_α, with $f(U_p)$ contained in a local coordinate neighborhood V_β such that $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ defines a holomorphic mapping from $\varphi_\alpha(U_p)$ into $\psi_\beta(V_\beta)$. The definition is easily seen to be independent of the choice of the local coordinate systems.

If f is a holomorphic mapping between two complex manifolds M and N of equal dimensions such that f is one-to-one, onto and the inverse mapping f^{-1} is also holomorphic, then f will be called a biholomorphic map or a biholomorphism from M onto N.

1.3 Tangent Spaces and the Hermitian Metric

Let $\mathbb{C}^n$ be identified with $\mathbb{R}^{2n}$ via the map $(z_1, \cdots, z_n) \mapsto (x_1, y_1, \cdots, x_n, y_n)$. For any point $p \in \mathbb{C}^n$ the tangent space $T_p(\mathbb{C}^n)$ is spanned by

$$\left(\frac{\partial}{\partial x_1}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \cdots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_n}\right)_p.$$

Define an $\mathbb{R}$-linear map J from $T_p(\mathbb{C}^n)$ onto itself by

$$J\left(\frac{\partial}{\partial x_j}\right)_p = \left(\frac{\partial}{\partial y_j}\right)_p, \quad J\left(\frac{\partial}{\partial y_j}\right)_p = -\left(\frac{\partial}{\partial x_j}\right)_p,$$

for all $j = 1, \cdots, n$. Obviously, we have $J^2 = -1$, and J is called the complex structure on $\mathbb{C}^n$.

The complex structure J induces a natural splitting of the complexified tangent space $\mathbb{C}T_p(\mathbb{C}^n) = T_p(\mathbb{C}^n) \otimes_{\mathbb{R}} \mathbb{C}$. First we extend J to the whole complexified tangent space by $J(x \otimes \alpha) = (Jx) \otimes \alpha$. It follows that J is a $\mathbb{C}$-linear map from $\mathbb{C}T_p(\mathbb{C}^n)$ onto itself with $J^2 = -1$, and the eigenvalues of J are i and $-i$. Denote by $T_p^{1,0}(\mathbb{C}^n)$ and $T_p^{0,1}(\mathbb{C}^n)$ the eigenspaces of J corresponding to i and $-i$ respectively. It is easily verified that $T_p^{0,1}(\mathbb{C}^n) = \overline{T_p^{1,0}(\mathbb{C}^n)}$ and $T_p^{1,0}(\mathbb{C}^n) \cap T_p^{0,1}(\mathbb{C}^n) = \{0\}$, and that $T_p^{1,0}(\mathbb{C}^n)$ is spanned by

$$\left(\frac{\partial}{\partial z_1}\right)_p, \cdots, \left(\frac{\partial}{\partial z_n}\right)_p,$$

where $(\partial/\partial z_j)_p = \frac{1}{2}(\partial/\partial x_j - i\partial/\partial y_j)_p$ for $1 \le j \le n$. Any vector $v \in T_p^{1,0}(\mathbb{C}^n)$ is called a vector of type $(1,0)$, and we call $\overline{v} \in T_p^{0,1}(\mathbb{C}^n)$ a vector of type $(0,1)$. The space $T_p^{1,0}(\mathbb{C}^n)$ is called the *holomorphic tangent space* at p.

Let $\mathbb{C}T_p^*(\mathbb{C}^n)$ be the dual space of $\mathbb{C}T_p(\mathbb{C}^n)$. By duality, J also induces a splitting on

$$\mathbb{C}T_p^*(\mathbb{C}^n) = \Lambda_p^{1,0}(\mathbb{C}^n) \oplus \Lambda_p^{0,1}(\mathbb{C}^n),$$

where $\Lambda_p^{1,0}(\mathbb{C}^n)$ and $\Lambda_p^{0,1}(\mathbb{C}^n)$ are eigenspaces corresponding to the eigenvalues i and $-i$ respectively. It is easy to see that the vectors $(dz_1)_p, \cdots, (dz_n)_p$ span $\Lambda_p^{1,0}(\mathbb{C}^n)$ and the space $\Lambda_p^{0,1}(\mathbb{C}^n)$ is spanned by $(d\overline{z}_1)_p, \cdots, (d\overline{z}_n)_p$.

Let M be a complex manifold of complex dimension n and p be a point of M. Let $(z_1, \cdots, z_n)$ be a local coordinate system near p with $z_j = x_j + iy_j, j = 1, \cdots, n$. Then the real tangent space $T_p(M)$ is spanned by

$$(\partial/\partial x_1)_p, (\partial/\partial y_1)_p, \cdots, (\partial/\partial x_n)_p, (\partial/\partial y_n)_p.$$

Define as before an $\mathbb{R}$-linear map J from $T_p(M)$ onto itself by

$$J\left(\frac{\partial}{\partial x_j}\right)_p = \left(\frac{\partial}{\partial y_j}\right)_p, \quad J\left(\frac{\partial}{\partial y_j}\right)_p = -\left(\frac{\partial}{\partial x_j}\right)_p,$$

for $1 \le j \le n$. We observe that the definition of J is independent of the choice of the local coordinates $(z_1, \cdots, z_n)$ and that $J^2 = -1$. Therefore, an argument similar to the one given above shows that

$$\mathbb{C}T_p(M) = T_p(M) \otimes_{\mathbb{R}} \mathbb{C} = T_p^{1,0}(M) \oplus T_p^{0,1}(M),$$

and

$$\mathbb{C}T_p^*(M) = \Lambda_p^{1,0}(M) \oplus \Lambda_p^{0,1}(M).$$

Next we introduce a Hermitian metric on M. By that, we mean at each point $p \in M$, a Hermitian inner product $h_p(u, v)$ is defined for $u, v \in T_p^{1,0}(M)$. If $(z_1, \cdots, z_n)$ is a local coordinate system on a neighborhood U of p, then

$$h_{ij}(p) = h_p\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$$

is a complex-valued function on U, and $(h_{ij}(p))_{i,j=1}^n$ is a positive definite Hermitian matrix defined for each point p of U. We shall assume the metric is smooth; namely, that all the h_{ij}'s vary smoothly on M. Then, we extend the metric to the whole complexified tangent space in a natural way by requiring $T^{1,0}(M)$ to be orthogonal to $T^{0,1}(M)$. If a complex manifold M is equipped with a Hermitian metric h, we shall call (M, h) a Hermitian manifold.

1.4 Vector Bundles

Let M be a smooth manifold of real dimension n. The union of all the tangent spaces $T_p(M), p \in M$, inherits a natural geometric structure called the vector bundle.

Definition 1.4.1. *Let E and M be two smooth manifolds. E is called a vector bundle over M of rank k if there exists a smooth mapping π, called the projection map, from E onto M such that the following conditions are satisfied:*

(1) *For each $p \in M$, $E_p = \pi^{-1}(p)$ is a vector space over $\mathbb{R}$ of dimension k. E_p is called the fibre space over p.*

(2) *For each $p \in M$, there exists an open neighborhood U containing p and a diffeomorphism*

$$h : \pi^{-1}(U) \to U \times \mathbb{R}^k,$$

such that $h(\pi^{-1}(q)) = \{q\} \times \mathbb{R}^k$ and the restriction $h_q : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^k \equiv \mathbb{R}^k$ is a linear isomorphism, for every $q \in U$.

The pair (U, h) is called a local trivialization.

When $k = 1$, E is also called a line bundle over M.

For a vector bundle $\pi : E \to M$, the manifold E is called the total space and M is called the base space, and E is called a vector bundle over M. Notice that if two local trivializations (U_α, h_α) and (U_β, h_β) have nonempty intersection, i.e., $U_\alpha \cap U_\beta \neq \emptyset$, then a map $g_{\alpha\beta}$ is induced on $U_\alpha \cap U_\beta$:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}),$$

such that

$$g_{\alpha\beta}(p) = (h_\alpha)_p \circ (h_\beta)_p^{-1} : \mathbb{R}^k \to \mathbb{R}^k.$$

The matrices $g_{\alpha\beta}$'s are called transition matrices. Clearly, they are smooth and satisfy the following conditions:

(1) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$,

(2) $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = I$,

where I is the identity matrix of rank k.

Let E be a vector bundle over M, and let U be an open subset of M. Any smooth mapping s from U to E such that $\pi \circ s = id_U$, where id_U is the identity mapping on U, will be called a section over U. The space of all the sections over U will be denoted by $\Gamma(U, E)$.

Notice that the concept of vector bundle can obviously be defined for other categories. For instance, if E and M are complex manifolds and the fibers are complex vector spaces, then one can define a holomorphic vector bundle E over M by requiring the morphisms and the transition matrices to be holomorphic mappings.

Here are some typical examples of vector bundles.

Example 1.4.2 (Tangent Bundles). Let M be a manifold of real dimension n. The set formed by the disjoint union of all tangent spaces $T_p(M)$ of $p \in M$, namely,

$$T(M) = \underset{p \in M}{\cup} T_p(M),$$

has a natural vector bundle structure of rank n over M. The local coordinate neighborhoods of $T(M)$ and the local trivializations of the bundle are given by the local coordinate systems of M as follows: let $(x_1, \cdots, x_n)$ be a local coordinate system on U of M, and let $p \in U$. Then any tangent vector v at p can be written as

$$v = \sum_{i=1}^{n} v_i(x) \left(\frac{\partial}{\partial x_i} \right)_p .$$

Thus, we obtain a map ϕ from $\pi^{-1}(U)$ onto $U \times \mathbb{R}^n$ by

$$\begin{aligned} \phi : \pi^{-1}(U) &\to U \times \mathbb{R}^n, \\ (p, v) &\mapsto (p, v_1(x), \cdots, v_n(x)). \end{aligned}$$

If two local coordinate systems have nontrivial intersection, then the transition matrix is clearly defined by the Jacobian matrix, with respect to these two local coordinate systems, which by definition is smooth. It is also clear that any global section s in $\Gamma(M, T(M))$ is a smooth vector field X defined on M.

Next, we can also form a new vector bundle from a given one. The most important examples of such algebraically derived vector bundles are those originating from the tangent bundle $T(M)$. For instance, by considering the dual space and the exterior algebra of the tangent space $T_p(M)$, we obtain the following new vector bundles:

Example 1.4.3 (Cotangent Bundle). Let M be a smooth manifold of real dimension n. The fibre of the cotangent bundle, $T^*(M)$, at each point $p \in M$ is the $\mathbb{R}$-linear dual space of $T_p(M)$, denoted by $T_p^*(M)$. Clearly, $T^*(M)$ is a vector bundle of rank n over M. A section s of this bundle over an open set U of M is called a smooth 1-form over U. We also have the complexified cotangent bundle, denoted by $\Lambda^1(M) = T^*(M) \otimes_{\mathbb{R}} \mathbb{C}$, over M.

Example 1.4.4 (Exterior Algebra Bundles). Let M be a complex manifold of complex dimension n. Then the exterior algebra bundles over M are the vector bundles $\Lambda^r(M)$ whose fibers at each point $z_0 \in M$ are the wedge product of degree r of $\Lambda_p^1(M)$, and

$$\Lambda(M) = \bigoplus_{r=0}^{2n} \Lambda^r(M).$$

Any smooth section s of $\Lambda^r(M)$ over an open subset U of M is a smooth r-form on U. If, at each point z_0 of M, we take the wedge product of p copies of $\Lambda^{1,0}(M)$ and q copies of $\Lambda^{0,1}(M)$, where $p \leq n$ and $q \leq n$, we obtain the vector bundle of bidegree (p, q), denoted by $\Lambda^{p,q}(M)$, and we have

$$\Lambda^r(M) = \bigoplus_{p+q=r} \Lambda^{p,q}(M).$$

Smooth sections of $\Lambda^{p,q}(M)$, denoted by $C^\infty_{(p,q)}(M)$, are called (p, q)-forms on M.

1.5 Exterior Derivatives and the Cauchy-Riemann Complex

Let M be a complex manifold of complex dimension n, and let $(z_1, \cdots, z_n)$ be a local coordinate system on an open neighborhood U of a point p of M, with $z_j = x_j + iy_j$ for $1 \leq j \leq n$. Let f be a C^1 complex-valued function defined on M. Then, locally on U one can express df as

$$\begin{aligned} df &= \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j + \sum_{j=1}^n \frac{\partial f}{\partial y_j} dy_j \\ &= \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j, \end{aligned} \tag{1.5.1}$$

where we have used the notation

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right), \quad \frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right),$$

and

$$dz_j = dx_j + idy_j, \quad d\overline{z}_j = dx_j - idy_j,$$

for $1 \le j \le n$. Define the operators ∂ and $\overline{\partial}$ on functions by

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j, \quad \text{and} \quad \overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

Then, (1.5.1) can be written as

$$df = \partial f + \overline{\partial} f.$$

This means that the differential df of a C^1 function f on U can be decomposed into the sum of a $(1,0)$-form ∂f and $(0,1)$-form $\overline{\partial} f$.

It is easily verified that the definitions of ∂ and $\overline{\partial}$ are invariant under holomorphic change of coordinates. Hence, the operators ∂ and $\overline{\partial}$ are well defined for functions on a complex manifold. A C^1 complex-valued function on a complex manifold is *holomorphic* if and only if

$$\overline{\partial} f = 0.$$

Next we extend the definition of the operators ∂ and $\overline{\partial}$ to differential forms of arbitrary degree. Let f be a (p,q)-form on U. Write f as

$$f = \sum_{|I|=p, |J|=q} f_{IJ} dz^I \wedge d\overline{z}^J,$$

where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$ are multiindices of length p and q respectively, $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, d\overline{z}^J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$.

The exterior derivative df of f is then defined by

$$\begin{aligned} df &= \sum_{I,J} df_{I,J} \wedge dz^I \wedge d\overline{z}^J \\ &= \partial f + \overline{\partial} f, \end{aligned}$$

where ∂f and $\overline{\partial} f$ are defined by

$$\partial f = \sum_{I,J} \partial f_{I,J} \wedge dz^I \wedge d\overline{z}^J, \quad \overline{\partial} f = \sum_{I,J} \overline{\partial} f_{I,J} \wedge dz^I \wedge d\overline{z}^J,$$

which are of type $(p+1, q)$ and $(p, q+1)$ respectively.

Since the transition matrices of a complex manifold M are holomorphic, the operators ∂ and $\overline{\partial}$ are well defined for (p,q)-forms on M, and we have

$$d = \partial + \overline{\partial}.$$

Since

$$0 = d^2 f = \partial^2 f + (\partial\overline{\partial} + \overline{\partial}\partial) f + \overline{\partial}^2 f$$

and all terms are of different types, we obtain

$$\partial^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0, \quad \overline{\partial}^2 = 0. \tag{1.5.2}$$

Since

$$\overline{\partial}^2 = 0, \tag{1.5.3}$$

It follows that the sequence

$$0 \to \Lambda^{p,0}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Lambda^{p,n-1}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,n}(M) \to 0,$$

for $0 \le p \le n$, is a complex. This is called the Cauchy-Riemann complex. Denote $\overline{\partial}_{p,q} = \overline{\partial} : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)$. It follows that the image of $\overline{\partial}_{p,q}$ lies in the kernel of $\overline{\partial}_{p,q+1}$. To measure the exactness of the sequence, we have to solve the following inhomogeneous equation

$$\overline{\partial} u = f, \tag{1.5.4}$$

under the compatibility condition

$$\overline{\partial} f = 0. \tag{1.5.5}$$

The solvability of the $\overline{\partial}$-equation as well as the smoothness of the solution is one of the main issues throughout this book.

1.6 The Frobenius Theorem

Let U be an open neighborhood of the origin in $\mathbb{R}^n$, and let k be an integer with $1 \le k < n$. Then, the set $N_c = \{x = (x_1, \cdots, x_n) \in U |\ x_{k+1} = c_{k+1}, \cdots, x_n = c_n\}$, where $c = (c_{k+1}, \cdots, c_n) \in \mathbb{R}^{n-k}$ is a constant vector, forms a k dimensional submanifold of U. By a submanifold we mean that N_c is a closed subset of U and N_c forms a manifold itself. Notice that $N_{c_1} \cap N_{c_2} = \emptyset$ if $c_1 \neq c_2$. Also $\cup_{c \in \mathbb{R}^{n-k}} N_c = U$. With such a submanifold structure, we shall say that U is foliated by k dimensional submanifolds N_c, and call N_c a leaf of the foliation.

Let $X_1, \cdots, X_k$ be k linearly independent vector fields on U such that they are tangent to some N_c, $c \in \mathbb{R}^{n-k}$, everywhere. Since the restriction of the vector field $X_i, 1 \le i \le k$ to each N_c defines a vector field on N_c, it is easily seen that the commutator $[X_i, X_j] = X_i X_j - X_j X_i, 1 \le i, j \le k$, is still a smooth vector field tangent to N_c everywhere on U. It follows that on U we have

$$[X_i, X_j] = \sum_{l=1}^{k} a_{ijl}(x) X_l, \tag{1.6.1}$$

where $a_{ijl}(x) \in C^\infty(U)$.

In this section we shall show that condition (1.6.1) is also sufficient for a manifold to be foliated locally by submanifolds whose tangent vectors are spanned by $X_i's$. Since the result is purely local, we shall formulate the theorem in an open neighborhood U of the origin in $\mathbb{R}^n$.

Theorem 1.6.1 (Frobenius). *Let $X_1, \cdots, X_k, 1 \le k < n$, be smooth vector fields defined in an open neighborhood U of the origin in $\mathbb{R}^n$. If*

(1) $X_1(0), \cdots, X_k(0)$ *are linearly independent, and*

(2) $[X_i, X_j] = \sum_{l=1}^k a_{ijl}(x) X_l,\ 1 \le i, j \le k,$ *for some* $a_{ijl}(x) \in C^\infty(U)$,

then there exist new local coordinates $(y_1, \cdots, y_n)$ in some open neighborhood V of the origin such that

$$X_i = \sum_{j=1}^k b_{ij}(y) \frac{\partial}{\partial y_j}, \quad i = 1, \cdots, k,$$

where $(b_{ij}(y))$ is an invertible matrix. In other words, V is foliated by the k-dimensional submanifolds $\{y \in V \mid y_i = c_i,\ i = k+1, \cdots, n\}$.

Proof. The theorem will be proved by induction on the dimension n of the ambient space. When $n = 1$, the assertion is obviously true. Let us assume that the assertion is valid up to dimension $n-1$.

First, we may simplify the vector field $X_1(x) = (a_1(x), \cdots, a_n(x))$. From the basic existence theorem for a system of first order ordinary differential equation, through every point p in a small open neighborhood of the origin, there exists exactly one integral curve $\gamma(t) = (\gamma_1(t), \cdots, \gamma_n(t))$, where $t \in (-\delta, \delta)$ for some real number $\delta > 0$, such that

$$\frac{d\gamma_i}{dt}(t) = a_i(\gamma(t)), \quad i = 1, \cdots, n.$$

It follows that for any smooth function f in a small neighborhood V of 0,

$$X_1 f(\gamma(t)) = \sum_{i=1}^k a_i(\gamma(t)) \frac{\partial f(\gamma(t))}{\partial x_i} = \frac{\partial f(\gamma(t))}{\partial t}.$$

Locally one can introduce new independent variables, also denoted by $(x_1, \cdots, x_n)$, which straighten out the integral curves so that $X_1 = \partial/\partial x_1$. Next, by subtracting a multiple of X_1 from X_i for $2 \le i \le k$, we may also assume that

$$X_i(x) = \sum_{l=2}^n c_{il}(x) \frac{\partial}{\partial x_l}, \quad i = 2, \cdots, k.$$

Denote $x' = (x_2, \cdots, x_n)$. On the submanifold $V \cap \{x = (x_1, x') \mid x_1 = 0\}$, these vector fields $X_2(0, x'), \cdots, X_k(0, x')$, satisfy both conditions (1) and (2) in an open neighborhood of the origin in $\mathbb{R}^{n-1}$. Hence, by the induction hypotheses, there exist new local coordinates near the origin in $\mathbb{R}^{n-1}$, denoted also by $x' = (x_2, \cdots, x_n)$, such that $c_{il}(0, x') = 0$ for $2 \le i \le k$ and $l > k$. For $2 \le i \le k$ and $2 \le l \le n$, we have

$$\begin{aligned}
\frac{\partial c_{il}}{\partial x_1}(x) &= X_1 X_i(x_l) = [X_1, X_i](x_l) \\
&= \sum_{\alpha=2}^k a_{1i\alpha}(x) X_\alpha(x_l) \\
&= \sum_{\alpha=2}^k a_{1i\alpha}(x) c_{\alpha l}(x).
\end{aligned}$$

Hence, the uniqueness part of the Cauchy problem for a system of first order ordinary differential equations implies that $c_{il}(x) \equiv 0$ for $l > k$ in an open neighborhood V of the origin. This completes the proof of the theorem.

It should be pointed out that the existence of the local coordinates $(y_1, \cdots, y_n)$ guaranteed by the Frobenius theorem is not unique. Suppose that there are k smooth vector fields $X_1, \cdots, X_k$ defined in some open neighborhood of the origin in $\mathbb{R}^n$ such that conditions (1) and (2) of Theorem 1.6.1 are satisfied. Let us consider a system of overdetermined partial differential equations

$$X_j u = f_j, \quad j = 1, \cdots, k, \tag{1.6.2}$$

where the data f_j's are smooth functions given in an open neighborhood of the origin. It is clear from condition (2) that the system (1.6.2) is solvable only if the given data satisfy the following compatibility condition

$$X_i f_j - X_j f_i = \sum_{l=1}^{k} a_{ijl}(x) f_l(x), \quad 1 \le i, j \le k. \tag{1.6.3}$$

With the aid of Theorem 1.6.1, the next theorem shows that condition (1.6.3) is, in fact, also sufficient for the solvability of (1.6.2).

Theorem 1.6.2. *Under the same hypotheses as in Theorem 1.6.1, let $f_1, \cdots, f_k$ be smooth functions defined on U. Then, the system (1.6.2) has a smooth solution u in an open neighborhood of the origin if and only if the compatibility conditions (1.6.3) are satisfied. Furthermore, if H is a closed submanifold of U through the origin of dimension $n-k$ such that the tangent plane of H at the origin is complementary to the space spanned by $X_1(0), \cdots, X_k(0)$, then given any smooth function u_h on H, there exists an unique solution u to the equations (1.6.2) in an open neighborhood of the origin with $u|_H = u_h$.*

Notice that in the language of partial differential equations, the hypothesis on H is equivalent to stating that the manifold H is noncharacteristic with respect to $X_1, \cdots, X_k$, or that, geometrically, H is transversal to the leaves of the foliation defined by the vector fields $X_1, \cdots, X_k$ in some open neighborhood of the origin.

Proof. Notice first that conditions (1) and (2) of Theorem 1.6.1 and equations (1.6.2),(1.6.3) are invariant if we change variables or make linear combinations of the equations. Hence, by Theorem 1.6.1, we may thus assume that $X_j(x) = \partial/\partial x_j, j = 1, \cdots, k$. It follows that $a_{ijl}(x) \equiv 0$ for all $1 \le i, j, l \le k$, and equation (1.6.3) is reduced to

$$\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = 0, \quad 1 \le i, j \le k. \tag{1.6.4}$$

Let us write the local coordinates $x = (x', x'')$ with $x' = (x_1, \cdots, x_k)$ and $x'' = (x_{k+1}, \cdots, x_n)$. Then, locally in some open neighborhood of the origin one may express H as a graph over an open subset V containing the origin in $\mathbb{R}^{n-k}$, namely,

H is defined by $x' = h(x'')$ where $h(x'')$ is a smooth function on V. Now, for each fixed $x'' \in V$,

$$\sum_{j=1}^{k} f_j(x', x'')dx_j$$

is a differential of x' which in turn by (1.6.4) is closed. Hence, the line integral

$$u(x) = \int_{h(x'')}^{x'} \sum_{j=1}^{k} f_j(x', x'')dx_j + u_h(h(x''), x''), \tag{1.6.5}$$

is well-defined, i.e., independent of the paths in x'-space from $h(x'')$ to x'. Obviously, (1.6.5) defines the unique solution u which is equal to the initial datum u_h on H to the equations (1.6.2). This completes the proof of the theorem.

Now we turn to the complex analog of the Frobenius theorem. Let $L_1, \cdots, L_k$, $1 \le k < n$, be type $(1,0)$ vector fields defined in some open neighborhood U of the origin in $\mathbb{C}^n$ such that $L_1, \cdots, L_k$ are linearly independent over $\mathbb{C}$ on U. If there exist local holomorphic coordinates $(z_1, \cdots, z_n)$ on U such that $L_1, \cdots, L_k$ are tangent to the k dimensional complex submanifolds $N_c = \{(z_1, \cdots, z_n) \in \mathbb{C}^n | z_{k+1} = c_1, \cdots, z_n = c_{n-k}\}$ for $c = (c_1, \cdots, c_{n-k}) \in \mathbb{C}^{n-k}$, then we see immediately that the subbundle $\mathbb{E}$ spanned by $L_1, \cdots, L_k$ is closed under the Lie bracket operation, and so is the subbundle $\mathbb{E} \oplus \overline{\mathbb{E}}$.

Conversely, if both the subbundles $\mathbb{E}$ and $\mathbb{E} \oplus \overline{\mathbb{E}}$ are closed respectively under the Lie bracket operation, then locally on U one may introduce new holomorphic coordinates $(w_1, \cdots, w_n)$ so that U is foliated by the complex submanifolds $N_c = \{(w_1, \cdots, w_n) \in \mathbb{C}^n | \quad w_{k+1} = c_1, \cdots, w_n = c_{n-k}\}$ for $c = (c_1, \cdots, c_{n-k}) \in \mathbb{C}^{n-k}$, and $\mathbb{E} = T^{1,0}(N_c)$. This is the so-called complex Frobenius theorem which can be deduced from the Newlander-Nirenberg theorem proved in Chapter 5. When $k = 1$, this will be proved in Chapter 2.

1.7 Inequivalence between the Ball and the Polydisc in $\mathbb{C}^n$

In one complex variable the Riemann mapping theorem states that any simply connected region not equal to the whole complex plane is biholomorphically equivalent to the unit disc.

However, the situation is completely different in higher dimensional spaces. Let $\triangle = \{z \in \mathbb{C} | \; |z| < 1\}$ and $B_n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n | \; |z_1|^2 + \cdots + |z_n|^2 < 1\}$. The following theorem shows that an analog of the Riemann mapping theorem in several variables is impossible.

Theorem 1.7.1 (Poincaré). *There exists no biholomorphic map*

$$f : \triangle^n \to B_n, \quad \text{for } n \ge 2,$$

where $\triangle^n$ is the Cartesian product of n copies of $\triangle$ in $\mathbb{C}^n$.

Proof. We shall assume that $n = 2$. The proof is the same for $n > 2$. Suppose that $f = (f_1, f_2) : \triangle^2 \to B_2$ is a biholomorphism. Let (z, w) be the coordinates in $\mathbb{C}^2$. For any point sequence $\{z_j\}$ in $\triangle$ with $|z_j| \to 1$ as $j \to \infty$, the sequence $g_j(w) = f(z_j, w) : \triangle \to B_2$ is uniformly bounded. Hence, by Montel's theorem, there is a subsequence, still denoted by $g_j(w)$, that converges uniformly on compact subsets of $\triangle$ to a holomorphic map $g(w) = (g_1(w), g_2(w)) : \triangle \to \overline{B}_2$. Since f is a biholomorphism, we must have $|g(w)|^2 = 1$ for all $w \in \triangle$. Hence, $|g'(w)| = 0$ for all $w \in \triangle$ which implies $g'(w) \equiv 0$ on $\triangle$. It follows that

$$\lim_{j\to\infty} f_w(z_j, w) = g'(w) \equiv 0. \tag{1.7.1}$$

Equation (1.7.1) implies that for each fixed $w \in \triangle$, $f_w(z, w)$, when viewed as a function of z alone, is continuous up to the boundary with boundary value identically equal to zero. Therefore, by the maximum modulus principle we get

$$f_w(z, w) \equiv 0, \quad \text{for all } (z, w) \in \triangle^2.$$

This implies f is independent of w, a contradiction to the fact that f is a biholomorphic map. This completes the proof of the theorem.

Thus, according to Theorem 1.7.1, the classification problem in several variables is considerably more complicated than in one variable. An approach towards the classification of certain domains in $\mathbb{C}^n$, $n \geq 2$, will be discussed in Section 6.3.

NOTES

For a general background on complex manifolds, the reader may consult books by S. S. Chern [Cher 2], J. Morrow and K. Kodaira [MoKo 1] and R. O. Wells [Wel 1]. For a proof of the complex Frobenius theorem, the reader is referred to [Nir 1]. See also [Hör 5]. The inequivalence between the polydisc and the unit ball was first discovered by H. Poincaré by counting the dimensions of the automorphism groups of both domains. The proof of Theorem 1.7.1 that we present here is based on the ideas of R. Remmert and K. Stein [ReSt 1]. See also [Nar 1] and [Ran 6].

CHAPTER 2

THE CAUCHY INTEGRAL FORMULA AND ITS APPLICATIONS

The main task of this chapter is to study the solvability and regularity of the Cauchy-Riemann operator on the complex plane. We will first show that the solution to the Cauchy-Riemann operator can be obtained via the Cauchy integral formula. Then we shall prove the Plemelj jump formula associated with the Cauchy transform. As an application of the Cauchy integral formula, given a (p, q)-form f on a polydisc satisfying the compatibility condition $\overline{\partial} f = 0$, we will solve the inhomogeneous $\overline{\partial}$-equation, $\overline{\partial} u = f$, on a relatively smaller polydisc in several complex variables.

Next we shall present the Bochner-Martinelli formula which can be viewed as a generalization of the Cauchy integral formula in several variables. Then, in a similar manner, we will prove the jump formula associated with the Bochner-Martinelli transform.

In Section 2.3, we will determine when a first-order partial differential equation in two real variables is locally equivalent to the Cauchy-Riemann equation.

2.1 The Cauchy Integral Formula

All functions in this chapter are complex-valued unless otherwise stated. Then the following formula, known as Cauchy's integral formula, holds:

Theorem 2.1.1. *Let D be a bounded open set in $\mathbb{C}$ with C^1 boundary bD. If $u \in C^1(\overline{D})$, we have*

$$u(z) = \frac{1}{2\pi i}\left(\int_{bD} \frac{u(\zeta)}{\zeta - z}\, d\zeta + \iint_D \frac{\frac{\partial u}{\partial \overline{\zeta}}}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}\right) \tag{2.1.1}$$

for any $z \in D$.

Proof. The proof is an easy consequence of Stokes' theorem. Let ϵ be any small positive number less than the distance from z to the boundary of D. Denote by $B_\epsilon(z)$ the open disc centered at z with radius ϵ. Applying Stokes' theorem to the form $u(\zeta)d\zeta/(\zeta - z)$ on the punctured domain $D_\epsilon = D \setminus \overline{B_\epsilon(z)}$, we obtain

$$\int_{bD} \frac{u(\zeta)}{\zeta - z}\, d\zeta - i\int_0^{2\pi} u(z + \epsilon e^{i\theta})\, d\theta = \iint_{D_\epsilon} \frac{\frac{\partial u}{\partial \overline{\zeta}}}{\zeta - z}\, d\overline{\zeta} \wedge d\zeta.$$

Letting $\epsilon \to 0$, we have (2.1.1).

Next we show how to apply the Cauchy integral formula to solve the Cauchy-Riemann equation.

Theorem 2.1.2. *Let D be a bounded domain in $\mathbb{C}$, and let $f \in C^k(\overline{D})$ for $k \geq 1$. Define*

$$u(z) = \frac{1}{2\pi i} \iint_D \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}. \tag{2.1.2}$$

Then $u(z)$ is in $C^k(D)$ and satisfies

$$\frac{\partial u}{\partial \overline{z}} = f(z) \tag{2.1.3}$$

on D. When $k = 0$, u defined by (2.1.2) is in $C(\overline{D})$ and satisfies (2.1.3) in the distribution sense.

Proof. For the case $k \geq 1$, we first assume $f \in C_0^k(\mathbb{C})$. Setting $-\eta = \zeta - z$, we have

$$u(z) = \frac{-1}{2\pi i} \iint_{\mathbb{C}} \frac{f(z-\eta)}{\eta}\, d\eta \wedge d\overline{\eta}.$$

Differentiation under the integral sign gives that $u \in C^k(\mathbb{C})$. Using Theorem 2.1.1 we obtain

$$\frac{\partial u}{\partial \overline{z}}(z) = \frac{1}{2\pi i} \iint \frac{\frac{\partial f}{\partial \overline{\zeta}}}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} = f(z).$$

For the general situation, let $z_0 \in D$, and let χ be a cut-off function, $0 \leq \chi \leq 1$, $\chi \equiv 1$ in some neighborhood V of z_0 and supp$\chi \subset D$. Thus,

$$\begin{aligned} u(z) &= \frac{1}{2\pi i} \iint_D \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} \\ &= \frac{1}{2\pi i} \iint_D \frac{\chi(\zeta) f(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} + \frac{1}{2\pi i} \iint_D \frac{(1-\chi(\zeta)) f(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} \\ &\equiv u_1(z) + u_2(z). \end{aligned}$$

It is easy to see that $u_2(z)$ is holomorphic in V. Hence, from the previous argument for $D = \mathbb{C}$, we obtain

$$\frac{\partial u}{\partial \overline{z}} = \frac{\partial u_1}{\partial \overline{z}} + \frac{\partial u_2}{\partial \overline{z}} = \chi(z) f(z) = f(z),$$

for $z \in V$.

To prove the case for $k = 0$, we observe that $1/(\zeta - z)$ is an integrable kernel after changing to polar coordinates. The following estimate holds for u defined by (2.1.2):

$$\| u \|_\infty \leq C \| f \|_\infty .$$

Approximate f by $f_n \in C^1(\overline{D})$ in the sup norm on $\overline{D}$. Define u_n by (2.1.2) with respect to f_n. Then u_n converges to u uniformly on D. This shows that $u \in C(\overline{D})$. Also, in the distribution sense, we have $\partial u / \partial \overline{z} = f$, by letting n pass to infinity. This proves the theorem.

We recall that a function f defined in some domain D contained in $\mathbb{R}^n$ is said to be *Hölder continuous* of order λ, $0<\lambda<1$, denoted by $f\in C^\lambda(D)$, if for any two distinct points x_1 and x_2 in D, we have

$$|f(x_1)-f(x_2)|\le K|x_1-x_2|^\lambda,$$

where the constant K is independent of x_1 and x_2. When λ is equal to 1, f is called *Lipschitz*. The space of Lipschitz continuous function on D is denoted by $\Lambda^1(D)$. Notice that $C^1(\overline{D})\subset\Lambda^1(D)\subset C^\lambda(D)\subset C(\overline{D})$. A function f is said to be Hölder continuous of order $k+\lambda$ with $k\in\mathbb{N}$ and $0<\lambda<1$, if all the partial derivatives of f of order k are Hölder continuous of order λ.

For any continuous function f on the boundary, the Cauchy transform of f, i.e.,

$$F(z)=\frac{1}{2\pi i}\int_{bD}\frac{f(\zeta)}{\zeta-z}\,d\zeta\quad\text{for } z\in\mathbb{C}\setminus bD, \tag{2.1.4}$$

defines a holomorphic function $F(z)$ off the boundary. The Cauchy transform $F(z)$ and the given data f on the boundary are related by the so-called Plemelj jump formula as shown in the following theorem.

Theorem 2.1.3 (Jump formula). *Let D be a bounded domain in $\mathbb{C}$ with C^{k+1} boundary, $k\in\mathbb{N}$, such that $\mathbb{C}\setminus D$ is connected, and let f be a C^k function defined on the boundary. Define $F(z)$ as in (2.1.4), and set $F_-(z)=F(z)$ for $z\in D$ and $F_+(z)=F(z)$ for $z\notin\overline{D}$. Then, for any given $0<\epsilon<1$, $F_-(z)\in C^{k-\epsilon}(\overline{D})\cap\mathcal{O}(D)$ and $F_+(z)\in C^{k-\epsilon}(\mathbb{C}\setminus D)\cap\mathcal{O}(\mathbb{C}\setminus\overline{D})$ and*

$$f(z)=F_-(z)-F_+(z)\quad\textit{for } z\in bD. \tag{2.1.5}$$

In particular, if D has C^∞ boundary and f is smooth on bD, then both $F_-(z)$ and $F_+(z)$ are smooth up to the boundary.

Proof. First we prove the identity (2.1.5). Let $f_e(z)$ be any C^k extension of f to the whole complex plane. Then, we have, for $z\in D$,

$$F_-(z)-f_e(z)=\frac{1}{2\pi i}\int_{bD}\frac{f(\zeta)-f_e(z)}{\zeta-z}\,d\zeta, \tag{2.1.6}$$

and, for $z\in\mathbb{C}\setminus\overline{D}$,

$$F_+(z)=\frac{1}{2\pi i}\int_{bD}\frac{f(\zeta)-f_e(z)}{\zeta-z}\,d\zeta. \tag{2.1.7}$$

Since f is, at least, of class C^1 on the boundary, the integral on the right-hand sides of (2.1.6) and (2.1.7) defines a continuous function on the whole complex plane. Therefore, letting z approach the same point on the boundary from either sides, we obtain

$$f(z)=F_-(z)-F_+(z)\quad\text{for } z\in bD.$$

This proves (2.1.5).

For the regularity of $F_-(z)$ and $F_+(z)$ near the boundary we shall invoke the Hardy-Littlewood lemma (see Theorem C.1 in the Appendix). It is clear that, without loss of generality, we may assume that f is compactly supported in a boundary coordinate chart, with $\partial r/\partial z \neq 0$ on this coordinate chart, where r is a C^{k+1} defining function for D. When $k = 1$, using (2.1.6), we have

$$
\begin{aligned}
|d(F_-(z) - f_e(z))| &\lesssim \int_{bD} \frac{|df_e(z)|}{|z-\zeta|}\, ds(\zeta) + \int_{bD} \frac{|f(\zeta) - f_e(z)|}{|z-\zeta|^2}\, ds(\zeta) \\
&\lesssim \int_{bD} \frac{1}{|z-\zeta|}\, ds(\zeta).
\end{aligned} \tag{2.1.8}
$$

Here, $A \lesssim B$ means there is an universal constant C, independent of A and B, such that $A \leq CB$. For any given $\epsilon > 0$, to show $F_-(z) \in C^{1-\epsilon}(\overline{D})$, it suffices to estimate (2.1.8) over a small neighborhood U of $\pi(z)$ on the boundary, where $\pi(z)$ is the projection of z on the boundary. Let $d(z)$ be the distance from z to the boundary. If z is sufficiently close to the boundary, it is easily seen that, for $\zeta \in U$, $|z - \zeta|$ is equivalent to $d(z) + s(\zeta)$, where $s(\zeta)$ is the distance from ζ to $\pi(z)$ along the boundary. It follows that

$$
\begin{aligned}
\int_U \frac{ds(\zeta)}{|z-\zeta|} &\leq d(z)^{-\epsilon} \int_U \frac{ds(\zeta)}{|z-\zeta|^{1-\epsilon}} \\
&\lesssim d(z)^{-\epsilon} \int_0^1 \frac{ds(\zeta)}{(d(z)+s(\zeta))^{1-\epsilon}} \\
&\lesssim d(z)^{-\epsilon}.
\end{aligned}
$$

This proves that $F_-(z) \in C^{1-\epsilon}(\overline{D})$. Similarly, we have $F_+(z) \in C^{1-\epsilon}(\mathbb{C} \setminus D)$.

For $k > 1$, observe that

$$
T_z = \frac{\partial}{\partial z} - \frac{\partial r}{\partial z}\left(\frac{\partial r}{\partial \overline{z}}\right)^{-1} \frac{\partial}{\partial \overline{z}} \tag{2.1.9}
$$

satisfies $T_z(r) = 0$. Hence, T_z is a tangential vector field with C^k coefficients along the level sets of r. Then, integration by parts shows, for $z \in D$,

$$
\begin{aligned}
\frac{\partial}{\partial z} F_-(z) &= \frac{1}{2\pi i} \int_{bD} f(\zeta) \frac{\partial}{\partial z}\left(\frac{1}{\zeta - z}\right) d\zeta \\
&= -\frac{1}{2\pi i} \int_{bD} f(\zeta) \frac{\partial}{\partial \zeta}\left(\frac{1}{\zeta - z}\right) d\zeta \\
&= -\frac{1}{2\pi i} \int_{bD} f(\zeta) T_\zeta\left(\frac{1}{\zeta - z}\right) d\zeta \\
&= \frac{1}{2\pi i} \int_{bD} T_\zeta^* f(\zeta)\left(\frac{1}{\zeta - z}\right) d\zeta,
\end{aligned}
$$

where T_ζ^* is a first order differential operator with C^{k-1} coefficients on the boundary. It follows that

$$
\frac{\partial^k}{\partial z^k} F_-(z) = \frac{1}{2\pi i} \int_{bD} (T_\zeta^*)^k f(\zeta)\left(\frac{1}{\zeta - z}\right) d\zeta. \tag{2.1.10}
$$

Since f is of class C^k on the boundary, a similar argument shows that, for any small $\epsilon > 0$, we have

$$\left| \frac{\partial^k}{\partial z^k} F_-(z) \right| \lesssim d(z)^{-\epsilon}.$$

This proves $F_-(z) \in C^{k-\epsilon}(\overline{D})$ from the Hardy-Littlewood lemma. Similarly, we have $F_+(z) \in C^{k-\epsilon}(\mathbb{C} \setminus D)$. The proof of the theorem is now complete.

Corollary 2.1.4. *Under the same hypotheses as in Theorem 2.1.3, f is the restriction of a holomorphic function $F \in C^{k-\epsilon}(\overline{D}) \cap \mathcal{O}(D)$ if and only if f is orthogonal to $\{\overline{z}^m\}_{m=0}^{\infty}$ on the boundary, namely,*

$$\int_{bD} f(z) z^m \, dz = 0 \quad \textit{for } m \in \{0\} \cup \mathbb{N}. \tag{2.1.11}$$

Proof. Assume that f is orthogonal to $\{\overline{z}^m\}_{m=0}^{\infty}$ on the boundary. If z satisfies $|z| > |\zeta|$ for all $\zeta \in bD$, we have

$$\begin{aligned} F_+(z) &= \frac{1}{2\pi i} \int_{bD} \frac{f(\zeta)}{\zeta - z} \, d\zeta \\ &= \frac{-1}{2\pi i} \sum_{m=0}^{\infty} \left(\int_{bD} f(\zeta) \zeta^m \, d\zeta \right) z^{-m-1} \\ &= 0. \end{aligned}$$

Since $F_+(z)$ is holomorphic on $\mathbb{C} \setminus \overline{D}$ from Theorem 2.1.3, the identity theorem shows that $F_+(z) \equiv 0$ for all $z \in \mathbb{C} \setminus D$. It is now clear from the jump formula that $F_-(z)$ is a $C^{k-\epsilon}$ holomorphic extension of f to $\overline{D}$.

Conversely, if f is the restriction of a function in $C^{k-\epsilon}(\overline{D}) \cap \mathcal{O}(D)$, we must have $F_+(z) \equiv 0$ on $\mathbb{C} \setminus D$. Thus, by reversing the above arguments for z outside a large disc centered at the origin, we have

$$\sum_{m=0}^{\infty} \left(\int_{bD} f(\zeta) \zeta^m \, d\zeta \right) z^{-m-1} = 0.$$

This implies

$$\int_{bD} f(\zeta) \zeta^m \, d\zeta = 0 \quad \text{for } m \in \{0\} \cup \mathbb{N},$$

and hence proves the corollary.

Combining with Theorem 2.1.2, the arguments for proving Corollary 2.1.4 can be applied, almost verbatim, to obtain necessary and sufficient conditions for solving the $\overline{\partial}$-equation with compactly supported solution, via the Cauchy integral formula in $\mathbb{C}$.

Corollary 2.1.5. *Let the domain D be as in Theorem 2.1.3, and let $f \in C^k(\overline{D})$, $k \geq 1$. Define $u(z)$ by (2.1.2). Then $u(z)$ satisfies $\partial u / \partial \overline{z} = f$ in $\mathbb{C}$ and is supported in $\overline{D}$ if and only if*

$$\iint_D f(\zeta) \zeta^m \, d\zeta \wedge d\overline{\zeta} = 0 \quad \textit{for } m \in \{0\} \cup \mathbb{N}.$$

As an application, we will apply the Cauchy integral formula to solve the $\overline{\partial}$-equation on a polydisc in $\mathbb{C}^n$, $n \geq 2$. By a polydisc $P(\zeta; r)$ centered at $\zeta = (\zeta_1, \cdots, \zeta_n)$ with multiradii $r = (r_1, \cdots, r_n)$ in $\mathbb{C}^n$, we mean $P(\zeta; r) = \prod_{j=1}^n D_{r_j}(\zeta_j)$ where $D_{r_j}(\zeta_j) = \{z \in \mathbb{C} | \ |z - \zeta_j| < r_j\}$. Let $P'(\zeta; r') = \prod_{j=1}^n D_{r'_j}(\zeta_j)$ be another polydisc with $r'_j < r_j$ for $1 \leq j \leq n$. Then, we have the following result:

Theorem 2.1.6. *Let P and P' be defined as above with $\zeta = 0$, and let f be a smooth $(p, q+1)$-form, $p \geq 0$, $q \geq 0$, defined on P, which satisfies the compatibility condition $\overline{\partial} f = 0$. Then there exists a smooth (p, q)-form u on P' such that $\overline{\partial} u = f$.*

Note that we have solved the $\overline{\partial}$-equation on any slightly smaller subdomain P'. In fact, it will be clear later that the $\overline{\partial}$-equation can be solved on the whole polydisc.

Proof. Write f as

$$f = \sum_{|I|=p, |J|=q+1}{}' f_{IJ} \ dz^I \wedge d\overline{z}^J,$$

where the prime means that we sum over only increasing multiindices. We shall inductively prove the following statement:

S_k : The assertion holds if f involves only $(0,1)$-forms from the set $\{d\overline{z}_1, \cdots, d\overline{z}_{k-1}$ and $d\overline{z}_k\}$.

When $k = n$, S_n gives the desired result.

S_k obviously holds when $0 \leq k \leq q$, since f is of type $(p, q+1)$. Hence, we assume the statement is valid up to S_{k-1} for some k with $k - 1 \geq q$, and we proceed to prove the statement S_k. Write

$$f = d\overline{z}_k \wedge \beta + \alpha,$$

where β is a (p, q)-form and α is a $(p, q+1)$-form, and both α and β involve only $(0,1)$-forms from $d\overline{z}_1, \cdots, d\overline{z}_{k-1}$. Express

$$\beta = \sum_{|I|=p, |J|=q}{}' \beta_{IJ} \ dz^I \wedge d\overline{z}^J.$$

It is easy to see by type consideration that $\partial \beta_{IJ} / \partial \overline{z}_j = 0$ for $j > k$ and all I, J. Now choose a cut-off function $\chi(z_k) \in C_0^\infty(D_{r_k})$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in some open neighborhood of $\overline{D}_{r'_k}$. Then Theorem 2.1.2 shows that, for each I, J, the function

$$B_{IJ}(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\chi(\zeta_k) \beta_{IJ}(z_1, \cdots, z_{k-1}, \zeta_k, z_{k+1}, \cdots, z_n)}{\zeta_k - z_k} \ d\zeta_k \wedge d\overline{\zeta}_k$$

is smooth and solves the $\overline{\partial}$-equation

$$\frac{\partial B_{IJ}}{\partial \overline{z}_k} = \chi(z_k) \beta_{IJ}(z) = \beta_{IJ}(z)$$

on some neighborhood of $\overline{P'}$. We also have, for $j > k$ and all I, J,

$$\frac{\partial B_{IJ}}{\partial \overline{z}_j}(z) = 0.$$

Put

$$B = \sum_{|I|=p,|J|=q}{}' B_{IJ}\, dz^I \wedge d\overline{z}^J,$$

then

$$\begin{aligned}\overline{\partial} B &= \sum_{|I|=p,|J|=q}{}' \left(\sum_{j=1}^{n} \frac{\partial B_{IJ}}{\partial \overline{z}_j}\, d\overline{z}_j\right) \wedge dz^I \wedge d\overline{z}^J \\ &= d\overline{z}_k \wedge \beta + \alpha_0,\end{aligned}$$

where α_0 is a $(p, q+1)$-form that involves only $(0,1)$-forms from $d\overline{z}_1, \cdots, d\overline{z}_{k-1}$. Hence,

$$f - \overline{\partial} B = \alpha - \alpha_0$$

is a smooth $(p, q+1)$-form which is $\overline{\partial}$-closed and involves only $(0,1)$-forms from $d\overline{z}_1, \cdots, d\overline{z}_{k-1}$. It follows now from the induction hypotheses that there exists a smooth (p, q)-form u_0 that satisfies

$$\overline{\partial} u_0 = \alpha - \alpha_0 = f - \overline{\partial} B.$$

Clearly, $u = u_0 + B$ is a solution of $\overline{\partial} u = f$, and the proof is complete.

Theorem 2.1.7 (Cauchy integral formula for polydiscs). *Let $P(\eta; r)$ be a polydisc in $\mathbb{C}^n$, $n \geq 2$. Suppose that f is continuous on $\overline{P}(\eta; r)$ and holomorphic in $P(\eta; r)$. Then for any $z \in P(\eta; r)$,*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - \eta_n| = r_n} \cdots \int_{|\zeta_1 - \eta_1| = r_1} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}\, d\zeta_1 \cdots d\zeta_n.$$

Proof. It is easily seen that the integral representation of $f(z)$ is obtained by repeated application of the Cauchy integral formula in one variable. This proves the theorem.

Here are some easy consequences of Theorem 2.1.7:

Theorem 2.1.8 (Cauchy estimates). *Under the same hypotheses as in Theorem 2.1.7. Suppose that $|f| \leq M$ for all $z \in \overline{P}(\eta; r)$. Then*

$$\left| \left(\frac{\partial}{\partial z}\right)^{\alpha} f(\eta) \right| \leq \frac{M \alpha!}{r_1^{\alpha_1} \cdots r_n{}^{\alpha_n}},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multiindex with $\alpha_j \in \{0\} \cup \mathbb{N}$, $\alpha! = \alpha_1! \cdots \alpha_n!$ and $\left(\frac{\partial}{\partial z}\right)^{\alpha} = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}$.

Theorem 2.1.9. *If f is holomorphic in $D \subset \mathbb{C}^n$, then locally near any point w in D, f has a power series representation. In particular, f is real analytic.*

By a power series representation for f near w, we mean

$$f(z) = \sum_{\alpha} a_\alpha (z-w)^\alpha$$

such that the series converges absolutely in some open neighborhood of w. Here the summation is over multiindices α and $(z-w)^\alpha = (z_1 - w_1)^{\alpha_1} \cdots (z_n - w_n)^{\alpha_n}$. It follows now from the power series expansion of holomorphic functions, we have

Theorem 2.1.10 (Identity Theorem). *Let f and g be two holomorphic functions defined on a connected open set $D \subset \mathbb{C}^n$. If f and g coincide on an open subset of D, then $f = g$ on D.*

2.2 The Bochner-Martinelli Formula

In this section, we shall extend the Cauchy kernel from the complex plane to higher dimensional space. Define the Bochner-Martinelli kernel by

$$B(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \frac{1}{|\zeta - z|^{2n}} \sum_{j=1}^{n} (\overline{\zeta}_j - \overline{z}_j) d\zeta_j \wedge (\underset{k \neq j}{\wedge} d\overline{\zeta}_k \wedge d\zeta_k), \tag{2.2.1}$$

for $\zeta = (\zeta_1, \cdots, \zeta_n)$, $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $\zeta \neq z$. $B(\zeta, z)$ is a form of type $(n, n-1)$ in ζ. It is clear that when $n = 1$,

$$B(\zeta, z) = \frac{1}{2\pi i} \frac{1}{\zeta - z} d\zeta$$

which is the Cauchy kernel in $\mathbb{C}$.

The following theorem is a generalized version of the Cauchy integral formula in several variables.

Theorem 2.2.1. *Let D be a bounded domain with C^1 boundary in $\mathbb{C}^n$, $n \geq 2$, and let $f \in C^1(\overline{D})$. Then*

$$f(z) = \int_{bD} f(\zeta) B(\zeta, z) - \int_D \overline{\partial} f \wedge B(\zeta, z) \quad \textit{for } z \in D, \tag{2.2.2}$$

and

$$0 = \int_{bD} f(\zeta) B(\zeta, z) - \int_D \overline{\partial} f \wedge B(\zeta, z) \quad \textit{for } z \notin \overline{D}. \tag{2.2.3}$$

Proof. A direct calculation shows that $\overline{\partial}_\zeta B(\zeta, z) = 0$ for $\zeta \neq z$. Since $B(\zeta, z)$ is of type $(n, n-1)$ in ζ, by Stokes' theorem we have, for $z \in D$,

$$\begin{aligned}\int_{bD} f(\zeta) B(\zeta, z) &= \int_{D_\epsilon(z)} d(f(\zeta) B(\zeta, z)) + \int_{bB_\epsilon(z)} f(\zeta) B(\zeta, z) \\ &= \int_{D_\epsilon(z)} \overline{\partial} f(\zeta) \wedge B(\zeta, z) + \int_{bB_\epsilon(z)} f(\zeta) B(\zeta, z),\end{aligned}$$

where $B_\epsilon(z) = \{\zeta \in \mathbb{C}^n | \, |\zeta - z| < \epsilon\}$ for small $\epsilon > 0$ and $D_\epsilon(z) = D \setminus \overline{B_\epsilon(z)}$. Using homogeneity of the kernel and Stokes' theorem, we easily get

$$\begin{aligned}\int_{bB_\epsilon(z)} B(\zeta, z) &= \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} \int_{bB_\epsilon(0)} \frac{\overline{\zeta}_j}{|\zeta|^{2n}}\, d\zeta_j \wedge (\underset{k\neq j}{\wedge} d\overline{\zeta}_k \wedge d\zeta_k) \\ &= \frac{(n-1)!}{(2\pi i)^n} \frac{1}{\epsilon^{2n}} \sum_{j=1}^{n} \int_{B_\epsilon(0)} d\overline{\zeta}_j \wedge d\zeta_j \wedge (\underset{k\neq j}{\wedge} d\overline{\zeta}_k \wedge d\zeta_k) \\ &= 1,\end{aligned}$$

for all $\epsilon > 0$. Now, letting $\epsilon \to 0$, we obtain

$$f(z) = \int_{bD} f(\zeta) B(\zeta, z) - \int_D \overline{\partial} f(\zeta) \wedge B(\zeta, z).$$

This proves (2.2.2).

Now for the proof of (2.2.3), since $z \notin \overline{D}$, the kernel is regular on $\overline{D}$. Hence, an application of Stokes' theorem gives

$$\int_{bD} f(\zeta) B(\zeta, z) = \int_D \overline{\partial} f(\zeta) \wedge B(\zeta, z).$$

This proves (2.2.3) and hence the theorem.

An immediate consequence of Theorem 2.2.1 is the following reproducing property of the Bochner-Martinelli kernel for holomorphic functions:

Corollary 2.2.2. *Let D be a bounded domain with C^1 boundary in $\mathbb{C}^n$, $n \geq 2$. For any $f \in \mathcal{O}(D) \cap C(\overline{D})$, we have*

$$f(z) = \int_{bD} f(\zeta) B(\zeta, z) \quad \textit{for } z \in D. \tag{2.2.4}$$

The integral (2.2.4) is zero if $z \notin \overline{D}$.

Proof. First we assume that $f \in C^1(\overline{D})$. Then the assertion follows immediately from Theorem 2.2.1. The general case now follows from approximation. This proves the corollary.

Thus, the Bochner-Martinelli kernel also enjoys the reproducing property for holomorphic functions, although $B(\zeta, z)$ is no longer holomorphic in z.

A more systematic treatment of kernels in several variables will be given in Chapter 11 where a reproducing kernel holomorphic in z variables will be constructed for convex domains.

Theorem 2.2.3 (Jump formula). *Let D be a bounded domain in $\mathbb{C}^n, n \geq 2$, with connected C^1 boundary, and let f be a C^1 function defined on the boundary. Define*

$$F(z) = \int_{bD} f(\zeta) B(\zeta, z) \quad \textit{for } z \in \mathbb{C}^n \setminus bD,$$

and let $F_-(z) = F(z)$ *for* $z \in D$ *and* $F_+(z) = F(z)$ *for* $z \notin \overline{D}$. *Then, for any small* $\epsilon > 0$, *we have* $F_-(z) \in C^{1-\epsilon}(\overline{D})$, $F_+(z) \in C^{1-\epsilon}(\mathbb{C}^n \setminus D)$ *and*

$$f(z) = F_-(z) - F_+(z) \quad \textit{for } z \in bD. \tag{2.2.5}$$

Equation (2.2.5) is the so-called jump formula associated with the Bochner-Martinelli transform. When $n = 1$, this is the Plemelj jump formula proved in Theorem 2.1.3 where F_- and F_+ are also holomorphic.

Proof. Let $f_e(z)$ be any C^1 extension of f to the whole space. Then, for any $z \in D$, we have

$$F_-(z) - f_e(z) = \int_{bD} (f(\zeta) - f_e(z))B(\zeta, z). \tag{2.2.6}$$

Since

$$|B(\zeta, z)| \lesssim |z - \zeta|^{1-2n}$$

from the definition of Bochner-Martinelli kernel, the right-hand side of (2.2.6) defines a continuous function on the whole space. Thus, we have $F_-(z) \in C(\overline{D})$. For $z \notin \overline{D}$, we get

$$F_+(z) = \int_{bD} (f(\zeta) - f_e(z))B(\zeta, z). \tag{2.2.7}$$

Letting z tend to the same point on the boundary from either side, we obtain

$$f(z) = F_-(z) - F_+(z) \quad \text{for } z \in bD.$$

This proves (2.2.5).

For the regularity of $F_-(z)$ and $F_+(z)$ we again use the Hardy-Littlewood lemma (see Theorem C.1 in the Appendix). Thus, we need to estimate the differential of $F_-(z)$. An easy exercise shows that

$$\begin{aligned}
&|d(F_-(z) - f_e(z))| \\
&\quad \le \int_{bD} |df_e(z)||B(\zeta, z)| + \int_{bD} |f(\zeta) - f_e(z)||dB(\zeta, z)| \\
&\quad \lesssim \int_{bD} \frac{1}{|z - \zeta|^{2n-1}}\, d\sigma(\zeta) \\
&\quad \lesssim d(z)^{-\epsilon},
\end{aligned}$$

for any small $\epsilon > 0$, where $d(z)$ is the distance from z to the boundary of D. This proves that $F_-(z) \in C^{1-\epsilon}(\overline{D})$. Using (2.2.7), we obtain through a similar argument the same assertion for $F_+(z)$. This proves the theorem.

2.3 The Cauchy-Riemann Operator in $\mathbb{C}$

Let

$$X = X_1 + iX_2 \tag{2.3.1}$$

be a first order partial differential operator defined in some open neighborhood U of the origin in $\mathbb{R}^2$, where

$$X_j = a_j(x,y)\frac{\partial}{\partial x} + b_j(x,y)\frac{\partial}{\partial y}, \quad j = 1,2, \tag{2.3.2}$$

and $a_j(x,y), b_j(x,y)$ are real-valued functions on U. We wish to study the solvability of such operator. If X_1 and X_2 are linearly dependent everywhere on U, then X_1 and X_2 will be multiples of the same first order operator X_0 with real coefficients in some neighborhood of the origin. It follows that X is reduced to

$$X = \lambda(z)X_0, \tag{2.3.3}$$

and the solvability of (2.3.3) will then follow from the basic theory of the ordinary differential equations.

Thus, let us assume that X_1 and X_2 are linearly independent everywhere on U. The most famous operator of this type is the Cauchy-Riemann operator,

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{2.3.4}$$

In Theorem 2.1.2, we have shown how to obtain a solution for the inhomogeneous Cauchy-Riemann equation via the Cauchy integral formula.

As another application of the Cauchy integral formula, we shall show that, under certain regularity hypotheses on the coefficients $a_j(x,y)$ and $b_j(x,y)$, locally one may introduce a new holomorphic coordinate w so that X can be converted to the Cauchy-Riemann operator in w. Hence, one can deduce the solvability of X from the knowledge of the Cauchy-Riemann operator. More precisely, we will prove the following theorem.

Theorem 2.3.1. *Let X be given as in (2.3.1) and (2.3.2) in some open neighborhood U of the origin in $\mathbb{R}^2$. Suppose that the coefficients $a_j(x,y)$ and $b_j(x,y)$, $j = 1,2$, are Hölder continuous of order λ, $0 < \lambda < 1$. Then there exists a new local holomorphic coordinate w in some neighborhood of the origin so that*

$$X(w) = \frac{\partial}{\partial \overline{w}}.$$

By a linear transformation on x, y with constant coefficients we may assume that the operator X takes the following form

$$X(z) = \frac{\partial}{\partial \bar{z}} - a(z)\frac{\partial}{\partial z}, \tag{2.3.5}$$

with $a(0) = 0$, and that $a(z)$ is Hölder continuous of order λ. Hence, the assertion of Theorem 2.3.1 is equivalent to the existence of a solution $w(z)$ to the equation

$$w_{\overline{z}}(z) - a(z)w_z(z) = 0,$$

or, in terms of the partial differential operator Z,

$$w_{\overline{z}}(z) = Zw(z), \tag{2.3.6}$$

where $Z = a(z)(\partial/\partial z)$, with $w_z(0) \neq 0$.

We shall use an iteration process to construct a solution to equation (2.3.6) for the remaining parts of this section. This is where one needs Hölder regularity for the coefficients of X. We shall first prove some lemmas and estimates that are needed in the sequel. We denote by $D_R = B_R(0)$ the disc centered at the origin with radius R in $\mathbb{R}^2$.

Lemma 2.3.2. *Let $\zeta = (a, b) \in D_R$. Put $r = |z - \zeta|$. Then, for any $\lambda > 0$, we have*

$$\iint_{D_R} \frac{r^\lambda}{r^2}\, dxdy \leq \frac{2\pi}{\lambda}(2R)^\lambda. \tag{2.3.7}$$

Proof. This is obvious if we apply polar coordinates to the disc centered at ζ with radius $2R$.

Lemma 2.3.3. *Let $0 < \epsilon_1, \epsilon_2 \leq 1$ with $\epsilon_1 + \epsilon_2 \neq 2$. Then, for any two distinct points ζ_1 and ζ_2 contained in $\overline{D}_R$, we have*

$$\frac{i}{2}\iint_{D_R} \frac{dz \wedge d\overline{z}}{|z - \zeta_1|^{2-\epsilon_1}|z - \zeta_2|^{2-\epsilon_2}} \leq c(\epsilon_1, \epsilon_2)\frac{1}{|\zeta_1 - \zeta_2|^{2-\epsilon_1-\epsilon_2}}.$$

What is essential in this lemma is that the constant $c(\epsilon_1, \epsilon_2)$ depends only on ϵ_1 and ϵ_2, but not on ζ_1 and ζ_2.

Proof. By changing to polar coordinates it is easy to see that the integral exists. For the estimate of the integral, let $2\delta = |\zeta_1 - \zeta_2|$ and

$$\begin{aligned}
\triangle_1 &= \{z \in D_R |\ |z - \zeta_1| < \delta\},\\
\triangle_2 &= \{z \in D_R |\ |z - \zeta_2| < \delta\},\\
\triangle_3 &= D_R \setminus \{\triangle_1 \cup \triangle_2\}.
\end{aligned}$$

Then, by changing to polar coordinates, the integral over $\triangle_1$ can be estimated as follows:

$$\frac{i}{2}\iint_{\triangle_1} \frac{dz \wedge d\overline{z}}{|z - \zeta_1|^{2-\epsilon_1}|z - \zeta_2|^{2-\epsilon_2}} \leq 2\pi\delta^{\epsilon_2-2}\int_0^\delta r^{\epsilon_1-1}dr = \frac{2\pi}{\epsilon_1}\delta^{\epsilon_1+\epsilon_2-2}.$$

Similarly, we have the estimate over $\triangle_2$,

$$\frac{i}{2}\iint_{\triangle_2} \frac{dz \wedge d\overline{z}}{|z - \zeta_1|^{2-\epsilon_1}|z - \zeta_2|^{2-\epsilon_2}} \leq \frac{2\pi}{\epsilon_2}\delta^{\epsilon_1+\epsilon_2-2}.$$

Both estimates are of the desired form. For the estimate over $\triangle_3$, note that the function $(z-\zeta_1)/(z-\zeta_2)$ is smooth on $\overline{\triangle}_3$. Hence, we obtain

$$\frac{1}{3} \le \left|\frac{z-\zeta_1}{z-\zeta_2}\right| \le 3, \quad \text{for } z \in \triangle_3.$$

It follows we have

$$\begin{aligned}
\frac{i}{2}\iint_{\triangle_3} \frac{dz\wedge d\overline{z}}{|z-\zeta_1|^{2-\epsilon_1}|z-\zeta_2|^{2-\epsilon_2}} &\le 3^{2-\epsilon_2}\iint_{\triangle_3}\frac{dxdy}{|z-\zeta_1|^{4-\epsilon_1-\epsilon_2}}\\
&\le 3^{2-\epsilon_2}\iint_{\mathbb{C}\setminus\triangle_1}\frac{dxdy}{|z-\zeta_1|^{4-\epsilon_1-\epsilon_2}}\\
&\le 2\pi 3^{2-\epsilon_2}\int_\delta^\infty r^{\epsilon_1+\epsilon_2-3}dr\\
&= \frac{2\pi 3^{2-\epsilon_2}}{2-\epsilon_1-\epsilon_2}\delta^{\epsilon_1+\epsilon_2-2}.
\end{aligned}$$

This completes the proof of Lemma 2.3.3.

The following lemma is the key for the regularity of the $\overline{\partial}$-equation.

Lemma 2.3.4. *Let $f(z)$ be a complex-valued continuous function defined on $\overline{D}_R$ which satisfies*

$$|f(z_1)-f(z_2)| \le B|z_1-z_2|^\lambda, \tag{2.3.8}$$

for any two points $z_1, z_2 \in D_R$, where λ, B are positive constants with $0<\lambda<1$. Define the function $F(\zeta)$ for $\zeta \in D_R$ by

$$F(\zeta) = \frac{1}{2\pi i}\iint_{D_R}\frac{f(z)}{z-\zeta}\, dz\wedge d\overline{z}.$$

Then $F \in C^{1+\lambda}(D_R)$. If $|f(z)| \le A$ for all $z \in D_R$, then we have

(1) *F_ζ and $F_{\overline{\zeta}}$ exist, and $F_{\overline{\zeta}}(\zeta) = f(\zeta)$, $\zeta \in D_R$.*
(2) *$|F(\zeta)| \le 4RA$, $\zeta \in D_R$.*
(3) *$|F_\zeta(\zeta)| \le (\frac{2^{\lambda+1}}{\lambda})R^\lambda B$, $\zeta \in D_R$.*
(4) *$|F(\zeta_1)-F(\zeta_2)| \le 2(A+(\frac{2^{\lambda+1}}{\lambda})R^\lambda B)|\zeta_1-\zeta_2|$, $\zeta_1,\zeta_2 \in D_R$.*
(5) *$|F_\zeta(\zeta_1)-F_\zeta(\zeta_2)| \le \mu(\lambda)B|\zeta_1-\zeta_2|^\lambda$, $\zeta_1,\zeta_2 \in D_R$.*

where $\mu(\lambda)>0$ is independent of ζ_1 and ζ_2.

Proof. The existence of $F_{\overline{\zeta}}$ and the equality $F_{\overline{\zeta}}(\zeta) = f(\zeta)$ are guaranteed by Theorem 2.1.2. To prove the existence of $F_\zeta(\zeta)$, we write

$$F(\zeta) = \frac{1}{2\pi i}\iint_{D_R}\frac{f(z)-f(\zeta)}{z-\zeta}dz\wedge d\overline{z} + \frac{f(\zeta)}{2\pi i}\iint_{D_R}\frac{1}{z-\zeta}\, dz\wedge d\overline{z}.$$

Note that by Cauchy's integral formula (2.1.1), we get

$$\overline{\zeta} = \frac{1}{2\pi i}\iint_{D_R}\frac{1}{z-\zeta}\, dz\wedge d\overline{z}.$$

Hence, if $f \in C^1(\overline{D})$, we clearly have

$$F_\zeta(\zeta) = \frac{1}{2\pi i}\iint_{D_R} \frac{f(z)-f(\zeta)}{(z-\zeta)^2}\, dz \wedge d\overline{z}. \tag{2.3.9}$$

In general, if f is only Hölder continuous of order λ, $0 < \lambda < 1$, we approximate f by functions in $C^1(\overline{D})$ to get (2.3.9). This proves (1).

Estimate (2) then follows from (2.1.2) and Lemma 2.3.2 with $\lambda = 1$ since

$$|F(\zeta)| \le \frac{A}{\pi}\iint_{D_R} \frac{1}{|z-\zeta|}\, dxdy \le 4RA.$$

Similarly, from (2.3.8) and (2.3.9) we have

$$|F_\zeta(\zeta)| \le \frac{B}{\pi}\iint_{D_R} \frac{|z-\zeta|^\lambda}{|z-\zeta|^2}\, dxdy \le \left(\frac{2^{\lambda+1}}{\lambda}\right)R^\lambda B.$$

This gives (3). Now (4) follows immediately from the Mean Value Theorem and estimate (3).

Finally, we estimate (5). Let ζ_1 and ζ_2 be two fixed distinct points in D_R, and set

$$\beta = \frac{f(\zeta_2)-f(\zeta_1)}{\zeta_1-\zeta_2}.$$

By the assumption (2.3.8) on $f(\zeta)$, we have

$$|\beta| \le B|\zeta_1-\zeta_2|^{\lambda-1}.$$

Set $\hat{f}(z) = f(z) + \beta z$, then we have

$$\widehat{F}(\zeta) \equiv \frac{1}{2\pi i}\iint_{D_R} \frac{\hat{f}(z)}{z-\zeta}\, dz \wedge d\overline{z} = F(\zeta) - \beta R^2 + \beta|\zeta|^2.$$

It follows that

$$\widehat{F}_\zeta(\zeta) = F_\zeta(\zeta) + \beta\overline{\zeta}.$$

Note also that the definition of β gives the following:

(1) $\hat{f}(\zeta_1) = \hat{f}(\zeta_2)$,
(2) $\hat{f}(z) - \hat{f}(\zeta_1) = f(z) - f(\zeta_1) + \beta(z-\zeta_1)$,
(3) $\hat{f}(z) - \hat{f}(\zeta_2) = f(z) - f(\zeta_2) + \beta(z-\zeta_2)$,

Based on these observations, we obtain

$$\begin{aligned}
&2\pi i(\widehat{F}_\zeta(\zeta_1) - \widehat{F}_\zeta(\zeta_2))\\
&\quad = \iint_{D_R}\left(\frac{\hat{f}(z)-\hat{f}(\zeta_1)}{(z-\zeta_1)^2} - \frac{\hat{f}(z)-\hat{f}(\zeta_2)}{(z-\zeta_2)^2}\right) dz \wedge d\overline{z}\\
&\quad = \iint_{D_R} \frac{(\hat{f}(z)-\hat{f}(\zeta_1))(\zeta_1-\zeta_2)((z-\zeta_1)+(z-\zeta_2))}{(z-\zeta_1)^2(z-\zeta_2)^2}\, dz \wedge d\overline{z}
\end{aligned}$$

$$
\begin{aligned}
&= (\zeta_1 - \zeta_2) \iint_{D_R} \frac{f(z) - f(\zeta_2)}{(z-\zeta_1)(z-\zeta_2)^2}\, dz \wedge d\bar{z} \\
&\quad + (\zeta_1 - \zeta_2) \iint_{D_R} \frac{f(z) - f(\zeta_1)}{(z-\zeta_1)^2(z-\zeta_2)}\, dz \wedge d\bar{z} \\
&\quad + 2\beta(\zeta_1 - \zeta_2) \iint_{D_R} \frac{1}{(z-\zeta_1)(z-\zeta_2)}\, dz \wedge d\bar{z} \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

By Lemma 2.3.3 the term I_1 can be estimated by

$$|I_1| \le 2B|\zeta_1 - \zeta_2| \iint_{D_R} \frac{dxdy}{|z-\zeta_2|^{2-\lambda}|z-\zeta_1|} \le c(\lambda)B|\zeta_1 - \zeta_2|^{\lambda},$$

where the constant $c(\lambda)$ depends only on λ. A similar estimate holds for I_2. The term I_3 can be written as

$$
\begin{aligned}
I_3 &= 2\beta(\zeta_1 - \zeta_2) \iint_{D_R} \frac{1}{(z-\zeta_1)(z-\zeta_2)}\, dz \wedge d\bar{z} \\
&= 2\beta \iint_{D_R} \left(\frac{1}{z-\zeta_1} - \frac{1}{z-\zeta_2} \right) dz \wedge d\bar{z} \\
&= (4\pi i)\beta(\overline{\zeta}_1 - \overline{\zeta}_2).
\end{aligned}
$$

Hence, we have

$$|I_3| \le 4\pi B|\zeta_1 - \zeta_2|^{\lambda}.$$

These estimates together show that

$$
\begin{aligned}
|F_\zeta(\zeta_1) - F_\zeta(\zeta_2)| &\le |\widehat{F}_\zeta(\zeta_1) - \widehat{F}_\zeta(\zeta_2)| + |\beta||\zeta_1 - \zeta_2| \\
&\le \mu(\lambda)B|\zeta_1 - \zeta_2|^{\lambda}.
\end{aligned}
$$

The constant $\mu(\lambda)$ is obviously independent of ζ_1 and ζ_2, and the proof of Lemma 2.3.4 is now complete.

With the aid of Lemma 2.3.4 we now prove the following existence and uniqueness theorem of an integro-differential equation from which Theorem 2.3.1 will follow.

Proposition 2.3.5. *Let*

$$Zw = a(z)w_z, \tag{2.3.10}$$

be a partial differential operator whose coefficient $a(z)$ is Hölder continuous of order λ, $0 < \lambda < 1$, and vanishes at $z = 0$. Then the equation

$$2\pi i w(\zeta) - \iint_{D_R} \frac{Zw(z)}{z - \zeta}\, dz \wedge d\bar{z} = \sigma(\zeta), \quad \zeta \in D_R, \tag{2.3.11}$$

where $\sigma(\zeta)$ is a holomorphic function with $\sigma(0) = 0$, has exactly one solution $w(z) \in C^{1+\lambda}(D_R)$, provided that R is sufficiently small.

Before proceeding to the proof of this proposition, based on Lemma 2.3.4, we shall first make some further estimates of integrals. The hypotheses of the proposition imply the existence of a number $M > 0$ so that the following estimates hold,

$$1 < M, \quad |\sigma(\zeta)| \leq M, \tag{2.3.12}$$

$$|h(\zeta_1) - h(\zeta_2)| \leq M|\zeta_1 - \zeta_2|^\lambda, \tag{2.3.13}$$

$$|Z\sigma(\zeta_1) - Z\sigma(\zeta_2)| \leq \frac{M^2}{2^\lambda}|\zeta_1 - \zeta_2|^\lambda, \tag{2.3.14}$$

where ζ, ζ_1, ζ_2 denote any three points of D_R and $h(z)$ stands for the functions $a(z)$ and $\sigma(z)$. Since $a(z)$ and $Z\sigma(z)$ both vanish at $z = 0$, we have

$$|a(\zeta)| \leq M|\zeta|^\lambda \leq MR^\lambda, \tag{2.3.15}$$

and

$$|Z\sigma(\zeta)| \leq M^2 R^\lambda. \tag{2.3.16}$$

Now we consider the function $F(\zeta)$ defined in Lemma 2.3.4 and using the notation of the lemma, we obtain

$$|ZF(\zeta)| = |a(\zeta)F_\zeta| \leq \left(\frac{2^{\lambda+1}}{\lambda}\right) MR^{2\lambda}B, \tag{2.3.17}$$

and

$$\begin{aligned} &|ZF(\zeta_1) - ZF(\zeta_2)| \\ &\quad = |a(\zeta_1)F_\zeta(\zeta_1) - a(\zeta_2)F_\zeta(\zeta_2)| \\ &\quad \leq M|\zeta_1 - \zeta_2|^\lambda \left(\frac{2^{\lambda+1}}{\lambda}\right) R^\lambda B + MR^\lambda \mu(\lambda) B |\zeta_1 - \zeta_2|^\lambda \\ &\quad \leq M|\zeta_1 - \zeta_2|^\lambda g(R) B, \end{aligned} \tag{2.3.18}$$

where

$$g(R) = \left(\left(\frac{2^{\lambda+1}}{\lambda}\right) + \mu(\lambda)\right) R^\lambda,$$

is a function of R which tends to zero as R approaches zero.

Proof of Proposition 2.3.5. Based on the estimates obtained above, we shall first prove the existence of a solution to (2.3.11) by successive approximations. In order to make the iteration converge, we shall choose the radius R to be sufficiently small so that it satisfies

$$4R^\lambda \leq 1 \quad \text{and} \quad 2^{2-\lambda}\left(1 + \frac{2}{\lambda}\right) R^{1-\lambda} \leq 1. \tag{2.3.19}$$

Denote by $c > 0$ another universal constant such that

$$\mu(\lambda) + \frac{2^{\lambda+1}}{\lambda} \leq c. \tag{2.3.20}$$

Now we construct a sequence of functions $\{w_j(\zeta)\}_{j=0}^{\infty}$ to generate a solution of (2.3.11). We first set

$$2\pi i w_0(\zeta) = \sigma(\zeta), \tag{2.3.21}$$

and inductively define for $\zeta \in D_R$,

$$2\pi i w_{n+1}(\zeta) = \iint_{D_R} \frac{Zw_n(z)}{z-\zeta}\, dz \wedge d\bar{z}, \quad n = 0, 1, 2, \cdots . \tag{2.3.22}$$

Claim 2.3.6. *The functions $\{w_j(\zeta)\}_{j=0}^{\infty}$ satisfy the following estimates:*

(1) $|w_n(\zeta)| \leq M(cMR^\lambda)^n$, $\zeta \in D_R$.
(2) $|Zw_n(\zeta)| \leq M(cMR^\lambda)^{n+1}$, $\zeta \in D_R$.
(3) $|w_n(\zeta_1) - w_n(\zeta_2)| \leq M(cMR^\lambda)^n|\zeta_1 - \zeta_2|^\lambda$, $\zeta_1, \zeta_2 \in D_R$.
(4) $|Zw_n(\zeta_1) - Zw_n(\zeta_2)| \leq (\frac{cM^2}{2^\lambda})(cMR^\lambda)^n|\zeta_1 - \zeta_2|^\lambda$. $\zeta_1, \zeta_2 \in D_R$.

In particular, estimate (4) of the claim implies that the function under the integral sign in (2.3.22) is Hölder continuous of order λ, thus allowing the definition of the next integral, and the iteration continues.

Proof of the claim. The claim will be proved by an induction on n. The initial step $n = 0$ follows easily from (2.3.12),(2.3.13),(2.3.14) and (2.3.16). Hence, let us assume that the claim is valid up to step n and proceed to prove the statement for $n + 1$.

By estimates (2) and (4) of Lemma 2.3.4, the choice of R in (2.3.19) and the induction hypotheses, we obtain

$$|w_{n+1}(\zeta)| \leq 4RM(cMR^\lambda)^{n+1} \leq M(cMR^\lambda)^{n+1},$$

and

$$\begin{aligned}
&|w_{n+1}(\zeta_1) - w_{n+1}(\zeta_2)| \\
&\quad \leq 2|\zeta_1 - \zeta_2| \left(M(cMR^\lambda)^{n+1} + \left(\frac{2^{\lambda+1}}{\lambda}\right) R^\lambda \left(\frac{cM^2}{2^\lambda}\right)(cMR^\lambda)^n \right) \\
&\quad = M(cMR^\lambda)^{n+1}|\zeta_1 - \zeta_2|^\lambda \left(1 + \frac{2}{\lambda}\right) 2|\zeta_1 - \zeta_2|^{1-\lambda} \\
&\quad \leq M(cMR^\lambda)^{n+1}|\zeta_1 - \zeta_2|^\lambda \left(2^{2-\lambda}\left(1 + \frac{2}{\lambda}\right) R^{1-\lambda}\right) \\
&\quad \leq M(cMR^\lambda)^{n+1}|\zeta_1 - \zeta_2|^\lambda.
\end{aligned}$$

This proves (1) and (3) of the claim for $n + 1$.

Next, we apply (2.3.17), (2.3.19) and (2.3.20) and the induction hypotheses to get (2) for $n+1$,

$$\begin{aligned}|Zw_{n+1}(\zeta)| &\le \left(\frac{2^{\lambda+1}}{\lambda}\right) MR^{2\lambda}\left(\frac{cM^2}{2^\lambda}\right)\left(cMR^\lambda\right)^n \\ &= M(cMR^\lambda)^{n+2}\left(\frac{2}{\lambda c}\right) \\ &\le M(cMR^\lambda)^{n+2}.\end{aligned}$$

Estimate (4) of the claim for $n+1$ can be obtained from (2.3.18), (2.3.19) and (2.3.20) as follows:

$$\begin{aligned}|Zw_{n+1}(\zeta_1) - Zw_{n+1}(\zeta_2)| &\le \left(\frac{cM^2}{2^\lambda}\right)(cMR^\lambda)^{n+1}|\zeta_1-\zeta_2|^\lambda\left(\frac{1}{cR^\lambda}\right)g(R) \\ &\le \left(\frac{cM^2}{2^\lambda}\right)(cMR^\lambda)^{n+1}|\zeta_1-\zeta_2|^\lambda.\end{aligned}$$

This completes the induction procedure, and hence the proof of the claim.

We return to the proof of Proposition 2.3.5. In addition to (2.3.19), let the radius R of the domain be chosen so small that it also satisfies $cMR^\lambda < 1$. It follows that the series

$$\sum_{j=0}^{\infty} w_j(z) \tag{2.3.23}$$

converges absolutely and uniformly in D_R, and defines a solution $w(z) \in C^{1+\lambda}(D_R)$ which satisfies (2.3.11).

For the uniqueness of the solution when R is sufficiently small, let $\eta(z)$ be another solution of (2.3.11) such that $Z\eta(z)$ satisfies a Hölder condition of order λ. Then the function

$$\widehat{w}(z) = w(z) - \eta(z)$$

satisfies the equation

$$2\pi i\widehat{w}(\zeta) = \iint_{D_R} \frac{Z\widehat{w}(z)}{z-\zeta}\, dz \wedge d\overline{z}. \tag{2.3.24}$$

Put

$$A = A(R) = \sup_{\zeta\in D_R} |Z\widehat{w}(\zeta)|,$$

and

$$B = B(R) = \sup_{\substack{\zeta_1,\zeta_2\in D_R \\ \zeta_1\neq\zeta_2}} \frac{|Z\widehat{w}(\zeta_1) - Z\widehat{w}(\zeta_2)|}{|\zeta_1-\zeta_2|^\lambda}.$$

Obviously, A and B will in general depend on R, and both decrease as R tends to zero. Then, from (2.3.17) and (2.3.18) we obtain

$$A \le \left(\frac{2^{\lambda+1}}{\lambda}\right) MR^{2\lambda} B,$$

and

$$B \le Mg(R)B.$$

Since $g(R)$ approaches zero when R tends to zero, we must have $B = 0$ for sufficiently small R. This implies $A = 0$ when R is sufficiently small. Hence $w(z) = \eta(z)$ on D_R for some sufficiently small R. This proves the uniqueness part of the proposition, and the proof of Proposition 2.3.5 is now complete.

Proof of Theorem 2.3.1. We are now in a position to prove Theorem 2.3.1. To solve equation (2.3.6) we set $\sigma(z) = z$ in the statement of Proposition 2.3.5. Then there exists a unique solution $w(z) \in C^{1+\lambda}(D_R)$ to the equation

$$2\pi i w(\zeta) - \iint_{D_R} \frac{Zw(z)}{z-\zeta}\, dz \wedge d\bar{z} = \zeta, \quad \text{for } \zeta \in D_R,$$

with $w_\zeta(0) \neq 0$ if R is sufficiently small. Since $Zw(z)$ satisfies a Hölder condition of order λ, we see by Lemma 2.3.4 that $w(z)$ satisfies equation (2.3.6). This proves Theorem 2.3.1.

NOTES

The Plemelj jump formula associated with the Cauchy transform was proved in [Ple 1]. Theorem 2.1.6 which is the analog of the Poincaré lemma for the $\overline{\partial}$ operator is often known as the Dolbeault-Grothendieck lemma (see [Dol 1,2]).

Theorem 2.2.1 is a special case of the so-called Bochner-Martinelli-Koppelman formula due to W. Koppelman [Kop 1]. Corollary 2.2.2 concerning the reproducing property of the Bochner-Martinelli kernel for holomorphic functions was discovered independently by S. Bochner [Boc 1] and E. Martinelli [Mar 1]. The jump formula stated in Theorem 2.2.3, which extends the jump formula associated with the Cauchy transform on the complex plane, can be found in [HaLa 1]. See also the book by R. M. Range [Ran 6] for more discussions.

Theorem 2.3.1 is known to geometers as the theorem of Korn and Lichtenstein which states that given a Riemannian metric

$$ds^2 = g_{11}(x,y)dx^2 + 2g_{12}(x,y)dxdy + g_{22}(x,y)dy^2,$$

in some open neighborhood U of the origin in $\mathbb{R}^2$, where the coefficient functions $g_{ij}(x,y), 1 \le i,j \le 2$, are Hölder continuous of order λ, $0 < \lambda < 1$, we have near every point there is a neighborhood whose local coordinates are isothermal parameters. By isothermal parameters we mean that, under new local coordinates, the metric ds^2 takes the following normal form

$$ds^2 = \lambda(u,v)(du^2 + dv^2),$$

for some $\lambda(u,v) > 0$. If the coefficient functions $g_{ij}(x,y), 1 \le i,j \le 2$, are assumed to be continuous only, then the Riemannian metric ds^2 cannot always be transformed to the normal form. A counterexample was found by P. Hartman and A. Wintner [HaWi 1]. The proof we present here for Theorem 2.3.1 is essentially taken from [Cher 1] and [Ber 1]. See also Chapter IV of the book, Volume II, by R. Courant and D. Hilbert [CoHi 1].

CHAPTER 3

HOLOMORPHIC EXTENSION AND PSEUDOCONVEXITY

Let M be a C^k hypersurface in $\mathbb{C}^n$, and let p be a point on M, where $k \in \mathbb{N}$. By this we mean that there exists a C^k real-valued defining function ρ and an open neighborhood U of p such that $M \cap U = \{z \in U | \rho(z) = 0\}$ and $d\rho(z) \neq 0$ on $M \cap U$. M divides U into two sides, U_+ and U_-, where $U_+ = \{z \in U | \rho(z) > 0\}$ and $U_- = \{z \in U | \rho(z) < 0\}$. Define by $\overline{L}$, a type $(0,1)$ vector field on M, such that

$$\overline{L} = \sum_{j=1}^{n} a_j(z) \frac{\partial}{\partial \overline{z}_j} \quad \text{on } M \cap U,$$

where the $a_j's$ satisfy

$$\sum_{j=1}^{n} a_j(z) \frac{\partial \rho}{\partial \overline{z}_j}(z) = 0 \quad \text{for all } z \in M \cap U.$$

Any such vector field $\overline{L}$ on M is called a tangential Cauchy-Riemann equation. Suppose that $f \in C^1(\overline{U}_-) \cap \mathcal{O}(U_-)$. By continuity, we see that $\overline{L}f = 0$ on $M \cap U$. This shows that the restriction of a holomorphic function f to a hypersurface will automatically satisfy the homogeneous tangential Cauchy-Riemann equations.

Definition 3.0.1. *Let M be a C^1 hypersurface in $\mathbb{C}^n$, $n \geq 2$. A C^1 function f on M is called a CR function if f satisfies the homogeneous tangential Cauchy-Riemann equations*

$$\sum_{j=1}^{n} a_j \frac{\partial f}{\partial \overline{z}_j}(z) = 0, \quad z \in M,$$

for all $a = (a_1, \cdots, a_n) \in \mathbb{C}^n$ with $\sum_{j=1}^{n} a_j (\partial\rho/\partial\overline{z}_j)(z) = 0$, $z \in M$, where $\rho(z)$ is a C^1 defining function for M.

The restriction of a holomorphic function f to a hypersurface is a CR function. Conversely, given any CR function f on M, can one extend f holomorphically into one side of M? This is the so-called holomorphic extension of CR functions. In general, the converse part is not true.

For instance, let M be defined by $y_1 = 0$ in $\mathbb{C}^n$, where $z_j = x_j + iy_j, 1 \leq j \leq n$. Consider a real-valued smooth function $f(x_1, z_2, \cdots, z_n) = f(x_1)$, which is independent of $z_2, \cdots, z_n$, in some open neighborhood of the origin. Suppose that

$f(x_1)$ is not real analytic at the origin. Note that $f(x_1, z_2, \cdots, z_n)$ is annihilated by the tangential type $(0,1)$ vector fields $\partial/\partial\bar{z}_2, \cdots, \partial/\partial\bar{z}_n$, hence f is a CR function on M. Still, f can not be holomorphically extended to some open neighborhood of the origin, or to just one side of the hypersurface M.

In this chapter, we first consider the problem of global holomorphic extension of a CR function on a compact hypersurface. We then study the local one-sided holomorphic extension of a CR function. In Sections 4 and 5, we define plurisubharmonic functions, pseudoconvex domains and domains of holomorphy. We study their relations with each other, and give several equivalent definitions of pseudoconvexity. Finally, we discuss the Levi problem and its relations with the $\bar{\partial}$-equation.

3.1 The Hartogs Extension Theorem

One of the major differences between one and several complex variables is the so-called Hartogs extension theorem, which states that if a bounded domain D in $\mathbb{C}^n$, $n \geq 2$, has connected boundary, then any holomorphic function $f(z)$ defined in some open neighborhood of the boundary bD can be holomorphically extended to the entire domain D. This sort of extension phenomenon fails in one complex variable. For instance, $f(z) = 1/z$ is holomorphic on the entire complex plane except the origin, but there is no way to extend it as an entire function.

We consider the inhomogeneous Cauchy-Riemann equations in $\mathbb{C}^n$

$$\bar{\partial}u = f, \tag{3.1.1}$$

where f is a (0,1)-form of class C^k with $k \geq 1$. Write f as $f = \sum_{j=1}^n f_j d\bar{z}_j$. Since $\bar{\partial}$ is a complex, a necessary condition for solving the $\bar{\partial}$-equation is $\bar{\partial}f = 0$. More explicitly, the equation (3.1.1) is overdetermined. In order to solve (3.1.1) for some function u, it is necessary that the f_i's satisfy the following compatibility conditions:

$$\frac{\partial f_j}{\partial \bar{z}_k} = \frac{\partial f_k}{\partial \bar{z}_j}, \tag{3.1.2}$$

for all $1 \leq j < k \leq n$.

First we prove the following theorem:

Theorem 3.1.1. *Let $f_j \in C_0^k(\mathbb{C}^n)$, $n \geq 2$, $j = 1, \cdots, n$, and let $k \geq 1$ be a positive integer such that (3.1.2) is satisfied. Then there is a function $u \in C_0^k(\mathbb{C}^n)$ satisfying (3.1.1). When $k = 0$, if (3.1.2) is satisfied in the distribution sense, then there exists a function $u \in C_0(\mathbb{C}^n)$ such that u satisfies (3.1.1) in the distribution sense.*

Proof. For $k \geq 1$, set

$$\begin{aligned} u(z) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta, z_2, \cdots, z_n)}{\zeta - z_1} \, d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta + z_1, z_2, \cdots, z_n)}{\zeta} \, d\zeta \wedge d\bar{\zeta}. \end{aligned}$$

It is easily seen that $u \in C^k(\mathbb{C}^n)$ from differentiation under the integral sign. We also have $u(z) = 0$ when $|z_2| + \cdots + |z_n|$ is sufficiently large, since f vanishes on the set.

By Theorem 2.1.2, we have

$$\frac{\partial u}{\partial \bar{z}_1} = f_1(z).$$

For $j > 1$, using the compatibility condition (3.1.2), we obtain

$$\frac{\partial u}{\partial \bar{z}_j} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial f_j}{\partial \bar{\zeta}}(\zeta, z_2, \cdots, z_n)}{\zeta - z_1} \, d\zeta \wedge d\bar{\zeta} = f_j(z).$$

Hence, $u(z)$ is a solution to the $\bar{\partial}$-equation (3.1.1). In particular, u is holomorphic on the unbounded component of the complement of the support of f. Since $u(z) = 0$ when $|z_2| + \cdots + |z_n|$ is sufficiently large, we see from the Identity Theorem for holomorphic functions that u must be zero on the unbounded component of the complement of the support of f. This completes the proof of the theorem for $k \geq 1$.

When $k = 0$, define $u(z)$ by the same equation. We see that $u \in C^0(\mathbb{C}^n)$, and that $u(z) = 0$ when $|z_2| + \cdots + |z_n|$ is sufficiently large.

By Theorem 2.1.2,

$$\frac{\partial u}{\partial \bar{z}_1} = f_1(z).$$

For $j > 1$, let $\phi \in C_0^\infty(\mathbb{C}^n)$. Then, using $(\ ,\ )_{\mathbb{C}^n}$ to denote the pairing between distributions and test functions, we have

$$\begin{aligned}
\left(\frac{\partial u}{\partial \bar{z}_j}, \phi\right)_{\mathbb{C}^n} &= \left(u, -\frac{\partial \phi}{\partial \bar{z}_j}\right)_{\mathbb{C}^n} \\
&= -\int_{\mathbb{C}^n} \left(\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta + z_1, z_2, \cdots, z_n)}{\zeta} \, d\zeta \wedge d\bar{\zeta}\right) \frac{\partial \phi}{\partial \bar{z}_j} \, d\lambda(z) \\
&= \frac{-1}{2\pi i} \int_{\mathbb{C}} \frac{1}{\zeta} \left(\int_{\mathbb{C}^n} f_1(\zeta + z_1, z_2, \cdots, z_n) \frac{\partial \phi}{\partial \bar{z}_j} \, d\lambda(z)\right) d\zeta \wedge d\bar{\zeta} \\
&= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{1}{\zeta} \left(\frac{\partial f_1}{\partial \bar{z}_j}(\zeta + z_1, z_2, \cdots, z_n), \phi\right)_{\mathbb{C}^n} d\zeta \wedge d\bar{\zeta} \\
&= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{1}{\zeta} \left(\frac{\partial f_j}{\partial \bar{z}_1}(\zeta + z_1, z_2, \cdots, z_n), \phi\right)_{\mathbb{C}^n} d\zeta \wedge d\bar{\zeta} \\
&= \frac{-1}{2\pi i} \int_{\mathbb{C}} \frac{1}{\zeta} \left(\int_{\mathbb{C}^n} f_j(\zeta + z_1, z_2, \cdots, z_n) \frac{\partial \phi}{\partial \bar{z}_1} \, d\lambda(z)\right) d\zeta \wedge d\bar{\zeta} \\
&= -\int_{\mathbb{C}^n} \left(\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_j(\zeta + z_1, z_2, \cdots, z_n)}{\zeta} \, d\zeta \wedge d\bar{\zeta}\right) \frac{\partial \phi}{\partial \bar{z}_1} \, d\lambda(z) \\
&= \left(\frac{\partial}{\partial \bar{z}_1} \left(\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_j(\zeta, z_2, \cdots, z_n)}{\zeta - z_1} \, d\zeta \wedge d\bar{\zeta}\right), \phi\right)_{\mathbb{C}^n} \\
&= (f_j, \phi)_{\mathbb{C}^n},
\end{aligned}$$

where the last equality is again guaranteed by Theorem 2.1.2. Hence,

$$\frac{\partial u}{\partial \bar{z}_j} = f_j(z),$$

for $j > 1$ in the distribution sense. One shows, similarly, that u must vanish on the unbounded component of the complement of the support of f. This proves the theorem.

Theorem 3.1.2 (Hartogs). *Let D be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$, and let K be a compact subset of D so that $D \setminus K$ is connected. Then any holomorphic function f defined on $D \setminus K$ can be extended holomorphically to D.*

Proof. Choose a cut-off function $\chi \in C_0^\infty(D)$ such that $\chi = 1$ in some open neighborhood of K. Then $-f(\overline{\partial}\chi) \in C^\infty_{(0,1)}(\mathbb{C}^n)$ satisfies the compatibility conditions, and it has compact support. By Theorem 3.1.1 there is a $u \in C_0^\infty(\mathbb{C}^n)$ such that

$$\overline{\partial} u = -f\overline{\partial}\chi,$$

and that $u = 0$ in some open neighborhood of $\mathbb{C}^n \setminus D$. Then, it is easily seen that

$$F = (1 - \chi)f - u$$

is the desired holomorphic extension of f.

Theorem 3.1.1 is the key for proving the Hartogs extension theorem. The hypothesis $n \geq 2$ made in Theorem 3.1.1 is crucial. Using Corollary 2.1.5 it is clear that in general, we cannot solve the equation $\partial u/\partial \overline{z} = f$ for a solution with compact support in $\mathbb{C}$ when the given function f has compact support.

Next, we prove another version of the holomorphic extension theorem which is an easy application of the Cauchy integral formula.

Theorem 3.1.3. *Let f be a continuous function on a domain D in $\mathbb{C}^n$, and let S be a smooth real hypersurface in $\mathbb{C}^n$. Suppose that f is holomorphic in $D \setminus S$. Then f is holomorphic on D.*

Proof. It suffices to show f is holomorphic near each $p \in D \cap S$. Let us fix such a point p. We may assume that p is the origin, and that locally near p, the hypersurface S is realized as a graph which can be represented as

$$S = \{(z_1 = x + iy, z') \in \mathbb{C} \times \mathbb{C}^{n-1} |\ y = \phi(x, z')\},$$

for some smooth function ϕ such that $\phi(0) = 0$ and $d\phi(0) = 0$.

Hence, for any small $\beta > 0$, there exists a $\delta_\beta > 0$ and a polydisc U_β in $\mathbb{C}^{n-1}$ centered at the origin, such that

$$|\phi(x, z')| < \beta$$

for all $|x| < \delta_\beta$ and $z' \in U_\beta$. Let $\beta_1 > 0$ be sufficiently small, and let $\beta_2 > \beta_1$ be a positive number sufficiently close to β_1, then we may assume that $V_0 \times U_{\beta_1}$, where

$$V_0 = \{z_1 \in \mathbb{C} |\ |x| < \delta_{\beta_1} \text{ and } \beta_1 < y < \beta_2\}$$

is contained in $D \setminus S$. Thus, $f \in \mathcal{O}(V_0 \times U_{\beta_1})$.

Next, for each $z' \in U_{\beta_1}$, $f(z_1, z')$ is continuous on

$$V = \{z_1 \in \mathbb{C} |\ |x| < \delta_{\beta_1} \text{ and } |y| < \beta_2\}$$

and holomorphic on V except for the smooth curve $y = \phi(x, z')$. By Morera's theorem in one complex variable, $f(z_1, z')$ is holomorphic in z_1 on V. Now choose a contour of integration Γ in U_{β_1}. Namely, let $\Gamma = \Gamma_2 \times \cdots \times \Gamma_n$, where $\Gamma_j = \{z_j \in \mathbb{C} |\ |z_j| = r_j\}$, for $2 \leq j \leq n$, so that $\Gamma \subset U_{\beta_1}$.

Define

$$F(z_1, z') = \frac{1}{(2\pi i)^{n-1}} \int_\Gamma \frac{f(z_1, \zeta')}{(\zeta_2 - z_2)\cdots(\zeta_n - z_n)}\, d\zeta_2 \cdots d\zeta_n.$$

It is easily seen that $F(z_1, z')$ is holomorphic on $V \times U$, where $U = D_2 \times \cdots \times D_n$ and $D_j = \{z_j \in \mathbb{C} |\ |z_j| < r_j\}$. Since for $(z_1, z') \in V_0 \times U$, we have

$$F(z_1, z') = f(z_1, z').$$

Since f is continuous, it follows from the Identity Theorem that f is holomorphic on $V \times U$. The proof of the theorem is now complete.

3.2 The Holomorphic Extension Theorem from a Compact Hypersurface

In this section we shall prove a generalized version of the Hartogs extension theorem. The following lemma is useful.

Lemma 3.2.1. *Let M be a C^k hypersurface with a C^k defining function r, $k \geq 1$. Then any CR function of class C^k on M can be extended to a C^{k-1} function $\tilde{f}$ in some open neighborhood of M such that $\bar\partial \tilde{f} = 0$ on M.*

Proof. We first extend f to a C^k function in some open neighborhood of M, still denoted by f. Since f is CR on M, we have

$$\overline{L}_i f = 0, \quad \text{on } M,\ i = 1, \cdots, n-1,$$

where $\overline{L}_1, \cdots, \overline{L}_{n-1}$ forms a basis of the tangential Cauchy Riemann equations. Let r be a defining function for M such that $|dr| = 1$ on M. Then we simply modify f to be $\tilde{f} = f - 4r(\overline{L}_n f)$, where $L_n = \sum_{j=1}^n (\partial r/\partial \bar{z}_j)(\partial/\partial z_j)$ is the type $(1,0)$ vector field transversal to the boundary everywhere. When $k \geq 2$, we have that $r(\overline{L}_n f)$ is a C^{k-1} function and

$$\overline{L}_n \tilde{f} = (\overline{L}_n f) - 4(\overline{L}_n r)(\overline{L}_n f) = 0 \quad \text{on } M. \tag{3.2.1}$$

This proves $\bar\partial \tilde{f} = 0$ on M.

When $k = 1$, $(\overline{L}_n f)$ is only a C^0 function but it is easy to check, from the definition, that $r(\overline{L}_n f)$ is C^1 on M and (3.2.1) still holds. This proves the lemma.

Theorem 3.2.2. *Let D be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, with connected C^1 boundary. Let f be a CR function of class C^1 defined on bD. Then, for any small $\epsilon > 0$, f extends holomorphically to a function $F \in C^{1-\epsilon}(\overline{D}) \cap \mathcal{O}(D)$ such that $F|_{bD} = f$.*

Proof. We define $F_-(z)$ and $F_+(z)$ as the Bochner-Martinelli transform of f on D and $\mathbb{C}^n \setminus D$. From Theorem 2.2.3, we have $F_- \in C^{1-\epsilon}(\overline{D})$, $F_+ \in C^{1-\epsilon}(\mathbb{C}^n \setminus D)$ and

$$f(z) = F_-(z) - F_+(z) \quad \text{for } z \in bD.$$

We will first show that $F_-(z) \in \mathcal{O}(D)$ and $F_+(z) \in \mathcal{O}(\mathbb{C}^n \setminus \overline{D})$. Define $B(\zeta, z)$ by (2.2.1) and

$$\begin{aligned} & B_1(\zeta, z) \\ & = -\frac{n-1}{(2\pi i)^n} \left(\sum_{j=1}^n \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^{2n}} \, d\zeta_j \right) \wedge \left(\sum_{j=1}^n d\overline{\zeta}_j \wedge d\zeta_j \right)^{n-2} \wedge \left(\sum_{j=1}^n d\overline{z}_j \wedge d\zeta_j \right). \end{aligned}$$

Then we have

$$\overline{\partial}_z B(\zeta, z) + \overline{\partial}_\zeta B_1(\zeta, z) = 0, \quad \text{for } \zeta \neq z. \tag{3.2.2}$$

Identity (3.2.2) is proved by a straightforward calculation as follows: For $\zeta \neq z$,

$$\begin{aligned} & -\overline{\partial}_z B(\zeta, z) \\ & = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^n \frac{d\overline{z}_j \wedge d\zeta_j}{|\zeta - z|^{2n}} \wedge (\underset{k \neq j}{\wedge} d\overline{\zeta}_k \wedge d\zeta_k) \\ & \quad - \frac{n!}{(2\pi i)^n} \sum_{j=1}^n \left(\sum_{l=1}^n \frac{(\overline{\zeta}_j - \overline{z}_j)(\zeta_l - z_l)}{|\zeta - z|^{2n+2}} \, d\overline{z}_l \wedge d\zeta_j \right) \wedge (\underset{k \neq j}{\wedge} d\overline{\zeta}_k \wedge d\zeta_k) \\ & = -\frac{n-1}{(2\pi i)^n} \frac{1}{|\zeta - z|^{2n}} \left(\sum_{j=1}^n d\overline{z}_j \wedge d\zeta_j \right) \wedge \left(\sum_{j=1}^n d\overline{\zeta}_j \wedge d\zeta_j \right)^{n-1} \\ & \quad - \frac{n(n-1)}{(2\pi i)^n} \left(\sum_{j \neq l} \frac{(\overline{\zeta}_j - \overline{z}_j)(\zeta_l - z_l)}{|\zeta - z|^{2n+2}} \, d\overline{z}_l \wedge d\zeta_j \wedge d\overline{\zeta}_l \wedge d\zeta_l \right) \\ & \qquad \wedge \left(\sum_{j=1}^n d\overline{\zeta}_j \wedge d\zeta_j \right)^{n-2} \\ & \quad + \frac{n!}{(2\pi i)^n} \sum_{j=1}^n \left(\frac{1}{|\zeta - z|^{2n}} - \frac{|\zeta_j - z_j|^2}{|\zeta - z|^{2n+2}} \right) \, d\overline{z}_j \wedge d\zeta_j \wedge (\underset{k \neq j}{\wedge} d\overline{\zeta}_k \wedge d\zeta_k) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{n-1}{(2\pi i)^n}\frac{1}{|\zeta-z|^{2n}}\left(\sum_{j=1}^n d\overline{z}_j\wedge d\zeta_j\right)\wedge(\sum_{j=1}^n d\overline{\zeta}_j\wedge d\zeta_j)^{n-1}\\
&\quad+\frac{n(n-1)}{(2\pi i)^n}\left(\sum_{j\neq l}\frac{(\overline{\zeta}_j-\overline{z}_j)(\zeta_l-z_l)}{|\zeta-z|^{2n+2}}\ d\overline{\zeta}_l\wedge d\zeta_j\wedge d\overline{z}_l\wedge d\zeta_l\right)\\
&\qquad\qquad\wedge\left(\sum_{j=1}^n d\overline{\zeta}_j\wedge d\zeta_j\right)^{n-2}\\
&\quad+\frac{n(n-1)}{(2\pi i)^n}\left(\sum_{j=1}^n\frac{|\zeta_j-z_j|^2}{|\zeta-z|^{2n+2}}\ d\overline{\zeta}_j\wedge d\zeta_j\right)\wedge\left(\sum_{j=1}^n d\overline{z}_j\wedge d\zeta_j\right)\\
&\qquad\qquad\wedge\left(\sum_{j=1}^n d\overline{\zeta}_j\wedge d\zeta_j\right)^{n-2}\\
&= -\frac{n-1}{(2\pi i)^n}\frac{1}{|\zeta-z|^{2n}}\left(\sum_{j=1}^n d\overline{\zeta}_j\wedge d\zeta_j\right)^{n-1}\wedge\left(\sum_{j=1}^n d\overline{z}_j\wedge d\zeta_j\right)\\
&\quad+\frac{n(n-1)}{(2\pi i)^n}\left(\sum_{j=1}^n\sum_{l=1}^n\frac{(\overline{\zeta}_j-\overline{z}_j)(\zeta_l-z_l)}{|\zeta-z|^{2n+2}}d\overline{\zeta}_l\wedge d\zeta_j\right)\wedge\left(\sum_{j=1}^n d\overline{\zeta}_j\wedge d\zeta_j\right)^{n-2}\\
&\qquad\qquad\wedge\left(\sum_{j=1}^n d\overline{z}_j\wedge d\zeta_j\right)\\
&= \overline{\partial}_\zeta B_1(\zeta,z).
\end{aligned}
$$

This proves (3.2.2).

Since f is a CR function of class C^1 on bD, using Lemma 3.2.1, one can extend f to be a continuous function in some open neighborhood of the boundary so that f is differentiable at bD and $\overline{\partial}f = 0$ on bD. It follows now from (3.2.2) that for $z\in D$, we have

$$
\begin{aligned}
\overline{\partial}_z F_-(z) &= \int_{bD} f(\zeta)\overline{\partial}_z B(\zeta,z)\\
&= -\int_{bD} f(\zeta)\overline{\partial}_\zeta B_1(\zeta,z)\\
&= -\int_{bD}\overline{\partial}_\zeta(f(\zeta)B_1(\zeta,z))\\
&= -\int_{bD} d_\zeta(f(\zeta)B_1(\zeta,z))\\
&= 0.
\end{aligned}
$$

Thus, $F_-(z)\in\mathcal{O}(D)$ and, similarly, $F_+(z)\in\mathcal{O}(\mathbb{C}^n\setminus\overline{D})$.

Finally, we claim $F_+(z)\equiv 0$. We let $|z_2|+\cdots+|z_n|$ be sufficiently large, then $F_+(\cdot,z_2,\cdots,z_n)$ is an entire function in z_1 which tends to zero as $|z_1|$ tends to infinity. Thus, by Liouville's theorem $F_+(z)\equiv 0$ if $|z_2|+\cdots+|z_n|$ is large enough.

It follows now from the identity theorem that $F_+(z) \equiv 0$. Setting $F = F_-$, we have $F_-|_{bD} = f$ and this proves the theorem.

3.3 A Local Extension Theorem

The result in this section deals with the local one-sided holomorphic extension of a smooth CR function defined on some neighborhood of p on M. It turns out that this question is related to the geometry of the domain. In particular, it is related to the so-called Levi form of the domain.

Definition 3.3.1. *Let D be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$, and let r be a C^2 defining function for D. The Hermitian form*

$$\mathcal{L}_p(r;t) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p) t_j \bar{t}_k, \qquad p \in bD, \tag{3.3.1}$$

defined for all $t = (t_1, \cdots, t_n) \in \mathbb{C}^n$ with $\sum_{j=1}^n t_j (\partial r/\partial z_j)(p) = 0$ is called the Levi form of the function r at the point p, denoted by $\mathcal{L}_p(r;t)$.

If ρ is another C^2 defining function for D, then $\rho = hr$ for some C^1 function h with $h > 0$ on some open neighborhood of bD. Hence, for $p \in bD$ and $t = (t_1, \cdots, t_n) \in \mathbb{C}^n$ with $\sum_{j=1}^n t_j(\partial r/\partial z_j)(p) = 0$, we have

$$\begin{aligned}
\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) t_j \bar{t}_k &= \sum_{j,k=1}^{n} \frac{\partial r}{\partial z_j}(p) \frac{\partial h}{\partial \overline{z}_k}(p) t_j \bar{t}_k + \sum_{j,k=1}^{n} \frac{\partial h}{\partial z_j}(p) \frac{\partial r}{\partial \overline{z}_k}(p) t_j \bar{t}_k \\
&\quad + h(p) \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p) t_j \bar{t}_k \\
&= h(p) \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p) t_j \bar{t}_k.
\end{aligned}$$

This shows that the Levi form associated with D is independent of the defining function up to a positive factor. In particular, the number of positive and negative eigenvalues of the Levi form are independent of the choice of the defining function.

For $p \in bD$, let

$$T_p^{1,0}(bD) = \{t = (t_1, \cdots, t_n) \in \mathbb{C}^n | \sum_{j=1}^{n} t_j (\partial r/\partial z_j)(p) = 0\}.$$

Then $T_p^{1,0}(bD)$ is the space of type $(1,0)$ vector fields which are tangent to the boundary at the point p. Smooth sections in $T^{0,1}(bD)$ are the tangential Cauchy-Riemann operators defined in the introduction. By definition, the Levi form is applied only to the tangential type $(1,0)$ vector fields. We now state and prove the local extension theorem for CR functions.

Theorem 3.3.2. *Let r be a C^2 defining function for a hypersurface M in a neighborhood U of p where $p \in M$. Assume that the Levi form $\mathcal{L}_p(r;t) < 0$ for some $t \in T_p^{1,0}(M)$. Then there exists a neighborhood $U' \subset U$ of p such that for any CR function $f(z)$ of class C^2 on $M \cap U'$, one can find an $F(z) \in C^0(\overline{U}'_+)$, where $\overline{U}'_+ = \{z \in U' | r(z) \geq 0\}$, so that $F = f$ on $M \cap U'$ and $\overline{\partial} F = 0$ on $U'_+ = \{z \in U' | r(z) > 0\}$.*

Proof. First we introduce new local coordinates near p. By a linear coordinate change we may assume that $p = 0$ and that the Taylor expansion at 0 gives

$$r(z) = y_n + A(z) + O(|z|^3),$$

where $z_n = x_n + iy_n$ and

$$A(z) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(0) z_j \overline{z}_k + \mathrm{Re}\bigg(\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(0) z_j z_k \bigg).$$

Consider the following holomorphic coordinate change. Let

$$\begin{aligned} w_j &= z_j \quad \text{for} \quad 1 \leq j \leq n-1, \\ w_n &= z_n + i \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(0) z_j z_k. \end{aligned}$$

Then the Taylor expansion becomes

$$r(w) = \mathrm{Im} w_n + \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial w_j \partial \overline{w}_k}(0) w_j \overline{w}_k + O(|w|^3).$$

Therefore, we may assume that we are working in a local coordinate system $z = (z_1, \cdots, z_n)$ so that

$$r(z) = \mathrm{Im} z_n + \sum_{j,k=1}^{n} M_{jk} z_j \overline{z}_k + O(|z|^3),$$

where (M_{jk}) is a Hermitian symmetric matrix. The hypothesis on the Levi form implies that the submatrix $(M_{jk})_{j,k=1}^{n-1}$ is not positive semidefinite. Hence, by another linear change of coordinates, we may assume that $M_{11} < 0$. Notice that

$$r(z_1, 0, \cdots, 0) = M_{11} |z_1|^2 + O(|z_1|^3).$$

Thus, we can first choose $\delta > 0$, and then $\epsilon > 0$ so that

$$\frac{\partial^2 r}{\partial z_1 \partial \overline{z}_1}(z) < 0$$

on

$$U' = \{z \in \mathbb{C}^n |\ |z_1| < \delta \text{ and } |z_2| + \cdots + |z_n| < \epsilon\} \subset U,$$

and $r(z) < 0$ on the part of the boundary where $|z_1| = \delta$. For any fixed $z' = (z_2, \cdots, z_n)$ with $|z_2| + \cdots + |z_n| < \epsilon$, the set of all z_1 with $|z_1| < \delta$ such that $r(z_1, z') < 0$ must be connected. Otherwise, $r(z_1, z')$ will attain a local minimum at some point $|z_1| < \delta$ and we will have $\triangle_{z_1} r(z_1, z') \geq 0$. This is a contradiction.

Consider now, a CR function f of class C^2 on $U' \cap M$. Using Lemma 3.2.1, extend f to U'_+, also denoted by f, so that $f \in C^1(\overline{U}'_+)$ and satisfies $\overline{\partial} f = 0$ on $U' \cap M$. If we write

$$\overline{\partial} f = g = \sum_{j=1}^{n} g_j d\overline{z}_j,$$

then $g_j \in C(\overline{U}'_+)$ and $g_j = 0$ on the boundary $U' \cap M$.

The $g_j{}'s$, extending by zero outside U'_+, will be viewed as functions defined on $W = \mathbb{C} \times V$, where $V = \{z' = (z_2, \cdots, z_n) \in \mathbb{C}^{n-1} | |z_2| + \cdots + |z_n| < \epsilon\}$. For any $z' \in V$, define

$$G(z_1, z') = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta, z')}{\zeta - z_1} \, d\zeta \wedge d\overline{\zeta}. \tag{3.3.2}$$

We have immediately that $G(z) \in C^0(W)$. Since $\overline{\partial} g = 0$ in the sense of distribution, Theorem 3.1.1 implies $\overline{\partial} G = g$ in the distribution sense.

Similarly, $(\partial G / \partial \overline{z}_j)(z) = 0$ on $W \setminus \overline{U}'_+$ for $1 \leq j \leq n$. It follows that G is holomorphic on $W \setminus \overline{U}'_+$. Also, notice that for any sufficiently small positive number $0 < \eta << 1$, there is a small open neighborhood V_0 of $(0, \cdots, 0, -i\eta)$ in $\mathbb{C}^{n-1} = \{z' = (z_2, \cdots, z_n)\}$ such that $\mathbb{C} \times V_0$ is contained in W and $r(z_1, z') < 0$ on $\mathbb{C} \times V_0$. It implies that $G(z) \equiv 0$ on $\mathbb{C} \times V_0$. Hence, by the identity theorem, $G(z) \equiv 0$ on $W \setminus \overline{U}'_+$. In particular, $G(z) = 0$ on $\{z \in U' |\, r(z) = 0\}$. Now the function $F = f - G$ is in $C^0(\overline{U}'_+)$ with $F = f$ on $\{z \in U' |\, r(z) = 0\}$ and satisfies $\overline{\partial} F = 0$ on U'_+. This proves the theorem.

Theorem 3.3.2 states that if the Levi form associated with the hypersurface M has one nonzero eigenvalue, then we have one-sided holomorphic extension of the CR functions. In particular, if the Levi form has eigenvalues of opposite signs, then the given CR function $f(z)$ on M can be extended holomorphically to both sides, say, $F_+(z)$ and $F_-(z)$ respectively, such that $F_+(z)|_M = F_-(z)|_M = f(z)$ on M. Hence, by Theorem 3.1.3, $F_+(z)$ and $F_-(z)$ can be patched together to form a holomorphic function defined in some open neighborhood of the reference point p in $\mathbb{C}^n$.

3.4 Pseudoconvexity

Let D be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$. In this section we define the concept of pseudoconvexity. We also discuss the relations between pseudoconvexity and plurisubharmonic functions.

Definition 3.4.1. *Let D be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$, and let r be a C^2 defining function for D. D is called pseudoconvex, or Levi pseudoconvex, at $p \in bD$,* *if the Levi form*

$$\mathcal{L}_p(r;t)=\sum_{j,k=1}^{n}\frac{\partial^2 r}{\partial z_j\partial\bar{z}_k}(p)t_j\bar{t}_k\geq 0 \tag{3.4.1}$$

for all $t\in T_p^{1,0}(bD)$. The domain D is said to be strictly (or strongly) pseudoconvex at p, if the Levi form (3.4.1) is strictly positive for all such $t\neq 0$. D is called a (Levi) pseudoconvex domain if D is (Levi) pseudoconvex at every boundary point of D. D is called a strictly (or strongly) pseudoconvex domain if D is strictly (or strongly) pseudoconvex at every boundary point of D.

Note that Definition 3.4.1 is clearly independent of the choice of the defining function r.

Definition 3.4.2. *A function ϕ defined on an open set $D\subset\mathbb{C}^n$, $n\geq 2$, with values in $[-\infty,+\infty)$ is called plurisubharmonic if*

(1) *ϕ is upper semicontinuous,*
(2) *for any $z\in D$ and $w\in\mathbb{C}^n$, $\phi(z+\tau w)$ is subharmonic in $\tau\in\mathbb{C}$ whenever $\{z+\tau w|\ \tau\in\mathbb{C}\}\subset D$.*

Theorem 3.4.3. *A C^2 real-valued function ϕ on D is plurisubharmonic if and only if*

$$\sum_{j,k=1}^{n}\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}(z)t_j\bar{t}_k\geq 0, \tag{3.4.2}$$

for all $t=(t_1,\cdots,t_n)\in\mathbb{C}^n$ and all $z\in D$.

Proof. The assertion follows immediately from the nonnegativeness of the Laplacian of the subharmonic function $\phi(z+\tau w)$ in $\tau\in\mathbb{C}$ whenever it is defined.

If (3.4.2) is strictly positive, we shall call ϕ a strictly plurisubharmonic function. It is obvious from Definition 3.4.2 that any plurisubharmonic function satisfies the submean value property on each complex line where it is defined. The following theorem shows that there always exists a strictly plurisubharmonic defining function for any strongly pseudoconvex domain.

Theorem 3.4.4. *Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n, n\geq 2$, with a C^k $(2\leq k\leq\infty)$ defining function $r(z)$. Then there exists a C^k $(2\leq k\leq\infty)$ strictly plurisubharmonic defining function for D.*

Proof. For any $\lambda>0$, set

$$\rho(z)=e^{\lambda r}-1\quad\text{for } z\in\overline{D}.$$

We will show that $\rho(z)$ is the desired strictly plurisubharmonic defining function for D if λ is chosen to be sufficiently large.

First, ρ is a C^k defining function for D since

$$\nabla\rho(z)=\lambda\nabla r(z)\neq 0\quad\text{for } z\in bD.$$

Next, we calculate the Levi form of $\rho(z)$ for $z \in bD$ to get that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) t_j \bar{t}_k = \lambda \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(z) t_j \bar{t}_k + \lambda^2 \Big| \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(z) t_j \Big|^2,$$

for $t \in \mathbb{C}^n$. By homogeneity, we may assume that $|t| = 1$. Since D is of strong pseudoconvexity, by continuity there exists an $\epsilon > 0$ such that

$$\sum_{j,k=1}^{n} (\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k})(z) t_j \bar{t}_k > 0$$

on the set $\{(z,t) \mid z \in bD,\ t \in \mathbb{C}^n, |t| = 1,\ |\sum_{j=1}^n (\partial r/\partial z_j)(z) t_j| < \epsilon\}$. On the other hand, if $t \in \mathbb{C}^n$ is of unit length and satisfies $|\sum_{j=1}^n (\partial r/\partial z_j)(z) t_j| \geq \frac{\epsilon}{2}$ for $z \in bD$, we may also achieve $\sum_{j,k=1}^n (\partial^2 \rho/\partial z_j \partial \overline{z}_k)(z) t_j \bar{t}_k > 0$ simply by choosing λ to be sufficiently large. This shows that ρ is strictly plurisubharmonic near the boundary by continuity if λ is large enough. This proves the theorem.

We recall that a bounded domain $D \subset \mathbb{R}^N$ with C^2 boundary is called strictly convex if there is a C^2 defining function ρ for D such that

$$\sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(p) t_j t_k > 0, \qquad p \in bD,$$

for all $t = (t_1, \cdots, t_N) \in \mathbb{R}^N$ with $\sum_{j=1}^N \partial \rho/\partial x_j(p) t_j = 0$.

Corollary 3.4.5. *Let D be a bounded pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$, $n \geq 2$. Then D is strongly pseudoconvex if and only if D is locally biholomorphically equivalent to a strictly convex domain near every boundary point.*

Proof. Suppose first that D is strongly pseudoconvex. By Theorem 3.4.4 there is a C^2 strictly plurisubharmonic defining function $r(z)$ for D. Let p be a boundary point. After a holomorphic coordinate change as we did in Theorem 3.3.2, we may assume that p is the origin and the defining function takes the following form

$$r(z) = \mathrm{Re} z_n + \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(0) z_j \overline{z}_k + O(|z|^3).$$

Since the quadratic term is positive by hypothesis for any $z \neq 0$, it is now easy to see that D is strictly convex near p. The other direction is trivially true. This proves the corollary.

Definition 3.4.6. *A function $\varphi : D \to \mathbb{R}$ on an open subset D in $\mathbb{R}^n$ is called an exhaustion function for D if for every $c \in \mathbb{R}$ the set $\{x \in D|\ \varphi(x) < c\}$ is relatively compact in D.*

Clearly, if φ is an exhaustion function for D, then $\varphi(x) \to \infty$ as $x \to bD$. This condition is also sufficient if the domain D is bounded. Next, we show the existence of a smooth strictly plurisubharmonic exhaustion function on a pseudoconvex domain. Let $d_D(z)$ denote the Euclidean distance from $z \in D$ to bD.

Theorem 3.4.7. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with a C^2 boundary. Then $-log(d_D(z))$ is plurisubharmonic near the boundary.*

Proof. First set

$$r(z) = \begin{cases} -d_D(z) = -\text{dist}(z, bD), & \text{for } z \in D, \\ \text{dist}(z, bD), & \text{for } z \notin D. \end{cases} \tag{3.4.3}$$

Then it follows from the implicit function theorem that $r(z)$ is a C^2 defining function for D in some small open neighborhood of the boundary. Hence the Levi form defined by r is positive semidefinite.

If $-\log(d_D(z))$ is not plurisubharmonic near the boundary, then

$$\frac{\partial^2}{\partial\tau\partial\overline{\tau}}\log d_D(z+\tau w)|_{\tau=0} > 0$$

for some $w \in \mathbb{C}^n$ and z close to the boundary where $d_D(z)$ is C^2. Expand $\log d_D(z+\tau w)$ at $\tau = 0$ to get

$$\log d_D(z+\tau w) = \log(d_D(z)) + \text{Re}(\alpha\tau + \beta\tau^2) + \gamma|\tau|^2 + O(|\tau|^3),$$

for small τ. Here $\alpha, \beta \in \mathbb{C}$, $\gamma > 0$ are constants. Choose $\eta \in \mathbb{C}^n$ such that $z+\eta \in bD$ and $|\eta| = d_D(z)$. Then consider the analytic disc

$$\triangle_\delta = \{z(\tau) = z + \tau w + \eta e^{\alpha\tau+\beta\tau^2} \mid |\tau| \leq \delta\}$$

for some sufficiently small $\delta > 0$. Using Taylor's expansion, for $|\tau| \leq \delta$, $\tau \neq 0$, we get

$$\begin{aligned} d_D(z(\tau)) &\geq d_D(z+\tau w) - |\eta||e^{\alpha\tau+\beta\tau^2}| \\ &\geq |\eta|(e^{\frac{\gamma}{2}|\tau|^2} - 1)|e^{\alpha\tau+\beta\tau^2}| \\ &> 0, \end{aligned}$$

if δ is small enough. Since $z(0) = z + \eta \in bD$, this implies that $\triangle_\delta$ is tangent to the boundary at $z(0)$. Hence,

$$\frac{\partial}{\partial\tau}d_D(z(\tau))|_{\tau=0} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial\tau\partial\overline{\tau}}d_D(z(\tau))|_{\tau=0} > 0.$$

From the definition of r, this means

$$\sum_{k=1}^n \frac{\partial r}{\partial z_k}(z+\eta)z_k'(0) = 0 \quad \text{and} \quad \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial\overline{z}_k}(z+\eta)z_j'(0)\overline{z_k'(0)} < 0.$$

This contradicts the nonnegativeness of the Levi form at $z(0) = z + \eta$. Hence $-\log(d_D(z))$ is plurisubharmonic near the boundary. This proves the theorem.

Corollary 3.4.8. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with a C^2 boundary. Then there exists a smooth strictly plurisubharmonic exhaustion function on D.*

Proof. By Theorem 3.4.7, $-\log(d_D(z))$ is a C^2 plurisubharmonic function for $z \in D$ near the boundary. Let $\eta(z)$ be a C^2 function on D such that $\eta(z) = -\log(d_D(z))$ on $U \cap D$, where U is an open neighborhood of bD. We may assume that $-\log(d_D(z))$ is plurisubharmonic on $U \cap D$. Then it is easily seen that

$$\lambda(z) = \eta(z) + M|z|^2$$

is a C^2 strictly plurisubharmonic exhaustion function on D if M is chosen large enough.

The next step is to regularize $\lambda(z)$. For each $j \in \mathbb{N}$, we set $D_j = \{z \in D | \lambda(z) < j\}$, then $D_j \subset\subset D$. Choose a function $\chi(z) = \chi(|z|) \in C_0^\infty(B(0;1))$ such that $\chi(z) \geq 0$ and $\int \chi(z)dV = 1$. Set $\chi_\epsilon(z) = \epsilon^{-2n}\chi(z/\epsilon)$. For $z \in D_j$, the function

$$\lambda_\epsilon(z) = \int \lambda(z-\zeta)\chi_\epsilon(\zeta)dV(\zeta) = \int \lambda(z-\epsilon\zeta)\chi(\zeta)dV(\zeta)$$

is defined and smooth on D_j if ϵ is sufficiently small. Since λ is strictly plurisubharmonic of class C^2, it is clear that $\lambda_\epsilon(z)$ is strictly plurisubharmonic and, by the submean value property, $\lambda_{\epsilon_1} \leq \lambda_{\epsilon_2}$ if $\epsilon_1 < \epsilon_2$, and $\lambda_\epsilon(z)$ converges uniformly to $\lambda(z)$ on any compact subset of D.

Therefore, by extending $\lambda_\epsilon(z)$ in a smooth manner to D, we see that there are functions $\lambda_{\epsilon_j}(z) \in C^\infty(D)$ for $j \in \mathbb{N}$ such that $\lambda_{\epsilon_j}(z)$ is strictly plurisubharmonic on D_{j+2}, $\lambda(z) < \lambda_{\epsilon_1}(z) < \lambda(z) + 1$ on $\overline{D}_2$ and $\lambda(z) < \lambda_{\epsilon_j}(z) < \lambda(z) + 1$ on D_j for $j \geq 2$. It follows that

$$\lambda_{\epsilon_j}(z) - j + 1 < 0 \text{ on } D_{j-2} \text{ for } j \geq 3,$$

and

$$\lambda_{\epsilon_j}(z) - j + 1 > 0 \text{ on } \overline{D}_j - D_{j-1} \text{ for } j \geq 3.$$

Now choose a $\beta(x) \in C^\infty(\mathbb{R})$ with $\beta(x) = 0$ for $x \leq 0$ and $\beta(x)$, $\beta'(x)$, $\beta''(x)$ positive for $x > 0$. Then, $\beta(\lambda_{\epsilon_j}(z) - j + 1) \geq 0$ and $\beta(\lambda_{\epsilon_j}(z) - j + 1) \equiv 0$ on D_{j-2}. A direct computation shows that $\beta(\lambda_{\epsilon_j}(z) - j + 1)$ is plurisubharmonic on D_{j+2} and strictly plurisubharmonic on $\overline{D}_j - D_{j-1}$. Thus, one may choose inductively $m_j \in \mathbb{N}$ so that, for $k \geq 3$,

$$\varphi_k(z) = \lambda_{\epsilon_1}(z) + \sum_{j=3}^{k} m_j \beta(\lambda_{\epsilon_j}(z) - j + 1)$$

is strictly plurisubharmonic and $\varphi_k(z) \geq \lambda(z)$ on D_k. Clearly, $\varphi_k(z) = \varphi_{k-1}(z)$ on D_{k-2}. Thus, $\varphi(z) = \lim_{k\to\infty} \varphi_k(z)$ is the desired smooth strictly plurisubharmonic exhaustion function on D. This completes the proof of Corollary 3.4.8.

Now if D is a bounded pseudoconvex domain in $\mathbb{C}^n$ with a C^2 boundary, according to Corollary 3.4.8, there exists a smooth strictly plurisubharmonic exhaustion function $\varphi(z)$ on D. Define $D_c = \{z \in D|\ \varphi(z) < c\}$ for every $c \in \mathbb{R}$. It follows from Sard's Theorem that, for almost every $c \in \mathbb{R}$, D_c is a strictly pseudoconvex domain with smooth boundary. In other words, any bounded pseudoconvex domain D in $\mathbb{C}^n$ with a C^2 boundary can be exhausted by a sequence of smooth bounded strictly pseudoconvex domains D_c.

When the domain D does not have smooth boundary or D is not bounded, we define pseudoconvexity by the following

Definition 3.4.9. *An open domain D in $\mathbb{C}^n$ is called pseudoconvex if there exists a smooth strictly plurisubharmonic exhaustion function $\varphi(z)$ on D.*

Theorem 3.4.10. *Let D be a pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, in the sense of Definition 3.4.9. Then $-log(d_D(z))$ is plurisubharmonic and continuous on D.*

Proof. Let φ be a smooth plurisubharmonic exhaustion function on D. We shall show that if z_0 is a point in D and $w \in \mathbb{C}^n$ is a nonzero vector, then $-\log d_D(z_0+\tau w)$ is subharmonic in $\tau \in \mathbb{C}$ whenever $z_0 + \tau w \in D$. Choose $\delta > 0$ so that

$$\triangle_0 = \{z_0 + \tau w|\ |\tau| \leq \delta\} \subseteq D,$$

and let $f(\tau)$ be a holomorphic polynomial such that

$$-\log d_D(z_0 + \tau w) \leq \mathrm{Re} f(\tau) \quad \text{for } |\tau| = \delta. \tag{3.4.4}$$

We want to show that

$$-\log d_D(z_0 + \tau w) \leq \mathrm{Re} f(\tau) \quad \text{for } |\tau| \leq \delta.$$

Equation (3.4.4) is equivalent to

$$d_D(z_0 + \tau w) \geq |e^{-f(\tau)}| \quad \text{for } |\tau| = \delta. \tag{3.4.5}$$

Now, for any $\eta \in \mathbb{C}^n$ with $|\eta| < 1$, we consider the mapping with $0 \leq t \leq 1$,

$$\tau \mapsto z_0 + \tau w + t\eta e^{-f(\tau)} \quad \text{for } |\tau| \leq \delta. \tag{3.4.6}$$

The image of (3.4.6) is an analytic disc. Let $\triangle_t = \{z_0 + \tau w + t\eta e^{-f(\tau)}|\ |\tau| \leq \delta\}$.

Set $E = \{t \in [0,1]|\ \triangle_t \subseteq D\}$. Clearly, $0 \in E$ and E is open. To show that E is closed, set $K = \cup_{0 \leq t \leq 1} b\triangle_t$. Estimate (3.4.5) implies that K is a compact subset of D. Now, if $\triangle_t \subseteq D$ for some t, $\varphi(z_0 + \tau w + t\eta e^{-f(\tau)})$ would define a subharmonic function in some open neighborhood of the closure of the unit disc in $\mathbb{C}$. Therefore, by the maximum principle for subharmonic functions and the exhaustion property of φ, we see that $\triangle_t$ must be contained in $\{z \in D|\ \varphi(z) \leq \sup_K \varphi\}$, a compact subset of D. It follows that E is closed, and hence $E = [0,1]$. This implies, for any $\eta \in \mathbb{C}^n$ with $|\eta| < 1$ and $|\tau| \leq \delta$, that

$$z_0 + \tau w + \eta e^{-f(\tau)} \in D.$$

Thus, we have

$$d_D(z_0 + \tau w) \geq |e^{-f(\tau)}| \quad \text{for } |\tau| \leq \delta,$$

or equivalently,

$$-\log d_D(z_0 + \tau w) \leq \mathrm{Re} f(\tau) \quad \text{for } |\tau| \leq \delta.$$

Hence, $-\log d_D(z_0 + \tau w)$ is subharmonic in $\tau \in \mathbb{C}$ whenever $z_0 + \tau w \in D$. This proves the theorem.

The equivalence between Definitions 3.4.1 and 3.4.9 on domains with smooth boundaries is proved in the following theorem.

Theorem 3.4.11. *Let D be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, with C^2 boundary. Then D is Levi pseudoconvex if and only if D is pseudoconvex according to Definition 3.4.9.*

Proof. If D is Levi pseudoconvex, then by Corollary 3.4.8 D is pseudoconvex in the sense of Definition 3.4.9.

On the other hand, assume D is pseudoconvex according to Definition 3.4.9. Define r by(3.4.3). Then $r(z)$ is a C^2 defining function for D in some small open neighborhood of the boundary.

Now Theorem 3.4.10 asserts that $-\log(d_D(z))$ is a C^2 plurisubharmonic function if $z \in D$ is sufficiently close to the boundary. Thus, following from the plurisubharmonicity of $-\log(d_D(z))$, we obtain that

$$\sum_{j,k=1}^{n} \left(-\frac{1}{d_D} \frac{\partial^2 d_D}{\partial z_j \overline{z}_k} a_j \overline{a}_k \right) + \frac{1}{d_D^2} \left| \sum_{j=1}^{n} \frac{\partial d_D}{\partial z_j} a_j \right|^2 \geq 0,$$

for any $a \in \mathbb{C}^n$ and $z \in D$ sufficiently close to bD. Therefore,

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \overline{z}_k}(z) a_j \overline{a}_k \geq 0 \quad \text{if} \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(z) a_j = 0.$$

Passing to the limit, we obtain the desired assertion. This proves the theorem.

We note that by Definition 3.4.9 every pseudoconvex domain D can be exhausted by strictly pseudoconvex domains, i.e.,

$$D = \cup D_\nu,$$

where $D_\nu \subset\subset D_{\nu+1} \subset\subset D$ and each D_ν is a strictly pseudoconvex domain.

To end this section we show that there always exists a bounded strictly plurisubharmonic exhaustion function on any smooth bounded pseudoconvex domain.

Theorem 3.4.12. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a smooth bounded pseudoconvex domain. Let r be a smooth defining function for D. Then there exist constants $K > 0$ and $0 < \eta_0 < 1$, such that for any η with $0 < \eta \leq \eta_0$, $\rho = -(-re^{-K|z|^2})^\eta$ is a smooth bounded strictly plurisubharmonic exhaustion function on D.*

Note that ρ is continuous on $\overline{D}$ and vanishes on bD.

Proof. We first assume that $z \in D$ with $|r(z)| \leq \epsilon$ for some small $\epsilon > 0$. With $\rho = -(-re^{-K|z|^2})^\eta$, a direct calculation shows, for $t \in \mathbb{C}^n$,

$$\begin{aligned}
\mathcal{L}_z(\rho; t) = \eta(-r)^{\eta-2} e^{-\eta K|z|^2} \Bigg(& K r^2 \left(|t|^2 - \eta K \left| \sum_{j=1}^{n} z_j \bar{t}_j \right|^2 \right) \\
& + (-r) \left(\mathcal{L}_z(r; t) - 2\eta K \mathrm{Re} \left(\sum_{i=1}^{n} \frac{\partial r}{\partial z_i} t_i \right) \left(\sum_{j=1}^{n} z_j \bar{t}_j \right) \right) \\
& + (1-\eta) \left| \sum_{i=1}^{n} \frac{\partial r}{\partial z_i} t_i \right|^2 \Bigg).
\end{aligned}$$

For each z with $|r(z)| \leq \epsilon$ and $t = (t_1, \cdots, t_n) \in \mathbb{C}^n$, write $t = t^\tau + t^\nu$, where $t^\nu = (t_1^\nu, \cdots, t_n^\nu)$ with

$$t_k^\nu = \left(\frac{\sum_{j=1}^n t_j \frac{\partial r}{\partial z_j}(z)}{\sum_{j=1}^n |\frac{\partial r}{\partial z_j}(z)|^2} \right) \frac{\partial r}{\partial \overline{z}_k}(z),$$

and $t^\tau = (t_1^\tau, \cdots, t_n^\tau) \in T_z^\tau = \{a \in \mathbb{C}^n | \ \sum_{j=1}^n (\partial r/\partial z_j)(z) a_j = 0\}$. Such a decomposition is clearly smooth when ϵ is sufficiently small. Also, let $\pi(z)$ be the projection of z along the normal on the boundary. Obviously, π is smooth for small ϵ. Then the Levi form of r at z is

$$\begin{aligned}\mathcal{L}_z(r; t^\tau) &= \sum_{i,j=1}^n \frac{\partial^2 r}{\partial z_i \partial \overline{z}_j}(z) t_i^\tau(z) \overline{t}_j^\tau(z) \\ &= \sum_{i,j=1}^n \frac{\partial^2 r}{\partial z_i \partial \overline{z}_j}(z)(t_i - t_i^\nu(z))\overline{(t_j - t_j^\nu(z))} \\ &= \sum_{i,j=1}^n b_{ij}(z) t_i \overline{t}_j,\end{aligned}$$

where $b_{ij}(z)$ is defined by the last equality. Hence, by pseudoconvexity of the domain, we have

$$\begin{aligned}\mathcal{L}_z(r; t^\tau(z)) &\geq \mathcal{L}_z(r; t^\tau(z)) - \mathcal{L}_{\pi(z)}(r; t^\tau(\pi(z))) \\ &= \sum_{i,j=1}^n (b_{ij}(z) - b_{ij}(\pi(z))) t_i \overline{t}_j \\ &\geq -C|r(z)||t|^2,\end{aligned} \tag{3.4.7}$$

for some constant $C > 0$. Since

$$|t^\nu| = O\left(\left| \sum_{i=1}^n \frac{\partial r}{\partial z_i} t_i \right| \right), \tag{3.4.8}$$

(3.4.7) and (3.4.8) together imply

$$\mathcal{L}_z(r; t) \geq -C|r(z)||t|^2 - C|t| \left| \sum_{j=1}^n \frac{\partial r}{\partial z_j} t_j \right|.$$

Hence

$$\begin{aligned}\mathcal{L}_z(\rho; t) \geq \eta(-r)^{\eta-2} e^{-\eta K|z|^2} \Bigg(&Kr^2(1 - C\eta K)|t|^2 \\ &- Cr^2|t|^2 + Cr|t| \left| \sum_{j=1}^n \frac{\partial r}{\partial z_j} t_j \right| + (1-\eta) \left| \sum_{j=1}^n \frac{\partial r}{\partial z_j} t_j \right|^2 \Bigg),\end{aligned}$$

for some constant $C > 0$. Since

$$C|r||t|\left|\sum_{j=1}^{n}\frac{\partial r}{\partial z_j}t_j\right| \le \frac{1}{4}\left|\sum_{j=1}^{n}\frac{\partial r}{\partial z_j}t_j\right|^2 + C_1 r^2|t|^2,$$

we have

$$\begin{aligned}\mathcal{L}_z(\rho;t) \ge \eta(-r)^{\eta-2}e^{-\eta K|z|^2}\bigg(&Kr^2(1-C\eta K)|t|^2 \\ &- (C+C_1)r^2|t|^2 + \left(\frac{3}{4}-\eta\right)\left|\sum_{j=1}^{n}\frac{\partial r}{\partial z_j}t_j\right|^2\bigg).\end{aligned}$$

Now, if we first choose $K > 2(C+C_1)+10$ and then η to be sufficiently small so that $\eta < 1/4$ and $C\eta K < 1/2$, we have

$$\mathcal{L}_z(\rho;t) > 0, \quad \text{for } t \in \mathbb{C}^n \setminus \{0\}.$$

For this case we may take $\eta_0 = \min(1/4, 1/(2CK))$.

If $|r(z)| \ge \epsilon$, the situation is even simpler. This proves the theorem.

3.5 Domains of Holomorphy

Throughout this section, D will denote a domain in $\mathbb{C}^n$, $n \ge 1$. Here we give the definition of a domain of holomorphy.

Definition 3.5.1. *A domain D in $\mathbb{C}^n$ is called a domain of holomorphy if we cannot find two nonempty open sets D_1 and D_2 in $\mathbb{C}^n$ with the following properties:*

(1) *D_1 is connected, $D_1 \not\subset D$ and $D_2 \subset D_1 \cap D$.*

(2) *For every $f \in \mathcal{O}(D)$ there is a $\tilde{f} \in \mathcal{O}(D_1)$ satisfying $f = \tilde{f}$ on D_2.*

According to Hartogs' theorem (Theorem 3.1.2), if we remove a compact subset K from the unit ball $B(0;1)$ in $\mathbb{C}^n$, $n \ge 2$, such that $B(0;1)\setminus K$ is connected, then the remaining set $B(0;1)\setminus K$ is not a domain of holomorphy. Also, from Theorem 3.3.2, if the Levi form of a smooth bounded domain D in $\mathbb{C}^n$, $n \ge 2$, has one negative eigenvalue, then D is not a domain of holomorphy.

In this section, we shall characterize the domain of holomorphy in $\mathbb{C}^n$ for $n \ge 2$. Let K be a compact subset of D. Define the holomorphically convex hull $\widehat{K}_D$ of K in D by

$$\widehat{K}_D = \{z \in D \mid |f(z)| \le \sup_K |f|, \text{ for all } f \in \mathcal{O}(D)\}. \tag{3.5.1}$$

A compact subset K of D is called holomorphically convex if $\widehat{K}_D = K$. We obviously have $\widehat{K}_D = \widehat{\widehat{K}}_D$. Using $f(z) = \exp(a_1z_1 + \cdots + a_nz_n)$ with $a_i \in \mathbb{C}$ for $i = 1, \cdots, n$, it is clear that $\widehat{K}_D$ must be contained in the geometrically convex hull of K, and is a closed subset of D. However, $\widehat{K}_D$ in general is not a closed subset of $\mathbb{C}^n$, i.e., $\widehat{K}_D$ in general is not a compact subset of D. In one complex variable $\widehat{K}_D$ is obtained from K by filling up all the bounded components of the complement K^c. For higher dimensional spaces, the situation is more subtle. In addition to the concept of holomorphically convex hull, we define:

Definition 3.5.2. *A domain D in $\mathbb{C}^n$ is called holomorphically convex if $\widehat{K}_D$ is relatively compact in D for every compact subset K of D.*

The main task of this section is to prove the following characterization of domains of holomorphy.

Theorem 3.5.3. *Let D be a domain in $\mathbb{C}^n$, $n \geq 2$. The following statements are equivalent:*

(1) *D is a domain of holomorphy.*
(2) *$dist(K, D^c) = dist(\widehat{K}_D, D^c)$ for every compact subset K in D, where $dist(K, D^c)$ denotes the distance between K and $D^c = \mathbb{C}^n \setminus D$.*
(3) *D is holomorphically convex.*
(4) *There exists a holomorphic function f on D which is singular at every boundary point of D.*

Proof. (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are obvious. We need to show (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4).

If $P(0; r)$ is a polydisc centered at zero with multiradii $r = (r_1, \cdots, r_n)$, for each $z \in D$, we set

$$d_r(z) = \sup\{\lambda > 0 |\ \{z\} + \lambda P(0; r) \subset D\}.$$

To prove (1) $\Rightarrow$ (2), we first show:

Lemma 3.5.4. *Let K be a compact subset of a domain D in $\mathbb{C}^n$, and let $f \in \mathcal{O}(D)$. Suppose that*

$$|f(z)| \leq d_r(z) \quad \textit{for } z \in K.$$

Let ζ be a fixed point in $\widehat{K}_D$. Then any $h \in \mathcal{O}(D)$ extends holomorphically to $D \cup \{\{\zeta\} + |f(\zeta)|P(0; r)\}$.

Proof. For each $0 < t < 1$, the union of the polydiscs with centers at $z \in K$

$$K_t = \bigcup_{z \in K} \{\{z\} + t|f(z)|\overline{P(0; r)}\} \tag{3.5.2}$$

is a compact subset of D. Hence, for any $h \in \mathcal{O}(D)$, there exists $M_t > 0$ such that $|h(z)| \leq M_t$ on K_t. Using Cauchy's estimates of h, we obtain

$$\frac{|\frac{\partial^\alpha h}{\partial z^\alpha}(z)| t^{|\alpha|} |f(z)|^{|\alpha|} r^\alpha}{\alpha!} \leq M_t \tag{3.5.3}$$

for $z \in K$ and all multiindices $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Since $(\partial^\alpha h / \partial z^\alpha)(z) f(z)^{|\alpha|}$ is holomorphic on D, by definition, (3.5.3) also holds for $z \in \widehat{K}_D$. Letting t tend to one, we see that $h(z)$ extends holomorphically to $D \cup \{\{\zeta\} + |f(\zeta)|P(0; r)\}$. This proves the lemma.

We write

$$\begin{aligned} \mathrm{dist}(z, D^c) = \sup\{r > 0 \mid\ & z + aw \in D, \text{ for all } w \in \mathbb{C}^n, |w| \leq 1 \text{ and} \\ & a \in \mathbb{C}, |a| < r\} \\ = \inf_{|w| \leq 1} & d_w(z), \end{aligned}$$

where

$$d_w(z) = \sup\{r > 0 \mid \ z + aw \in D, \ \text{for all} \ \ a \in \mathbb{C}, \ |a| < r\}.$$

Fix a w, we may assume that $w = (1, 0, \cdots, 0)$. Denote by $P_j = P(0; r(j))$ the polydisc with multiradii $r(j) = (1, 1/j, \cdots, 1/j)$ for $j \in \mathbb{N}$. Then it is easily seen that

$$\lim_{j\to\infty} d_{r(j)}(z) = d_w(z).$$

Thus, given $\epsilon > 0$, if j is sufficiently large, we have

$$\text{dist}(K, D^c) \le (1+\epsilon) d_{r(j)}(z), \quad z \in K. \tag{3.5.4}$$

We let $f(z) = \text{dist}(K, D^c)/(1+\epsilon)$ be the constant function. Since D is a domain of holomorphy, using estimate (3.5.4), Lemma 3.5.4 shows that

$$\text{dist}(K, D^c) \le (1+\epsilon) d_{r(j)}(\zeta) \le (1+\epsilon) d_w(\zeta), \quad \text{for all } \zeta \in \widehat{K}_D.$$

Letting ϵ tend to zero, we get

$$\begin{aligned} \text{dist}(K, D^c) &\le \inf_{\zeta \in \widehat{K}_D} \big(\inf_{|w|\le 1} d_w(\zeta)\big) \\ &= \inf_{\zeta \in \widehat{K}_D} \text{dist}(\zeta, D^c) \\ &= \text{dist}(\widehat{K}_D, D^c). \end{aligned}$$

This proves that (1) $\Rightarrow$ (2).

Finally, we show (3) $\Rightarrow$ (4). Assume that D is holomorphically convex. Let $\mathcal{P}$ be the set containing all points in D with rational coordinates. Clearly, $\mathcal{P}$ is countable and dense in D. Let $\{\zeta_i\}_{i=1}^\infty$ be a sequence of points in D such that every point belonging to $\mathcal{P}$ appears infinitely many times in the sequence. Now, we exhaust D by a sequence of increasingly holomorphically convex compact subsets $\{K_j\}_{j=1}^\infty$ of D with $K_j \subset \overset{o}{K}_{j+1}$, where $\overset{o}{K}_{j+1}$ is the interior of K_{j+1}. For each i, denote by P_{ζ_i} the largest polydisc of the form $P_{\zeta_i} = \{\zeta_i\} + \eta P(0;1)$ that is contained in D, where $\eta > 0$ and $P(0;1)$ is the polydisc centered at the origin with multiradii $r = (1, \cdots, 1)$. Then, inductively for each j, pick a $z_j \in (P_{\zeta_j} \setminus K_{n_j}) \cap \overset{o}{K}_{n_{j+1}}$, where $\{K_{n_j}\}$ is a suitable subsequence of $\{K_j\}$, and a $f_j(z) \in \mathcal{O}(D)$ satisfying

$$|f_j(z)| < \frac{1}{2^j}, \quad z \in K_{n_j},$$

and

$$|f_j(z_j)| \ge \sum_{i=1}^{j-1} |f_i(z_j)| + j + 1.$$

It follows that

$$h(z) = \sum_{j=1}^{\infty} f_j(z)$$

defines a holomorphic function on D and that

$$|h(z_j)| \geq |f_j(z_j)| - \sum_{i=1}^{j-1} |f_i(z_j)| - \sum_{i=j+1}^{\infty} |f_i(z_j)| \geq j,$$

which implies $h(z)$ is singular at every boundary point of D. Otherwise, if $h(z)$ extends holomorphically across some boundary point, then $h(z)$ would be bounded on $\overline{P}_{\zeta_i}$ for some ζ_i. Obviously, it contradicts the construction of h. This proves (3) $\Rightarrow$ (4), and, hence the theorem.

We see from Theorem 3.5.3 that the concept of domains of holomorphy is equivalent to that of holomorphic convexity. With this characterization, the next theorem shows that a domain of holomorphy is pseudoconvex.

Theorem 3.5.5. *If D is a domain of holomorphy, then D is pseudoconvex in the sense of Definition 3.4.9.*

Proof. Let D be a domain of holomorphy and $\{K_j\}_{j=1}^{\infty}$ be a sequence of increasingly holomorphically convex compact subsets of D which exhausts D. We may assume that $K_j \subset \overset{o}{K}_{j+1}$ for all j. Then, by hypothesis, for each $j \in \mathbb{N}$ there exist $f_{j1}, \cdots, f_{jm_j} \in \mathcal{O}(D)$ such that the function $\phi_j(z) = \sum_{k=1}^{m_j} |f_{jk}(z)|^2$ satisfies

$$\phi_j(z) < \frac{1}{2^j} \quad \text{for } z \in K_j, \tag{3.5.5}$$

and

$$\phi_j(z) > j \quad \text{for } z \in K_{j+2} \setminus \overset{o}{K}_{j+1}.$$

Hence,

$$\varphi(z) = \sum_{j=1}^{\infty} \phi_j(z)$$

is a continuous exhaustion function defined on D. In fact, $\varphi(z)$ is real analytic. It can be seen easily from (3.5.5) that the series

$$\sum_{j=1}^{\infty} \Bigg(\sum_{k=1}^{m_j} f_{jk}(z) \overline{f_{jk}(\overline{w})} \Bigg)$$

converges uniformly on compact subsets of $D \times D^*$, where $D^* = \{\overline{z} \mid z \in D\}$ denotes the conjugate domain of D. Thus the series defines a holomorphic function on $D \times D^*$. By substituting $\overline{z}$ for w in the above series, we obtain the real analyticity of $\varphi(z)$ on D, and that one can differentiate $\varphi(z)$ term by term. Obviously, $\varphi(z)$ is plurisubharmonic on D. It follows that $|z|^2 + \varphi(z)$ is a smooth strictly plurisubharmonic exhaustion function on D, and by definition, D is pseudoconvex. This proves the theorem.

3.6 The Levi Problem and the $\bar\partial$ Equation

Let D be a pseudoconvex domain in $\mathbb{C}^n$ with $n \geq 2$. One of the major problems in complex analysis is to show that a pseudoconvex domain D is a domain of holomorphy. Near each boundary point $p \in bD$, one must find a holomorphic function $f(z)$ on D which cannot be continued holomorphically near p. This problem is called the Levi problem for D at p. It involves the construction of a holomorphic function with certain specific local properties.

If the domain D is strongly pseudoconvex with C^∞ boundary bD and $p \in bD$, one can construct a *local* holomorphic function f in an open neighborhood U of p, such that f is holomorphic in $U \cap D$, $f \in C(\overline{D} \cap U \setminus \{p\})$ and $f(z) \to \infty$ as $z \in D$ approaches p. In fact f can be easily obtained as follows: let r be a strictly plurisubharmonic defining function for D and we assume that $p = 0$. Let

$$F(z) = -2\sum_{i=1}^n \frac{\partial r}{\partial z_i}(0)z_i - \sum_{i,j=1}^n \frac{\partial^2 r}{\partial z_i \partial z_j}(0)z_i z_j.$$

$F(z)$ is holomorphic, and it is called the Levi polynomial of r at 0. Using Taylor's expansion at 0, there exists a sufficiently small neighborhood U of 0 and $C > 0$ such that for any $z \in \overline{D} \cap U$,

$$\mathrm{Re}F(z) = -r(z) + \sum_{i,j=1}^n \frac{\partial^2 r}{\partial z_i \partial \bar z_j}(0)z_i \bar z_j + O(|z|^3) \geq C|z|^2.$$

Thus, $F(z) \neq 0$ when $z \in \overline{D} \cap U \setminus \{0\}$. Setting

$$f = \frac{1}{F},$$

it is easily seen that f is locally a holomorphic function which cannot be extended holomorphically across 0.

Global holomorphic functions cannot be obtained simply by employing smooth cut-off functions to patch together the local holomorphic data, since the cut-off functions are no longer holomorphic. Let χ be a cut-off function such that $\chi \in C_0^\infty(U)$ and $\chi = 1$ in a neighborhood of 0. We note that χf is not holomorphic in D. However, if χf can be corrected by solving a $\bar\partial$-equation, then the Levi problem will be solved.

Let us consider the (0,1)-form g defined by

$$g = \bar\partial(\chi f) = (\bar\partial \chi) f.$$

This form g can obviously be extended smoothly up to the boundary. It is easy to see that g is a $\bar\partial$-closed form in D and $g \in C^\infty_{(0,1)}(\overline{D})$. If we can find a solution $u \in C^\infty(\overline{D})$ such that

$$\bar\partial u = g \quad \text{in} \quad D, \tag{3.6.1}$$

then we define for $z \in D$,

$$h(z) = \chi(z)f(z) - u(z).$$

It follows that h is holomorphic in D, $h \in C^\infty(\overline{D} \setminus \{0\})$ and h is singular at 0. Thus one can solve the Levi problem for strongly pseudoconvex domains provided one can solve equation (3.6.1) with solutions smooth up to the boundary.

Problems of this sort are among the most difficult in complex analysis and they are the main topics of the next three chapters. In Chapter 4, we will solve the Levi problem using the L^2 estimate method for $\bar{\partial}$ (Hörmander's solution) on pseudoconvex domains (Theorem 4.5.2). In Chapter 5, we study the boundary regularity for $\bar{\partial}$ on strongly pseudoconvex domains. This gives another solution (Kohn's solution) of the Levi problem on complex manifolds (Theorem 5.3.11). In Chapter 6, we further investigate the boundary regularity of $\bar{\partial}$ on pseudoconvex domains with smooth boundaries for other applications.

NOTES

Theorem 3.1.2 is a theorem due to F. Hartogs [Har 1]. The present proof of Theorem 3.1.2 as pointed out by L. Ehrenpreis [Ehr 2] is essentially based on Theorem 3.1.1, i.e., the existence of compactly supported solutions to the Cauchy-Riemann equation. The proof of Theorem 3.1.3 is based on an idea of F. Hartogs [Har 1].

Using a more delicate proof found in Harvey and Lawson [HaLa 1], one can prove Theorem 3.2.2 in an optimal way. Namely, if the domain D has C^k $(1 \le k \le \infty)$ boundary and f is a CR function of class C^k on the boundary, then the holomorphic extension F is also in $C^k(\overline{D})$. See also the book by R. M. Range [Ran 6].

Theorem 3.3.2 is concerned with the local one-sided holomorphic extension of CR functions which is essentially due to H. Lewy [Lew 1]. Another way to prove the local extension theorem for the CR functions is to invoke the result discovered by Baouendi and Treves [BaTr 1]. This is the so-called analytic disc method. See the books by A. Boggess [Bog 2] and M. S. Baouendi, P. Ebenfelt and L. P. Rothschild [BER 1] for details and the references therein.

Corollary 3.4.5 in general is false for weakly pseudoconvex domains. A counterexample was discovered by J. J. Kohn and L. Nirenberg [KoNi 3]. The concept of plurisubharmonicity (Definition 3.4.2) was first introduced in two variables by K. Oka [Oka 1], and by K. Oka [Oka 2] and P. Lelong [Lel 1] in arbitrary dimension. It was K. Oka [Oka 1] who first proved the plurisubharmonicity of $-\log(d_D(z))$ for a pseudoconvex domain (Theorem 3.4.10) in $\mathbb{C}^2$. Later, similar results were obtained independently by K. Oka [Oka 2], P. Lelong [Lel 2] and H. Bremermann [Bre 2] in $\mathbb{C}^n$. The existence of a Hölder bounded strictly plurisubharmonic exhaustion function on a C^2 bounded pseudoconvex domain was first proved by K. Diederich and J. E. Fornaess [DiFo 2]. The proof we present here for Theorem 3.4.12, based on an idea of J. J. Kohn [Koh 6], is due to R. M. Range [Ran 4].

The characterization of domains of holomorphy in Theorem 3.5.3 is due to H. Cartan and P. Thullen [CaTh 1]. For more discussion on pseudoconvexity and domains of holomorphy, we refer the reader to the books by L. Hörmander [Hör 9] and R. M. Range [Ran 6].

CHAPTER 4

L^2 THEORY FOR $\bar{\partial}$ ON PSEUDOCONVEX DOMAINS

Let D be a domain in $\mathbb{C}^n$. We study the existence of solutions of the Cauchy-Riemann equations

$$\bar{\partial} u = f \qquad \text{in} \quad D, \tag{4.0.1}$$

where f is a (p,q)-form and u is a $(p,q-1)$-form on D, $0 \le p \le n$, $1 \le q \le n$. Since $\bar{\partial}^2 = 0$, it is necessary that

$$\bar{\partial} f = 0 \qquad \text{in} \quad D \tag{4.0.2}$$

in order for equation (4.0.1) to be solvable.

In this chapter, we prove Hörmander's L^2 existence theorems for the $\bar{\partial}$ operator on pseudoconvex domains in $\mathbb{C}^n$. To study Equation (4.0.1), Hilbert space techniques are used in the context of the $\bar{\partial}$-Neumann problem. First, we set up the $\bar{\partial}$-Neumann problem with weights and derive the basic *a priori* estimates of Morrey-Kohn-Hörmander. We then choose suitable weight functions in order to obtain existence theorems with L^2 estimates.

The L^2 existence theorems for $\bar{\partial}$ also give existence theorems for the $\bar{\partial}$-Neumann operator. We will conclude the chapter with a discussion of existence theorems in other function spaces. The solution of the Levi problem will be given at the end.

4.1 Unbounded Operators in Hilbert Spaces

We shall use Hilbert space techniques to study the $\bar{\partial}$ operator. To do this we need to formulate the $\bar{\partial}$ operator as a linear, closed, densely defined operator from one Hilbert space to another. This will be done in the next section. We first discuss some basic facts for unbounded operators in Hilbert spaces.

Let H_1 and H_2 be two Hilbert spaces and let $T : H_1 \to H_2$ be a linear, closed, densely defined operator. We recall that T is closed if and only if the graph of T is closed. The domain of definition for T is denoted by $\mathrm{Dom}(T)$. If T is an unbounded operator, $\mathrm{Dom}(T)$ is a proper subset of H_1 by the closed graph theorem. The norms in H_1, H_2 are denoted by $\|\ \|_1$, $\|\ \|_2$, respectively. Then the adjoint of T, $T^* : H_2 \to H_1$ is also a linear, closed, densely defined operator and $T^{**} = T$. (See [RiNa 1].)

We use $\mathrm{Ker}(T)$ and $\mathcal{R}(T)$ to denote the kernel and the range of T respectively. Since T is a closed operator, $\mathrm{Ker}(T)$ is closed. Let $\overline{\mathcal{R}(T)}$ denote the closure of the range of T. By the definition of the adjoint operator, it is easy to see that

$$H_1 = \mathrm{Ker}(T) \oplus \overline{\mathcal{R}(T^*)} \tag{4.1.1}$$

and

$$H_2 = \mathrm{Ker}(T^*) \oplus \overline{\mathcal{R}(T)}. \tag{4.1.2}$$

In later applications, the operator T will be a system of differential operators associated with the Cauchy-Riemann equations and H_1, H_2 will be spaces of forms with L^2 coefficients. To solve Equation (4.0.1) in the Hilbert space sense is to show that the range of T is closed. Using (4.1.2), the range of T is then equal to $\mathrm{Ker}(T^*)^{\perp}$.

In order to show that the range of T is closed, we use the following lemma for unbounded operators in Hilbert spaces to reduce the proof to verifying an estimate.

Lemma 4.1.1. *Let $T : H_1 \to H_2$ be a linear, closed, densely defined operator. The following conditions on T are equivalent:*

(1) *$\mathcal{R}(T)$ is closed.*
(2) *There is a constant C such that*

$$\| f \|_1 \le C \| Tf \|_2 \quad \textit{for all } f \in \mathit{Dom}(T) \cap \overline{\mathcal{R}(T^*)}. \tag{4.1.3}$$

(3) *$\mathcal{R}(T^*)$ is closed.*
(4) *There is a constant C such that*

$$\| f \|_2 \le C \| T^* f \|_1 \quad \textit{for all } f \in \mathit{Dom}(T^*) \cap \overline{\mathcal{R}(T)}. \tag{4.1.4}$$

The best constants in (4.1.3) and (4.1.4) are the same.

Proof. We assume that (1) holds. From (4.1.1),

$$T : \mathrm{Dom}(T) \cap \overline{\mathcal{R}(T^*)} \to \mathcal{R}(T)$$

is one-to-one, and its inverse

$$T^{-1} : \mathcal{R}(T) \to \mathrm{Dom}(T) \cap \overline{\mathcal{R}(T^*)}$$

is well-defined and is also a closed operator. Thus from the closed graph theorem, T^{-1} is continuous and this proves (2). It is obvious that (2) implies (1). Similarly, (3) and (4) are equivalent.

To prove that (2) implies (4), notice that

$$|(g, Tf)_2| = |(T^* g, f)_1| \le C \| T^* g \|_1 \| Tf \|_2,$$

for $g \in \mathrm{Dom}(T^*)$ and $f \in \mathrm{Dom}(T) \cap \overline{\mathcal{R}(T^*)}$. Thus

$$|(g, h)_2| \le C \| T^* g \|_1 \| h \|_2, \ \text{for } g \in \mathrm{Dom}(T^*) \text{ and } h \in \mathcal{R}(T),$$

which implies (4). Similarly, (4) implies (2).

4.2 The $\overline{\partial}$-Neumann Problem

Let D be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, not necessarily with a smooth boundary. Let $C^\infty_{(p,q)}(D)$ denote the smooth (p,q)-forms on D, where $0 \leq p \leq n,\ 0 \leq q \leq n$. We use $C^\infty_{(p,q)}(\overline{D})$ to denote the smooth (p,q)-forms on $\overline{D}$, i.e., the restriction of smooth (p,q)-forms in $\mathbb{C}^n$ to $\overline{D}$. Let $(z_1, \cdots, z_n)$ be the complex coordinates for $\mathbb{C}^n$. Then any (p,q)-form $f \in C^\infty_{(p,q)}(D)$ can be expressed as

$$f = \sum_{I,J}{}' f_{I,J} dz^I \wedge d\bar{z}^J, \tag{4.2.0}$$

where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$ are multiindices and $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. The notation $\sum'$ means the summation over strictly increasing multiindices and the $f_{I,J}$'s are defined for arbitrary I and J so that they are antisymmetric. The operator

$$\bar{\partial} = \bar{\partial}_{(p,q)} : C^\infty_{(p,q)}(D) \to C^\infty_{(p,q+1)}(D)$$

is defined by

$$\bar{\partial} f = \sum_{I,J}{}' \sum_{k=1}^{n} \frac{\partial f_{I,J}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz^I \wedge d\bar{z}^J. \tag{4.2.1}$$

Let $L^2(D)$ denote the space of square integrable functions on D with respect to the Lebesgue measure in $\mathbb{C}^n$ such that the volume element is $dV = i^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$. This volume element differs from the usual Euclidean measure by a factor of 2^n and it is more suitable for our purpose. We use $L^2_{(p,q)}(D)$ to denote the space of (p,q)-forms whose coefficients are in $L^2(D)$. If $f = \sum'_{I,J} f_{I,J} dz^I \wedge d\bar{z}^J$, $g = \sum'_{I,J} g_{I,J} dz^I \wedge d\bar{z}^J$ are two (p,q)-forms in $L^2_{(p,q)}(D)$, we define

$$\langle f, g \rangle = \sum_{I,J}{}' \langle f_{I,J}, g_{I,J} \rangle,\ |f|^2 = \langle f, f \rangle = \sum_{I,J}{}' |f_{I,J}|^2,$$

$$\| f \|^2 = \int_D \langle f, f \rangle dV = \sum_{I,J}{}' \int_D |f_{I,J}|^2 dV.$$

We use $(\ ,\ )_D$ to denote the inner product in $L^2_{(p,q)}(D)$ and when there is no danger of confusion, we drop the subscript D in the notation. If ϕ is a continuous function in D, then $L^2(D, \phi)$ is the space of functions in D which are square integrable with respect to the weight function $e^{-\phi}$. The norm in $L^2_{(p,q)}(D, \phi)$ is defined by

$$\|f\|^2_\phi = \int_D |f|^2 e^{-\phi} dV, \qquad f \in L^2_{(p,q)}(D, \phi).$$

The inner product in $L^2_{(p,q)}(D,\phi)$ is denoted by $(\ ,\ )_\phi$. Notice that the space $L^2_{(p,q)}(D,\phi)$ is equal to $L^2_{(p,q)}(D)$ if ϕ is continuous on $\overline{D}$. Let $L^2(D,\mathrm{loc})$ denote the space of locally square integrable functions. A function f is in $L^2(D,\mathrm{loc})$ if and only if f is in $L^2(\mathcal{K})$ for every compact subset $\mathcal{K}$ of D. $L^2_{(p,q)}(D,\mathrm{loc})$ is defined similarly. When there is no danger of confusion, we also use $L^2(D)$ to denote $L^2_{(p,q)}(D)$.

The formal adjoint of $\bar{\partial} : C^\infty_{(p,q-1)}(D) \to C^\infty_{(p,q)}(D)$, $1 \le q \le n$, under the usual L^2 norm is denoted by ϑ, where

$$\vartheta = \vartheta_{(p,q)} : C^\infty_{(p,q)}(D) \to C^\infty_{(p,q-1)}(D).$$

The operator ϑ is defined by the requirement that

$$(\vartheta f, g) = (f, \bar{\partial} g) \tag{4.2.2}$$

for all smooth $g \in C^\infty_{(p,q-1)}(\overline{D})$ with compact support in D. If f is expressed by (4.2.0) and $g = \sum'_{|I|=p,|K|=q-1} g_{I,K}\, dz^I \wedge d\bar{z}^K$, we have

$$\begin{aligned}(f,\ \bar{\partial} g) &= (-1)^p \sum_{I,K}{}' \sum_{k=1}^n \left(f_{I,kK},\ \frac{\partial g_{I,K}}{\partial \bar{z}_k}\right) \\ &= (-1)^{p+1} \sum_{I,K}{}' \sum_{k=1}^n \left(\frac{\partial f_{I,kK}}{\partial z_k},\ g_{I,K}\right) \\ &= (\vartheta f,\ g).\end{aligned}$$

Therefore, ϑ can be expressed explicitly by

$$\vartheta f = (-1)^{p-1} \sum_{I,K}{}' \sum_{j=1}^n \frac{\partial f_{I,jK}}{\partial z_j} dz^I \wedge d\bar{z}^K, \tag{4.2.3}$$

where I and K are multiindices with $|I| = p$ and $|K| = q-1$. It is easy to check that $\bar{\partial}^2 = \vartheta^2 = 0$. Let ϑ_ϕ be the formal adjoint of $\bar{\partial}$ under the $L^2(D,\phi)$ norm, i.e.,

$$(\vartheta_\phi f, g)_\phi = (f, \bar{\partial} g)_\phi \tag{4.2.4}$$

for every compactly supported $g \in C^\infty_{(p,q-1)}(D)$. We have the following relation between ϑ and ϑ_ϕ: for any $f \in C^\infty_{(p,q)}(D)$,

$$\vartheta_\phi f = e^\phi \vartheta(e^{-\phi} f). \tag{4.2.5}$$

Thus $\bar{\partial}$, ϑ and ϑ_ϕ are systems of first order differential operators.

We take the (weak) L^2 closure of the unbounded differential operator $\bar{\partial}$, still denoted by $\bar{\partial}$. Let

$$\bar{\partial} = \bar{\partial}_{p,q-1} : L^2_{(p,q-1)}(D,\phi) \to L^2_{(p,q)}(D,\phi)$$

be the maximal closure of the Cauchy-Riemannn operator defined as follows: an element $u \in L^2_{(p,q-1)}(D,\phi)$ is in the domain of $\bar{\partial}$ if $\bar{\partial} u$, defined in the distribution sense, belongs to $L^2_{(p,q)}(D,\phi)$. Then $\bar\partial$ defines a linear, closed, densely defined operator. $\bar\partial$ is closed since differentiation is a continuous operation in distribution theory. It is densely defined since $\mathrm{Dom}(\bar\partial)$ contains all the compactly supported smooth $(p,q-1)$-forms. If D is bounded, any $f\in C^\infty_{(p,q-1)}(\overline{D})$ is in $\mathrm{Dom}(\bar\partial)$.

The Hilbert space adjoint of $\bar\partial$, denoted by $\bar\partial^*_\phi$, is a linear, closed, densely defined operator and

$$\bar\partial^*_\phi : L^2_{(p,q)}(D,\phi)\to L^2_{(p,q-1)}(D,\phi).$$

When $\phi=0$, we denote the adjoint by $\bar\partial^*$. Let $\mathrm{Dom}(\bar\partial^*)$ and $\mathrm{Dom}(\bar\partial^*_\phi)$ denote the domains for $\bar\partial^*$ and $\bar\partial^*_\phi$, respectively. An element f belongs to $\mathrm{Dom}(\bar\partial^*_\phi)$ if there exists a $g\in L^2_{(p,q-1)}(D,\phi)$ such that for every $\psi\in\mathrm{Dom}(\bar\partial)\cap L^2_{(p,q-1)}(D,\phi)$, we have

$$(f,\bar\partial\psi)_\phi=(g,\psi)_\phi.$$

We then define $\bar\partial^*_\phi f=g$. Note that if $f\in\mathrm{Dom}(\bar\partial^*_\phi)$, then it follows from (4.2.4) that $\bar\partial^*_\phi f=\vartheta_\phi f$ where ϑ_ϕ is defined in the distribution sense in D.

If D is bounded, we have $C^\infty_{(p,q-1)}(\overline{D})\subset\mathrm{Dom}(\bar\partial)$. However, not every element in $C^\infty_{(p,q)}(\overline{D})$ is in $\mathrm{Dom}(\bar\partial^*_\phi)$. Any element in $\mathrm{Dom}(\bar\partial^*)$ (or $\mathrm{Dom}(\bar\partial^*_\phi)$) must satisfy certain boundary conditions in the weak sense. If D has C^1 boundary bD, then any $f\in\mathrm{Dom}(\bar\partial^*_\phi)\cap C^1_{(p,q)}(\overline{D})$ must satisfy the following:

Lemma 4.2.1. *Let D be a bounded domain with C^1 boundary bD and ρ be a C^1 defining function for D. For any $f\in Dom(\bar\partial^*_\phi)\cap C^1_{(p,q)}(\overline{D})$, where $\phi\in C^1(\overline{D})$, f must satisfy the boundary condition*

$$\sigma(\vartheta,d\rho)f(z)=0,\qquad z\in bD, \tag{4.2.6}$$

where $\sigma(\vartheta,d\rho)f(z)=\vartheta(\rho f)(z)$ denotes the symbol of ϑ in the $d\rho$ direction evaluated at z. More explicitly, if f is expressed as in (4.2.0), then f must satisfy

$$\sum_k f_{I,kK}\frac{\partial\rho}{\partial z_k}=0\quad\text{on } bD\quad\text{for all } I,K, \tag{4.2.6$'$}$$

where $|I|=p$ and $|K|=q-1$.

Proof. We first assume that $\phi=0$. Note that (4.2.6) and (4.2.6′) are independent of the defining function ρ. We normalize ρ such that $|d\rho|=1$ on bD.

Let f be a (0,1)-form and $f=\sum_{i=1}^n f_i d\bar z_i$. Using integration by parts and (4.2.3), we have for any $\psi\in C^\infty(\overline{D})\subset\mathrm{Dom}(\bar\partial)$,

$$\begin{aligned}(\vartheta f,\psi)&=\sum_{i=1}^n\left(-\frac{\partial f_i}{\partial z_i},\psi\right)\\&=\sum_{i=1}^n\left(f_i,\frac{\partial\psi}{\partial\bar z_i}\right)-\sum_{i=1}^n\int_{bD}f_i\frac{\partial\rho}{\partial z_i}\bar\psi dS\\&=(f,\bar\partial\psi)+\int_{bD}\langle\sigma(\vartheta,d\rho)f,\psi\rangle dS,\end{aligned}$$

where dS is the surface measure of bD. Similarly, for a (p,q)-form f and $\psi \in C^\infty_{(p,q-1)}(\overline{D}) \subset \mathrm{Dom}(\bar{\partial})$, using integration by parts, we obtain

$$(\vartheta f, \psi) = (f, \bar{\partial}\psi) + \int_{bD} \langle \sigma(\vartheta, d\rho) f, \psi \rangle dS. \tag{4.2.7}$$

If, in addition, ψ has compact support in D, we have

$$(\bar{\partial}^* f, \psi) = (\vartheta f, \psi) = (f, \bar{\partial}\psi),$$

where the first equality follows from $f \in \mathrm{Dom}(\bar{\partial}^*) \cap C^1_{(p,q)}(\overline{D})$. Since compactly supported smooth $(p, q-1)$-forms are dense in $L^2_{(p,q-1)}(D)$, we must have

$$\int_{bD} \langle \sigma(\vartheta, d\rho) f, \psi \rangle dS = 0, \qquad \text{for any } \psi \in C^\infty_{(p,q-1)}(\overline{D}).$$

This implies that $\sigma(\vartheta, d\rho) f(z) = 0$ for $z \in bD$.

If f is expressed by (4.2.0), one can easily show that (4.2.6) implies that (4.2.6′) holds on bD for each I, K. The case for $\phi \neq 0$ can be proved similarly and is left to the reader. This proves the lemma.

Another way to express condition (4.2.6) or (4.2.6′) is as follows. Let $\vee$ be the interior product defined as the dual of the wedge product. For any (p,q)-form f, $\bar{\partial}\rho \vee f$ is defined as the $(p, q-1)$-form satisfying

$$\langle g \wedge \bar{\partial}\rho, f \rangle = \langle g, \bar{\partial}\rho \vee f \rangle, \quad g \in C^\infty_{(p,q-1)}(\mathbb{C}^n).$$

Using this notation, condition (4.2.6) or (4.2.6′) can be expressed as

$$\bar{\partial}\rho \vee f = 0 \quad \text{on} \quad bD. \tag{4.2.6$''$}$$

It is also easy to see that $f \in C^1_{(p,q)}(\overline{D}) \cap \mathrm{Dom}(\bar{\partial}^*_\phi)$ if and only if f satisfies one of the three equivalent conditions (4.2.6), (4.2.6′) or (4.2.6″).

For a fixed $0 \le p \le n,\ 1 \le q \le n$, we define the Laplacian of the $\bar{\partial}$ complex

$$L^2_{(p,q-1)}(D) \underset{\bar{\partial}^*_{(p,q)}}{\overset{\bar{\partial}_{(p,q-1)}}{\rightleftarrows}} L^2_{(p,q)}(D) \underset{\bar{\partial}^*_{(p,q+1)}}{\overset{\bar{\partial}_{(p,q)}}{\rightleftarrows}} L^2_{(p,q+1)}(D).$$

Definition 4.2.2. *Let* $\Box_{(p,q)} = \bar{\partial}_{(p,q-1)}\bar{\partial}^*_{(p,q)} + \bar{\partial}^*_{(p,q+1)}\bar{\partial}_{(p,q)}$ *be the operator from* $L^2_{(p,q)}(D)$ *to* $L^2_{(p,q)}(D)$ *such that* $Dom(\Box_{(p,q)}) = \{f \in L^2_{(p,q)}(D) \mid f \in Dom(\bar{\partial}_{(p,q)}) \cap Dom(\bar{\partial}^*_{(p,q)});\ \bar{\partial}_{(p,q)} f \in Dom(\bar{\partial}^*_{(p,q+1)})$ *and* $\bar{\partial}^*_{(p,q)} f \in Dom(\bar{\partial}_{(p,q-1)})\}$.

Proposition 4.2.3. $\Box_{(p,q)}$ *is a linear, closed, densely defined self-adjoint operator.*

Proof. $\Box_{(p,q)}$ is densely defined since $\mathrm{Dom}(\Box_{(p,q)})$ contains all smooth forms with compact support. To show that $\Box_{(p,q)}$ is closed, one needs to prove that for every sequence $f_n \in \mathrm{Dom}(\Box_{(p,q)})$ such that $f_n \to f$ in $L^2_{(p,q)}(D)$ and $\Box_{(p,q)} f_n$ converges, we have $f \in \mathrm{Dom}(\Box_{(p,q)})$ and $\Box_{(p,q)} f_n \to \Box_{(p,q)} f$. Since $f_n \in \mathrm{Dom}(\Box_{(p,q)})$,

$$\begin{aligned}(\Box_{(p,q)} f_n, f_n) &= (\bar\partial\bar\partial^* f_n, f_n) + (\bar\partial^*\bar\partial f_n, f_n)\\ &= \|\bar\partial^* f_n\|^2 + \|\bar\partial f_n\|^2,\end{aligned}$$

thus $\bar\partial^* f_n$ and $\bar\partial f_n$ converge in $L^2_{(p,q-1)}(D)$ and $L^2_{(p,q+1)}(D)$, respectively. Since $\bar\partial$ and $\bar\partial^*$ are closed operators, we have $f \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$ and

$$\bar\partial f_n \to \bar\partial f \quad \text{and} \quad \bar\partial^* f_n \to \bar\partial^* f \quad \text{in} \quad L^2.$$

To show that $\bar\partial f \in \mathrm{Dom}(\bar\partial^*)$ and $\bar\partial^* f \in \mathrm{Dom}(\bar\partial)$, we note that since $\Box_{(p,q)} f_n = \bar\partial\bar\partial^* f_n + \bar\partial^*\bar\partial f_n$ converges, both $\bar\partial\bar\partial^* f_n$ and $\bar\partial^*\bar\partial f_n$ converge. This follows from the fact that $\bar\partial\bar\partial^* f_n$ and $\bar\partial^*\bar\partial f_n$ are orthogonal to each other since

$$(\bar\partial\bar\partial^* f_n, \bar\partial^*\bar\partial f_n) = (\bar\partial^2\bar\partial^* f_n, \bar\partial f_n) = 0.$$

It follows again from the fact that $\bar\partial$ and $\bar\partial^*$ are closed operators that

$$\bar\partial\bar\partial^* f_n \to \bar\partial\bar\partial^* f \quad \text{and} \quad \bar\partial^*\bar\partial f_n \to \bar\partial^*\bar\partial f.$$

Therefore, we have proved that $\Box_{(p,q)} f_n \to \Box_{(p,q)} f$ and $\Box_{(p,q)}$ is a closed operator.

Let $\Box^*_{(p,q)}$ be the Hilbert space adjoint of $\Box_{(p,q)}$. It is easy to see that $\Box_{(p,q)} = \Box^*_{(p,q)}$ on $\mathrm{Dom}(\Box_{(p,q)}) \cap \mathrm{Dom}(\Box^*_{(p,q)})$. To show that $\mathrm{Dom}(\Box_{(p,q)}) = \mathrm{Dom}(\Box^*_{(p,q)})$, define

$$L_1 = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial + I = \Box_{(p,q)} + I \quad \text{on} \quad \mathrm{Dom}(\Box_{(p,q)}).$$

We shall prove that L_1^{-1} is self-adjoint. By a theorem of Von Neumann [RiNa 1],

$$(I + \bar\partial\bar\partial^*)^{-1} \quad \text{and} \quad (I + \bar\partial^*\bar\partial)^{-1}$$

are bounded self-adjoint operators. We define

$$Q_1 = (I + \bar\partial\bar\partial^*)^{-1} + (I + \bar\partial^*\bar\partial)^{-1} - I.$$

Then Q_1 is bounded and self-adjoint. We claim that $Q_1 = L_1^{-1}$. Since

$$\begin{aligned}(I + \bar\partial\bar\partial^*)^{-1} - I &= (I - (I + \bar\partial\bar\partial^*))(I + \bar\partial\bar\partial^*)^{-1}\\ &= -\bar\partial\bar\partial^*(I + \bar\partial\bar\partial^*)^{-1},\end{aligned}$$

we have that $\mathcal{R}(I + \bar\partial\bar\partial^*)^{-1} \subset \mathrm{Dom}(\bar\partial\bar\partial^*)$. Similarly, we have $\mathcal{R}(I + \bar\partial^*\bar\partial)^{-1} \subset \mathrm{Dom}(\bar\partial^*\bar\partial)$ and

$$Q_1 = (I + \bar\partial^*\bar\partial)^{-1} - \bar\partial\bar\partial^*(I + \bar\partial\bar\partial^*)^{-1}.$$

Since $\bar\partial^2 = 0$, we have $\mathcal{R}(Q_1) \subset \mathrm{Dom}(\bar\partial^*\bar\partial)$ and

$$\bar\partial^*\bar\partial Q_1 = \bar\partial^*\bar\partial(I + \bar\partial^*\bar\partial)^{-1}.$$

Similarly, we have $\mathcal{R}(Q_1) \subset \mathrm{Dom}(\bar\partial\bar\partial^*)$ and

$$\bar\partial\bar\partial^* Q_1 = \bar\partial\bar\partial^*(I + \bar\partial\bar\partial^*)^{-1}.$$

Thus, $\mathcal{R}(Q_1) \subset \mathrm{Dom}(L_1)$ and

$$L_1 Q_1 = \bar\partial\bar\partial^*(I + \bar\partial\bar\partial^*)^{-1} + \bar\partial^*\bar\partial(I + \bar\partial^*\bar\partial)^{-1} + Q_1 = I.$$

Since L_1 is injective, we have that $Q_1 = L_1^{-1}$. This proves that L_1 is self-adjoint which implies $\Box_{(p,q)} = L_1 - I$ is self-adjoint. The proposition is proved.

The following proposition shows that smooth forms in $\mathrm{Dom}(\Box_{(p,q)})$ must satisfy two sets of boundary conditions, namely, the $\bar\partial$-Neumann boundary conditions.

Proposition 4.2.4. *Let D be a bounded domain with C^1 boundary and ρ be a C^1 defining function. If $f \in C^2_{(p,q)}(\overline{D})$, then*

$$f \in Dom(\Box_{(p,q)})$$

if and only if

$$\sigma(\vartheta, d\rho)f = 0 \quad and \quad \sigma(\vartheta, d\rho)\overline{\partial} f = 0 \quad on\ bD.$$

If $f = \sum'_{I,J} f_{I,J} dz^I \wedge d\bar{z}^J \in C^2_{(p,q)}(\overline{D}) \cap Dom(\Box_{(p,q)})$, we have

$$\Box_{(p,q)} f = -\frac{1}{4}\sum_{I,J}{}' \triangle f_{I,J} dz^I \wedge d\bar{z}^J, \tag{4.2.8}$$

where $\triangle = 4\sum_{k=1}^n \partial^2/\partial z_k \partial \bar{z}_k = \sum_{k=1}^n (\partial^2/\partial x_k^2 + \partial^2/\partial y_k^2)$ is the usual Laplacian on functions.

Proof. If $f \in C^2_{(p,q)}(\overline{D}) \cap \mathrm{Dom}(\Box_{(p,q)})$, then $f \in \mathrm{Dom}(\overline{\partial}^*)$ and $\overline{\partial} f \in \mathrm{Dom}(\overline{\partial}^*)$. Thus from the same arguments as in Lemma 4.2.1, f must satisfy $\sigma(\vartheta, d\rho)f = \sigma(\vartheta, d\rho)\overline{\partial} f = 0$ on bD. Conversely, if $\sigma(\vartheta, d\rho)f = \sigma(\vartheta, d\rho)\overline{\partial} f = 0$, then $f \in \mathrm{Dom}(\overline{\partial}^*)$ and $\overline{\partial} f \in \mathrm{Dom}(\overline{\partial}^*)$ from integration by parts. Also, it is easy to see that f and $\bar{\partial}^* f = \vartheta f$ are in $\mathrm{Dom}(\bar{\partial})$. Thus $f \in \mathrm{Dom}(\Box_{(p,q)})$.

If $f \in C^2_{(p,q)}(\overline{D}) \cap \mathrm{Dom}(\Box_{(p,q)})$, we have

$$\Box_{(p,q)} f = (\bar{\partial}\vartheta + \vartheta\bar{\partial})f.$$

A direct calculation, using (4.2.1) and (4.2.3), gives us that

$$\begin{aligned}\vartheta\bar{\partial} f = &-\sum_{I,J}{}'\sum_k \frac{\partial^2 f_{I,J}}{\partial z_k \partial \bar{z}_k} dz^I \wedge d\bar{z}^J \\ &+ (-1)^p \sum_{I,K}{}' \sum_k \sum_j \frac{\partial^2 f_{I,jK}}{\partial z_j \partial \bar{z}_k} d\bar{z}_k \wedge dz^I \wedge d\bar{z}^K,\end{aligned} \tag{4.2.9}$$

and

$$\bar{\partial}\vartheta f = (-1)^{p-1} \sum_{I,K}{}' \sum_j \sum_k \frac{\partial f_{I,jK}}{\partial \bar{z}_k \partial z_j} d\bar{z}_k \wedge dz^I \wedge d\bar{z}^K. \tag{4.2.10}$$

Adding (4.2.9) and (4.2.10), we get that

$$\Box_{(p,q)} f = -\sum_{I,J}{}' \sum_k \frac{\partial^2 f_{I,J}}{\partial z_k \partial \bar{z}_k} dz^I \wedge d\bar{z}^J = -\frac{1}{4}\sum_{I,J}{}' \triangle f_{I,J} dz^I \wedge d\bar{z}^J.$$

Example. *Let D be a smooth bounded domain in $\mathbb{C}^n$ with the origin $0 \in bD$. We assume that for some neighborhood U of 0*

$$D \cap U = \{Im\ z_n = y_n < 0\} \cap U.$$

Let $f = \sum_k f_k d\bar{z}_k \in C^2_{(0,1)}(\overline{D})$ and the support of f lies in $U \cap \overline{D}$. Then f is in $Dom(\square_{(0,1)})$ if and only if f satisfies

$$\text{(a)} \qquad f_n = 0 \qquad on\ bD \cap U,$$

$$\text{(b)} \qquad \frac{\partial f_i}{\partial \bar{z}_n} = 0 \quad on\ bD \cap U, \quad i = 1, \cdots, n-1.$$

Proof. (a) follows from the condition that $f \in \mathrm{Dom}(\bar{\partial}^*)$. To see that (b) holds, we note that $\bar{\partial} f \in \mathrm{Dom}(\bar{\partial}^*)$, implying

$$\frac{\partial f_i}{\partial \bar{z}_n} - \frac{\partial f_n}{\partial \bar{z}_i} = 0 \quad \text{on } bD \cap U.$$

From (a), we have $\partial f_n / \partial \bar{z}_i = 0$ on $bD \cap U$ for $i = 1, \cdots, n-1$ since each $\partial / \partial \bar{z}_i$ is tangential. This proves (b).

We note that the first boundary condition (a) is just the Dirichlet boundary value problem. The second condition (b) is the complex normal derivative $\partial / \partial \bar{z}_n$ on each f_i instead of the usual normal derivative $\partial / \partial y_n$. It is the second boundary condition which makes the system noncoercive, i.e., it is not an elliptic boundary value problem (One can check easily that it does not satisfy the Lopatinski's conditions, see e.g., Treves [Tre 1]).

There are two objectives to the study of the $\bar{\partial}$-Neumann problem: one is to show that the range of $\square_{(p,q)}$ is closed in L^2 and that there exists a bounded inverse of the operator $\square_{(p,q)}$ on any bounded pseudoconvex domain; the other is to study the regularity of the solution of $\square_{(p,q)}$ up to the boundary. In the next sections we shall prove L^2 existence theorems for $\overline{\partial}$ and $\square_{(p,q)}$. We discuss the boundary regularity for the solution of $\square_{(p,q)}$ in Chapters 5 and 6.

4.3 L^2 Existence Theorems for $\overline{\partial}$ in Pseudoconvex Domains

In this section we prove the L^2 estimates and existence theorems for the $\bar{\partial}$ operator with precise bounds on any bounded pseudoconvex domains.

Let D be a domain with C^2 boundary bD. Let ρ be a C^2 defining function in a neighborhood of D such that $D = \{z \mid \rho(z) < 0\}$ and $|d\rho| = 1$ on bD. For each $\ell \in \mathbb{N}$, we set

$$\mathcal{D}^{\ell}_{(p,q)} = \mathrm{Dom}(\bar{\partial}^*) \cap C^{\ell}_{(p,q)}(\overline{D})$$

and

$$\mathcal{D}_{(p,q)} = \mathrm{Dom}(\bar{\partial}^*) \cap C^{\infty}_{(p,q)}(\overline{D}).$$

Let $\phi \in C^2(\overline{D})$ be a fixed function. Let

$$\mathcal{D}^{\phi}_{(p,q)} = \mathrm{Dom}(\bar{\partial}^*_{\phi}) \cap C^{\infty}_{(p,q)}(\overline{D}).$$

It is easy to see from the arguments in the proof of Lemma 4.2.1 that $f \in \mathcal{D}^{\phi}_{(p,q)}$ if and only if $\sigma(\vartheta, d\rho)f(z) = 0$ for any $z \in bD$, a condition independent of ϕ. Thus we have

$$\mathcal{D}^{\phi}_{(p,q)} = \mathcal{D}_{(p,q)},$$

which is also independent of ϕ. Similarly, we also have

$$\mathrm{Dom}(\bar{\partial}^*_{\phi}) \cap C^{\ell}_{(p,q)}(\overline{D}) = \mathcal{D}^{\ell}_{(p,q)}.$$

Let Q^{ϕ} be the form on $\mathcal{D}_{(p,q)}$ defined by

$$Q^{\phi}(f,f) = \|\bar{\partial} f\|^2_{\phi} + \|\bar{\partial}^*_{\phi} f\|^2_{\phi}.$$

We shall first prove the following basic *a priori* identity:

Proposition 4.3.1 (Morrey-Kohn-Hörmander). *Let $D \subset\subset \mathbb{C}^n$ be a domain with C^2 boundary bD and ρ be a C^2 defining function for D such that $|d\rho| = 1$ on bD. Let $\phi \in C^2(\overline{D})$. For any $f = \sum'_{|I|=p,|J|=q} f_{I,J} dz^I \wedge d\bar{z}^J \in \mathcal{D}^1_{(p,q)}$,*

$$\begin{aligned} Q^{\phi}(f,f) &= \| \bar{\partial} f \|^2_{\phi} + \| \vartheta_{\phi} f \|^2_{\phi} \\ &= \sideset{}{'}\sum_{|I|=p,|K|=q-1} \sum_{i,j} \int_D \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} f_{I,iK} \bar{f}_{I,jK} e^{-\phi} dV \\ &\quad + \sideset{}{'}\sum_{|I|=p,|J|=q} \sum_k \int_D \left| \frac{\partial f_{I,J}}{\partial \bar{z}_k} \right|^2 e^{-\phi} dV \\ &\quad + \sideset{}{'}\sum_{|I|=p,|K|=q-1} \sum_{i,j} \int_{bD} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} f_{I,iK} \bar{f}_{I,jK} e^{-\phi} dS. \end{aligned} \tag{4.3.1}$$

Proof. From (4.2.1), (4.2.3) and (4.2.5), we have

$$\overline{\partial} f = \sideset{}{'}\sum_{I,J} \sum_j \frac{\partial f_{I,J}}{\partial \overline{z}_j} d\bar{z}_j \wedge dz^I \wedge d\overline{z}^J \tag{4.3.2}$$

and

$$\vartheta_{\phi} f = (-1)^{p-1} \sideset{}{'}\sum_{I,K} \sum_j \delta^{\phi}_j f_{I,jK} dz^I \wedge d\overline{z}^K, \tag{4.3.3}$$

where $\delta^{\phi}_j u = e^{\phi} \dfrac{\partial}{\partial z_j}(e^{-\phi} u)$. Thus, setting $\overline{L}_j = \partial/\partial \bar{z}_j$, we get

$$\begin{aligned} \|\overline{\partial} f\|^2_{\phi} + \|\vartheta_{\phi} f\|^2_{\phi} = &\sideset{}{'}\sum_{I,J,L} \sum_{j,\ell} \epsilon^{jJ}_{\ell L} (\overline{L}_j(f_{I,J}),\ \overline{L}_{\ell}(f_{I,L}))_{\phi} \\ &+ \sideset{}{'}\sum_{I,K} \sum_{j,k} (\delta^{\phi}_j f_{I,jK},\ \delta^{\phi}_k f_{I,kK})_{\phi}, \end{aligned} \tag{4.3.4}$$

where $\epsilon^{jJ}_{\ell L} = 0$, unless $j \notin J, \ell \notin L$ and $\{j\} \cup J = \{\ell\} \cup L$, in which case $\epsilon^{jJ}_{\ell L}$ is the sign of permutation $\binom{jJ}{\ell L}$. Rearranging the terms in (4.3.4) gives

$$\begin{aligned}\|\overline{\partial} f\|^2_\phi \ + \ \|\vartheta_\phi f\|^2_\phi \ &= \sum_{I,J}{}' \sum_j \ \|\overline{L_j} f_{I,J}\|^2_\phi \\ &\quad - \sum_{I,K}{}' \sum_{j,k} \ (\overline{L}_k f_{I,jK}, \ \overline{L}_j f_{I,kK})_\phi \\ &\quad + \sum_{I,K}{}' \sum_{j,k} \ (\delta^\phi_j f_{I,jK}, \ \delta^\phi_k f_{I,kK})_\phi .\end{aligned} \tag{4.3.5}$$

We now apply integration by parts to the last term in (4.3.4). Note that for each $u, \ v \in C^2(\overline{D})$,

$$(u, \ \delta^\phi_j v)_\phi = -(\overline{L}_j u, \ v)_\phi + \int_{bD} \frac{\partial \rho}{\partial \bar{z}_j} \ u\overline{v} e^{-\phi} dS$$

and

$$[\delta^\phi_j, \overline{L}_k] u = \delta^\phi_j \overline{L}_k u - \overline{L}_k \delta^\phi_j u = u \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k}.$$

Thus, we find that

$$\begin{aligned}&(\delta^\phi_j u, \ \delta^\phi_k v)_\phi \\ &= (-\overline{L}_k \delta^\phi_j u, \ v)_\phi + \int_{bD} \frac{\partial \rho}{\partial \overline{z}_k} (\delta^\phi_j u) \overline{v} e^{-\phi} dS \\ &= (-\delta^\phi_j \overline{L}_k u, \ v)_\phi + ([\delta^\phi_j, \overline{L}_k] u, \ v)_\phi + \int_{bD} \frac{\partial \rho}{\partial \overline{z}_k} (\delta^\phi_j u) \overline{v} e^{-\phi} dS \\ &= (\overline{L}_k u, \ \overline{L}_j v)_\phi \ + (\frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k} u, \ v)_\phi \\ &\quad + \int_{bD} \frac{\partial \rho}{\partial \overline{z}_k} (\delta^\phi_j u) \overline{v} e^{-\phi} dS - \int_{bD} \frac{\partial \rho}{\partial z_j} (\overline{L}_k u) \overline{v} e^{-\phi} dS.\end{aligned} \tag{4.3.6}$$

When $u, \ v$ are in $C^1(\overline{D})$, (4.3.6) also holds by approximation since $C^2(\overline{D})$ is a dense subset in $C^1(\overline{D})$. Using (4.3.6) for each fixed I, K, it follows that

$$\begin{aligned}&\sum_{j,k} (\delta^\phi_j f_{I,jK}, \ \delta^\phi_k f_{I,kK})_\phi \\ &= \sum_{j,k} (\overline{L}_k f_{I,jK}, \ \overline{L}_j f_{I,kK})_\phi \ + \ \sum_{j,k} (\frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k} f_{I,jK}, \ f_{I,kK})_\phi \\ &\quad + \sum_{j,k} \int_{bD} \frac{\partial \rho}{\partial \overline{z}_k} \ (\delta^\phi_j f_{I,jK}) \bar{f}_{I,kK} e^{-\phi} dS \\ &\quad - \sum_{j,k} \int_{bD} \frac{\partial \rho}{\partial z_j} \left(\frac{\partial}{\partial \overline{z}_k} f_{I,jK} \right) \bar{f}_{I,kK} e^{-\phi} dS.\end{aligned} \tag{4.3.7}$$

If $f \in \mathcal{D}^1{}_{(p,q)}$, Lemma 4.2.1 and (4.2.6′) show that

$$\sum_k \frac{\partial \rho}{\partial \bar{z}_k} \ \bar{f}_{I,kK} \ = \ 0 \quad \text{on } bD \tag{4.3.8}$$

for each I, K. Since $\sum_k \bar{f}_{I,kK} \frac{\partial}{\partial \bar{z}_k}$ is tangential to bD, we conclude from (4.3.8) that

$$\sum_k \bar{f}_{I,kK} \frac{\partial}{\partial \bar{z}_k} \left(\sum_j \frac{\partial \rho}{\partial z_j} f_{I,jK} \right) = 0 \quad \text{on } bD \text{ for each } I, K.$$

This implies

$$\sum_k \sum_j \bar{f}_{I,kK} \frac{\partial \rho}{\partial z_j} \frac{\partial f_{I,jK}}{\partial \bar{z}_k} + \sum_k \sum_j f_{I,jK} \bar{f}_{I,kK} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} = 0 \tag{4.3.9}$$

on bD. Combining (4.3.5)-(4.3.9), we have proved (4.3.1) and the proposition.

In order to pass from *a priori* estimates (4.3.1) to the real estimates, the following density lemma is crucial:

Lemma 4.3.2 (A density lemma). *Let D be a bounded domain with $C^{\ell+1}$ boundary bD, $\ell \geq 1$ and $\phi \in C^2(\overline{D})$. Then $\mathcal{D}^{\ell}_{(p,q)}$ is dense in Dom$(\bar{\partial}) \cap$ Dom$(\bar{\partial}^*_\phi)$ in the graph norm*

$$f \to \|f\|_\phi + \|\bar{\partial} f\|_\phi + \|\bar{\partial}^*_\phi f\|_\phi.$$

Proof. The proof is essentially a variation of Friedrichs' lemma (see Appendix D). We divide the proof of the lemma into three steps.

(i). *$C^\infty_{(p,q)}(\overline{D})$ is dense in Dom$(\bar{\partial}) \cap$ Dom$(\bar{\partial}^*_\phi)$ in the graph norm.*

By this we mean that if $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*_\phi)$, one can construct a sequence $f_n \in C^\infty_{(p,q)}(\overline{D})$ such that $f_n \to f$, $\bar{\partial} f_n \to \bar{\partial} f$ and $\vartheta_\phi f_n \to \vartheta_\phi f$ in $L^2(D, \phi)$. We first show that this can be done on a compact subset in D from the usual regularization by convolution.

Let $\chi \in C^\infty_0(\mathbb{C}^n)$ be a function such that $\chi \geq 0, \int \chi dV = 1,\ \chi(z)$ depends only on $|z|$ and vanishes when $|z| \geq 1$. We define $\chi_\varepsilon(z) = \varepsilon^{-2n} \chi(z/\varepsilon)$ for $\varepsilon > 0$. Extending f to be 0 outside D, we define for $\varepsilon > 0$ and $z \in \mathbb{C}^n$,

$$\begin{aligned} f_\varepsilon(z) &= f * \chi_\varepsilon(z) \\ &= \int f(z') \chi_\varepsilon(z - z') dV(z') = \int f(z - \varepsilon z') \chi(z') dV(z'), \end{aligned}$$

where the convolution is performed on each component of f. In the first integral defining f_ε, we can differentiate under the integral sign to show that f_ε is $C^\infty(\mathbb{C}^n)$. From Young's inequality for convolution, we have

$$\| f_\varepsilon \| \leq \| f \| .$$

Since $f_\varepsilon \to f$ uniformly when $f \in C^\infty_0(\mathbb{C}^n)$, a dense subset of $L^2(\mathbb{C}^n)$, we have that

$$f_\varepsilon \to f \quad \text{in } L^2(\mathbb{C}^n) \quad \text{for every } f \in L^2(\mathbb{C}^n).$$

Obviously, this implies that $f_\varepsilon \to f$ in $L^2(D, \phi)$.

Let δ_ν be a sequence of small numbers with $\delta_\nu \searrow 0$. For each δ_ν, we define $D_{\delta_\nu} = \{z \in D|\ \rho(z) < -\delta_\nu\}$. Then D_{δ_ν} is a sequence of relatively compact open subsets of D with union equal to D. Using similar arguments as before, for any first order differential operator D_i with constant coefficients, if $D_i f \in L^2(D,\phi)$, we have

$$D_i f_\varepsilon = D_i(f * \chi_\varepsilon) = D_i f * \chi_\varepsilon \to D_i f \quad \text{in } L^2(D_{\delta_\nu},\phi)$$

as $\varepsilon \to 0$. Since $\vartheta_\phi = \vartheta + A_0$ where A_0 is an operator of degree 0, we have $\bar\partial f_\varepsilon \to \bar\partial f$ and $\vartheta_\phi f_\varepsilon \to \vartheta_\phi f$ in $L^2(D_{\delta_\nu},\phi)$ on D_{δ_ν}, where $f_\varepsilon \in C^\infty_{(p,q)}(\overline{D}_{\delta_\nu})$.

To see that this can be done up to the boundary, we first assume that the domain D is star-shaped and $0 \in D$ is a center. We approximate f first by dilation componentwise. Let $D^\epsilon = \{(1+\epsilon)z \mid z \in D\}$ and

$$f^\epsilon = f\left(\frac{z}{1+\epsilon}\right),$$

where the dilation is performed for each component of f. Then $D \subset\subset D^\epsilon$ and $f^\epsilon \in L^2(D^\epsilon)$. Also $\bar\partial f^\epsilon \to \bar\partial f \in L^2(D)$ and $\vartheta_\phi f^\epsilon \to \vartheta_\phi f \in L^2(D)$. By regularizing f^ϵ componentwise as before, we can find a family of $f_{(\epsilon)} \in C^\infty_{(p,q)}(\overline{D})$ defined by

$$f_{(\epsilon)} = f\left(\frac{z}{1+\epsilon}\right) * \chi_{\delta_\epsilon}, \tag{4.3.10}$$

where $\delta_\epsilon \searrow 0$ as $\epsilon \searrow 0$ and δ_ϵ is chosen sufficiently small. We have $f_{(\epsilon)} \to f$ in $L^2(D,\phi)$, $\bar\partial f_{(\epsilon)} \to \bar\partial f$ and $\vartheta_\phi f_{(\epsilon)} \to \vartheta_\phi f$ in $L^2(D,\phi)$. Thus, $C^\infty_{(p,q)}(\overline{D})$ is dense in the graph norm when D is star-shaped. The general case follows by using a partition of unity since we assume our domain has at least C^2 boundary. (In fact, C^1 boundary will suffice in this step).

(ii). *Compactly supported smooth forms (i.e., forms with coefficients in $C_0^\infty(D)$) are dense in Dom$(\bar\partial^*_\phi)$ in the graph norm*

$$f \to \|f\|_\phi + \|\bar\partial^*_\phi f\|_\phi.$$

We first assume that $\phi = 0$. Since $\bar\partial$ is the maximal closure (i.e., the domain of $\bar\partial$ contains all elements in $C^\infty_{(p,q)}(\overline{D})$) of the Cauchy-Riemann operator, its L^2 adjoint, $\bar\partial^*$, is minimal. This means that if $f \in \mathrm{Dom}(\bar\partial^*)$ and we extend f to $\tilde f$ on the whole space $\mathbb{C}^n$ by setting $\tilde f$ to be zero in D^c, then $\vartheta \tilde f \in L^2(\mathbb{C}^n)$ in the distribution sense. In fact, for $f \in \mathrm{Dom}(\bar\partial^*)$, we have

$$\vartheta \tilde f = \widetilde{\vartheta f}$$

where $\widetilde{\vartheta f} = \vartheta f$ in D and $\widetilde{\vartheta f} = 0$ in D^c. This can be checked from the definition of $\bar\partial^*$, since for any $v \in C^\infty_{(p,q-1)}(\mathbb{C}^n)$,

$$(\tilde f, \bar\partial v)_{\mathbb{C}^n} = (f, \bar\partial v)_D = (\bar\partial^* f, v)_D = (\vartheta f, v)_D = (\widetilde{\vartheta f}, v)_{\mathbb{C}^n}.$$

Again, we can assume that D is star-shaped and 0 is a center. The general case can be proved using a partition of unity. We first approximate $\tilde f$ by

$$\tilde f^{-\epsilon} = \tilde f\left(\frac{z}{1-\epsilon}\right).$$

Then $\tilde f^{-\epsilon}$ has compact support in D and $\vartheta\tilde f^{-\epsilon} \to \vartheta\tilde f$ in $L^2(\mathbb{C}^n)$. Regularizing $\tilde f^{-\epsilon}$ by convolution as before, we define

$$f_{(-\epsilon)} = \tilde f\left(\frac{z}{1-\epsilon}\right) * \chi_{\delta_\epsilon}. \tag{4.3.11}$$

Then the $f_{(-\epsilon)}$ are (p,q)-forms with coefficients in $C_0^\infty(D)$ such that $f_{(-\epsilon)} \to f$ in $L^2(D)$ and $\vartheta f_{(-\epsilon)} \to \vartheta f$ in $L^2(D)$. This proves (ii) when $\phi = 0$. Again, in this step we only require that the boundary be C^1. The case for $\bar\partial_\phi^*$ can be proved similarly.

However, compactly supported smooth forms are not dense in Dom $(\bar\partial)$ in the graph norm $f \to \|f\|_\phi + \|\bar\partial f\|_\phi$. Nevertheless, we have:

(iii). $\mathcal{D}^\ell_{(p,q)}$ *is dense in Dom*$(\bar\partial)$ *in the graph norm*

$$f \to \|f\|_\phi + \|\bar\partial f\|_\phi.$$

To prove (iii), we must use Friedrichs' lemma in a more subtle way. From (i), it suffices to show that for any $f \in C^\infty_{(p,q)}(\overline{D})$ that one can find a sequence $f_n \in \mathcal{D}^\ell_{(p,q)}$ such that $f_n \to f$ in $L^2(D,\phi)$ and $\bar\partial f_n \to \bar\partial f$ in $L^2(D,\phi)$. We may assume $\phi = 0$ and the general case is similar.

We regularize near a boundary point $z_0 \in bD$. Let U be a small neighborhood of z_0. By a partition of unity, we may assume that $D \cap U$ is star-shaped and f is supported in $U \cap \overline{D}$. Let ρ be a $C^{\ell+1}$ defining function such that $|d\rho| = 1$ on bD. Shrinking U if necessary, we can choose a special boundary chart $(t_1, t_2, \cdots, t_{2n-1}, \rho)$ where $(t_1, t_2, \cdots, t_{2n-1})$, when restricted to bD, forms a coordinate system on bD. Let $\bar w^1, \cdots, \bar w^n$ be an orthonormal basis for (0,1)-forms on U such that $\bar\partial\rho = \bar w^n$. Written in this basis,

$$f = \sum_{|I|=p,|J|=q}{}' f_{I,J} w^I \wedge \bar w^J,$$

where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$ are increasing multiindices, and $w^I = w^{i_1} \wedge \cdots \wedge w^{i_p}$, $\bar w^J = \bar w^{j_1} \wedge \cdots \wedge \bar w^{j_q}$. Each $f_{I,J}$ is a function in $C^\ell(\overline{D})$. We note that both $\bar\partial$ and ϑ are first order differential operators with variable coefficients which are in $C^\ell(\overline{D})$ when computed in the special frame $w^1, \cdots, w^n$. We write

$$f = f^\tau + f^\nu,$$

where

$$f^\tau = \sum_{|I|=p,|J|=q,\ n\notin J}{}' f_{I,J} w^I \wedge \bar w^J,$$

$$f^\nu = \sum_{|I|=p,|J|=q,\ n\in J}{}' f_{I,J} w^I \wedge \bar w^J.$$

f^τ is the complex tangential part of f, and f^ν is the complex normal part of f. From Lemma 4.2.1 and (4.2.6$''$), it follows that

$$f \in \mathcal{D}^\ell_{(p,q)} \quad \text{if and only if} \quad f^\nu = 0 \quad \text{on } bD.$$

We also observe that from integration by parts, for $f \in C^\infty_{(p,q)}(\overline{D})$, $g \in C^\infty_{(p,q+1)}(\overline{D})$,

$$(\bar\partial f, g) = (f, \vartheta g) + \int_{bD} \langle \sigma(\bar\partial, d\rho) f, g\rangle dS, \tag{4.3.12}$$

where dS is the surface measure of bD and

$$\sigma(\bar\partial, d\rho) f = \bar\partial(\rho f) = \bar\partial\rho \wedge f = \bar\partial\rho \wedge f^\tau \qquad \text{on} \quad bD.$$

Thus, when we do integration by parts for $\bar\partial f$, only the tangential part f^τ will appear in the boundary term. This is called the Cauchy data of f with respect to the operator $\bar\partial$. The Cauchy data of f with respect to $\bar\partial$ contains the tangential part of f, and it does not contain the complex normal part f^ν (From (4.2.7), it is easy to see that the Cauchy data of f for ϑ is the complex normal part f^ν).

We regularize only the complex normal part of f and leave the complex tangential part f^τ unchanged. Let $\tilde f^\nu$ be the extension of f^ν to $\mathbb{C}^n$ by setting $\tilde f^\nu$ equal to zero outside D. We approximate $\tilde f^\nu$ by the dilation and regularization by convolution as in (4.3.11),

$$f^\nu_{(-\epsilon)} = \tilde f^\nu\left(\frac{z}{1-\epsilon}\right) * \chi_{\delta_\epsilon}.$$

Thus, $f^\nu_{(-\epsilon)}$ is smooth and supported in a compact subset in $D \cap U$. By this, we approximate f^ν by $f^\nu_{(-\epsilon)} \in C^\infty_0(D\cap U)$ in the L^2 norm. Furthermore, by extending $\bar\partial f^\nu$ to be zero outside $D \cap U$ and denoting the extension by $\widetilde{\bar\partial f^\nu}$, we have

$$\bar\partial \tilde f^\nu = \widetilde{\bar\partial f^\nu} \qquad \text{in } L^2(\mathbb{C}^n)$$

in the distribution sense. This follows from (4.3.12) since $f^\nu \in C^\ell(\overline{D})$ and for any $g \in C^\infty_{(p,q+1)}(\mathbb{C}^n)$,

$$(\tilde f^\nu, \vartheta g)_{\mathbb{C}^n} = (\bar\partial f^\nu, g)_D - \int_{bD} \langle \bar\partial\rho \wedge f^\nu, g\rangle dS = (\widetilde{\bar\partial f^\nu}, g)_{\mathbb{C}^n}.$$

Since $\bar\partial$ is a first order differential operator with variable coefficients, using the arguments for (ii), but now applying Friedrichs' lemma (see Appendix D), we have

$$\bar\partial f^\nu_{(-\epsilon)} \to \bar\partial \tilde f^\nu \quad \text{in } L^2(\mathbb{C}^n). \tag{4.3.13}$$

We set

$$f_{(-\epsilon)} = f^\tau + f^\nu_{(-\epsilon)}.$$

It follows $f_{(-\epsilon)} \in \mathcal{D}^\ell_{(p,q)}$ since each coefficient and w^i is in $C^\ell(\overline{D\cap U})$. Also we see that

$$f_{(-\epsilon)} \in \mathcal{D}^\ell_{(p,q)} \text{ and } f_{(-\epsilon)} \to f \quad \text{in } L^2(D).$$

To see that $\bar\partial f_{(-\epsilon)} \to \bar\partial f$ in the $L^2(D)$ norm, using (4.3.13), we find that

$$\bar\partial f_{(-\epsilon)} = \bar\partial f^\tau + \bar\partial f^\nu_{(-\epsilon)} \to \bar\partial f \quad \text{ in } L^2(D) \quad \text{as } \epsilon \to 0.$$

Thus, $\mathcal{D}^\ell_{(p,q)}$ is dense in $\mathrm{Dom}(\bar\partial)$ in the graph norm $f \to \|f\|_\phi + \|\bar\partial f\|_\phi$. This proves (iii).

To finish the proof of the lemma, we assume that $\phi = 0$. For any $f \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$, we use a partition of unity and the same notation as before to regularize f in each small star-shaped neighborhood near the boundary. We regularize the complex tangential and normal part separately by setting

$$\tilde f_{(\epsilon)} = f^\tau_{(\epsilon)} + f^\nu_{(-\epsilon)},$$

where $f^\tau_{(\epsilon)}$ is the regularization defined by (4.3.10) for each coefficient in the complex tangent component and $f^\nu_{(-\epsilon)}$ is the regularization defined by (4.3.11) for each coefficient in the complex normal component. It follows that for sufficiently small $\epsilon > 0$, $f^\nu_{(-\epsilon)}$ has coefficients in $C^\infty_0(D)$ and $f^\tau_{(\epsilon)}$ has coefficients in $C^\infty(\overline{D})$. Thus we see that

$$\tilde f_{(\epsilon)} \in \mathcal{D}^\ell_{(p,q)}, \qquad \tilde f_{(\epsilon)} \to f \quad \text{in} \quad L^2(D).$$

It follows from steps (i), (iii) and Friedrichs's lemma that

$$\bar\partial \tilde f_{(\epsilon)} \to \bar\partial f \qquad \text{in} \quad L^2(D).$$

Also, from steps (i) and (ii), it follows that

$$\vartheta \tilde f_{(\epsilon)} \to \vartheta \tilde f \qquad \text{in} \quad L^2(\mathbb{C}^n),$$

where $\tilde f$ is the extension of f to be zero outside D. This shows that $\mathcal{D}^\ell_{(p,q)}$ is dense in $\mathrm{Dom}(\bar\partial)\cap\mathrm{Dom}(\bar\partial^*)$ in the graph norm $f \to \|f\| + \|\bar\partial f\| + \|\bar\partial^* f\|$. Thus, the lemma is proved for $\phi = 0$. For $\phi \neq 0$ the proof is similar and the density lemma is proved.

Proposition 4.3.3. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary and $\phi \in C^2(\overline{D})$. We have for every $f \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*_\phi)$,*

$$\text{(4.3.14)} \qquad \sum_{I,K}{}' \sum_{j,k} \int_D \frac{\partial^2 \phi}{\partial z_j \, \partial \bar z_k} \; f_{I,jK} \bar f_{I,kK} \; e^{-\phi} dV \le \| \, \overline{\partial} f \, \|^2_\phi + \| \, \overline{\partial}^*_\phi f \, \|^2_\phi \, .$$

Proof. From the assumption that D is pseudoconvex and has C^2 boundary, we have for any $f \in \mathcal{D}^1_{(p,q)}$,

$$\sum_{i,j} \int_{bD} \frac{\partial^2 \rho}{\partial z_i \partial \bar z_j} \; f_{I,iK} \bar f_{I,jK} e^{-\phi} dS \ge 0,$$

since f satisfies (4.2.6′). The proposition follows directly from Proposition 4.3.1 and Lemma 4.3.2.

Theorem 4.3.4 (L^2 existence theorems for $\bar\partial$). *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. For every $f \in L^2_{(p,q)}(D)$, where $0 \le p \le n$, $1 \le q \le n$ with $\overline{\partial} f = 0$, one can find $u \in L^2_{(p,q-1)}(D)$ such that $\overline{\partial} u = f$ and*

$$q \int\limits_D |\, u\,|^2 \, dV \le e\delta^2 \int\limits_D |\, f \,|^2 \, dV, \tag{4.3.15}$$

where $\delta = \sup_{z,z' \in D} |z - z'|$ is the diameter of D.

Proof. We first prove the theorem for D with C^2 boundary. Without loss of generality, we may assume that $0 \in D$. We shall choose $\phi = t|z|^2$ for some positive number t. From Proposition 4.3.3, we have for any $g \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*_\phi)$,

$$tq \int_D |\, g\,|^2 \, e^{-t|z|^2} dV \le \|\, \overline{\partial} g \,\|^2_\phi + \|\, \overline{\partial}^*_\phi g \,\|^2_\phi \,. \tag{4.3.16}$$

Since $\bar\partial^2 = 0$, we have

$$\overline{\mathcal{R}(\bar\partial)} \subset \mathrm{Ker}(\bar\partial) \text{ and } \overline{\mathcal{R}(\overline{\partial}^*_\phi)} \subset \mathrm{Ker}(\overline{\partial}^*_\phi). \tag{4.3.17}$$

It follows from (4.3.16) that for any $g \in \mathrm{Dom}(\overline{\partial}^*_\phi) \cap \mathrm{Ker}(\overline{\partial})$,

$$tq \int_D |\, g\,|^2 \, e^{-t|z|^2} dV \le \|\, \overline{\partial}^*_\phi g \,\|^2_\phi \,. \tag{4.3.18}$$

Using Lemma 4.1.1, we have that $\mathcal{R}(\bar\partial)$ is closed in $L^2_{(p,q)}(D, \phi)$. To show that

$$\mathcal{R}(\bar\partial) = \mathrm{Ker}(\bar\partial), \tag{4.3.19}$$

we claim that for any $f \in L^2_{(p,q)}(D)$ with $\overline{\partial} f = 0$, there exists a constant $C > 0$ such that

$$|\, (f, g)_\phi \,| \le C \,\|\, \overline{\partial}^*_\phi g \,\|_\phi, \quad \text{for all } g \in \mathrm{Dom}(\overline{\partial}^*_\phi). \tag{4.3.20}$$

Using Lemma 4.1.1, $\mathcal{R}(\overline{\partial}^*_\phi)$ is also closed. From (4.1.1), we have

$$L^2_{(p,q)}(D, \phi) = \mathrm{Ker}(\bar\partial) \oplus \mathrm{Ker}(\bar\partial)^\perp = \mathrm{Ker}(\bar\partial) \oplus \mathcal{R}(\overline{\partial}^*_\phi).$$

For any $g_1 \in \mathrm{Dom}(\overline{\partial}^*_\phi) \cap \mathrm{Ker}(\overline{\partial})$, using (4.3.18),

$$|\, (f, g_1)_\phi \,| \le \|\, f \,\|_\phi \|\, g_1 \,\|_\phi \le \frac{1}{\sqrt{tq}} \,\|\, f \,\|_\phi \|\, \overline{\partial}^*_\phi g_1 \,\|_\phi \,.$$

If $g_2 \in \mathrm{Dom}(\overline{\partial}^*_\phi) \cap \mathrm{Ker}(\overline{\partial})^\perp$, we have

$$(f, g_2)_\phi = 0,$$

since $f \in \mathrm{Ker}(\overline{\partial})$. For any $g \in \mathrm{Dom}(\overline{\partial}^*_\phi)$, we write $g = g_1 + g_2$ where $g_1 \in \mathrm{Ker}(\overline{\partial})$ and $g_2 \in \mathrm{Ker}(\overline{\partial})^\perp = \mathcal{R}(\overline{\partial}^*_\phi) \subset \mathrm{Ker}(\overline{\partial}^*_\phi)$. Thus, $g_2 \in \mathrm{Dom}(\overline{\partial}^*_\phi)$ and $\overline{\partial}^*_\phi g_2 = 0$. This implies that $g_1 \in \mathrm{Dom}(\overline{\partial}^*_\phi)$ and $\overline{\partial}^*_\phi\, g = \overline{\partial}^*_\phi\, g_1$. Hence, we have for any $g \in \mathrm{Dom}(\overline{\partial}^*_\phi)$,

$$\begin{aligned} \mid (f,g)_\phi \mid \ &= \mid (f,g_1)_\phi \mid \\ &\le \frac{1}{\sqrt{tq}} \parallel f \parallel_\phi \parallel \overline{\partial}^*_\phi g_1 \parallel_\phi \\ &= \frac{1}{\sqrt{tq}} \parallel f \parallel_\phi \parallel \overline{\partial}^*_\phi g \parallel_\phi . \end{aligned}$$

This proves the claim (4.3.20). Using the Hahn-Banach theorem and the Riesz representation theorem applied to the antilinear functional $\overline{\partial}^*_\phi g \to (f,g)_\phi$, there exists $u \in L^2_{(p,q-1)}(D,\phi)$ such that for every $g \in \mathrm{Dom}(\overline{\partial}^*_\phi)$,

$$(f,g)_\phi = (u, \overline{\partial}^*_\phi g)_\phi,$$

and

$$\parallel u \parallel_\phi \le \frac{1}{\sqrt{tq}} \parallel f \parallel_\phi .$$

This implies that $\overline{\partial} u = f$ in the distribution sense and u satisfies

$$\begin{aligned} q\int_D \mid u \mid^2 dV &\le q e^{t\delta^2} \int_D \mid u \mid^2 e^{-t|z|^2} dV \\ &\le \frac{1}{t} e^{t\delta^2} \int_D \mid f \mid^2 e^{-t|z|^2} dV \\ &\le \frac{1}{t} e^{t\delta^2} \int_D \mid f \mid^2 dV. \end{aligned}$$

Since the function $\frac{1}{t}e^{t\delta^2}$ achieves its minimum when $t = \delta^{-2}$, we have

$$q\int_D \mid u \mid^2 dV \le e\delta^2 \int_D \mid f \mid^2 dV.$$

This proves the theorem when the boundary bD is C^2.

For a general pseudoconvex domain, from Definition 3.4.9, one can exhaust D by a sequence of pseudoconvex domains with C^∞ boundary D_ν. We write

$$D = \bigcup_{\nu=1}^{\infty} D_\nu,$$

where each D_ν is a bounded pseudoconvex domain with C^∞ boundary and $D_\nu \subset D_{\nu+1} \subset D$ for each ν. Let δ_ν denote the diameter for D_ν. On each D_ν, there exists a $u_\nu \in L^2_{(p,q-1)}(D_\nu)$ such that $\overline{\partial} u_\nu = f$ in D_ν and

$$q\int_{D_\nu} \mid u_\nu \mid^2 dV \le e\delta_\nu^2 \int_{D_\nu} \mid f \mid^2 dV \le e\delta^2 \int_D \mid f \mid^2 dV.$$

We can choose a subsequence of u_ν, still denoted by u_ν, such that $u_\nu \rightharpoonup u$ weakly in $L^2_{(p,q-1)}(D)$. Furthermore, u satisfies the estimate

$$q\int\limits_D \mid u \mid^2 dV \le \liminf\ e\delta_\nu^2 \int\limits_{D_\nu} \mid f \mid^2 dV \le e\delta^2 \int\limits_D \mid f \mid^2 dV,$$

and $\overline{\partial} u = f$ in D in the distribution sense. Theorem 4.3.4 is proved.

Theorem 4.3.5. *Let D be a pseudoconvex domain in $\mathbb{C}^n$. For every $f \in L^2_{(p,q)}(D,$ loc), where $0 \le p \le n,\ 1 \le q \le n$ with $\overline{\partial} f = 0$, one can find $u \in L^2_{(p,q-1)}(D, loc)$ such that $\overline{\partial} u = f$.*

Proof. Since D is pseudoconvex, from Definition 3.4.9, there exists a C^∞ strictly plurisubharmonic exhaustion function σ for D. For any $f \in L^2_{(p,q)}(D, \mathrm{loc})$, we can choose a rapidly increasing convex function $\eta(t),\ t \in \mathbb{R}$ such that $\eta(t) = 0$ when $t \le 0$ and $f \in L^2_{(p,q)}(D, \eta(\sigma))$. Let $D_\nu = \{z \in D |\ \sigma(z) < \nu\}$, then

$$D = \bigcup_{\nu=1}^{\infty} D_\nu,$$

where each D_ν is a bounded pseudoconvex domain with C^∞ boundary and $D_\nu \subset D_{\nu+1} \subset D$ for each ν. Since $\eta(\sigma)$ is plurisubharmonic, the function $\phi = \eta(\sigma) + |z|^2$ is strictly plurisubharmonic with

$$\sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(z) a_j \bar{a}_k \ge |a|^2$$

for all $(a_1, \cdots, a_n) \in \mathbb{C}^n$ and all $z \in D$. Applying Proposition 4.3.3 to each D_ν we have for any $g \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*_\phi)$,

$$\begin{aligned} q \parallel g \parallel^2_{\phi(D_\nu)} \ &\le \int\limits_{D_\nu} \sum_{I,K}{}' \sum_{j,k} \frac{\partial^2 \phi}{\partial z_j\ \partial \bar{z}_k}\ g_{I,jK} \overline{g}_{I,kK}\ e^{-\phi} dV \\ &\le \parallel \overline{\partial} g \parallel^2_{\phi(D_\nu)} + \parallel \overline{\partial}^*_\phi g \parallel^2_{\phi(D_\nu)}\ . \end{aligned}$$

Repeating the same argument as in Theorem 4.3.4, there exists a $u_\nu \in L^2_{(p,q-1)}(D_\nu, \phi)$ such that $\bar{\partial} u_\nu = f$ in D_ν and

$$q\int\limits_{D_\nu} |u_\nu|^2 e^{-\phi} dV \le \int\limits_{D_\nu} |f|^2 e^{-\phi} dV \le \int\limits_D |f|^2 e^{-\phi} dV < \infty.$$

Taking a weak limit u of u_ν as $\nu \to \infty$, we have shown that there exists u such that $\bar{\partial} u = f$ in D and

$$q\int\limits_D |u|^2 e^{-\phi} dV \le \int\limits_D |f|^2 e^{-\phi} dV.$$

This proves the theorem.

4.4 L^2 Existence Theorems for the $\bar\partial$-Neumann Operator

We shall use the L^2 existence theorems for $\bar\partial$ in Section 4.3 to establish the existence theorem for the $\bar\partial$-Neumann operator on any bounded pseudoconvex domain D in $\mathbb{C}^n$. Using Proposition 4.2.3, the operator $\square_{(p,q)}$ is closed and self-adjoint. Thus, the kernel of $\square_{(p,q)}$, denoted by $\mathrm{Ker}(\square_{(p,q)})$, is closed. From the Hilbert space theory, we have the following weak Hodge decomposition

$$L^2_{(p,q)}(D) = \overline{\mathcal{R}(\square_{(p,q)})} \oplus \mathrm{Ker}(\square_{(p,q)}), \tag{4.4.1}$$

where $\mathcal{R}(\square_{(p,q)})$ denotes the range of $\square_{(p,q)}$. We shall show that $\mathcal{R}(\square_{(p,q)})$ is closed and $\mathrm{Ker}(\square_{(p,q)}) = \{0\}$. We claim that

$$\mathrm{Ker}(\square_{(p,q)}) = \mathrm{Ker}(\bar\partial) \cap \mathrm{Ker}(\bar\partial^*) = \{0\} \quad \text{for } q \geq 1. \tag{4.4.2}$$

For any $\alpha \in \mathrm{Ker}(\square_{(p,q)})$, we have $\alpha \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$ and

$$(\alpha, \square_{(p,q)}\alpha) = \|\bar\partial\alpha\|^2 + \|\bar\partial^*\alpha\|^2 = 0.$$

Thus, $\mathrm{Ker}(\square_{(p,q)}) \subset \mathrm{Ker}(\bar\partial) \cap \mathrm{Ker}(\bar\partial^*)$. On the other hand, if $\alpha \in \mathrm{Ker}(\bar\partial) \cap \mathrm{Ker}(\bar\partial^*)$, then $\alpha \in \mathrm{Dom}(\square_{(p,q)})$ and $\square_{(p,q)}\alpha = 0$. Thus, $\mathrm{Ker}(\square_{(p,q)}) \supset \mathrm{Ker}(\bar\partial) \cap \mathrm{Ker}(\bar\partial^*)$ and the first equality in (4.4.2) is proved. To see that $\mathrm{Ker}(\bar\partial) \cap \mathrm{Ker}(\bar\partial^*) = \{0\}$, we note that if $\alpha \in \mathrm{Ker}(\bar\partial)$, from Theorem 4.3.4, there exists $u \in L^2_{(p,q-1)}(D)$ such that $\alpha = \bar\partial u$. If α is also in $\mathrm{Ker}(\bar\partial^*)$, we have

$$0 = (\bar\partial^*\bar\partial u, u) = \|\bar\partial u\|^2$$

and $\alpha = 0$. This proves (4.4.2).

We shall show that $\mathcal{R}(\square_{(p,q)})$ is closed and the following L^2 existence theorem holds for the $\bar\partial$-Neumann operator.

Theorem 4.4.1. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. For each $0 \leq p \leq n$, $1 \leq q \leq n$, there exists a bounded operator $N_{(p,q)} : L^2_{(p,q)}(D) \to L^2_{(p,q)}(D)$ such that*

(1) *$\mathcal{R}(N_{(p,q)}) \subset Dom(\square_{(p,q)})$,*
   $N_{(p,q)}\square_{(p,q)} = \square_{(p,q)}N_{(p,q)} = I$ on $Dom(\square_{(p,q)})$.
(2) *For any $f \in L^2_{(p,q)}(D)$, $f = \bar\partial\bar\partial^* N_{(p,q)} f \oplus \bar\partial^*\bar\partial N_{(p,q)} f$.*
(3) *$\bar\partial N_{(p,q)} = N_{(p,q+1)}\bar\partial$ on $Dom(\bar\partial)$, $1 \leq q \leq n-1$.*
(4) *$\bar\partial^* N_{(p,q)} = N_{(p,q-1)}\bar\partial^*$ on $Dom(\bar\partial^*)$, $2 \leq q \leq n$.*
(5) *Let δ be the diameter of D. The following estimates hold for any $f \in L^2_{(p,q)}(D)$:*

$$\| N_{(p,q)} f \| \leq \frac{e\delta^2}{q} \| f \|,$$

$$\| \bar\partial N_{(p,q)} f \| \leq \sqrt{\frac{e\delta^2}{q}} \| f \|,$$

$$\| \bar\partial^* N_{(p,q)} f \| \leq \sqrt{\frac{e\delta^2}{q}} \| f \| .$$

Proof. Using Theorem 4.3.4, for any $f \in L^2_{(p,q)}(D)$, $q > 0$ with $\bar{\partial} f = 0$, there exists $u \in L^2_{(p,q-1)}(D)$ such that $\bar{\partial} u = f$ and u satisfies the estimate (4.3.15). Thus, $\mathcal{R}(\bar{\partial})$ is closed in every degree and is equal to $\mathrm{Ker}(\bar{\partial})$. It follows from Lemma 4.1.1 that $\mathcal{R}(\bar{\partial}^*)$ is closed also for every q, and we have the following orthogonal decomposition:

$$L^2_{(p,q)}(D) = \mathrm{Ker}(\bar{\partial}) \oplus \mathcal{R}(\bar{\partial}^*) = \mathcal{R}(\bar{\partial}) \oplus \mathcal{R}(\bar{\partial}^*). \tag{4.4.3}$$

For every $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$, we have

$$f = f_1 \oplus f_2$$

where $f_1 \in \mathcal{R}(\bar{\partial})$ and $f_2 \in \mathcal{R}(\bar{\partial}^*)$. Also, we have $f_1, f_2 \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$, $\bar{\partial} f = \bar{\partial} f_2$, $\bar{\partial}^* f = \bar{\partial}^* f_1$.

Using Theorem 4.3.4 and (4.1.3) and (4.1.4) in Lemma 4.1.1, we have the following estimates:

$$\|f_1\|^2 \le c_q \|\bar{\partial}^* f_1\|^2 \tag{4.4.4}$$

and

$$\|f_2\|^2 \le c_{q+1} \|\bar{\partial} f_2\|^2, \tag{4.4.5}$$

where the constant $c_q = e\delta^2/q$. Combining (4.4.4), (4.4.5) we have

$$\|f\|^2 = \|f_1\|^2 + \|f_2\|^2 \le c_q(\|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2) \tag{4.4.6}$$

for every $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$. Thus for any $f \in \mathrm{Dom}(\Box_{(p,q)})$, we have

$$\begin{aligned} \|f\|^2 &\le c_q[(\bar{\partial} f, \bar{\partial} f) + (\bar{\partial}^* f, \bar{\partial}^* f)] \\ &= c_q[(\bar{\partial}^* \bar{\partial} f, f) + (\bar{\partial} \bar{\partial}^* f, f)] \\ &= c_q(\Box_{(p,q)} f, f) \\ &\le c_q \|\Box_{(p,q)} f\| \|f\|. \end{aligned} \tag{4.4.7}$$

Hence,

$$\|f\| \le c_q \|\Box_{(p,q)} f\|. \tag{4.4.8}$$

It follows from Lemma 4.1.1 (since $\Box_{(p,q)}$ is a closed operator from Proposition 4.2.3) that the range of $\Box_{(p,q)}$ is closed. We have the strong Hodge decomposition

$$L^2_{(p,q)}(D) = \mathcal{R}(\Box_{(p,q)}) = \bar{\partial}\bar{\partial}^*(\mathrm{Dom}(\Box_{(p,q)})) \oplus \bar{\partial}^*\bar{\partial}(\mathrm{Dom}(\Box_{(p,q)})).$$

Also, from (4.4.8), $\Box_{(p,q)}$ is one to one and the range of $\Box_{(p,q)}$ is the whole space $L^2_{(p,q)}(D)$. There exists a unique inverse $N_{(p,q)} : L^2_{(p,q)}(D) \to \mathrm{Dom}(\Box_{(p,q)})$ such that $N_{(p,q)}\Box = \Box N_{(p,q)} = I$. Using (4.4.8), $N_{(p,q)}$ is bounded. The assertions (1) and (2) in Theorem 4.4.1 have been established.

To show that $\bar\partial N_{(p,q)} = N_{(p,q+1)}\bar\partial$ on $\mathrm{Dom}(\bar\partial)$, we note that from (2), $\bar\partial f = \bar\partial\bar\partial^*\bar\partial N_{(p,q)}f$ for $f \in \mathrm{Dom}(\bar\partial)$. It follows that

$$\begin{aligned}N_{(p,q+1)}\bar\partial f &= N_{(p,q+1)}\bar\partial\bar\partial^*\bar\partial N_{(p,q)}f\\ &= N_{(p,q+1)}(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\bar\partial N_{(p,q)}f\\ &= \bar\partial N_{(p,q)}f.\end{aligned}$$

If $2 \le q \le n$, one can prove $N_{(p,q-1)}\bar\partial^* = \bar\partial^* N_{(p,q)}$ on $\mathrm{Dom}(\bar\partial^*)$ similarly.

To prove (5), we see from (4.4.8) that

$$\| N_{(p,q)}f \| \le \frac{e\delta^2}{q} \| f \| \quad \text{for } f \in L^2_{(p,q)}(D).$$

Using (2), we have

$$\begin{aligned}&(\bar\partial N_{(p,q)}f, \bar\partial N_{(p,q)}f) + (\bar\partial^* N_{(p,q)}f, \bar\partial^* N_{(p,q)}f)\\ &= ((\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)N_{(p,q)}f, N_{(p,q)}f)\\ &= (f, N_{(p,q)}f)\\ &\le \|f\|\|N_{(p,q)}f\|\\ &\le \frac{e\delta^2}{q}\|f\|^2.\end{aligned}$$

This proves (5). The proof of Theorem 4.4.1 is complete.

Corollary 4.4.2. *Let D and $N_{(p,q)}$ be the same as in Theorem 4.4.1, where $0 \le p \le n,\ 1 \le q \le n$. For any $\alpha \in L^2_{(p,q)}(D)$ such that $\bar\partial\alpha = 0$, the $(p, q-1)$-form*

$$u = \bar\partial^* N_{(p,q)}\alpha \tag{4.4.9}$$

satisfies the equation $\bar\partial u = \alpha$ and the estimate

$$\|u\|^2 \le \frac{e\delta^2}{q}\|\alpha\|^2.$$

The solution u is called the canonical solution to the equation (4.0.1) and it is the unique solution which is orthogonal to $Ker(\bar\partial)$.

Proof. We have from (2) of Theorem 4.4.1,

$$\alpha = \bar\partial\bar\partial^* N_{(p,q)}\alpha + \bar\partial^*\bar\partial N_{(p,q)}\alpha.$$

Using (3) in Theorem 4.4.1, we have

$$\bar\partial N_{(p,q)}\alpha = N_{(p,q+1)}\bar\partial\alpha = 0,$$

since $\bar\partial\alpha = 0$. Thus we have $\alpha = \bar\partial\bar\partial^* N_{(p,q)}\alpha$. The estimate of u follows from (5) in Theorem 4.4.1. If v is another solution orthogonal to $\mathrm{Ker}(\bar\partial)$, then $u - v \in \mathrm{Ker}(\bar\partial) \cap \mathrm{Ker}(\bar\partial^*) = 0$. This proves the uniqueness of the canonical solution. Corollary 4.4.2 is proved.

The existence of the $\bar\partial$-Neumann operator for $q = 0$, $N_{(p,0)}$, is also important. Let $\Box_{(p,0)} = \bar\partial^*\bar\partial$ on $L^2_{(p,0)}(D)$. We define

$$\mathcal{H}_{(p,0)}(D) = \{f \in L^2_{(p,0)}(D) |\ \bar\partial f = 0\}.$$

$\mathcal{H}_{(p,0)}(D)$ is a closed subspace of $L^2_{(p,0)}$ since $\bar\partial$ is a closed operator. Let $H_{(p,0)}$ denote the projection from $L^2_{(p,0)}(D)$ onto the set $\mathcal{H}_{(p,0)}(D)$. We have the following theorem.

Theorem 4.4.3. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. There exists an operator $N_{(p,0)} : L^2_{(p,0)}(D) \to L^2_{(p,0)}(D)$ such that*

(1) $\mathcal{R}(N_{(p,0)}) \subset Dom(\Box_{(p,0)})$,
$N_{(p,0)}\Box_{(p,0)} = \Box_{(p,0)}N_{(p,0)} = I - H_{(p,0)}$.
(2) *For every* $f \in L^2_{(p,0)}(D)$, $f = \bar\partial^*\bar\partial N_{(p,0)} f \oplus H_{(p,0)} f$.
(3) $\bar\partial N_{(p,0)} = N_{(p,1)}\bar\partial$ *on* $Dom(\bar\partial)$,
$\bar\partial^* N_{(p,1)} = N_{(p,0)}\bar\partial^*$ *on* $Dom\ (\bar\partial^*)$.
(4) $N_{(p,0)} = \bar\partial^* N^2_{(p,1)}\bar\partial$.
(5) *Let δ be the diameter of D. For any $f \in L^2_{(p,0)}(D)$,*

$$\| N_{(p,0)} f \| \leq e\delta^2 \| f \|,$$

$$\| \bar\partial N_{(p,0)} f \| \leq \sqrt{e}\delta \| f \| .$$

Proof. Note that $\mathcal{H}_{(p,0)}(D) = \mathcal{H}_{(p,0)} = \mathrm{Ker}(\Box_{(p,0)})$. We first show that $\Box_{(p,0)}$ is bounded away from zero on $(\mathcal{H}_{(p,0)})^\perp$. Since $\bar\partial$ has closed range in every degree, $\bar\partial^*$ also has closed range by Lemma 4.1.1. If $f \in \mathrm{Dom}(\Box_{(p,0)}) \cap (\mathcal{H}_{(p,0)})^\perp$, we have $f \perp \mathrm{Ker}(\bar\partial)$ and $f \in \mathcal{R}(\bar\partial^*)$.

Let $\alpha = \bar\partial f$, then $\alpha \in L^2_{(p,1)}(D)$ since $f \in \mathrm{Dom}(\Box_{(p,0)})$. Using (4) in Theorem 4.4.1, we have that $\phi \equiv \bar\partial^* N_{(p,1)}\alpha$ is the unique solution satisfying $\bar\partial\phi = \alpha$ and $\phi \perp \mathrm{Ker}(\bar\partial)$. Thus, $\phi = f$. Applying Corollary 4.4.2, we have

$$\|f\|^2 \leq c\|\alpha\|^2 = c\|\bar\partial f\|^2 = c(\Box_{(p,0)} f, f) \leq c\|\Box_{(p,0)} f\|\|f\|,$$

where $c = e\delta^2$. This implies that

$$\|f\| \leq e\delta^2\|\Box_{(p,0)} f\| \quad \text{for } f \in \mathrm{Dom}(\Box_{(p,0)}) \cap (\mathcal{H}_{(p,0)})^\perp.$$

Using Lemma 4.1.1, we see that $\Box_{(p,0)}$ has closed range. From (4.1.1), the following strong Hodge decomposition holds:

$$L^2_{(p,0)}(D) = \mathcal{R}(\Box_{(p,0)}) \oplus \mathcal{H}_{(p,0)} = \bar\partial^*\bar\partial(\mathrm{Dom}(\Box_{(p,0)})) \oplus \mathcal{H}_{(p,0)}.$$

For any $\alpha \in \mathcal{R}(\Box_{(p,0)})$, there is a unique $N_{(p,0)}\alpha \perp \mathcal{H}_{(p,0)}$ such that $\Box_{(p,0)}N_{(p,0)}\alpha = \alpha$. Extending $N_{(p,0)}$ to $L^2_{(p,0)}(D)$ by requiring $N_{(p,0)}H_{(p,0)} = 0$, $N_{(p,0)}$ satisfies (1) and (2) in Theorem 4.4.3. That $N_{(p,0)}$ commutes with $\bar\partial$ and $\bar\partial^*$ is proved exactly as before, and we omit the details. If $f \in \mathrm{Dom}(\bar\partial)$, it follows that

$$N_{(p,0)} f = (I - H_{(p,0)})N_{(p,0)} f = N_{(p,0)}(\bar\partial^*\bar\partial)N_{(p,0)} f = \bar\partial^* N^2_{(p,1)}\bar\partial f. \tag{4.4.10}$$

Thus, (4) holds on $\mathrm{Dom}(\bar\partial)$. In fact, we can show that (4.4.10) holds on all of $L^2_{(p,0)}(D)$. From (5) in Theorem 4.4.1, we have

$$\| N_{(p,1)}\alpha \| \leq e\delta^2\|\alpha\|.$$

We obtain, for any $f \in C^\infty_{(p,0)}(\overline{D})$,

$$
\begin{aligned}
\|N_{(p,0)}f\|^2 &= (\bar\partial\bar\partial^* N^2_{(p,1)}\bar\partial f, N^2_{(p,1)}\bar\partial f) \\
&= (N_{(p,1)}\bar\partial f, N^2_{(p,1)}\bar\partial f) \\
&\le \|N_{(p,1)}\bar\partial f\|\|N^2_{(p,1)}\bar\partial f\| \\
&\le e\delta^2\|N_{(p,1)}\bar\partial f\|^2.
\end{aligned} \tag{4.4.11}
$$

On the other hand, we have

$$
\begin{aligned}
(N_{(p,1)}\bar\partial f, N_{(p,1)}\bar\partial f) &= (N^2_{(p,1)}\bar\partial f, \bar\partial f) \\
&= (\bar\partial^* N^2_{(p,1)}\bar\partial f, f) \\
&\le \|N_{(p,0)}f\|\|f\|.
\end{aligned} \tag{4.4.12}
$$

Combining (4.4.11) and (4.4.12), we have proved that

$$\|N_{(p,0)}f\| \le e\delta^2\|f\|.$$

Thus, $N_{(p,0)}$ defined by (4.4.10) is bounded on all smooth $(p,0)$-forms and it can be extended to $L^2_{(p,0)}(D)$ as a bounded operator. This proves (4). It follows from (1) that

$$
\begin{aligned}
\|\bar\partial N_{(p,0)}f\|^2 &= (\bar\partial^*\bar\partial N_{(p,0)}f, N_{(p,0)}f) \\
&= ((I - H_{(p,0)})f, N_{(p,0)}f) \\
&\le \|f\|\|N_{(p,0)}f\| \le e\delta^2\|f\|^2.
\end{aligned}
$$

Theorem 4.4.3 is proved.

Corollary 4.4.4. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. For any $f \in L^2_{(p,0)}(D)$, we have*

$$H_{(p,0)}f = f - \bar\partial^* N_{(p,1)}\bar\partial f. \tag{4.4.13}$$

Proof. For any $f \in \mathrm{Dom}(\bar\partial)$, this follows from (2) in Theorem 4.4.3, since

$$H_{(p,0)} = I - \bar\partial^*\bar\partial N_{(p,0)} = I - \bar\partial^* N_{(p,1)}\bar\partial.$$

From (3) in Theorem 4.4.3, $\bar\partial^* N_{(p,1)}\bar\partial = \bar\partial^*\bar\partial N_{(p,0)} = I - H_{(p,0)}$ is a bounded operator on $\mathrm{Dom}(\bar\partial)$, it can be extended to any $f \in L^2_{(p,0)}(D)$ by continuity. This proves the corollary.

Corollary 4.4.4 is especially important when $p = 0$, so we state it below as a theorem. Let P denote the projection onto the closed subspace of holomorphic square integrable functions

$$\mathcal{H}(D) = \{f \in L^2(D) \mid \bar\partial f = 0\}.$$

P is called the Bergman projection and $\mathcal{H}(D)$ is called the Bergman space.

Theorem 4.4.5 (Bergman projection). *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. For any $f \in L^2(D)$, the Bergman projection Pf is given by*

$$Pf = f - \bar\partial^* N_{(0,1)} \bar\partial f. \tag{4.4.14}$$

4.5 Pseudoconvexity and the Levi Problem

In this section we show that pseudoconvex domains are domains of holomorphy. We first examine the solvability of $\bar\partial$ in the $C^\infty(D)$ category.

Theorem 4.5.1. *Let D be a pseudoconvex domain in $\mathbb{C}^n$. For every $f \in C^\infty_{(p,q)}(D)$, where $0 \le p \le n,\ 1 \le q \le n$ with $\overline\partial f = 0$, one can find $u \in C^\infty_{(p,q-1)}(D)$ such that $\overline\partial u = f$.*

Proof. Let $f \in C^\infty_{(p,q)}(D)$. From Theorem 4.3.5, there exists a strictly plurisubharmonic function $\phi \in C^\infty(D)$ such that f is in $L^2_{(p,q)}(D, \phi)$ and there exists $v \in L^2_{(p,q-1)}(D,\, \phi)$ with $\bar\partial v = f$ and

$$\| v \|_\phi \le \| f \|_\phi\ .$$

Repeating the same arguments as in Section 4.4, there exists a weighted $\bar\partial$-Neumann operator N_ϕ such that for any $f \in L^2_{(p,q)}(D, \phi)$, we have

$$f = \bar\partial \bar\partial^*_\phi N_\phi f + \bar\partial^*_\phi \bar\partial N_\phi f.$$

Since $\bar\partial f = 0$, we have that $f = \bar\partial\bar\partial^*_\phi N_\phi f$. Setting $u = \bar\partial^*_\phi N_\phi f$, we shall show that $u \in C^\infty_{(p,q-1)}(D)$. Since $\bar\partial^*_\phi u = \vartheta u + A_0 u = 0$ for some zeroth order operator A_0, we have

$$\begin{cases} \bar\partial u = f, \\ \vartheta u = -A_0 u \in L^2(D, \mathrm{loc}). \end{cases}$$

However, $\bar\partial \oplus \vartheta$ is an elliptic system. By this we mean that for any $\alpha \in C^\infty_{(p,q)}(\overline{D})$ such that α has compact support in D, the following inequality holds:

$$\| \alpha \|_1 \le C(\| \bar\partial \alpha \| + \| \vartheta\alpha \| + \| \alpha \|). \tag{4.5.1}$$

Inequality (4.5.1) is called Gårding's inequality. To prove (4.5.1), we use Proposition 4.2.4 and (4.2.8) to get

$$4(\| \bar\partial\alpha \|^2 + \| \vartheta\alpha \|^2) = 4(\Box\alpha, \alpha) = (-\triangle\alpha, \alpha) = \| \nabla\alpha \|^2, \tag{4.5.2}$$

where $\triangle$ is the real Laplacian and ∇ is the gradient, both act on α componentwise. When $q = 0$, (4.5.2) also holds since $\Box = \vartheta\bar\partial$ is also equal to $-\frac{1}{4}\triangle$. Thus (4.5.1) holds for any compactly supported smooth form α. Let $\tilde u = \zeta u$ where $\zeta \in C^\infty_0(D)$ and define $u_\epsilon = \tilde u * \chi_\epsilon$ where χ and χ_ϵ are the same as in Lemma 4.3.2. It follows that $\| u_\epsilon \| \le \| \tilde{u} \|$, $\bar\partial u_\epsilon = \bar\partial \tilde{u} * \chi_\epsilon$ and $\vartheta u_\epsilon = \vartheta \tilde{u} * \chi_\epsilon$. Substituting u_ϵ into (4.5.1), we have

$$\begin{aligned} \| u_\epsilon \|_1 &\le C(\| \bar\partial u_\epsilon \| + \| \vartheta u_\epsilon \| + \| u_\epsilon \|) \\ &\le C(\| \bar\partial \tilde{u} \| + \| \vartheta \tilde{u} \| + \| \tilde{u} \|). \end{aligned} \tag{4.5.3}$$

Thus, u_ϵ converges in $W^1(D)$ to $\tilde{u}$, and we have $u \in W^1(D, \text{loc})$. Continuing this process to $D^k \tilde{u}$ where D^k is any kth order differential operator with constant coefficients, we conclude by induction that $u \in W^{k+1}(D, \text{loc})$ for any $k \in \mathbb{N}$. The theorem follows from the Sobolev embedding theorem (see Theorem A.7 in the Appendix).

The following theorem unifies domains of holomorphy, pseudoconvexity and existence theorems for the Cauchy-Riemann equations:

Theorem 4.5.2. *Let D be a domain in $\mathbb{C}^n$, $n \ge 1$. Then the following conditions are equivalent:*

(1) *D is pseudoconvex.*
(2) *D is a domain of holomorphy.*
(3) *For every $f \in C^\infty_{(p,q)}(D)$, where $0 \le p \le n$, $1 \le q \le n$ with $\overline\partial f = 0$, one can find $u \in C^\infty_{(p,q-1)}(D)$ such that $\overline\partial u = f$.*

Proof. We have already proved in Theorem 3.5.5 that (2) implies (1). That (1) implies (3) follows from Theorem 4.5.1.

To prove that (3) implies (2), we use an induction argument. This is obviously true for $n = 1$, since any open set in $\mathbb{C}$ is a domain of holomorphy. We shall show that if (3) implies (2) for $n - 1$, then it is true for n.

To prove this, for any $z_0 \in bD$, we need to construct a holomorphic function in D which cannot be extended holomorphically across any neighborhood containing z_0. It suffices to prove this in a dense subset of bD.

Let z_0 be a boundary point such that there exists a complex $(n-1)$-dimensional hyperplane Σ and $z_0 \in b(\Sigma \cap D)$. Such boundary points are dense in bD. To see this, we note that for almost every boundary point z_0, one can find a ball $B \subset D$ such that $z_0 \in b(B \cap D)$. At such a z_0, we obviously can find a complex hyperplane Σ passing through the center of the ball and z_0. It is easy to see that $z_0 \in b(\Sigma \cap D)$. By a linear transformation, we may assume that $z_0 = 0$ and $\Sigma_0 = D \cap \{z_n = 0\}$ is nonempty.

We shall show that on Σ_0, (3) is fulfilled. Let f be a smooth $\bar\partial$-closed (p, q)-form on Σ_0, where $0 \le p \le n-1$, $1 \le q \le n-1$. We claim that f can be extended to be a smooth $\bar\partial$-closed form in D. We first extend f to $\tilde{f}$ in D such that $\tilde{f} \in C^\infty(D)$ and $\tilde{f}(z) = \tilde{f}(z_1, \cdots, z_n) = f(z', 0)$ in an open neighborhood of Σ_0. This can be done as follows: Let $\pi : D \to \mathbb{C}^{n-1}$ be the projection such that $\pi(z) = (z_1, \cdots, z_{n-1}, 0)$. Then the set $D_0 = D \setminus \pi^{-1}(\Sigma_0)$ is a closed subset of D. Since Σ_0 and D_0 are closed (with respect to D) disjoint subsets of D, using Urysohn's lemma, we see that there exists a function $\eta \in C^\infty(D)$ such that $\eta = 1$ in a neighborhood of Σ_0 and $\eta = 0$ in a neighborhood of D_0. Then we can choose our $\tilde{f} = \eta \pi^* f(z')$, where π^* is the pull-back of the form f. Let

$$F(z) = \tilde{f}(z) - z_n u(z)$$

where $u(z)$ is chosen such that

$$\bar{\partial}u(z) = \frac{\bar{\partial}\tilde{f}}{z_n}. \tag{4.5.4}$$

We note that the right-hand side of (4.5.4) is $\bar{\partial}$-closed and is in $C^\infty_{(p,q+1)}(D)$, since $\tilde{f}$ is $\bar{\partial}$-closed in a neighborhood of Σ_0. Thus, from (3), there exists $u \in C^\infty_{(p,q)}(D)$ satisfying the Equation (4.5.4). This implies that F is $\bar{\partial}$-closed on D and $F = f$ on Σ_0. Thus, any $\bar{\partial}$-closed form f on Σ_0 can be extended to a $\bar{\partial}$ closed form F on D. This is also true for $q = 0$.

From (3), we can find $U(z) \in C^\infty_{(p,q-1)}(D)$ such that $\bar{\partial}U = F$ in D. Restricting U to Σ_0, we have shown that (3) is fulfilled on Σ_0.

By the induction hypothesis, Σ_0 is a domain of holomorphy. Hence, there exists a holomorphic function $f(z') = f(z_1, \cdots, z_{n-1})$ such that f is singular at 0. Since $\bar{\partial}f = 0$ in Σ_0, repeating the same argument above for $q = 0$, there exists F in D such that $\bar{\partial}F = 0$ in D and $F = f$ on Σ_0. $F(z)$ is holomorphic in D and is equal to $f(z', 0)$ on Σ_0. Thus, it is a holomorphic function in D which cannot be extended across 0. This shows that D is a domain of holomorphy. Thus, (3) implies (2) and the theorem is proved.

Theorem 4.5.2 solves the Levi problem on pseudoconvex domains in $\mathbb{C}^n$.

NOTES

The $\bar{\partial}$-Neumann problem was suggested by P. R. Garabedian and D. C. Spencer [GaSp 1] to study the Cauchy-Riemann equations. This approach generalizes the Hodge-de Rham theorem from compact manifolds to complex manifolds with boundaries. The basic *a priori* estimates were first proved for $(0,1)$-forms by C. B. Morrey [Mor 1]. J. J. Kohn [Koh 1] has derived the general estimates and has proved the boundary regularity for the $\bar{\partial}$-Neumann operator on strongly pseudoconvex manifolds. This latter result is actually required in Kohn's approach to the $\bar{\partial}$-Neumann problem which will be discussed in Chapter 5. The use of weighted L^2 estimates which depend on a parameter, combined with the basic estimates of Morrey and Kohn, to study the overdetermined system was introduced by L. Hörmander [Hör 3] in order to bypass the boundary regularity problem. Related arguments are used in A. Andreotti and E. Vesentini [AnVe 1].

Lemma 4.1.1 and much of the material on L^2 existence theorems presented in Section 4.3 are taken from the paper of L. Hörmander [Hör 3]. Theorem 4.3.4 is a special case of Theorem 2.2.3 in [Hör 3] where the precise bounds are given. The density lemma 4.3.2 was also proved in [Hör 3] in a much more general setting. Using three different weight functions which are singular near the boundary, L. Hörmander [Hör 9] gives another approach to L^2 existence theorems. The canonical solution formula given by (4.4.9) and the Bergman projection formula (4.4.14) are due to J. J. Kohn [Koh 1]. The proof of Theorem 4.5.2 is due to K. Oka [Oka 2] and H. Bremermann [Bre 1], and F. Norguet [Nor 1] and our presentation follows that of Section 4.2 in [Hör 9].

CHAPTER 5

THE $\bar{\partial}$-NEUMANN PROBLEM ON STRONGLY PSEUDOCONVEX MANIFOLDS

In this chapter we study boundary regularity for the $\bar{\partial}$-Neumann problem on a strongly pseudoconvex domain Ω. Let ρ be a C^2 defining function for Ω. The $\bar{\partial}$-Neumann problem for (p,q)-forms, $0 \le p \le n,\ 1 \le q \le n$, in Ω is the boundary value problem:

$$\tag{5.0.1} \begin{cases} \Box u = f & \text{in } \Omega, \\ \bar{\partial}\rho \vee u = 0 & \text{on } b\Omega, \\ \bar{\partial}\rho \vee \bar{\partial} u = 0 & \text{on } b\Omega, \end{cases}$$

where u, f are (p,q)-forms, $\Box = \bar{\partial}\vartheta + \vartheta\bar{\partial}$, $\vee$ denotes the interior product of forms.

On any bounded pseudoconvex domain Ω in $\mathbb{C}^n$, we have derived the following estimates: for any $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$,

$$\tag{5.0.2} \| f \|_{\Omega}^2 \le \frac{e\delta^2}{q}(\| \bar{\partial} f \|_{\Omega}^2 + \| \bar{\partial}^* f \|_{\Omega}^2),$$

where δ is the diameter of Ω (see (4.4.6)). It follows from Theorem 4.4.1 that the $\bar{\partial}$-Neumann operator $N_{(p,q)}$ exists in Ω, which solves (5.0.1) in the Hilbert space sense.

The $\Box$ operator is elliptic in the interior, but the boundary conditions are not coercive except when $q = n$. It only satisfies Gårding's inequality in the interior, but not near the boundary. However, under the assumption of strong pseudoconvexity, we will show that it satisfies subelliptic 1/2-estimates near the boundary.

In Section 5.1, we prove that the following subelliptic 1/2-estimate holds on a strongly pseudoconvex domain Ω in $\mathbb{C}^n$: for any $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$,

$$\tag{5.0.3} \| f \|_{\frac{1}{2}(\Omega)}^2 \le C(\| \bar{\partial} f \|_{\Omega}^2 + \| \bar{\partial}^* f \|_{\Omega}^2),$$

where $\| \ \|_{\frac{1}{2}(\Omega)}$ is the Sobolev norm in the Sobolev space $W^{1/2}_{(p,q)}(\Omega)$.

The regularity of the $\bar{\partial}$-Neumann operator in other Sobolev spaces when the boundary $b\Omega$ is C^∞ is discussed in Section 5.2. In Section 5.3, we discus the existence and regularity of the $\bar{\partial}$-Neumann operator in an open subset of a complex manifold with a Hermitian metric. In particular, a solution to the Levi problem on strongly pseudoconvex manifolds is obtained using the $\bar{\partial}$-Neumann operator. Finally, the Newlander-Nirenberg theorem is proved by using the solution for $\bar{\partial}$ on an almost complex manifold in the last section.

5.1 Subelliptic Estimates for the $\bar\partial$-Neumann Operator

In this section we shall derive the subelliptic 1/2-estimate for the $\bar\partial$-Neumann operator when Ω is strongly pseudoconvex with C^2 boundary. We shall use N instead of $N_{(p,q)}$ to simplify the notation. We also use $L^2(\Omega)$, $W^s(\Omega)$ and $W^s(\Omega, \mathrm{loc})$ to denote the spaces $L^2_{(p,q)}(\Omega)$, $W^s_{(p,q)}(\Omega)$ and $W^s_{(p,q)}(\Omega, \mathrm{loc})$ respectively, where $W^s(\Omega)$ is the Sobolev space, $s \in \mathbb{R}$. (See Appendix A for its definition and basic properties.) The norm in W^s is denoted by $\| \ \|_{s(\Omega)}$.

We first observe that the first order system

$$\bar\partial \oplus \vartheta : \ C^\infty_{(p,q)}(\overline{\Omega}) \to \ C^\infty_{(p,q+1)}(\overline{\Omega}) \oplus \ C^\infty_{(p,q-1)}(\overline{\Omega})$$

is elliptic in the interior. This means that we have Gårding's inequality in the interior.

Proposition 5.1.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $\eta \in C_0^\infty(\Omega)$. For any (p,q)-form $f \in L^2_{(p,q)}(\Omega)$ such that $\bar\partial f \in L^2_{(p,q+1)}(\Omega)$ and $\vartheta f \in L^2_{(p,q-1)}(\Omega)$, where $0 \le p \le n$ and $0 \le q \le n$, we have the following estimates:*

$$\| \eta f \|^2_{1(\Omega)} \le C(\| \bar\partial f \|^2_\Omega + \| \vartheta f \|^2_\Omega + \| f \|^2_\Omega), \tag{5.1.1}$$

where C is a constant depending only on η but not on f.

Proof. Using the basic estimates proved in Proposition 4.3.1, when $\phi = 0$, we have for any $\eta \in C_0^\infty(\Omega)$ and $f \in C^\infty_{(p,q)}(\Omega)$, $0 \le p \le n$ and $1 \le q \le n$,

$$\begin{aligned} \sum_{I,J}{}' \sum_k \| \frac{\partial(\eta f_{I,J})}{\partial \bar z_k} \|^2_\Omega &= (\| \bar\partial(\eta f) \|^2_\Omega + \| \vartheta(\eta f) \|^2_\Omega) \\ &\le C(\| \eta\bar\partial f \|^2_\Omega + \| \eta\vartheta f \|^2_\Omega + \| f \|^2_\Omega). \end{aligned} \tag{5.1.2}$$

(5.1.2) also holds trivially for $q = 0$ (in this case, $\vartheta f = 0$). But

$$\| \frac{\partial(\eta f_{I,J})}{\partial \bar z_k} \|^2_\Omega = \| \frac{\partial(\eta f_{I,J})}{\partial z_k} \|^2_\Omega \tag{5.1.3}$$

from integration by parts. We have for any smooth f,

$$\begin{aligned} \| \eta f \|^2_{1(\Omega)} &\le C \left(\sum_{I,J}{}' \sum_k \| \frac{\partial(\eta f_{I,J})}{\partial z_k} \|^2_\Omega + \sum_{I,J}{}' \sum_k \| \frac{\partial(\eta f_{I,J})}{\partial \bar z_k} \|^2_\Omega + \| f \|^2_\Omega \right) \\ &\le C(\| \eta\bar\partial f \|^2_\Omega + \| \eta\vartheta f \|^2_\Omega + \| f \|^2_\Omega). \end{aligned}$$

We note that one can also use (4.5.2) to prove the above *a priori* estimates. Estimate (5.1.1) follows from regularization similar to (4.5.3) and we omit the details.

If $f \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$, then $f \in W^1(\Omega, \mathrm{loc})$. The difficulty for the $\bar\partial$-Neumann problem is only on the boundary. The following subelliptic 1/2-estimates for a strongly pseudoconvex domain are of fundamental importance:

Theorem 5.1.2. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. The following estimate holds: for any $0 \le p \le n$ and $1 \le q \le n-1$, there exists $C > 0$ such that for any $f \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$,*

$$\| f \|^2_{\frac{1}{2}(\Omega)} \le C(\| \bar{\partial} f \|^2_\Omega + \| \bar{\partial}^* f \|^2_\Omega), \tag{5.1.4}$$

where C is independent of f.

Theorem 5.1.3. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. For any $0 \le p \le n$ and $1 \le q \le n-1$, the $\bar{\partial}$-Neumann operator N satisfies the estimates:*

$$\| Nf \|^2_{\frac{1}{2}(\Omega)} \le C \| f \|^2_{-\frac{1}{2}(\Omega)}, \qquad f \in L^2_{(p,q)}(\Omega), \tag{5.1.5}$$

where C is independent of f. N can be extended as a bounded operator from $W^{-1/2}_{(p,q)}(\Omega)$ into $W^{1/2}_{(p,q)}(\Omega)$. In particular, N is a compact operator on $L^2_{(p,q)}(\Omega)$.

We divide the proof of Theorem 5.1.2 into several lemmas. From Lemma 4.3.2, we have that $C^1_{(p,q)}(\overline{\Omega}) \cap \mathrm{Dom}(\bar{\partial}^*) = \mathcal{D}^1_{(p,q)}$ is dense in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$ in the graph norm. We only need to prove (5.1.4) for C^1 smooth forms f. The starting point is the following basic *a priori* estimate of Morrey-Kohn:

Lemma 5.1.4. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary $b\Omega$. There exists a constant $C > 0$, such that for any $f \in C^1_{(p,q)}(\overline{\Omega}) \cap Dom(\bar{\partial}^*) = \mathcal{D}^1_{(p,q)}$,*

$$\int_{b\Omega} |f|^2 dS \le C(\| \bar{\partial} f \|^2_\Omega + \| \vartheta f \|^2_\Omega), \tag{5.1.6}$$

where dS is the surface element on $b\Omega$ and C is independent of f.

Proof. Let $f = \sum'_{|I|=p,|J|=q} f_{I,J} dz^I \wedge d\bar{z}^J$, where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$ are increasing multiindices. Let ρ be a C^2 defining function for Ω normalized such that $|d\rho| = 1$ on $b\Omega$. Following Proposition 4.3.1 with $\phi = 0$, we have for each $f \in C^1_{(p,q)}(\overline{\Omega}) \cap \mathrm{Dom}(\bar{\partial}^*)$,

$$\begin{aligned} &\| \bar{\partial} f \|^2_\Omega + \| \vartheta f \|^2_\Omega \\ &= {\sum_{I,J}}' \sum_k \| \frac{\partial f_{I,J}}{\partial \bar{z}_k} \|^2_\Omega + {\sum_{I,K}}' \sum_{i,j=1}^n \int_{b\Omega} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} f_{I,iK} \bar{f}_{I,jK} dS, \end{aligned} \tag{5.1.7}$$

where K is an increasing multiindex and $|K| = q-1$.

Since $b\Omega$ is strongly pseudoconvex with C^2 boundary, there exists $C_0 > 0$ such that for any $z \in b\Omega$,

$$\sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j \ge C_0 |a|^2, \quad \text{ if } \sum_{i=1}^n a_i \frac{\partial \rho}{\partial z_i} = 0. \tag{5.1.8}$$

Since $f \in \mathrm{Dom}(\bar{\partial}^*) \cap C^1_{(p,q)}(\overline{\Omega})$, we have from Lemma 4.2.1,

$$\sum_{j=1}^{n} f_{I,jK} \frac{\partial \rho}{\partial z_j} = 0 \qquad \text{on } b\Omega \text{ for each } I, K.$$

Substituting (5.1.8) into (5.1.7), we have

$$\| \bar{\partial} f \|_\Omega^2 + \| \vartheta f \|_\Omega^2 \geq \sum_{I,J}{}' \sum_k \| \frac{\partial f_{I,J}}{\partial \bar{z}_k} \|_\Omega^2 + q\ C_0 \int_{b\Omega} |f|^2 dS.$$

This proves the lemma.

Lemma 5.1.5. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary $b\Omega$ and let ρ be a C^2 defining function for Ω. There exists a constant $C > 0$ such that for any $f \in C^2(\overline{\Omega})$*

$$\int_\Omega |\rho| |\nabla f|^2 dV \leq C \left(\int_{b\Omega} |f|^2 dS + \int_\Omega |f|^2 dV \right) + Re \int_\Omega \rho (\triangle f) \bar{f} dV, \tag{5.1.9}$$

where dS is the surface element on $b\Omega$ and $C > 0$ is independent of f.

Proof. By Green's theorem, for any $u, v \in C^2(\overline{\Omega})$,

$$\int_\Omega (u \triangle v - v \triangle u) dV = \int_{b\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS, \tag{5.1.10}$$

where $\partial/\partial n$ is the directional derivative along the outward normal. Let $u = -\rho$ and $v = |f|^2/2$, then $\triangle v = \mathrm{Re}(\bar{f} \triangle f) + |\nabla f|^2$. We have from (5.1.10)

$$\int_\Omega -\rho |\nabla f|^2 dV + \ \mathrm{Re} \int_\Omega -\rho (\triangle f) \bar{f} dV + \int_\Omega \frac{|f|^2}{2} \triangle \rho dV = \int_{b\Omega} \frac{|f|^2}{2} \frac{\partial \rho}{\partial n} dS. \tag{5.1.11}$$

Since ρ is in $C^2(\overline{\Omega})$, there exists $C > 0$ such that $|\triangle \rho| \leq C$ in Ω and $|\partial \rho / \partial n| \leq C$ on $b\Omega$. This implies that

$$\int_\Omega |\rho| |\nabla f|^2 dV \leq C \left(\int_{b\Omega} |f|^2 dS + \int_\Omega |f|^2 dV \right) + \mathrm{Re} \int_\Omega \rho (\triangle f) \bar{f} dV, \tag{5.1.12}$$

where C is independent of f. This proves the Lemma.

Lemma 5.1.6. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary $b\Omega$. There exists a constant $C > 0$ such that for any $f \in C^1_{(p,q)}(\overline{\Omega})$,*

$$\| f \|^2_{\frac{1}{2}(\Omega)} \leq C \left(\int_{b\Omega} |f|^2 dS + \| \bar{\partial} f \|_\Omega^2 + \| \vartheta f \|_\Omega^2 + \| f \|_\Omega^2 \right), \tag{5.1.13}$$

where C is independent of f.

Proof. When $f \in C^2_{(p,q)}(\overline{\Omega})$, one sees from Proposition 4.2.4 that $\Box f = -\frac{1}{4}\triangle f$, where $\Box = \bar{\partial}\vartheta + \vartheta\bar{\partial}$ and $\triangle$ is defined componentwise. Applying Lemma 5.1.5 to each component of f, we get

$$\int_\Omega |\rho||\nabla f|^2 dV \le C\left(\int_{b\Omega} |f|^2 dS + \int_\Omega |f|^2 dV\right) + 4|(\rho\Box f, f)_\Omega|. \tag{5.1.14}$$

Since $f \in C^1_{(p,q)}(\overline{\Omega})$, we have

$$\| f \|^2_{\frac{1}{2}(\Omega)} \le C\left(\int_\Omega |\rho||\nabla f|^2 dV + \int_\Omega |f|^2 dV\right). \tag{5.1.15}$$

(5.1.15) follows from Theorem C.2 in the Appendix applied to each component of f.

Now integration by parts gives that

$$\begin{aligned} |\,(\rho\Box f, f)_\Omega\,| &\le |\,(\bar{\partial}(\rho\vartheta f), f)_\Omega\,| + |\,([\bar{\partial}, \rho]\vartheta f, f)_\Omega\,| \\ &\quad + |\,(\vartheta(\rho\bar{\partial} f), f)_\Omega\,| + |\,([\vartheta, \rho]\bar{\partial} f, f)_\Omega\,| \\ &\le |\,(\rho\vartheta f, \vartheta f)_\Omega\,| + |\,(\rho\bar{\partial} f, \bar{\partial} f)_\Omega\,| \\ &\quad + \sup_\Omega|\nabla\rho|(\| \bar{\partial} f \|_\Omega + \| \vartheta f \|_\Omega) \| f \|_\Omega \\ &\le C(\| \bar{\partial} f \|^2_\Omega + \| \vartheta f \|^2_\Omega + \| f \|^2_\Omega). \end{aligned} \tag{5.1.16}$$

Combining (5.1.14)-(5.1.16), we see that (5.1.13) holds for any $f \in C^2_{(p,q)}(\overline{\Omega})$. An approximation argument shows that (5.1.13) holds for any $f \in C^1_{(p,q)}(\overline{\Omega})$ since $C^2(\overline{\Omega})$ is dense in $C^1(\overline{\Omega})$.

Proof of Theorem 5.1.2. From the assumption of strong pseudoconvexity and Lemma 5.1.4, we have for any $f \in C^1_{(p,q)}(\overline{\Omega}) \cap \mathrm{Dom}(\bar{\partial}^*)$,

$$\int_{b\Omega} |f|^2 dS \le C(\| \bar{\partial} f \|^2_\Omega + \| \vartheta f \|^2_\Omega). \tag{5.1.17}$$

Using (4.4.6), we also get

$$\| f \|^2_\Omega \le \frac{e\delta^2}{q}(\| \bar{\partial} f \|^2_\Omega + \| \vartheta f \|^2_\Omega). \tag{5.1.18}$$

Thus, Lemmas 5.1.4 and 5.1.6 show that (5.1.4) holds for all forms $f \in C^1_{(p,q)}(\overline{\Omega}) \cap \mathrm{Dom}(\bar{\partial}^*) = \mathcal{D}^1_{(p,q)}$. Since $\mathcal{D}^1_{(p,q)}$ is dense in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$ in the graph norm $\| \bar{\partial} f \|_\Omega + \| \bar{\partial}^* f \|_\Omega$ from Lemma 4.3.2, Theorem 5.1.2 is proved.

Proof of Theorem 5.1.3. By the definition of the space $W^{-1/2}(\Omega)$ (see Appendix A), we have

$$|(h, g)_\Omega| \le \| h \|_{\frac{1}{2}(\Omega)} \| g \|_{-\frac{1}{2}(\Omega)} \tag{5.1.19}$$

for any $h \in W^{1/2}(\Omega)$ and $g \in W^{-1/2}(\Omega)$. There exists a constant $C > 0$ such that for any $f \in L^2_{(p,q)}(\Omega) \cap \mathrm{Dom}(\Box)$, $0 \le p \le n$ and $1 \le q \le n-1$,

$$\begin{aligned} \| f \|^2_{\frac{1}{2}(\Omega)} &\le C(\| \bar\partial f \|^2_\Omega + \| \bar\partial^* f \|^2_\Omega) \\ &= C(\Box f, f)_\Omega \\ &\le C \| \Box f \|_{-\frac{1}{2}(\Omega)} \| f \|_{\frac{1}{2}(\Omega)}, \end{aligned} \tag{5.1.20}$$

where C is independent of f. Substituting Nf into (5.1.20), we have

$$\| Nf \|_{\frac{1}{2}(\Omega)} \le C \| \Box Nf \|_{-\frac{1}{2}(\Omega)} = C \| f \|_{-\frac{1}{2}(\Omega)} .$$

Thus, N can be extended as a bounded operator from $W^{-1/2}(\Omega)$ to $W^{1/2}(\Omega)$. Using the Rellich lemma (see Theorem A.8 in the Appendix), N is a compact operator on $W^{-1/2}(\Omega)$ and $L^2_{(p,q)}(\Omega)$. This proves Theorem 5.1.3.

Corollary 5.1.7. *Let Ω and f be as in Theorem 5.1.2. Then $\bar\partial^* N$ and $\bar\partial N$ are compact operators on $L^2_{(p,q)}(\Omega)$. Moreover, the following estimates hold: there exists a $C > 0$ such that for any $f \in L^2_{(p,q)}(\Omega)$,*

$$\| \bar\partial^* Nf \|_{\frac{1}{2}(\Omega)} \le C \| f \|_\Omega, \quad 1 \le q \le n, \tag{5.1.21}$$

$$\| \bar\partial Nf \|_{\frac{1}{2}(\Omega)} \le C \| f \|_\Omega, \quad 0 \le q \le n-1, \tag{5.1.22}$$

where C is independent of f.

Proof. Estimate (5.1.22) follows easily from (5.1.4). Since $\bar\partial Nf$ is in $\mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$ when $0 \le q \le n-1$, substituting $\bar\partial Nf$ into (5.1.4), we have

$$\| \bar\partial Nf \|^2_{\frac{1}{2}(\Omega)} \le C(\| \bar\partial^* \bar\partial Nf \|^2_\Omega + \| \bar\partial \bar\partial Nf \|^2_\Omega) \le C \| f \|^2_\Omega .$$

When $q > 1$, (5.1.21) can be proved similarly since in this case, we also have $\bar\partial^* Nf$ is in $\mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$. When $q = 1$, $\bar\partial^* Nf$ is a $(p,0)$-form and (5.1.4) does not hold for $q = 0$. The proof of (5.1.21) for $q = 1$ is much more involved, and will be postponed. A more general result will be proved in the next section (see Theorem 5.2.6).

Remark. When $q = n$, the $\bar\partial$-Neumann problem is the classical Dirichlet problem

$$\begin{cases} \Box u = f & \text{in } \Omega, \\ u = 0 & \text{on } b\Omega, \end{cases}$$

where u and f are (p,n)-forms. In this case, we have Gårding's inequality in place of (5.1.4):

$$\| f \|^2_{1(\Omega)} \le C \| \bar\partial^* f \|^2_\Omega, \qquad f \in \mathrm{Dom}(\bar\partial^*),$$

where C is independent of f. Since for any $f \in L^2_{(p,n)}(\Omega)$, $Nf \in \mathrm{Dom}(\bar\partial^*)$ and we obtain

$$\begin{aligned} \| Nf \|^2_{1(\Omega)} &\le C \| \bar\partial^* Nf \|^2_\Omega = C(\bar\partial \bar\partial^* Nf, Nf)_\Omega = C(f, Nf)_\Omega \\ &\le C \| f \|_{-1(\Omega)} \| Nf \|_{1(\Omega)} . \end{aligned}$$

$N_{(p,n)}$ can be extended as a bounded operator from $W^{-1}(\Omega)$ into $W_0^1(\Omega)$ (see Appendix A for its definition) for any bounded domain with C^2 boundary.

5.2 Boundary Regularity for N and $\bar{\partial}^* N$

In this section, we assume that Ω is strongly pseudoconvex and its boundary $b\Omega$ is C^∞; i.e., there exists a C^∞ defining function ρ. We normalize ρ such that $|d\rho| = 1$ on $b\Omega$. First we derive estimates for the $\bar{\partial}$-Neumann operator N in Sobolev spaces for $s \geq 0$. As observed in Proposition 5.1.1, the system $\bar{\partial} \oplus \vartheta$ is elliptic in the interior. The interior regularity follows from the usual elliptic theory (see e.g. Lions-Magenes [LiMa 1], Bers-John-Schechter [BJS 1] or Treves [Tre 1]). Based on Gårding's inequality (5.1.1), we have for every $\eta \in C_0^\infty(\Omega)$ and $s \geq 0$,

$$\eta N f \in W^{s+2}_{(p,q)}(\Omega), \quad \text{for any} \quad f \in W^s_{(p,q)}(\Omega).$$

The main result in this section is to prove the following estimates on boundary regularity for N.

Theorem 5.2.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^∞ boundary. The $\bar{\partial}$-Neumann operator N is a bounded operator from $W^s_{(p,q)}(\Omega)$ to $W^{s+1}_{(p,q)}(\Omega)$ where $s \geq -\frac{1}{2}$, $0 \leq p \leq n$ and $1 \leq q \leq n-1$, and N satisfies the estimate: there exists a constant C_s such that for any $f \in W^s_{(p,q)}(\Omega)$,*

$$\| Nf \|^2_{s+1(\Omega)} \leq C_s \| f \|^2_{s(\Omega)}, \tag{5.2.1}$$

where C_s is independent of f.

In order to obtain the boundary regularity, we shall distinguish the tangential derivatives from the normal derivatives. Restricting to a small neighborhood U near a boundary point, we shall choose special boundary coordinates $t_1, \cdots, t_{2n-1}, \rho$ such that $t_1, \cdots, t_{2n-1}$ restricted to $b\Omega$ are coordinates for $b\Omega$. Let $D_{t_j} = \partial/\partial t_j$, $j = 1, \cdots, 2n-1$, and $D_\rho = \partial/\partial\rho$. Thus D_{t_j}'s are the tangential derivatives on $b\Omega$, and D_ρ is the normal derivative. For a multiindex $\alpha = (\alpha_1, \cdots, \alpha_{2n-1})$, where each α_i is a nonnegative integer, D_t^α denotes the product of D_{t_j}'s with order $|\alpha| = \alpha_1 + \cdots + \alpha_{2n-1}$, i.e.,

$$D_t^\alpha = D_{t_1}^{\alpha_1} \cdots D_{t_{2n-1}}^{\alpha_{2n-1}}.$$

For any $u \in C_0^\infty(U \cap \overline{\Omega})$, we define the tangential Fourier transform for u in a special boundary chart by

$$\tilde{u}(\tau, \rho) = \int_{\mathbb{R}^{2n-1}} e^{-i\langle t, \tau \rangle} u(t, \rho) dt,$$

where $\tau = (\tau_1, \cdots, \tau_{2n-1})$ and $\langle t, \tau \rangle = t_1\tau_1 + \cdots + t_{2n-1}\tau_{2n-1}$. For each fixed $-\epsilon < 0$, we define $\Gamma_\epsilon = \{z \in \Omega |\ \rho(z) = -\epsilon\}$ and set

$$\| u(\cdot, -\epsilon) \|^2_{s(\Gamma_\epsilon)} = \int_{\mathbb{R}^{2n-1}} (1 + |\tau|^2)^s |\tilde{u}(\tau, -\epsilon)|^2 d\tau.$$

We define the tangential Sobolev norms $|||\ |||_s$ by

$$|||u|||_s^2 = \int_{\mathbb{R}^{2n-1}} \int_{-\infty}^0 (1+|\tau|^2)^s |\tilde u(\tau,\rho)|^2 d\rho d\tau$$
$$= \int_{-\infty}^0 \| u(\cdot,\rho) \|^2_{s(\Gamma_\rho)}\, d\rho.$$

As usual, tangential norms for forms are defined as the sum of the norms of the components. The operator Λ_t^s for any $s \in \mathbb{R}$ is given by

$$\Lambda_t^s u(t,\rho) = (2\pi)^{1-2n} \int_{\mathbb{R}^{2n-1}} e^{i\langle t,\tau\rangle} (1+|\tau|^2)^{\frac{s}{2}} \tilde u(\tau,\rho) d\tau.$$

Using this notation and Plancherel's theorem (see Theorem A.5 in the Appendix), we have

$$|||u|||_s = \| \Lambda_t^s u \| .$$

The tangential norms have the following properties:

Properties for the tangential norms. *Let W_t^s denote the completion of $u \in C_0^\infty(U \cap \overline{\Omega})$ under the $|||\ |||_s$ norm. Then the following hold:*

(1) When s is a nonnegative integer, we have

$$\text{(5.2.2 i)} \qquad |||u|||_s \approx \sum_{0\le|\alpha|\le s} \| D_t^\alpha u \|,$$

where $\approx$ means that the two norms are equivalent.

(2) For any $s \in \mathbb{R}$, there exists a constant C such that

$$\text{(5.2.2 ii)} \qquad |||D_t^\alpha f|||_s \le C|||f|||_{k+s} \quad \textit{for } |\alpha| = k,$$

where C is independent of f.

(3) The Schwarz inequality holds; for any $s \ge 0$, $f \in W_t^s$ and $g \in W_t^{-s}$, we have

$$\text{(5.2.2 iii)} \qquad |(f,g)| \le C|||f|||_s|||g|||_{-s},$$

where $C = (2\pi)^{1-2n}$.

(4) Given two spaces $W_t^{s_1}$ and $W_t^{s_2}$, where s_1 and s_2 are real numbers, $s_1 > s_2$, the interpolation space $[W_t^{s_1}, W_t^{s_2}]_\theta = W_t^{s_1(1-\theta)+s_2\theta}$ for any $0 < \theta < 1$.

Properties (1) and (2) are easily checked from Plancherel's Theorem (see Theorem A.5 in the Appendix). (3) follows immediately from the definition. The proof of (4) is the same as that for the usual Sobolev spaces (see Appendix B). The following simple fact will also play a crucial rule in proving the boundary regularity.

Commutator of two operators. *For any smooth differential operators A and B of order k_1 and k_2, k_1 and k_2 are nonnegative integers, the commutator $[A, B] = AB - BA$ is an operator of order $k_1 + k_2 - 1$.*

The proof follows directly from the definition. We shall denote $\Lambda_t^1 u = \Lambda_t u$ and define for any real s,

$$|||Du|||_s^2 = |||D_\rho u|||_s^2 + |||\Lambda_t u|||_s^2 = |||D_\rho u|||_s^2 + |||u|||_{s+1}^2.$$

The norm $|||Du|||_s$ is stronger than the norm $|||u|||_{s+1}$ in general, since the normal derivatives are not controlled in the tangential norms.

By fixing a partition of unity, we can also define $|||\ \ |||_{s(\Omega_\delta)}$ for some tubular neighborhood $\Omega_\delta = \{z \in \Omega \mid -\delta < \rho(z) < 0\}$. Let U_i, $i = 1, 2, \cdots, K$, be boundary coordinate patches such that each $U_i \cap b\Omega \neq \emptyset$, $\cup_{i=1}^K U_i$ covers Ω_δ and there exists a special boundary chart on each $U_i \cap \overline{\Omega}$. We choose $\eta_i \in C_0^\infty(U_i)$, $i = 1, \cdots, K$, such that

$$\sum_{i=1}^K \eta_i^2 = 1 \quad \text{in } \Omega_\delta.$$

For each fixed ϵ, the norm $\|\ \ \|_{s(\Gamma_\epsilon)}$ is defined by a partition of unity.

We set

$$|||u|||_{s(\Omega_\delta)}^2 = \int_{-\delta}^0 \| u(\cdot, \rho) \|_{s(\Gamma_\rho)}^2 \, d\rho.$$

We choose a special boundary frame such that $w^1, \cdots, w^n$ is an orthonormal basis for (1,0)-forms with $w^n = \partial\rho$ in a boundary patch U as before. Written in this basis, a smooth (p, q)-form supported in $U \cap \overline{\Omega}$ can be expressed as

$$f = \sum_{|I|=p, |J|=q}{}' f_{I,J} w^I \wedge \bar{w}^J,$$

where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$ are increasing multiindices and $w^I = w^{i_1} \wedge \cdots \wedge w^{i_p}$, $\bar{w}^J = \bar{w}^{j_1} \wedge \cdots \wedge \bar{w}^{j_q}$. From Lemma 4.2.1 and (4.2.6″), it follows that

$$f \in \mathcal{D}_{(p,q)} \ \text{ if and only if } \ f_{I,J} = 0 \ \text{ on } b\Omega, \ \text{whenever } n \in J,$$

where $\mathcal{D}_{(p,q)} = C^\infty_{(p,q)}(\overline{\Omega}) \cap \mathrm{Dom}(\bar{\partial}^*)$. Thus, the tangential derivatives preserve $\mathcal{D}_{(p,q)}$ in the following sense:

Lemma 5.2.2. *Let $f \in \mathcal{D}_{(p,q)}$. Assume that f is supported in $U \cap \overline{\Omega}$ and f is expressed in the special boundary frame. Let T be a first order tangential differential operator with smooth coefficients acting componentwise. Then $Tf \in \mathcal{D}_{(p,q)}$.*

In order to obtain estimates for the $\bar{\partial}$-Neumann operator on the boundary, our first observation is that when f satisfies an elliptic system, then there is no distinction between the tangential Sobolev norms and the Sobolev norms. We have the following lemma:

Lemma 5.2.3. *Let Ω be a bounded domain with C^∞ boundary and let U be a special boundary patch. Let $f \in Dom(\bar\partial) \cap Dom(\bar\partial^*)$ where the support of f lies in $U \cap \overline{\Omega}$. The following conditions are equivalent:*

(a) $\| f \|_{\frac{1}{2}(\Omega)} < \infty$.

(b) $|||f|||_{\frac{1}{2}(\Omega)} < \infty$.

(c) $|||Df|||_{-\frac{1}{2}(\Omega)} < \infty$.

(d) $\| f \|_{L^2(b\Omega)} < \infty$.

Proof. It is obvious from the definition that (a) implies (b).

From the density lemma 4.3.2, we can assume $f \in C^1(U \cap \overline{\Omega})$ and the other cases follow by approximation. To show that (b) implies (c), we note that $\bar\partial \oplus \vartheta$ is elliptic. We can express $D_\rho f$ by the sum of the components of $\bar\partial f$, ϑf and the tangential derivatives of f. Thus

$$
\begin{aligned}
|||D_\rho f|||_{-\frac{1}{2}} &\le C(|||\bar\partial f|||_{-\frac{1}{2}} + |||\vartheta f|||_{-\frac{1}{2}} + |||\Lambda_t f|||_{-\frac{1}{2}}) \\
&\le C(\| \bar\partial f \| + \| \vartheta f \| + |||f|||_{\frac{1}{2}}).
\end{aligned}
$$

To see that (c) implies (d), we use

$$
\begin{aligned}
| \tilde f(\tau, 0) |^2 &= \int_{-\infty}^0 D_\rho \mid \tilde f(\tau,\rho) \mid^2 d\rho = \mathrm{Re} \int_{-\infty}^0 2 D_\rho \tilde f(\tau,\rho) \overline{\tilde f(\tau,\rho)} d\rho \\
&\le A \int_{-\infty}^0 \mid \tilde f(\tau,\rho) \mid^2 d\rho + \frac{1}{A} \int_{-\infty}^0 \mid D_\rho \tilde f(\tau,\rho) \mid^2 d\rho,
\end{aligned}
$$

for any $A > 0$. Choosing $A = (1 + |\tau|^2)^{1/2}$ and integrating over $\mathbb{R}^{2n-1}$, we have from Plancherel's theorem,

$$
\begin{aligned}
\| f \|^2_{L^2(b\Omega)} &= \left(\frac{1}{2\pi}\right)^{2n-1} \int_{\mathbb{R}^{2n-1}} \mid \tilde f(\tau, 0) \mid^2 d\tau \\
&\le C \left(\int_{\mathbb{R}^{2n-1}} \int_{-\infty}^0 (1 + |\tau|^2)^{\frac{1}{2}} \mid \tilde f(\tau,\rho) \mid^2 d\rho d\tau \right) \\
&\quad + C \left(\int_{\mathbb{R}^{2n-1}} \int_{-\infty}^0 (1 + |\tau|^2)^{-\frac{1}{2}} \mid D_\rho \tilde f(\tau,\rho) \mid^2 d\rho d\tau \right) \\
&\le C \ |||Df|||^2_{-\frac{1}{2}(\Omega)}.
\end{aligned}
$$

Finally, (d) implies (a) follows from Lemma 5.1.6 and Lemma 4.3.2, since we have for any $f \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$,

$$
\| f \|^2_{\frac{1}{2}(\Omega)} \le C \left(\int_{b\Omega} |f|^2 dS + \| \bar\partial f \|^2_\Omega + \| \vartheta f \|^2_\Omega + \| f \|^2_\Omega \right).
$$

We remark that (c) implies (d) can be viewed as a version of the trace theorem for Sobolev spaces. In general, a function satisfying (a) does not necessarily satisfy (d) (see Theorem A.9 in the Appendix).

To prove Theorem 5.2.1, we first prove *a priori* estimates; i.e., we assume that Nf is smooth up to the boundary.

Lemma 5.2.4. *Let Ω be a bounded pseudoconvex domain with C^∞ boundary and let ρ be a C^∞ defining function. Choose $\delta > 0$, such that the tangential norms are defined on the fixed tubular neighborhood $\Omega_\delta = \{z \in \Omega \mid \rho(z) > -\delta\}$. Then there exists a constant C_k, $k = 1, 2, \cdots$ such that for any $f \in Dom(\Box) \cap C^\infty_{(p,q)}(\overline{\Omega})$,*

$$\| f \|_{k(\Omega)} \le C_k(\| \Box f \|_{k-1(\Omega)} + |||f|||_{k(\Omega_\delta)}), \tag{5.2.3}$$

where C_k is independent of f.

Proof. We use induction on k for $k = 1, 2, \cdots$. Since

$$\sum_{m=0}^{k} |||D^m_\rho f|||_{k-m} \approx \| f \|_k,$$

(5.2.3) is proved if we can show

$$\sum_{m=0}^{k} |||D^m_\rho f|||_{k-m} \le C_k(\| \Box f \|_{k-1} + |||f|||_k). \tag{5.2.4}$$

We first prove (5.2.4) under the assumption that f is supported in $\overline{\Omega} \cap U$ where U is a special coordinate chart near the boundary.

When $k = 1$, we use the same argument as in Lemma 5.2.2 to express $D_\rho f$ by the components of $\bar\partial f$, ϑf and the tangential derivatives of f. We have

$$\begin{aligned}
\| D_\rho f \|^2 &\le C \left(\| \bar\partial f \|^2 + \| \vartheta f \|^2 + \sum_{i=1}^{2n-1} \| D_{t_i} f \|^2 \right) \\
&\le C \left((\Box f, f) + |||f|||_1^2 \right) \\
&\le C(\| \Box f \|^2 + |||f|||_1^2).
\end{aligned}$$

This proves (5.2.4) for $k = 1$. Assuming the lemma holds for $k - 1$, we shall show that (5.2.4) holds for k. If $m = 1$, again we get

$$\begin{aligned}
|||D_\rho f|||_{k-1}^2 &\le C(|||\bar\partial f|||_{k-1}^2 + |||\vartheta f|||_{k-1}^2 + \sum_{i=1}^{2n-1} |||D_{t_i} f|||_{k-1}^2) \\
&\le C(|||\bar\partial f|||_{k-1}^2 + |||\vartheta f|||_{k-1}^2 + |||f|||_k^2).
\end{aligned} \tag{5.2.5}$$

Notice that

$$\begin{aligned}
|||\bar\partial f|||_{k-1}^2 + |||\vartheta f|||_{k-1}^2 &= \| \Lambda_t^{k-1} \bar\partial f \|^2 + \| \Lambda_t^{k-1} \vartheta f \|^2 \\
&\le C \left(\sum_{0 \le |\alpha| \le k-1} \| D_t^\alpha \bar\partial f \|^2 + \sum_{0 \le |\alpha| \le k-1} \| D_t^\alpha \vartheta f \|^2 \right).
\end{aligned}$$

For any nonnegative integer k, let T^k denote any tangential differential operator of the form D_t^α, where $|\alpha| = k$. Using Lemma 5.2.2, we find that $T^k f \in \mathcal{D}_{(p,q)}$ and

$T^k\bar\partial f \in \mathcal{D}_{(p,q+1)}$ since $f \in \mathrm{Dom}(\Box)$. We see that

$$\begin{aligned}
&(T^{k-1}\bar\partial f,\ T^{k-1}\bar\partial f) + (T^{k-1}\vartheta f,\ T^{k-1}\vartheta f)\\
&= (\bar\partial T^{k-1} f,\ T^{k-1}\bar\partial f) + (\left[T^{k-1},\ \bar\partial\right] f,\ T^{k-1}\bar\partial f)\\
&\quad + (\vartheta T^{k-1} f,\ T^{k-1}\vartheta f) + (\left[T^{k-1},\ \vartheta\right] f,\ T^{k-1}\vartheta f)\\
&= (T^{k-1} f,\ \vartheta T^{k-1}\bar\partial f) + (\left[T^{k-1},\ \bar\partial\right] f,\ T^{k-1}\bar\partial f)\\
&\quad + (T^{k-1} f,\ \bar\partial T^{k-1}\vartheta f) + (\left[T^{k-1},\ \vartheta\right] f,\ T^{k-1}\vartheta f)\\
&= (T^{k-1} f,\ T^{k-1}\vartheta\bar\partial f) + (\left[T^{k-1},\ \bar\partial\right] f,\ T^{k-1}\bar\partial f)\\
&\quad + (T^{k-1} f,\ T^{k-1}\bar\partial\vartheta f) + (\left[T^{k-1},\ \vartheta\right] f,\ T^{k-1}\vartheta f)\\
&\quad + (T^{k-1} f,\ \left[\vartheta,\ T^{k-1}\right]\bar\partial f) + (T^{k-1} f,\ \left[\bar\partial,\ T^{k-1}\right]\vartheta f)\\
&= (T^{k-1} f,\ T^{k-1}\Box f) + \mathcal{R} + O(\| f \|_{k-1}\,(|||\bar\partial f|||_{k-1} + |||\vartheta f|||_{k-1})),
\end{aligned} \tag{5.2.6}$$

where

$$\mathcal{R} = (T^{k-1} f,\ \left[\vartheta,\ T^{k-1}\right]\bar\partial f) + (T^{k-1} f,\ \left[\bar\partial,\ T^{k-1}\right]\vartheta f).$$

The term $(T^{k-1} f,\ \left[\vartheta,\ T^{k-1}\right]\bar\partial f)$ in $\mathcal{R}$ can be estimated by

$$\begin{aligned}
&(T^{k-1} f,\ \left[\vartheta,\ T^{k-1}\right]\bar\partial f)\\
&= (T^{k-1} f,\ \bar\partial\left[\vartheta,\ T^{k-1}\right] f) + (T^{k-1} f,\ \left[\left[\vartheta,\ T^{k-1}\right],\ \bar\partial\right] f)\\
&= (\vartheta T^{k-1} f,\ \left[\vartheta,\ T^{k-1}\right] f) + (T^{k-1} f,\ \left[\left[\vartheta,\ T^{k-1}\right],\ \bar\partial\right] f)\\
&= (T^{k-1}\vartheta f,\ \left[\vartheta,\ T^{k-1}\right] f) + (\left[\vartheta,\ T^{k-1}\right] f, \left[\vartheta,\ T^{k-1}\right] f)\\
&\quad + (T^{k-1} f,\ \left[\left[\vartheta,\ T^{k-1}\right],\ \bar\partial\right] f).
\end{aligned} \tag{5.2.7}$$

Similarly, one can estimate the term $(T^{k-1} f,\ \left[\bar\partial,\ T^{k-1}\right]\vartheta f)$ in $\mathcal{R}$. Thus $|\mathcal{R}|$ can be estimated by

$$C(\| f \|_{k-1}\ |||\bar\partial f|||_{k-1} + \| f \|_{k-1}\ |||\vartheta f|||_{k-1} + \| f \|^2_{k-1}). \tag{5.2.8}$$

If we apply (5.2.6), (5.2.7) to each term of the form $T^{|\alpha|} = D_t^\alpha$, where $0 \le |\alpha| \le k-1$, we can conclude that

$$\begin{aligned}
&\sum_{0\le|\alpha|\le k-1} (\| D_t^\alpha\bar\partial f \|^2 + \| D_t^\alpha\vartheta f \|^2)\\
&\quad \le C(\| \Lambda_t^{k-1} f \| \| \Lambda_t^{k-1}\Box f \| + \| f \|_{k-1}\,(|||\bar\partial f|||_{k-1} + |||\vartheta f|||_{k-1}) + \| f \|^2_{k-1}).
\end{aligned}$$

Using the inequality that $ab \le \epsilon a^2 + (1/\epsilon)b^2$ for any $\epsilon > 0$, we see from (5.2.5) that

$$\begin{aligned}
&|||\bar\partial f|||^2_{k-1} + |||\vartheta f|||^2_{k-1}\\
&\le C(\| \Lambda_t^{k-1} f \| \| \Lambda_t^{k-1}\Box f \|\\
&\quad + \| f \|_{k-1}\,(|||\bar\partial f|||_{k-1} + |||\vartheta f|||_{k-1}) + \| f \|^2_{k-1})\\
&\le C|||f|||_{k-1}|||\Box f|||_{k-1} + \epsilon(|||\bar\partial f|||^2_{k-1} + |||\vartheta f|||^2_{k-1}) + C_\epsilon \| f \|^2_{k-1}\ .
\end{aligned}$$

Choosing ϵ sufficiently small, the induction hypothesis yields

$$|||\bar{\partial} f|||^2_{k-1} + |||\vartheta f|||^2_{k-1} \le C(\| \Box f \|^2_{k-1} + |||f|||^2_{k-1}). \tag{5.2.9}$$

Substituting (5.2.9) back into (5.2.5), we have proved (5.2.4) when $m = 1$. For $m \ge 2$, we shall repeat the procedure and use induction on $1 \le m \le k$.

Since $\Box$ is a constant multiple of the Laplacian operator on each component, we can express $D^2_\rho f$ by the sum of the components of terms of the form

$$\Box f,\ D_\rho D_{t_j} f,\ D_{t_i} D_{t_j} f, \quad i, j = 1, \cdots, 2n-1,$$

and lower order terms. If $m > 2$, differentiation shows that one can express $D^m_\rho f$ by the sum of terms of the form

$$D^{m-2}_\rho \Box f,\ D^{m-1}_\rho D_{t_j} f,\ D^{m-2}_\rho D_{t_i} D_{t_j} f, \quad i, j = 1, \cdots, 2n-1,$$

and lower order terms. Assuming that (5.2.4) holds for $1, \cdots, m-1$, we use the induction hypothesis to show that

$$\begin{aligned}
|||D^m_\rho f|||_{k-m} &\le C(\ |||D^{m-2}_\rho \Box f|||_{k-m} + \sum_{i=1}^{2n-1} |||D^{m-1}_\rho D_{t_i} f|||_{k-m} \\
&\quad + \sum_{i,j=1}^{2n-1} |||D^{m-2}_\rho D_{t_i} D_{t_j} f|||_{k-m} + \| f \|_{k-1}) \\
&\le C(\| \Box f \|_{k-2} + |||D^{m-1}_\rho f|||_{k-m+1} \\
&\quad + \ |||D^{m-2}_\rho f|||_{k-m+2} + \| f \|_{k-1}) \\
&\le C(\| \Box f \|_{k-2} + \| \Box f \|_{k-1} + |||f|||_k + \| f \|_{k-1}) \\
&\le C(\| \Box f \|_{k-1} + |||f|||_k).
\end{aligned} \tag{5.2.10}$$

Thus, (5.2.4) holds for all $m \le k$. This finishes the proof of Lemma 5.2.4 when f has compact support in a coordinate patch. The general case follows from a partition of unity. We shall do this in detail for the case when $k = 1$. Letting $\eta \in C^\infty(\overline{\Omega} \cap U)$, we have that

$$\begin{aligned}
\| \eta f \|^2_1 &\le C(\ \| \bar{\partial}(\eta f) \|^2 + \| \vartheta(\eta f) \|^2 + |||f|||^2_{1(\Omega_\delta)}\ ) \\
&\le C(\ \| \eta \bar{\partial} f \|^2 + \| \eta \vartheta f \|^2 + \| f \|^2 + |||f|||^2_{1(\Omega_\delta)}\ ) \\
&\le C((\eta^2 \Box f, f)_\Omega + \epsilon(\| \bar{\partial}(\eta f) \|^2 + \| \vartheta(\eta f) \|^2) \\
&\quad + C_\epsilon \| f \|^2 + |||f|||^2_{1(\Omega_\delta)}).
\end{aligned}$$

If we choose ϵ sufficiently small, then the term

$$\epsilon(\| \bar{\partial}(\eta f) \|^2 + \| \vartheta(\eta f) \|^2) \le \epsilon \| \eta f \|^2_1$$

can be absorbed by the left-hand side. It follows that

$$\begin{aligned}
\| \eta f \|^2_1 &\le C(\| \eta \Box f \| \| \eta f \| + \| f \|^2 + |||f|||^2_{1(\Omega_\delta)}) \\
&\le C(\| \eta \Box f \|^2 + \| f \|^2 + |||f|||^2_{1(\Omega_\delta)}) \\
&\le C(\| \Box f \|^2 + |||f|||^2_{1(\Omega_\delta)}).
\end{aligned}$$

In the last inequality above, we have used $\| f \| \le C \| \Box f \|$ which was proved in Theorem 4.4.1 under the assumption of pseudoconvexity.

If $\eta_0 \in C_0^\infty(\Omega)$, using Proposition 5.1.1, one sees that

$$\| \eta_0 f \|_1^2 \le C \| \Box f \|^2 .$$

Summing over a partition of unity η_i, $i = 0, \cdots, K$ such that $\eta_0 \in C_0^\infty(\Omega)$, each η_i is supported in a boundary coordinate patch and $\sum_{i=0}^K \eta_i^2 = 1$, we have proved

$$\| f \|_1^2 \le C(\| \Box f \|^2 + |||f|||^2_{1(\Omega_\delta)}).$$

The lemma is proved when $k = 1$. The other cases are proved similarly by using a partition of unity and induction.

Proposition 5.2.5. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^∞ defining function. Choose $\delta > 0$, such that the tangential norms are defined on the fixed tubular neighborhood $\Omega_\delta = \{z \in \Omega \mid \rho(z) > -\delta\}$. For each $k = 0, 1, 2, \cdots$, there exists a constant C_k such that for any $f \in Dom(\Box) \cap C^\infty_{(p,q)}(\overline{\Omega})$,*

$$|||f|||^2_{k+\frac{1}{2}(\Omega_\delta)} \le C_k \| \Box f \|^2_{k-\frac{1}{2}(\Omega)}, \tag{5.2.11}$$

where C_k is independent of f.

Proof. We shall prove the proposition by induction on $k = 0, 1, \cdots$. When $k = 0$, this is already proved in Theorem 5.1.3. Thus we have

$$\| f \|_{\frac{1}{2}} \le C \| \Box f \|_{-\frac{1}{2}} . \tag{5.2.12}$$

(We even have the actual estimates instead of just *a priori* estimates.) Assume that (5.2.11) holds for $k-1$. Let U be a special coordinate patch. We first assume that f is supported in $\overline{\Omega} \cap U$ and written in the special frame as before. Let T^k be a kth order tangential differential operator of the form D_t^α where $|\alpha| = k$. From Lemma 5.2.2, we know that $T^k f \in \mathcal{D}_{(p,q)}$. Substituting $T^k f$ into the estimate (5.1.4), we see that

$$\| T^k f \|^2_{\frac{1}{2}} \le C(\| \bar{\partial} T^k f \|^2 + \| \vartheta T^k f \|^2). \tag{5.2.13}$$

Using arguments similar to those in (5.2.6) and (5.2.7), it follows that

$$\begin{aligned} &\| \bar{\partial} T^k f \|^2 + \| \vartheta T^k f \|^2 \\ &= \| T^k \bar{\partial} f \|^2 + \| T^k \vartheta f \|^2 + O(\| f \|_k^2) \\ &= (T^k f,\ T^k \Box f) + O(\| f \|_k\, |||\bar{\partial} f|||_k + \| f \|_k\, |||\vartheta f|||_k + \| f \|_k^2). \end{aligned} \tag{5.2.14}$$

Applying the inequality $ab \le \epsilon a^2 + \frac{1}{\epsilon} b^2$ again and (5.2.2 ii), (5.2.2 iii), we obtain

$$\begin{aligned} | (T^k f,\ T^k \Box f) | &\le C |||T^k f|||_{\frac{1}{2}}\ |||T^k \Box f|||_{-\frac{1}{2}} \\ &\le C_\epsilon |||\Box f|||^2_{k-\frac{1}{2}} + \epsilon |||T^k f|||^2_{\frac{1}{2}} \\ &\le C_\epsilon |||\Box f|||^2_{k-\frac{1}{2}} + \epsilon |||f|||^2_{k+\frac{1}{2}}. \end{aligned} \tag{5.2.15}$$

Now since

$$
\begin{aligned}
|||\bar\partial f|||_k + |||\vartheta f|||_k &\le C \sum_{|\alpha|\le k} (\| D_t^\alpha \bar\partial f \| + \| D_t^\alpha \vartheta f \|)\\
&\le C \sum_{|\alpha|\le k} (\| \bar\partial D_t^\alpha f \| + \| \vartheta D_t^\alpha f \|) + O(\| f \|_k),
\end{aligned}
$$

it follows from from (5.2.14) that

$$
\sum_{|\alpha|\le k} (\| \bar\partial D_t^\alpha f \|^2 + \| \vartheta D_t^\alpha f \|^2) \le C\Bigg(\sum_{|\alpha|\le k} (D_t^\alpha f,\ D_t^\alpha \Box f) + \| f \|_k^2 \Bigg). \tag{5.2.16}
$$

Combining (5.2.13)-(5.2.16) and summing up all the tangential derivatives of the form D_t^α, where $|\alpha| \le k$, we deduce that

$$
\begin{aligned}
|||f|||^2_{k+\frac12} &\le C\Bigg(\sum_{|\alpha|\le k} |||D_t^\alpha f|||_{\frac12}\ |||D_t^\alpha \Box f|||_{-\frac12} + \| f \|_k^2 \Bigg)\\
&\le \epsilon |||f|||^2_{k+\frac12} + C_\epsilon |||\Box f|||^2_{k-\frac12} + C \| f \|_k^2 .
\end{aligned} \tag{5.2.17}
$$

Using the interpolation inequality for Sobolev spaces (see Theorem B.2 in the Appendix), for any $\epsilon' > 0$ there exists a $C_{\epsilon'}$ such that

$$
|||f|||_k \le \epsilon' |||f|||_{k+\frac12} + C_{\epsilon'} |||f|||_{\frac12}. \tag{5.2.18}
$$

Applying Lemma 5.2.4, we observe that

$$
\begin{aligned}
\| f \|_k &\le C(\ |||f|||_k + \| \Box f \|_{k-1})\\
&\le C(\epsilon' |||f|||_{k+\frac12} + C_{\epsilon'} |||f|||_{\frac12} + \| \Box f \|_{k-1}).
\end{aligned}
$$

Choosing first ϵ and then ϵ' sufficiently small, using (5.2.17) and (5.2.18), one obtains

$$
|||f|||^2_{k+\frac12} \le C(\| \Box f \|^2_{k-\frac12} + \| f \|^2_{\frac12}).
$$

From (5.2.12), we have established

$$
|||f|||^2_{k+\frac12} \le C \| \Box f \|^2_{k-\frac12}, \tag{5.2.19}
$$

when f is supported in a special coordinate patch.

The general case will be derived from a partition of unity. Let $\eta,\ \eta' \in C_0^\infty(U)$ and $\eta' = 1$ on the support of η. We have

$$
\begin{aligned}
|||T^k \eta f|||^2_{\frac12} &\le C(\ |||\eta T^k f|||^2_{\frac12} + |||\eta' f|||^2_{k-\frac12})\\
&\le C(\| \bar\partial \eta T^k f \|^2 + \| \vartheta \eta T^k f \|^2 + |||\eta' f|||^2_{k-\frac12}).
\end{aligned}
$$

Repeating the previous argument with ηT^k substituted for T^k, we see that

$$
|||\eta f|||^2_{k+\frac12} \le \epsilon |||\eta' f|||^2_{k+\frac12} + C_\epsilon |||\Box f|||^2_{k-\frac12} + C \| \eta' f \|_k^2 .
$$

Now for any $\eta_0 \in C_0^\infty(\Omega)$, we already know that

$$\| \eta_0 f \|_{k+\frac{3}{2}} \leq C \| \Box f \|_{k-\frac{1}{2}} .$$

Summing over a partition of unity η_i, $i = 1, \cdots, K$ for the tubular neighborhood Ω_δ, yields that for some $\eta_0 \in C_0^\infty(\Omega)$,

$$|||f|||^2_{k+\frac{1}{2}(\Omega_\delta)} \leq C(\| \Box f \|^2_{k-\frac{1}{2}} + \| \eta_0 f \|^2_k) \leq C \| \Box f \|^2_{k-\frac{1}{2}} .$$

This proves the proposition.

Proof of Theorem 5.2.1. Using Theorem 5.1.3, we already know that Theorem 5.2.1 holds when $s = -1/2$ and

$$\text{(5.2.20)} \qquad \| Nf \|_{\frac{1}{2}(\Omega)} \leq C \| f \|_{-\frac{1}{2}(\Omega)}, \quad \text{for any } f \in W^{-\frac{1}{2}}_{(p,q)}(\Omega).$$

We shall prove the theorem for $s = k$ when $k \in \mathbb{N}$. Since $C^\infty_{(p,q)}(\overline{\Omega})$ is dense in $W^s_{(p,q)}(\Omega)$, it suffices to prove the following estimates:

$$\text{(5.2.21)} \qquad \| Nf \|_{s+\frac{1}{2}(\Omega)} \leq C \| f \|_{s-\frac{1}{2}(\Omega)}, \quad \text{for any } f \in C^\infty_{(p,q)}(\overline{\Omega}).$$

When s is a nonnegative integer, (5.2.21) has already been established in Proposition 5.2.5 and Lemma 5.2.4 assuming that Nf is smooth up to the boundary. To pass from *a priori* estimates to the real estimates, we can use the following elliptic regularization method:

Let Q be defined by

$$Q(g, g) = \|\bar{\partial} g\|^2 + \|\bar{\partial}^* g\|^2, \qquad g \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*).$$

We define

$$Q^\epsilon(g,\ g) = Q(g,\ g) + \epsilon \| \nabla g \|^2, \qquad g \in \overline{\mathcal{D}}^\epsilon,$$

where $\overline{\mathcal{D}}^\epsilon$ is the completion of $\mathcal{D}_{(p,q)}$ under the Q^ϵ norm. From inequality (4.4.6), we see that

$$\text{(5.2.22)} \qquad Q^\epsilon(g,\ g) \geq C \| g \|^2 \quad \text{for every } \ g \in \overline{\mathcal{D}}^\epsilon,$$

where $C > 0$ is independent of ϵ (C can be chosen as $e\delta^2/q$ where δ is the diameter of Ω). Thus, for any $f \in L^2_{(p,q)}(\Omega)$, $g \in \overline{\mathcal{D}}^\epsilon$, we can deduce that

$$\text{(5.2.23)} \qquad |(f,\ g)| \leq C^{-\frac{1}{2}} \| f \| Q^\epsilon(g, g)^{\frac{1}{2}}.$$

This implies that the map from $g \mapsto (f,\ g)$ is a bounded conjugate linear functional on $\overline{\mathcal{D}}^\epsilon$. By the Riesz representation theorem, there exists an element $N^\epsilon f \in \overline{\mathcal{D}}^\epsilon$ such that

$$(f,\ g) = Q^\epsilon(N^\epsilon f,\ g) \qquad \text{for every } g \in \overline{\mathcal{D}}^\epsilon.$$

Moreover, we have

$$\| N^\epsilon f \| \le C \| f \|,$$

where C is the same constant as in (5.2.22). Note that Q^ϵ satisfies Gårding's inequality

$$Q^\epsilon(f, f) \ge \epsilon \| f \|_1^2 \quad \text{for every} \quad f \in \overline{\mathcal{D}}^\epsilon.$$

Thus, the bilinear form Q^ϵ is elliptic on $\overline{\mathcal{D}}^\epsilon$ and we can use the theory for elliptic boundary value problems on a smooth domain to conclude that $N^\epsilon f \in C^\infty_{(p,q)}(\overline{\Omega})$ if $f \in C^\infty_{(p,q)}(\overline{\Omega})$. Applying the *a priori* estimates (5.2.11) to the form $N^\epsilon f$, we get

$$|||N^\epsilon f|||_{k+\frac{1}{2}(\Omega_\delta)} \le C_k \| f \|_{k-\frac{1}{2}(\Omega)}, \tag{5.2.24}$$

where C_k is independent of ϵ. The interpolation theorem for the operator N^ϵ on Sobolev spaces $W^s(\Omega)$ and tangential Sobolev spaces $W^s_t(\Omega_\delta)$ (see Theorem B.3 in the Appendix) gives

$$|||N^\epsilon f|||_{k(\Omega_\delta)} \le C \| f \|_{k-1(\Omega)} \qquad \text{for } k = 1, 2, \cdots .$$

Repeating the argument of Lemma 5.2.4, we obtain

$$\| N^\epsilon f \|_{k(\Omega)} \le C \| f \|_{k-1(\Omega)} \qquad \text{for } k = 1, 2, \cdots .$$

Thus, a subsequence of $N^\epsilon f$ converges weakly in $W^k_{(p,q)}(\Omega)$ to some element $\beta \in W^k_{(p,q)}(\Omega)$. We claim that $\beta = Nf$.

For any $g \in \mathcal{D}_{(p,q)}$,

$$(f,\ g) = Q(Nf,\ g) = Q^\epsilon(N^\epsilon f,\ g).$$

It follows from the definition of Q^ϵ that for any $g \in \mathcal{D}_{(p,q)}$,

$$|Q(N^\epsilon f - Nf,\ g)| \le \epsilon \| N^\epsilon f \|_1 \| g \|_1 \le \epsilon C \| f \| \| g \|_1 \to 0$$

as $\epsilon \to 0$. Since $\mathcal{D}_{(p,q)}$ is dense in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) = \overline{\mathcal{D}}$ from Lemma 4.3.2, we see that

$$Q(N^\epsilon f - Nf,\ g) \to 0 \qquad \text{for every } g \in \overline{\mathcal{D}}.$$

Thus, $N^\epsilon f$ converges to Nf weakly in the Q-norm. But (a subsequence of) $N^\epsilon f \to \beta$ weakly in the W^k norm. Therefore, we must have

$$Nf = \beta$$

and

$$\| Nf \|_k \le \liminf \| N^\epsilon f \|_k \le C \| f \|_{k-1}, \qquad k = 1, 2, \cdots .$$

Thus (5.2.21) is proved for $s = 0$ and $s = \frac{1}{2} + k$, $k = 0, 1, 2, \cdots$. Using the interpolation theorem for operators on Sobolev spaces again, we have for any $s \ge -\frac{1}{2}$,

$$\| Nf \|_{s+1} \le C_s \| f \|_s, \qquad f \in W^s_{(p,q)}(\Omega).$$

This proves Theorem 5.2.1.

Theorem 5.2.6. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^∞ boundary. Then $\bar{\partial}^* N$ and $\bar{\partial} N$ are bounded operators from $W^s_{(p,q)}(\Omega)$ into $W^{s+1/2}_{(p,q-1)}(\Omega)$ and $W^{s+1/2}_{(p,q+1)}(\Omega)$ respectively, where $s \geq -1/2$, $0 \leq p \leq n$ and $1 \leq q \leq n-1$. There exists a constant C_s such that for any $f \in W^s_{(p,q)}(\Omega)$,*

$$\| \bar{\partial}^* N f \|^2_{s+\frac{1}{2}(\Omega)} + \| \bar{\partial} N f \|^2_{s+\frac{1}{2}(\Omega)} \leq C_s \| f \|^2_{s(\Omega)}, \tag{5.2.25}$$

where C_s is independent of f.

Proof. When $s = -1/2$, we have shown in Theorem 5.1.3 that

$$\begin{aligned} \| \bar{\partial}^* N f \|^2 + \| \bar{\partial} N f \|^2 &= (\bar{\partial}\bar{\partial}^* N f,\ N f) + (\bar{\partial}^* \bar{\partial} N f,\ N f) = (f,\ N f) \\ &\leq \| f \|_{-\frac{1}{2}} \| N f \|_{\frac{1}{2}} \leq C \| f \|^2_{-\frac{1}{2}} . \end{aligned}$$

We shall prove by induction that

$$\| \bar{\partial}^* N f \|^2_k + \| \bar{\partial} N f \|^2_k \leq C_k \| f \|^2_{k-\frac{1}{2}}, \tag{5.2.26}$$

for $k = 1, 2, \cdots$. Let η be a cut-off function. If η is supported in a boundary coordinate patch, let T^k denote the same kth order tangential differential operator of the form ηD^α_t, $|\alpha| = k$. Observe that

$$\begin{aligned} &(T^k \bar{\partial}^* N f,\ T^k \bar{\partial}^* N f) + (T^k \bar{\partial} N f,\ T^k \bar{\partial} N f) \\ &= (T^k \bar{\partial}^* N f,\ \bar{\partial}^* T^k N f) + (T^k \bar{\partial} N f,\ \bar{\partial} T^k N f) \\ &\quad + (T^k \bar{\partial}^* N f,\ [T^k, \bar{\partial}^*] N f) + (T^k \bar{\partial} N f,\ [T^k, \bar{\partial}] N f) \\ &\leq C \left((T^k \bar{\partial}\bar{\partial}^* N f,\ T^k N f) + (T^k \bar{\partial}^* \bar{\partial} N f,\ T^k N f) + E \right) \\ &\leq C (T^k \Box N f,\ T^k N f) + E) \\ &\leq C (|||T^k f|||_{-\frac{1}{2}} |||T^k N f|||_{\frac{1}{2}} + E) \\ &\leq C (|||f|||_{k-\frac{1}{2}} |||N f|||_{k+\frac{1}{2}} + E) \\ &\leq C (|||f|||^2_{k-\frac{1}{2}} + E), \end{aligned} \tag{5.2.27}$$

where E denotes terms which can be bounded by

$$\begin{aligned} &C \left(\| N f \|_k \, (|||\bar{\partial}^* N f|||_k + |||\bar{\partial} N f|||_k) + \| N f \|^2_k \right) \\ &\leq \epsilon (|||\bar{\partial}^* N f|||^2_k + |||\bar{\partial} N f|||^2_k) + C_\epsilon \| N f \|^2_k . \end{aligned} \tag{5.2.28}$$

Using a partition of unity and summing up all the tangential derivatives up to order k in (5.2.27), we obtain, using (5.2.28), that

$$|||\bar{\partial}^* N f|||^2_k + |||\bar{\partial} N f|||^2_k \leq C (\| f \|^2_{k-1} + |||f|||^2_{k-\frac{1}{2}}),$$

if ϵ in (5.2.28) is chosen sufficiently small.

Again, using the fact $\bar{\partial} \oplus \vartheta$ is an elliptic system, the normal derivative can be expressed as the linear combination of terms which have been estimated before. The interior regularity is easier. This proves the inequality (5.2.26). The theorem follows from interpolation.

Corollary 5.2.7. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^∞ boundary and $0 \le p \le n$ and $0 \le q \le n-1$. Then N maps $C^\infty_{(p,q)}(\overline{\Omega})$ into $C^\infty_{(p,q)}(\overline{\Omega})$. In particular, the Bergman projection P maps $C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$. Also if $f \in C^\infty_{(p,q)}(\overline{\Omega})$ and $\bar{\partial} f$=0, the canonical solution $u = \bar{\partial}^* N f \in C^\infty_{(p,q-1)}(\overline{\Omega})$.*

Proof. The corollary is an easy consequence from the Sobolev embedding theorem for $q \ge 1$. For $q = 0$, we use (4) in Theorem 4.4.3. The regularity of the Bergman projection follows from the formula $Pf = f - \bar{\partial}^* N \bar{\partial} f = f - \vartheta N \bar{\partial} f$.

In Chapter 6, we prove a more precise result for the Bergman projection. In fact, P preserve $W^s(\Omega)$ for all $s \ge 0$ (see Theorem 6.2.2).

5.3 Function Theory on Manifolds

Let M be a complex manifold of dimension n (For the definition of complex manifolds, see Chapter 1). The decomposition of differential forms into forms of type (p,q), the definition of the $\bar{\partial}$ operator and the definition of plurisubharmonic functions for domains in $\mathbb{C}^n$ can immediately be extended to forms and functions on the complex manifold M. In order to study the operator $\bar{\partial}$ with Hilbert space techniques, we must equip M with a Hermitian metric such that $\mathbb{C}T(M) = T^{1,0}(M) \oplus T^{0,1}(M)$ and $T^{1,0}(M) \perp T^{0,1}(M)$. A Hermitian metric in local coordinates $z_1, \cdots, z_n$ is of the form

$$\sum_{i,j=1}^{n} h_{ij} dz_i \otimes d\bar{z}_j,$$

where h_{ij} is a positive definite Hermitian matrix with C^∞ coefficients. The existence of a Hermitian metric is trivial locally and is proved globally by a partition of unity. We fix a Hermitian metric in all that follows. This induces an inner product in $C^\infty_{(p,q)}(M)$ for each $p \in M$. If $\phi,\ \psi \in C^\infty_{(p,q)}(M)$, this inner product is denoted by $\langle \phi, \psi \rangle$. We have the following definition:

Definition 5.3.1. *Let $p \in M$ and $\phi \in C^2(M)$. If $L \in T^{1,0}_p(M)$, the complex Hessian of ϕ at p is defined to be the Hermitian form*

$$L \mapsto (\partial\bar{\partial}\phi)_p(L \wedge \overline{L}).$$

The function ϕ is called plurisubharmonic at p if the complex Hessian is positive semi-definite. ϕ is called strictly plurisubharmonic at p if the complex Hessian is positive definite.

Let Ω be an open subset in M whose closure is compact in M, i.e., Ω is relatively compact in M and denoted by $\Omega \subset\subset M$. Ω is called a complex manifold with C^k boundary $b\Omega$ if there exists a neighborhood V of $\overline{\Omega}$ and a real-valued function $\rho \in C^k(V)$ such that $\Omega = \{z \in V \mid \rho < 0\}$, $\rho > 0$ in $V \setminus \overline{\Omega}$ and $|d\rho| \ne 0$ on $b\Omega$. Let $\mathbb{C}T_p(b\Omega)$ be the complexified tangent bundle of $b\Omega$ at p.

Definition 5.3.2. *Let Ω be a complex manifold with C^2 boundary and ρ be a C^2 defining function. Ω is called pseudoconvex (strictly pseudoconvex) if for each $p \in b\Omega$, the restriction of the complex Hessian of ρ to $T_p^{1,0}(M) \cap \mathbb{C}T_p(b\Omega)$ is positive semi-definite (positive definite).*

In local coordinates, by the usual Gram-Schmidt orthogonalization process we can choose an orthonormal basis $w^1, \cdots, w^n$ for (1,0)-forms locally on a sufficiently small neighborhood U such that $\langle w^i, w^k \rangle = \delta_{ik},\ i,k = 1, \cdots, n$. Then written in this basis, for any $u \in C^1(U)$, we can write

$$du = \sum_{i=1}^{n} \frac{\partial u}{\partial w^i} w^i + \sum_{i=1}^{n} \frac{\partial u}{\partial \bar{w}^i} \bar{w}^i,$$

where the first order linear differential operators $\partial/\partial w^i$ and $\partial/\partial \bar{w}^i$ are duals of w^i and $\bar{w}^i$ respectively. Then we have

$$\bar{\partial} u = \sum_{i=1}^{n} \frac{\partial u}{\partial \bar{w}^i} \bar{w}^i.$$

If f is a (p,q)-form on U, then we can write f as

$$f = \sum_{|I|=p,|J|=q}{}' f_{I,J} w^I \wedge \bar{w}^J, \tag{5.3.1}$$

where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$ are multiindices and $w^I = w^{i_1} \wedge \cdots \wedge w^{i_p}$, $\bar{w}^J = \bar{w}^{j_1} \wedge \cdots \wedge \bar{w}^{j_q}$. If $u \in C^2(U)$, we set u_{ij} to be the coefficients of $\partial\bar{\partial} u$, i.e.,

$$\partial\bar{\partial} u = \sum_{i,j} u_{ij} w^i \wedge \bar{w}^j. \tag{5.3.2}$$

Let c^i_{jk} be the smooth functions such that

$$\bar{\partial} w^i = \sum_{j,k} c^i_{jk} \bar{w}^j \wedge w^k.$$

Then u_{ij} can be calculated as follows:

$$\partial\bar{\partial} u = \partial \left(\sum_k \frac{\partial u}{\partial \bar{w}^k} \bar{w}^k \right) = \sum_{j,k} \left(\frac{\partial^2 u}{\partial w^j \partial \bar{w}^k} + \sum_i \frac{\partial u}{\partial \bar{w}^i} \bar{c}^i_{jk} \right) w^j \wedge \bar{w}^k.$$

From the fact that $\partial\bar{\partial} + \bar{\partial}\partial = 0$, we have

$$u_{jk} = \frac{\partial^2 u}{\partial w^j \partial \bar{w}^k} + \sum_i \frac{\partial u}{\partial \bar{w}^i} \bar{c}^i_{jk} = \frac{\partial^2 u}{\partial \bar{w}^k \partial w^j} + \sum_i \frac{\partial u}{\partial w^i} c^i_{kj}. \tag{5.3.3}$$

A function $\phi \in C^2$ is *plurisubharmonic* (*strictly plurisubharmonic*) if the form

$$\sum_{j,k=1}^{n} \phi_{jk} a_j \bar{a}_k, \quad a = (a_1, \cdots, a_n) \in \mathbb{C}^n,$$

is positive semi-definite (positive definite).

We shall normalize ρ such that $|d\rho| = 1$ on $b\Omega$. Ω is pseudoconvex at a point z on $b\Omega$ if there exists a neighborhood U of z and a local (1,0) orthonormal frame $w^1, \cdots, w^n$ such that

$$\sum_{j,k=1}^{n} \rho_{jk} a_j \bar{a}_k \geq 0 \qquad \text{if} \quad \sum_{j=1}^{n} a_j \frac{\partial \rho(z)}{\partial w^j} = 0. \tag{5.3.4}$$

Here $a = (a_1, \cdots, a_n)$ is a vector in $\mathbb{C}^n$. If the Hermitian form is strictly positive for all such $a \neq 0$, the boundary is *strongly pseudoconvex* at z.

Note that these definitions are independent of the choice of the defining function ρ and are independent of the choice of $w^1, \cdots, w^n$. If we choose a special boundary chart such that $w^n = \partial\rho$, then $\partial/\partial w^i$, $i = 1, \cdots, n-1$ are tangential operators. We have, substituting ρ for u in (5.3.3),

$$\partial\bar{\partial}\rho = \overline{\bar{\partial} w^n} = \sum_{j,k} \rho_{jk} w^j \wedge \bar{w}^k, \tag{5.3.5}$$

where $(\rho_{jk}) = (\bar{c}^n_{jk}) = (c^n_{kj})$ is the Levi matrix. In this case, $b\Omega$ is pseudoconvex if and only if $(\rho_{jk})_{j,k=1}^{n-1}$ is positive semi-definite.

We shall use the same Hilbert space theory as that in Chapter 4 to study the function theory on pseudoconvex manifolds. We fix a function $\phi \in C^2(\overline{\Omega})$. Let $\bar{\partial} : L^2_{(p,q)}(\Omega, \phi) \to L^2_{(p,q+1)}(\Omega, \phi)$ be the closure of the Cauchy-Riemann operator and we define the Hilbert space adjoint for $\bar{\partial}^*_\phi$ as before. Let $z_0 \in b\Omega$ be a boundary point and U be an open neighborhood of z_0. We shall fix a special orthonormal boundary frame $w^1, \cdots, w^n = \partial\rho$. Writing $L_i = \partial/\partial w^i$, then $L_1, \cdots, L_n$ are dual to the (1,0)-forms $w^1, \cdots, w^n$ and we have

$$\begin{cases} L_i(\rho) = 0, & \text{when } z \in b\Omega \cap U,\ i = 1, \cdots, n-1, \\ L_n(\rho) = 1, & \text{when } z \in b\Omega \cap U. \end{cases} \tag{5.3.6}$$

We compute $\bar{\partial} f$ and ϑf in this special coordinate chart. We can write any $f \in C^\infty_{(p,q)}(\overline{\Omega} \cap U)$ as $f = \sum'_{|I|=p,|J|=q} f_{I,J} w^I \wedge \bar{w}^J$. Then

$$\begin{gathered} f \in C^\infty_0(\overline{\Omega} \cap U) \cap \mathrm{Dom}(\bar{\partial}^*_\phi) \\ \text{if and only if} \quad f_{I,J} = 0 \text{ whenever } n \in J, \end{gathered} \tag{5.3.7}$$

where $C^\infty_0(U \cap \overline{\Omega})$ denotes the space of functions in $C^\infty(\overline{\Omega})$ which are supported in $U \cap \overline{\Omega}$.

We denote the space $C^\infty_{(p,q)}(\overline{\Omega}) \cap \mathrm{Dom}(\bar{\partial}^*_\phi)$ by $\mathcal{D}_{(p,q)}$ and $C^\ell_{(p,q)}(\overline{\Omega}) \cap \mathrm{Dom}(\bar{\partial}^*_\phi)$ by $\mathcal{D}^\ell_{(p,q)}$, $\ell \in \mathbb{N}$, as before. It follows that

$$\bar{\partial} f = {\sum_{I,J}}' \sum_j \frac{\partial f_{I,J}}{\partial \overline{w}^j} \overline{w}^j \wedge w^I \wedge \overline{w}^J + \cdots = Af + \cdots, \tag{5.3.8}$$

and

$$\vartheta_\phi f = (-1)^{p-1} \sum_{I,K}{}' \sum_j \delta_j^\phi f_{I,jK} w^I \wedge \overline{w}^K + \cdots = Bf + \cdots , \tag{5.3.9}$$

where $\delta_j^\phi u = e^\phi L_j(e^{-\phi}u)$ and dots indicate terms where no derivatives of $f_{I,J}$ occur and which do not involve ϕ. The second equalities in (5.3.8) and (5.3.9) are definitions of A and B which are first order differential operators.

Thus, we have

$$\begin{aligned} \|\overline{\partial} f\|_\phi^2 \ + \ \|\vartheta_\phi f\|_\phi^2 = & \sum_{I,J,L}{}' \sum_{j,\ell} \epsilon_{\ell L}^{jJ} (\overline{L}_j(f_{I,J}),\ \overline{L}_\ell(f_{I,L}))_\phi \\ & + \sum_{I,K}{}' \sum_{j,k} (\delta_j^\phi f_{I,jK},\ \delta_k^\phi f_{I,kK})_\phi + R(f), \end{aligned} \tag{5.3.10}$$

where $R(f)$ involves terms that can be controlled by $O((\| Af \|_\phi + \| Bf \|_\phi) \| f \|_\phi)$ and $\epsilon_{\ell L}^{jJ}$ is defined as before. Rearranging the terms in (5.3.10), we have

$$\begin{aligned} \|\overline{\partial} f\|_\phi^2 \ + \ \|\vartheta_\phi f\|_\phi^2 \ = & \sum_{I,J}{}' \sum_j \|\overline{L}_j f_{I,J}\|_\phi^2 \\ & - \sum_{I,K}{}' \sum_{j,k} (\overline{L}_k f_{I,jK},\ \overline{L}_j f_{I,kK})_\phi \\ & + \sum_{I,K}{}' \sum_{j,k} (\delta_j^\phi f_{I,jK},\ \delta_k^\phi f_{I,kK})_\phi + R(f). \end{aligned} \tag{5.3.11}$$

We apply integration by parts to the terms $(\delta_j^\phi f_{I,jK},\ \delta_k^\phi f_{I,kK})_\phi$. For each $u,\ v \in C_0^1(\overline{\Omega} \cap U)$, Green's formula gives

$$(u,\ \delta_j^\phi v)_\phi = -(\overline{L}_j u,\ v)_\phi + (\sigma_j u,\ v)_\phi + \int_{b\Omega} (\overline{L}_j \rho) u \overline{v} e^{-\phi} dS, \tag{5.3.12}$$

where dS is the surface element on $b\Omega$ and σ_j is in $C^1(\overline{\Omega} \cap U)$. The boundary term in (5.3.12) will vanish if $j < n$ from (5.3.6). If $f \in \mathcal{D}_{(p,q)}$, when we apply (5.3.12) to the terms $(\delta_j^\phi f_{I,jK},\ \delta_k^\phi f_{I,kK})_\phi$, no boundary terms arise since

$$f_{I,J} = 0 \qquad \text{on } b\Omega, \quad \text{if } n \in J, \tag{5.3.13}$$

and if $j < n$ and $n \in J$,

$$L_j(\rho) = \overline{L}_j(\rho) = L_j(f_{I,J}) = \overline{L}_j(f_{I,J}) = 0 \quad \text{on } b\Omega. \tag{5.3.14}$$

In order to calculate the commutator $[\delta_j^\phi,\ \overline{L}_k]$, we use (5.3.2) and (5.3.3),

$$\begin{aligned} [\delta_j^\phi,\ \overline{L}_k]\ u &= [L_j - L_j(\phi),\ \overline{L}_k]u = [L_j,\ \overline{L}_k]u + \overline{L}_k L_j(\phi)u \\ &= \sum_i c_{kj}^i L_i(u) - \sum_i \overline{c}_{jk}^i \overline{L}_i(u) + \overline{L}_k L_j(\phi)u \\ &= \phi_{jk} u + \sum_i c_{kj}^i \delta_i^\phi(u) - \sum_i \overline{c}_{jk}^i \overline{L}_i(u). \end{aligned} \tag{5.3.15}$$

Using (5.3.12)-(5.3.15), for each fixed I, K, j, k, we have

$$
\begin{aligned}
&(\delta_j^\phi f_{I,jK},\ \delta_k^\phi f_{I,kK})_\phi \\
&\quad= (-\overline{L}_k\delta_j^\phi f_{I,jK},\ f_{I,kK})_\phi + (\delta_j^\phi f_{I,jK},\ \bar{\sigma}_k f_{I,kK})_\phi \\
&\quad= (\overline{L}_k f_{I,jK},\ \overline{L}_j f_{I,kK})_\phi + ([\delta_j^\phi,\ \overline{L}_k] f_{I,jK},\ f_{I,kK})_\phi \\
&\qquad - (\overline{L}_k f_{I,jK},\ \sigma_j f_{I,kK})_\phi + (\delta_j^\phi f_{I,jK},\ \bar{\sigma}_k f_{I,kK})_\phi.
\end{aligned}
\tag{5.3.16}
$$

In the above calculation, no boundary terms arise since $f \in \mathcal{D}_{(p,q)}$ and by (5.3.13) and (5.3.14). Introducing the notation

$$\| \overline{L} f \|_\phi^2 = \sum_{I,J}{}' \sum_j \| \overline{L}_j f_{I,J} \|_\phi^2 + \| f \|_\phi^2$$

and applying integration by parts to the last terms of (5.3.16), we see from (5.3.13), (5.3.14) that

$$|(\delta_j^\phi f_{I,jK}, \bar{\sigma}_k f_{I,kK})_\phi| \le C \| \overline{L} f \|_\phi \| f \|_\phi, \tag{5.3.17}$$

where C is a constant independent of ϕ. We shall use $O(\| \overline{L} f \|_\phi \| f \|_\phi)$ to denote terms which are bounded by $C \| \overline{L} f \|_\phi \| f \|_\phi$ where C is a constant independent of ϕ. Thus, (5.3.16) reads

$$
\begin{aligned}
&(\delta_j^\phi f_{I,jK},\ \delta_k^\phi f_{I,kK}) \\
&\quad= (\overline{L}_k f_{I,jK},\ \overline{L}_j f_{I,kK})_\phi + ([\delta_j^\phi,\ \overline{L}_k] f_{I,jK},\ f_{I,kK})_\phi \\
&\qquad + O(\| \overline{L} f \|_\phi \| f \|_\phi) \\
&\quad= (\overline{L}_k f_{I,jK},\ \overline{L}_j f_{I,kK})_\phi + (\phi_{jk} f_{I,jK},\ f_{I,kK})_\phi \\
&\qquad + \left(\sum_i c_{kj}^i \delta_i^\phi f_{I,jK},\ f_{I,kK} \right)_\phi + O(\| \overline{L} f \|_\phi \| f \|_\phi).
\end{aligned}
\tag{5.3.18}
$$

If $i < n$, integration by parts gives

$$|(c_{kj}^i \delta_i^\phi f_{I,jK}, f_{I,kK})_\phi| \le C \| \overline{L} f \|_\phi \| f \|_\phi\ .$$

If $i = n$, we get, using (5.3.5),

$$
\begin{aligned}
&(c_{kj}^n \delta_n^\phi f_{I,jK}, f_{I,kK})_\phi \\
&= \int_{b\Omega\cap U} c_{kj}^n f_{I,jK} \bar{f}_{I,kK} e^{-\phi} dS + O(\| \overline{L} f \|_\phi \| f \|_\phi) \\
&= \int_{b\Omega\cap U} \rho_{jk} f_{I,jK} \bar{f}_{I,kK} e^{-\phi} dS + O(\| \overline{L} f \|_\phi \| f \|_\phi).
\end{aligned}
\tag{5.3.19}
$$

Combining (5.3.11), (5.3.18) and (5.3.19), we obtain

$$
\begin{aligned}
&\|\overline{\partial} f\|_\phi^2 + \|\vartheta_\phi f\|_\phi^2 \\
&\quad= \sum_{I,J}{}' \sum_j \|\overline{L}_j f_{I,J}\|_\phi^2 + \sum_{I,K}{}' \sum_{j,k} (\phi_{jk} f_{I,jK},\ f_{I,kK})_\phi \\
&\qquad + \sum_{I,K}{}' \sum_{j,k} \int_{b\Omega\cap U} \rho_{jk} f_{I,jK} \bar{f}_{I,kK} e^{-\phi} dS + R(f) + E(f),
\end{aligned}
\tag{5.3.20}
$$

where $|E(f)| \le C(\| \overline{L}f \|_\phi \| f \|_\phi)$. Also for any $\epsilon > 0$, there exists a $C_\epsilon > 0$ such that

$$|R(f)| \le \epsilon(\|\overline{\partial} f\|_\phi^2 \ + \ \|\vartheta_\phi f\|_\phi^2) + C_\epsilon \|f\|_\phi^2, \tag{5.3.21}$$

where C_ϵ is independent of ϕ. Thus combining (5.3.20) and (5.3.21), we have proved the following proposition (notice only three derivatives of ρ are required).

Proposition 5.3.3. *Let $\Omega \subset\subset M$ be a complex manifold with C^3 boundary and ρ be a C^3 defining function for Ω. For any $f \in \mathcal{D}^2_{(p,q)}$ such that f vanishes outside a coordinate patch U near a boundary point in $b\Omega$ and $\phi \in C^2(\overline{\Omega})$, we have*

$$\begin{aligned} &\|\overline{\partial} f\|_\phi^2 \ + \ \|\vartheta_\phi f\|_\phi^2 \\ &\ge (1-\epsilon)\sum_{I,K}{}' \sum_{j,k=1}^n (\phi_{jk} f_{I,jK}, \ f_{I,kK})_\phi \\ &\quad + (1-\epsilon)\sum_{I,K}{}' \sum_{j,k=1}^n \int_{b\Omega\cap U} \rho_{jk} f_{I,jK} \bar{f}_{I,kK} e^{-\phi} dS \\ &\quad + (1-\epsilon) \sum_{I,J}{}' \sum_{j=1}^n \|\overline{L}_j f_{I,J}\|_\phi^2 + O(\epsilon)(\| \overline{L}f \|_\phi \| f \|_\phi), \end{aligned} \tag{5.3.22}$$

where $\epsilon > 0$ can be chosen arbitrarily small and $O(\epsilon)(\| \overline{L}f \|_\phi \| f \|_\phi)$ denotes terms which can be bounded by $C_\epsilon \| \overline{L}f \|_\phi \| f \|_\phi$ for some constant C_ϵ independent of ϕ.

Let λ be the smallest eigenvalue of the Hermitian symmetric form

$$\sum_{j,k=1}^n \phi_{jk} a_j \bar{a}_k, \quad a = (a_1, \cdots, a_n) \in \mathbb{C}^n. \tag{5.3.23}$$

Let μ be the smallest eigenvalue of the Levi form

$$\sum_{j,k=1}^n \rho_{jk} a_j \bar{a}_k, \qquad \text{where} \quad \sum_{i=1}^n a_i \frac{\partial \rho}{\partial w^i} = 0. \tag{5.3.24}$$

Note that λ and μ are independent of the choice of the basis $w^1, \cdots, w^n$. We have the following global *a priori* estimates:

Proposition 5.3.4. *Let $\Omega \subset\subset M$ be a complex manifold with C^3 boundary $b\Omega$ and $\phi \in C^2(\overline{\Omega})$. We have the following estimates: for every $f \in \mathcal{D}_{(p,q)}$,*

$$\|\overline{\partial} f\|_\phi^2 \ + \ \|\vartheta_\phi f\|_\phi^2 \ge \frac{1}{2}\left(\int_\Omega (\lambda - C)|f|^2 e^{-\phi} dV + \int_{b\Omega} \mu \ |f|^2 e^{-\phi} dS\right), \tag{5.3.25}$$

where λ is the smallest eigenvalue of the form (5.3.23), μ is the smallest eigenvalue of the Levi form (5.3.24) and C is a constant independent of ϕ.

Proof. Let $\{\eta_i\}_{i=0}^N$ be a partition of unity such that $\eta_0 \in C_0^\infty(\Omega)$ and each η_i, $1 \le i \le N$, is supported in a coordinate patch U_i, $\eta_i \in C_0^\infty(U_i)$, $\overline{\Omega} \subset \Omega \cup (\bigcup_i U_i)$, and

$$\sum_{i=0}^N \eta_i^2 = 1 \qquad \text{on} \quad \overline{\Omega}.$$

Since $f_{I,nK} = 0$ on $b\Omega$, we have, for $1 \le i \le N$,

$$\sum_{I,K}{}' \sum_{j,k} \int_{b\Omega\cap U_i} \rho_{jk} f_{I,jK} \bar{f}_{I,kK} e^{-\phi} dS \ge \int_{b\Omega\cap U_i} \mu\eta_i^2 |f|^2 e^{-\phi} dS.$$

Applying Proposition 5.3.3 to each $\eta_i f$, choosing ϵ sufficiently small, we have

$$\begin{aligned}(5.3.26)\qquad &\int_{b\Omega\cap U_i} \mu\eta_i^2|f|^2e^{-\phi}dS + \int_{U_i} \lambda\eta_i^2|f|^2e^{-\phi}dV \\ &\qquad \le 2(\| \eta_i\bar{\partial} f \|_\phi^2 + \| \eta_i \vartheta_\phi f \|_\phi^2) + C_i \int_{U_i\cap\Omega} |f|^2 e^{-\phi} dV.\end{aligned}$$

The constant C_i depends only on η_i but not on ϕ. Summing up over i, the proposition is proved.

From (5.3.25), we can repeat the same argument as in Chapter 4 to prove the following L^2 existence theorem for $\bar{\partial}$ if there exists a strictly plurisubharmonic function on $\overline{\Omega}$.

Theorem 5.3.5. *Let $\Omega \subset\subset M$ be a pseudoconvex manifold with C^3 boundary $b\Omega$ such that there exists a strictly plurisubharmonic function ϕ on $\overline{\Omega}$. Then for any $f \in L^2_{(p,q)}(\Omega)$ with $\bar{\partial} f = 0$, there exists $u \in L^2_{(p,q-1)}(\Omega)$ such that $\bar{\partial} u = f$.*

Proof. From pseudoconvexity, $\mu \ge 0$ on $b\Omega$ where μ is the smallest eigenvalue of the Levi form. We have that the last term in (5.3.25) is nonnegative. Since ϕ is strictly plurisubharmonic in Ω which is relatively compact in M, it follows that $\lambda > 0$ where λ is the smallest eigenvalue of the form (5.3.23). If we choose $t > 0$ such that $t\lambda \ge C + 2$, where C is as in (5.3.25), we see that for any $g \in \mathcal{D}^2_{(p,q)}$,

$$(5.3.27)\qquad \|g\|^2_{t\phi} \le \|\overline{\partial} g\|^2_{t\phi} + \|\bar{\partial}^*_{t\phi} g\|^2_{t\phi}.$$

Using the same arguments as in the proof of the density lemma, Lemma 4.3.2, we can show that $\mathcal{D}^2_{(p,q)}$ is dense in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*_{t\phi})$ in the graph norm $\|g\|_{t\phi} + \|\overline{\partial} g\|_{t\phi} + \|\bar{\partial}^*_{t\phi} g\|_{t\phi}$ and (5.3.27) holds for any $g \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*_{t\phi})$. Using Lemma 4.1.1, this implies that $\mathcal{R}(\bar{\partial})$ and $\mathcal{R}(\bar{\partial}^*_{t\phi})$ are closed. To show that $\mathcal{R}(\bar{\partial}) = \ker(\bar{\partial})$, we repeat the arguments of the proof of Theorem 4.3.4. For any $g \in L^2_{(p,q)}(\Omega) \cap \mathrm{Dom}(\bar{\partial}^*_{t\phi})$, one has

$$(5.3.28)\qquad |\ (f,g)_{t\phi}\ | \le \|\ f\ \|_{t\phi} \|\ g\ \|_{t\phi} \le \|\ f\ \|_{t\phi} \|\ \bar{\partial}^*_{t\phi} g\ \|_{t\phi}\ .$$

Thus, there exists $u \in L^2_{(p,q-1)}(\Omega)$ such that $\bar{\partial} u = f$ in Ω and

$$\|\ u\ \|_{t\phi} \le \|\ f\ \|_{t\phi}\ .$$

This proves the theorem.

We note that if $M = \mathbb{C}^n$, one can take $\phi = |z|^2$. However, on a general complex manifold, there does not always exist a plurisubharmonic function on M.

Let $\Box_{(p,q)} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ on $\mathrm{Dom}(\Box_{(p,q)})$ and $\mathrm{Dom}(\Box_{(p,q)})$ be defined as in Definition 4.2.2. The arguments of the proof of Proposition 4.2.3 can be applied to show that $\Box_{(p,q)}$ is a linear, closed, densely defined self-adjoint operator on $L^2_{(p,q)}(\Omega)$. Using the L^2 existence theorem 5.3.5, we can obtain the following existence theorem for the $\bar\partial$-Neumann operator on pseudoconvex manifolds.

Theorem 5.3.6. *Let $\Omega \subset\subset M$ be a pseudoconvex Hermitian manifold with C^3 boundary $b\Omega$ such that there exists a strictly plurisubharmonic function ϕ on $\overline{\Omega}$. For each p, q such that $0 \le p \le n, 1 \le q \le n$, there exists a bounded operator $N_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ such that*

(1) *$\mathcal{R}(N_{(p,q)}) \subset Dom(\Box_{(p,q)})$,*
$N_{(p,q)}\Box_{(p,q)} = \Box_{(p,q)}N_{(p,q)} = I$ on $Dom(\Box_{(p,q)})$.
(2) *For any $f \in L^2_{(p,q)}(\Omega)$, $f = \bar\partial\bar\partial^* N_{(p,q)} f \oplus \bar\partial^*\bar\partial N_{(p,q)} f$.*
(3) *$\bar\partial N_{(p,q)} = N_{(p,q+1)}\bar\partial$ on $Dom(\bar\partial)$, $1 \le q \le n-1$.*
(4) *$\bar\partial^* N_{(p,q)} = N_{(p,q-1)}\bar\partial^*$ on $Dom(\bar\partial^*)$, $2 \le q \le n$.*
(5) *For any $f \in L^2_{(p,q)}(\Omega)$ such that $\bar\partial f = 0$, $f = \bar\partial\bar\partial^* N_{(p,q)} f$.*

The proof is exactly the same as the proof for Theorem 4.4.1. One can also show the existence of $N_{(p,0)}$ for $q = 0$ following the same arguments as in Theorem 4.4.3 and we omit the details.

If $b\Omega$ is strongly pseudoconvex, there exists a $c > 0$ such that the smallest eigenvalue of the Levi form $\mu > c > 0$ on $b\Omega$, and we have from (5.3.25) (setting $\phi = 0$),

$$\|\bar\partial f\|^2 \ + \ \|\vartheta f\|^2 \ \ge \frac{c}{2}\int_{b\Omega} |f|^2 dS - C\|f\|^2. \tag{5.3.29}$$

When $b\Omega$ is a strongly pseudoconvex manifold with C^∞ boundary, we can also use the boundary term in the estimates (5.3.29) to obtain the existence and the regularity for the $\bar\partial$-Neumann operator. By using a partition of unity, the Sobolev spaces $W^s(\Omega)$ can be defined on a manifold Ω for any $s \in \mathbb{R}$ (see Appendix A). Using the same arguments as in Section 5.2, we have the following subelliptic estimates.

Theorem 5.3.7. *Let $\Omega \subset\subset M$ be a strongly pseudoconvex Hermitian manifold with C^3 boundary $b\Omega$. There exists a constant $C > 0$ such that for any $f \in Dom(\bar\partial) \cap Dom(\bar\partial^*)$, $0 \le p \le n$ and $1 \le q \le n-1$,*

$$\| f \|^2_{\frac{1}{2}(\Omega)} \le C(\| \bar\partial f \|^2_\Omega + \| \bar\partial^* f \|^2_\Omega + \| f \|^2_\Omega), \tag{5.3.30}$$

where C is independent of f.

The proof is similar to Theorem 5.1.2. We note that in each coordinate patch with a special frame, the operators $\bar\partial$ and ϑ given by (5.3.8) and (5.3.9) differ from (4.2.1) and (4.2.3) only by lower order terms. Thus, the arguments used in proving Theorem 5.1.2 can be easily modified to prove (5.3.30). We omit the details.

We can use (5.3.30) to prove that there exists a $\bar{\partial}$-Neumann operator which inverts $\square_{(p,q)}$. Let

$$\begin{aligned}\mathcal{H}_{(p,q)}(\Omega) &= \{f \in L^2_{(p,q)}(\Omega) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) |\ \bar{\partial} f = \bar{\partial}^* f = 0\} \\ &= \mathrm{Ker}(\bar{\partial}) \cap \mathrm{Ker}(\bar{\partial}^*) \\ &= \mathrm{Ker}(\square_{(p,q)}).\end{aligned}$$

The last equality can be verified in the same way as in the proof of (4.4.2). However, on a strongly pseudoconvex complex manifold, $\mathcal{H}_{(p,q)}(\Omega) = \mathcal{H}_{(p,q)}$ is not always trivial for $q \geq 1$. The following theorem shows that $\mathcal{H}_{(p,q)}$ is always finite dimensional when $q \geq 1$.

Theorem 5.3.8. *Let $\Omega \subset\subset M$ be a strongly pseudoconvex Hermitian manifold with a C^3 boundary $b\Omega$. For any $0 \leq p \leq n$ and $1 \leq q \leq n$, the space $\mathcal{H}_{(p,q)}$ is finite dimensional. Furthermore, the following estimate holds: for any $f \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*) \cap \mathcal{H}^{\perp}_{(p,q)}$,*

$$\| f \|^2_\Omega \leq C(\| \bar{\partial} f \|^2_\Omega + \| \bar{\partial}^* f \|^2_\Omega). \tag{5.3.31}$$

Proof. We have from (5.3.30),

$$\| f \|^2_{\frac{1}{2}(\Omega)} \leq C \| f \|^2_\Omega, \qquad f \in \mathcal{H}_{(p,q)}. \tag{5.3.32}$$

Since $W^{1/2}$ is compact in $L^2(\Omega)$ by the Rellich lemma (see Theorem A.8 in the Appendix), we have that the unit sphere in $\mathcal{H}_{(p,q)}$ is compact. Thus, $\mathcal{H}_{(p,q)}$ is finite dimensional.

If (5.3.31) does not hold, there exists a sequence f_n such that $f_n \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) \cap \mathcal{H}^{\perp}_{(p,q)}$,

$$\| f_n \|^2_\Omega \geq n(\| \bar{\partial} f_n \|^2_\Omega + \| \bar{\partial}^* f_n \|^2_\Omega).$$

Let $\theta_n = f_n / \| f_n \|_\Omega$. Then $\| \bar{\partial}\theta_n \|_\Omega + \| \bar{\partial}^*\theta_n \|_\Omega \to 0$ and $\| \theta_n \|_\Omega = 1$. From (5.3.30) we have $\| \theta_n \|_{\frac{1}{2}(\Omega)} \leq C$. Using the Rellich lemma, there exists a subsequence of θ_n which converges to some element $\theta \in L^2_{(p,q)} \cap \mathcal{H}^{\perp}_{(p,q)}$. However, $\bar{\partial}\theta = \bar{\partial}^*\theta = 0$ and $\| \theta \|_\Omega = 1$, giving a contradiction. This proves (5.3.31).

Let $H_{(p,q)}$ denote the projection onto the subspace $\mathcal{H}_{(p,q)}$ where $0 \leq p \leq n$ and $0 \leq q \leq n$. We note that when $q = 0$, $H_{(p,0)}$ is the projection onto L^2 holomorphic forms, i.e., forms with L^2 holomorphic coefficients. The following theorem gives the existence and regularity of the $\bar{\partial}$-Neumann operator on any strongly pseudoconvex Hermitian manifold.

Theorem 5.3.9. *Let $\Omega \subset\subset M$ be a strongly pseudoconvex Hermitian manifold with C^∞ boundary $b\Omega$. For each p, q such that $0 \leq p \leq n$, $0 \leq q \leq n$, there exists a compact operator $N_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ such that*

(1) $\mathcal{R}(N_{(p,q)}) \subset Dom(\square_{(p,q)})$,
$N_{(p,q)}\square_{(p,q)} = \square_{(p,q)}N_{(p,q)} = I -\ H_{(p,q)}$ *on* $Dom(\square_{(p,q)})$.
(2) *For any* $f \in L^2_{(p,q)}(\Omega)$, $f = \bar{\partial}\bar{\partial}^* N_{(p,q)} f \oplus \bar{\partial}^*\bar{\partial} N_{(p,q)} f \oplus H_{(p,q)} f$.

(3) $\bar{\partial} N_{(p,q)} = N_{(p,q+1)} \bar{\partial}$ *on* $Dom(\bar{\partial})$.
(4) $\bar{\partial}^* N_{(p,q)} = N_{(p,q-1)} \bar{\partial}^*$ *on* $Dom(\bar{\partial}^*)$.
(5) $N_{(p,q)}(C^\infty_{(p,q)}(\overline{\Omega})) \subset C^\infty_{(p,q)}(\overline{\Omega})$, $q \geq 0$.
$H_{(p,q)}(C^\infty_{(p,q)}(\overline{\Omega})) \subset C^\infty_{(p,q)}(\overline{\Omega})$, $q \geq 0$.

Proof. We first prove that $\mathcal{R}(\Box_{(p,q)})$ is closed for any $1 \leq q \leq n$. From (5.3.31), we know that for any $f \in \mathrm{Dom}(\Box_{(p,q)}) \cap \mathcal{H}^\perp_{(p,q)}$,

$$\begin{aligned} \| f \|^2_\Omega &\leq C(\| \bar{\partial} f \|^2_\Omega + \| \bar{\partial}^* f \|^2_\Omega) \\ &= C(\Box_{(p,q)} f, f)_\Omega \\ &\leq C \| \Box_{(p,q)} f \|_\Omega \| f \|_\Omega . \end{aligned}$$

Now applying Lemma 4.1.1, we see that $\mathcal{R}(\Box_{(p,q)})$ is closed and

$$L^2_{(p,q)}(\Omega) = \mathcal{R}(\Box_{(p,q)}) \oplus \mathrm{Ker}(\Box_{(p,q)}) = \mathcal{R}(\Box_{(p,q)}) \oplus \mathcal{H}_{(p,q)}.$$

Let $N_{(p,q)}$ to be the bounded inverse operator of $\Box_{(p,q)}$ on $\mathcal{R}(\Box_{(p,q)})$. We extend $N_{(p,q)}$ to $L^2_{(p,q)}(\Omega)$ by setting $N_{(p,q)} H_{(p,q)} = 0$. One can easily show that (1) and (2) hold. Using (5.3.30), we observe that for any $f \in L^2_{(p,q)}(\Omega)$,

$$\begin{aligned} \| N_{(p,q)} f \|^2_{\frac{1}{2}(\Omega)} &\leq C(\| \Box_{(p,q)} N_{(p,q)} f \|^2_\Omega + \| N_{(p,q)} f \|^2_\Omega) \\ &\leq C \| f \|^2_\Omega . \end{aligned} \tag{5.3.33}$$

The Rellich lemma implies that N is a compact operator. (3) and (4) can be verified by repeating the proofs of (3) and (4) in Theorem 4.4.1. We can establish (5) using the same arguments in the proof of Theorem 5.2.1. From the proof of Theorem 4.4.3, one can show that $N_{(p,0)}$ exists and can be expressed as

$$N_{(p,0)} = \vartheta N^2_{(p,1)} \bar{\partial}. \tag{5.3.34}$$

Also, $N_{(p,0)}$ is bounded. The compactness of $N_{(p,0)}$ follows since $\vartheta N_{(p,1)}$ is compact and $N_{(p,1)} \bar{\partial}$ is bounded (see (4.4.12)).

Using the same arguments as the proof of Theorems 5.2.1 and 5.2.6, we have the following more precise estimates.

Theorem 5.3.10. *Let* $\Omega \subset\subset M$ *be a strongly pseudoconvex Hermitian manifold with* C^∞ *boundary* $b\Omega$. *For* $q \geq 1$ *and each* $k = 0, 1, 2, \cdots$, *there exists a constant* $C_k > 0$ *such that for any* $f \in W^k_{(p,q)}(\Omega)$,

$$\|N_{(p,q)} f\|_{k+1} \leq C_k \|f\|_k, \tag{5.3.35}$$

$$\|\bar{\partial}^* N_{(p,q)} f\|_{k+\frac{1}{2}} + \|\bar{\partial} N_{(p,q)} f\|_{k+\frac{1}{2}} \leq C_k \|f\|_k. \tag{5.3.36}$$

Corollary 5.3.11. *Let $\Omega \subset\subset M$ be a strongly pseudoconvex Hermitian manifold with C^∞ boundary $b\Omega$. For any $f \in W^k_{(p,q)}(\Omega)$, $q \geq 1$ and $k \geq 0$, such that $\bar\partial f = 0$ in Ω and $H_{(p,q)}f = 0$, we have $u = \bar\partial^* N f$ is a solution of $\bar\partial u = f$ in Ω and*

$$\|u\|_{k+\frac{1}{2}} \leq C\|f\|_k,$$

where C is a constant independent of f. In particular, if $f \in C^\infty_{(p,q)}(\overline{\Omega})$, $\bar\partial f = 0$ and $H_{(p,q)}f = 0$, there exists a solution $u \in C^\infty_{(p,q-1)}(\overline{\Omega})$ such that $\bar\partial u = f$ in Ω.

The solution u is called the canonical solution (or Kohn's solution) to the equation $\bar\partial u = f$ and it is the unique solution which is orthogonal to $\mathrm{Ker}(\bar\partial)$.

An important consequence of Corollary 5.3.11 is the solution to the Levi problem on a strongly pseudoconvex manifold with smooth boundary. A complex manifold Ω with smooth boundary $b\Omega$ is called a domain of holomorphy if for every $p \in b\Omega$ there is a holomorphic function on Ω which is singular at p (c.f. Definition 3.5.1). In Theorem 4.5.2, we have already proved that pseudoconvex domains in $\mathbb{C}^n$ are domains of holomorphy. The next theorem shows that strongly pseudoconvex domains in complex manifolds are domains of holomorphy.

Theorem 5.3.12. *Let $\Omega \subset\subset M$ be a strongly pseudoconvex manifold with C^∞ boundary $b\Omega$. Then Ω is a domain of holomorphy.*

Proof. For each boundary point $p \in b\Omega$, we will construct a function $h(z)$ such that $h \in C^\infty(\overline{\Omega} \setminus \{p\})$, h is holomorphic in Ω and

$$\lim_{z \to p} h(z) = \infty.$$

Since Ω is strongly pseudoconvex with C^∞ boundary $b\Omega$, one can construct a *local* holomorphic function f on $\Omega \cap U$ where U is an open neighborhood of p such that f is singular at p. To do this, let $z_1, \cdots, z_n$ be a coordinate system in the neighborhood U of p with origin at p. Let r be a smooth defining function for Ω such that r is strictly plurisubharmonic near p. That such a defining function exists follows from the same arguments as before (see Theorem 3.4.4). Let

$$P(z) = -2\sum_{i=1}^n \frac{\partial r}{\partial z_i}(0) z_i - \sum_{i,j=1}^n \frac{\partial^2 r}{\partial z_i \partial z_j}(0) z_i z_j.$$

$P(z)$ is holomorphic in U. Using Taylor's expansion at 0, there exists a sufficiently small neighborhood $V \subset U$ of 0 and $C > 0$ such that for any $z \in \overline{\Omega} \cap V$,

$$\mathrm{Re}P(z) = -r(z) + \sum_{i,j=1}^n \frac{\partial^2 r}{\partial z_i \partial \bar z_j}(0) z_i \bar z_j + O(|z|^3) \geq C|z|^2.$$

Thus, $P(z) \neq 0$ when $z \in \overline{\Omega} \cap V \setminus \{0\}$. Setting

$$f = \frac{1}{P},$$

it is easily seen that f is locally a holomorphic function which cannot be extended holomorphically across 0.

Let χ be a cut-off function such that $\chi \in C_0^\infty(V)$ and $\chi = 1$ in a neighborhood of 0. We extend χ to be 0 on $\Omega \setminus V$. Let g be the $(0,1)$-form defined by

$$g = \bar\partial(\chi f) = (\bar\partial \chi) f \qquad \text{in } \Omega.$$

Obviously g can be extended smoothly up to the boundary. Thus, $g \in C^\infty_{(0,1)}(\overline{\Omega})$ and $\bar\partial g = 0$ in Ω. To show that $H_{(0,1)} g = 0$, notice that when $n \geq 3$, $\chi f \in L^2(\Omega)$. Thus, $g \in \mathcal{R}(\bar\partial) = \mathrm{Ker}(\bar\partial^*)^\perp$ and $g \perp \mathcal{H}_{(0,1)}$. When $n = 2$, we approximate f by $f_\epsilon = \frac{1}{P+\epsilon}$ for $\epsilon \searrow 0$. Then $\chi f_\epsilon \in C^\infty(\overline{\Omega})$ and $\bar\partial(\chi f_\epsilon) \to g$ in L^2. However, each $\bar\partial(\chi f_\epsilon)$ is in the $\mathcal{R}(\bar\partial)$ which is closed from Theorem 5.3.9. This implies that $g \in \mathcal{R}(\bar\partial)$ and $H_{(0,1)} g = 0$ for $n = 2$ also. We define

$$u = \bar\partial^* N_{(0,1)} g.$$

It follows from Corollary 5.3.11 that $u \in C^\infty(\overline{\Omega})$ and $\bar\partial u = g$ in Ω. Let h be defined by

$$h = \chi f - u.$$

Then, h is holomorphic in Ω, $h \in C^\infty(\overline{\Omega} \setminus \{p\})$ and h is singular at p. This proves the theorem.

Thus, the Levi problem for strongly pseudoconvex manifolds with smooth boundaries is solved.

5.4 Almost Complex Structures

In Chapter 2 we study when a complex vector field in $\mathbb{R}^2$ is actually a Cauchy-Riemann equation in other coordinates. In this section we study the n-dimensional analog of this problem.

Definition 5.4.1. *Let M be a real C^∞ manifold of dimension $2n$. An almost complex structure $T^{1,0}(M)$ is a subbundle of the complexified tangent bundle $\mathbb{C}T(M)$ such that*

(1) $\mathbb{C}T(M) = T^{1,0}(M) + T^{0,1}(M)$,

(2) $T^{1,0}(M) \cap T^{0,1}(M) = \{0\}$,

where $T^{0,1}(M) = \overline{T^{1,0}(M)}$. M is called an almost complex manifold with an almost complex structure $T^{1,0}(M)$.

When M is a complex manifold, there is a canonical $T^{1,0}$ defined on M, namely, the holomorphic vector bundle. In local holomorphic coordinates $z_1, \cdots, z_n$, in a neighborhood U, we have that

$$T^{1,0}(M \cap U) = \left\langle \frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n} \right\rangle, \tag{5.4.1}$$

where the right-hand side denotes the linear span by vector fields $\partial/\partial z_1, \cdots, \partial/\partial z_n$. Let $C^\infty(T^{1,0}(M))$ denote the smooth sections of $T^{1,0}(M)$. If M is a complex manifold, we have

$$[L, L'] \in C^\infty(T^{1,0}(M)), \quad \text{for any } L,\ L' \in C^\infty(T^{1,0}(M)). \tag{5.4.2}$$

Definition 5.4.2. *An almost complex structure $T^{1,0}(M)$ is called integrable if (5.4.2) is satisfied.*

A complex manifold is an integrable almost complex manifold. The Newlander-Nirenberg theorem states that the converse is also true. Before we state and prove the theorem, we first note that on an almost complex manifold, there is a notion of the Cauchy-Riemann equations and the $\bar{\partial}$ operator.

Let $\Pi_{1,0}, \Pi_{0,1}$ denote the projection from $\mathbb{C}T(M)$ onto $T^{1,0}(M)$ and $T^{0,1}(M)$ respectively. Then, one has

$$\Pi_{1,0} + \Pi_{0,1} = 1, \quad \Pi_{1,0}\Pi_{0,1} = 0, \quad \text{and} \quad \Pi_{0,1} = \overline{\Pi_{1,0}}.$$

The last equation means that $\Pi_{0,1}\zeta = \overline{\Pi_{1,0}\bar{\zeta}}$, for $\zeta \in \mathbb{C}T(M)$. Thus there is a natural splitting of the differential 1-forms $\Lambda^1(M)$ into (1,0)-forms, $\Lambda^{1,0}(M)$, and (0,1)-forms, $\Lambda^{0,1}(M)$, which are defined to be the dual of $T^{1,0}(M)$ and $T^{0,1}(M)$ respectively. We shall still use $\Pi_{1,0}$, $\Pi_{0,1}$ to denote the projection from $\Lambda^1(M)$ onto $\Lambda^{1,0}(M)$ and $\Lambda^{0,1}(M)$ respectively. For any smooth function u, we have $du = \Pi_{1,0}du + \Pi_{0,1}du$. On an almost complex manifold, one can define the Cauchy-Riemann equation by

$$\bar{\partial}u = \Pi_{0,1}du \quad \text{and} \quad \partial u = \Pi_{1,0}du,$$

where u is any smooth function on M. We can also extend this definition to (p,q)-forms and define ∂ and $\bar{\partial}$ on (p,q)-forms f to be the projection of the exterior derivative df into the space of $(p+1,q)$-forms and $(p,q+1)$-forms respectively. The integrability condition guarantees that $\bar{\partial}$ is a complex.

Lemma 5.4.3. *If an almost complex structure is integrable, then $d = \partial + \bar{\partial}$ and*

$$\partial^2 = \partial\bar{\partial} + \bar{\partial}\partial = \bar{\partial}^2 = 0.$$

Proof. If one can show that $d = \partial + \bar{\partial}$, then $\partial^2 = \partial\bar{\partial} + \bar{\partial}\partial = \bar{\partial}^2 = 0$ follows easily from degree consideration. It is obvious that $d = \partial + \bar{\partial}$ on functions. For 1-forms, we have $\Lambda^1(M) = \Lambda^{1,0}(M) + \Lambda^{0,1}(M)$. To verify for 1-forms, it suffices to prove for each (0,1)-form and each (1,0)-form. If f is a (0,1)-form, for any $L, L' \in C^\infty(T^{1,0}(M))$, we have

$$df(L, L') = \frac{1}{2}\left(L(f, L') - L'(f, L) - (f, [L, L'])\right) = 0$$

since $T^{1,0}(M)$ is integrable. This shows that df has no component of (2,0)-forms. Similarly, if f is a (1,0)-form, df has no component of (0,2)-forms. In each case, $df = \partial f + \bar{\partial} f$. The general case follows from the fact that each (p,q)-form can be written as linear combination of forms of the type

$$h = f_1 \wedge \cdots \wedge f_p \wedge g_1 \wedge \cdots \wedge g_q,$$

where f_i's are (1,0)-forms and g_j's are (0,1)-forms. Since dh is a sum of a type $(p+1,q)$-form and a type $(p,q+1)$-form, we have $dh = \partial h + \bar{\partial} h$. This proves the lemma.

Theorem 5.4.4 (Newlander-Nirenberg). *An integrable almost complex manifold is a complex manifold.*

Proof. This problem is purely local and we shall assume that M is a small neighborhood of 0 in $\mathbb{R}^{2n}$. Let $L_1, \cdots, L_n$ be a local basis for smooth sections of $T^{1,0}(M)$.

If we can find complex-valued functions $\zeta_1, \cdots, \zeta_n$ such that

$$\overline{L}_i \zeta_j = 0, \quad i, j = 1, \cdots, n, \tag{5.4.3}$$

where $d\zeta_1, \cdots, d\zeta_n$ are linearly independent, then the theorem will be proved since

$$\langle L_1, \cdots, L_n \rangle \;=\; \left\langle \frac{\partial}{\partial \zeta_1}, \cdots, \frac{\partial}{\partial \zeta_n} \right\rangle, \tag{5.4.4}$$

where $\langle L_1, \cdots, L_n \rangle$ denotes the linear span of $L_1, \cdots, L_n$ over $\mathbb{C}$.

Let $x_1, \cdots, x_{2n}$ be the real coordinates for M and we write $z_j = x_j + ix_{n+j}$. We can, after a quadratic change of coordinates, assume that

$$L_i = \frac{\partial}{\partial z_i} + \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial \bar{z}_j}, \quad i = 1, \cdots, n, \tag{5.4.5}$$

where the a_{ij}'s are smooth functions and $a_{ij}(0) = 0$ for all $i, j = 1, \cdots, n$. At the origin, L_i is the constant coefficient operator $\partial/\partial z_i$. We shall show that (5.4.4) can be solved in a small neighborhood of 0. Let

$$L_i^\epsilon = \frac{\partial}{\partial z_i} + \sum_{j=1}^{n} a_{ij}(\epsilon x) \frac{\partial}{\partial \bar{z}_j}, \quad i = 1, \cdots, n, \tag{5.4.6}$$

where $\epsilon > 0$ is small. Then

$$T_\epsilon^{0,1} = \; \langle L_1^\epsilon, \cdots, L_n^\epsilon \rangle$$

defines an almost complex structure that is integrable for each $\epsilon < \epsilon_0$ for some sufficiently small $\epsilon_0 > 0$.

From Lemma 5.4.3, there is a Cauchy-Riemann complex, denoted by $\bar{\partial}_\epsilon$, associated with each almost complex structure $T_\epsilon^{0,1}$. We shall equip the almost complex structure with a Hermitian metric. Then the existence and regularity theory developed for $\bar{\partial}$ in the previous section on any complex manifold can be applied to M with $\bar{\partial}$ substituted by $\bar{\partial}_\epsilon$. Let $\phi = \sum_{i=1}^n |z_i|^2 = |x|^2$, then at 0 we see that

$$\sum_{j,k=1}^{n} \phi_{jk} a_j \bar{a}_k = \sum_{i=1}^{n} |a_i|^2.$$

Thus, ϕ is a strictly plurisubharmonic function near 0. If we set

$$\Omega = \{x \in M \mid |x|^2 < \delta\}$$

for some small $\delta > 0$, then Ω is strongly pseudoconvex with respect to the almost complex structure $T_\epsilon^{0,1}(M)$. Using Corollary 5.3.11, there exists a solution u_i^ϵ on Ω such that

$$\bar{\partial}_\epsilon u_i^\epsilon = \bar{\partial}_\epsilon z_i \tag{5.4.7}$$

and

$$\| u_i^\epsilon \|_s \le C_s \| \bar{\partial}_\epsilon z_i \|_s, \tag{5.4.8}$$

where C_s can be chosen uniformly for $\epsilon < \epsilon_0$. Since

$$\overline{L}_i^\epsilon z_j = \bar{a}_{ij}(\epsilon x),$$

we have

$$D^\alpha \bar{\partial}_\epsilon z_i = O(\epsilon)$$

for any $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_{2n})^{\alpha_{2n}}$, where the α_i's are nonnegative integers. This implies that

$$\| \bar{\partial}_\epsilon z_i \|_s \to 0 \quad \text{if} \quad \epsilon \to 0. \tag{5.4.9}$$

Let

$$\zeta_i^\epsilon = z_i - u_i^\epsilon. \tag{5.4.10}$$

The Sobolev embedding theorem (see Theorem A.7 in the Appendix) shows that if we choose $s > n + 1$, then

$$|du_i^\epsilon(0)| \le \| u_i^\epsilon \|_s \le C_s \| \bar{\partial}_\epsilon z_i \|_s \to 0 \quad \text{if } \epsilon \to 0.$$

We have from (5.4.7) that $\bar{\partial}_\epsilon \zeta_i^\epsilon = 0$ in Ω and also $d\zeta_i^\epsilon(0) = dz_i - du_i^\epsilon(0)$ are linearly independent if ϵ is sufficiently small. If we pull back ζ_i^ϵ to $\epsilon\Omega$ by setting $\zeta_i = \zeta_i^\epsilon(x/\epsilon)$, then we have that $\bar{\partial}\zeta_i = 0$ and $d\zeta_i$ are linearly independent in $\epsilon\Omega$ provided we choose ϵ sufficiently small. This proves the theorem.

NOTES

The subelliptic 1/2-estimates and boundary regularity for the $\bar{\partial}$-Neumann operator on strongly pseudoconvex manifolds were proved in J. J. Kohn [Koh 1]. Much of the material concerning strong pseudoconvex domains in this chapter was obtained there. The use of a special boundary frame was due to M. E. Ash [Ash 1]. A simplification of the proof of the boundary regularity for subelliptic operators using pseudodifferential operators was given in J. J. Kohn and L. Nirenberg [KoNi 1] where the method of elliptic regularization was used in order to pass from *a priori* estimates to actual estimates. In [KoNi 1], a systematic treatment of the subelliptic boundary value problem with any subellipticity $0 < \epsilon \le 1/2$ was discussed.

Pseudodifferential operators and subelliptic estimates will be discussed in Chapter 8.

The proof of subelliptic 1/2-estimates in Theorem 5.1.2 follows the approach of J. Michel and M.-C. Shaw [MiSh 1]. The proof of the boundary regularity discussed in 5.2 is a variation of the proof used in [KoNi 1] since only commutators of differential operators are used but not pseudodifferential operators. The discussion of function theory on manifolds mainly follows that of L. Hörmander [Hör 3]. Global strictly plurisubharmonic functions do not always exists on general complex manifolds. If M is a Stein manifold, then there exists a strictly plurisubharmonic exhaustion function for M. Thus Theorem 5.3.5 can be applied to any relatively compact pseudoconvex manifold Ω which lies in a Stein manifold. For a detailed discussion of function theory on Stein manifolds, see Chapter 5 in L. Hörmander [Hör 9].

The Levi problem on a strongly pseudoconvex manifold (Theorem 5.3.12) was first solved by H. Grauert [Gra 1] using sheaf theory. The proof of Theorem 5.3.12 using the existence and the regularity of the $\bar\partial$-Neumann operator was due to J. J. Kohn [Koh 1]. Since a pseudoconvex domain in $\mathbb{C}^n$ by definition can be exhausted by strongly pseudoconvex domains, using a result of H. Behnke and K. Stein (see [BeSt 1] or [GuRo 1], one can deduce that any pseudoconvex domain in $\mathbb{C}^n$ is a domain of holomorphy (c.f. Theorem 4.5.2). This needs not be true for pseudoconvex domains in complex manifolds (see J. E. Fornaess [For 1]). More discussions on the Levi problem on pseudoconvex manifolds can be found in [For 3], [Siu 2].

Theorem 5.4.4 was first proved by A. Newlander and L. Nirenberg [NeNi 1]. B. Malgrange has given a totally different proof (see B. Malgrange [Mal 2] or L. Nirenberg [Nir 3]). There is yet another proof, due to S. Webster [Web 1], using integral kernel methods. Our proof was essentially given in J. J. Kohn [Koh 1] as an application of the $\bar\partial$-Neumann problem.

There is a considerable amount of literature on the $\bar\partial$-Neumann operator, the canonical solution and the Bergman projection on strongly pseudoconvex domains in other function spaces, including Hölder and L^p spaces (See R. Beals, P. C. Greiner and N. Stanton [BGS 1], R. Harvey and J. Polking [HaPo 3,4], I. Lieb and R. M. Range [LiRa 2,3,4], A. Nagel and E. M. Stein [NaSt 1], D. H. Phong and E. M. Stein [PhSt 1], R. M. Range [Ran 5] and the references therein). We also refer the reader to the article by M. Beals, C. Fefferman and R. Grossman [BFG 1] for more discussions on strongly pseudoconvex domains.

CHAPTER 6

BOUNDARY REGULARITY FOR $\overline{\partial}$ ON PSEUDOCONVEX DOMAINS

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary bD. In this chapter, we study the global regularity of the equation

$$\overline{\partial} u = f \quad \text{on} \quad D, \tag{6.0.1}$$

where $f \in C^\infty_{(p,q)}(\overline{D})$ with $0 \le p \le n$, $1 \le q \le n$ and f is $\overline{\partial}$-closed.

The existence theorems for $\overline{\partial}$ and the $\overline{\partial}$-Neumann operator N on any bounded pseudoconvex domain have been proved in Chapter 4 in L^2 spaces. In this chapter we are interested in the following questions:

(1) Can one solve equation (6.0.1) with a smooth solution $u \in C^\infty_{(p,q-1)}(\overline{D})$ if f is in $C^\infty_{(p,q)}(\overline{D})$?

(2) Does the canonical solution $\overline{\partial}^* Nf$ belong to $W^s_{(p,q-1)}(D)$ if f is in $W^s_{(p,q)}(D)$?

(3) Does the Bergman projection P preserve $C^\infty(\overline{D})$ or $W^s(D)$?

(4) Under what conditions can a biholomorphic mapping between two smooth bounded domains be extended smoothly up to the boundaries?

From the results in Chapter 5, both N and $\overline{\partial}^* N$ are regular in Sobolev spaces or the C^∞ category if we assume that bD is smooth and strongly pseudoconvex. In fact, the canonical solution has a "gain" of 1/2 derivative in the Sobolev spaces. Here, we study the global boundary regularity for $\overline{\partial}$ and the $\overline{\partial}$-Neumann operator N on a smooth bounded weakly pseudoconvex domain in $\mathbb{C}^n$.

In Section 6.1, we prove that the first question can be answered affirmatively. This result is proved using the weighted $\overline{\partial}$-Neumann problem. However, the smooth solution might not be the canonical solution. In Section 6.2, we study the global regularity for N when the domain has either a smooth plurisubharmonic defining function or certain transverse symmetry. We establish the Sobolev estimates for the $\overline{\partial}$-Neumann operator and the regularity of the canonical solution on such domains. This result implies the regularity of the Bergman projection and is used to prove the boundary regularity of biholomorphic mappings between pseudoconvex domains. In general, a smooth pseudoconvex domain does not necessarily have a plurisubharmonic defining function or transverse symmetry. A counterexample, known as the worm domain, is constructed in Section 6.4. Finally, we prove in Section 6.5 that, for any $s > 0$, there is a smooth bounded pseudoconvex domain on which the $\overline{\partial}$-Neumann operator fails to be regular in any Sobolev spaces W^k for $k \ge s$.

6.1 Global Regularity for $\overline{\partial}$ on Pseudoconvex Domains with Smooth Boundaries

The main result in the section is the following theorem:

Theorem 6.1.1. *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with $n \geq 2$. For every $f \in C^\infty_{(p,q)}(\overline{D})$, where $0 \leq p \leq n$, $1 \leq q \leq n$ with $\overline{\partial} f = 0$, one can find $u \in C^\infty_{(p,q-1)}(\overline{D})$ such that $\overline{\partial} u = f$.*

We will prove the theorem using the weighted $\overline{\partial}$-Neumann operator with respect to the weighted L^2 norm $L^2(D, \phi_t)$ introduced in Section 4.2, where $\phi_t = t|z|^2$ for some $t > 0$. Theorem 6.1.1 will be proved at the end of this section. We note that $L^2(D, \phi_t) = L^2(D)$. The existence for the weighted $\bar{\partial}$-Neumann operator on any pseudoconvex domain with smooth boundary follows from the discussion in Chapter 4. We briefly describe below.

From Proposition 4.3.3, we have for any (p,q)-form $f \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*_{\phi_t})$,

$$\int_D \sum_{I,K}{}' \sum_{j,k} \frac{\partial^2 \phi_t}{\partial z_j \partial \overline{z}_k} f_{I,jK} \overline{f}_{I,kK}\, e^{-\phi_t} dV \leq \| \overline{\partial} f \|^2_{\phi_t} + \| \overline{\partial}^*_{\phi_t} f \|^2_{\phi_t} .$$

Using the notation $\| \ \|_{(t)} = \| \ \|_{\phi_t}$ and $\overline{\partial}^*_t = \overline{\partial}^*_{\phi_t}$, we see that for any $f \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*_t)$,

$$tq \| f \|^2_{(t)} \leq \| \overline{\partial} f \|^2_{(t)} + \| \overline{\partial}^*_t f \|^2_{(t)} . \tag{6.1.1}$$

Let $\Box_t = \overline{\partial}\overline{\partial}^*_t + \overline{\partial}^*_t \overline{\partial}$. If $f \in \mathrm{Dom}(\Box_t)$, from (6.1.1), we have that

$$\begin{aligned} tq \| f \|^2_{(t)} &\leq \| \overline{\partial} f \|^2_{(t)} + \| \overline{\partial}^*_t f \|^2_{(t)} \\ &= (\Box_t f, f)_{(t)} \\ &\leq \| \Box_t f \|_{(t)} \| f \|_{(t)} . \end{aligned} \tag{6.1.2}$$

Applying Lemma 4.1.1, (6.1.2) implies that the range of $\Box_t$ is closed and $\Box_t$ is one-to-one. Thus, $\Box_t$ has a bounded inverse N_t, the $\bar{\partial}$-Neumann operator with weight ϕ_t. We can also show the following existence theorem of the weighted $\bar{\partial}$-Neumann operator on any bounded pseudoconvex domain by repeating the same argument as in Theorem 4.4.1:

Theorem 6.1.2. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. For each $0 \leq p \leq n$, $1 \leq q \leq n$ and $t > 0$, there exists a bounded operator $N_t : L^2_{(p,q)}(D) \to L^2_{(p,q)}(D)$, such that*

(1) *Range$(N_t) \subset$ Dom$(\Box_t)$. $N_t \Box_t = \Box_t N_t = I$ on Dom$(\Box_t)$.*
(2) *For any $f \in L^2_{(p,q)}(D)$, $f = \overline{\partial}\overline{\partial}^*_t N_t f \oplus \overline{\partial}^*_t \overline{\partial} N_t f$.*
(3) *$\overline{\partial} N_t = N_t \overline{\partial}$ on Dom$(\overline{\partial})$, $1 \leq q \leq n-1$,*
$\overline{\partial}^*_t N_t = N_t \overline{\partial}^*_t$ on Dom$(\overline{\partial}^*_t)$, $2 \leq q \leq n$.
(4) *The following estimates hold: For any $f \in L^2_{(p,q)}(D)$,*

$$tq \| N_t f \|_{(t)} \leq \| f \|_{(t)},$$

$$\sqrt{tq} \parallel \overline{\partial} N_t f \parallel_{(t)} \leq \parallel f \parallel_{(t)},$$

$$\sqrt{tq} \parallel \overline{\partial}^* N_t f \parallel_{(t)} \leq \parallel f \parallel_{(t)} .$$

(5) *If $f \in L^2_{(p,q)}(D)$ and $\overline{\partial} f = 0$ in D, then for each $t > 0$, there exists a solution $u_t = \overline{\partial}^*_t N_t f$ satisfying $\overline{\partial} u_t = f$ and the estimate*

$$tq \parallel u_t \parallel^2_{(t)} \leq \parallel f \parallel^2_{(t)} .$$

In Chapter 4, we have chosen $t = \delta^{-2}$, where δ is the diameter of D, to obtain the best constant for the bound of the $\bar{\partial}$-Neumann operator without weights. Our next theorem gives the regularity for N_t in the Sobolev spaces when t is large.

Theorem 6.1.3. *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. For every nonnegative integer k, there exists a constant $S_k > 0$ such that the weighted $\bar{\partial}$-Neumann operator N_t maps $W^k_{(p,q)}(D)$ boundedly into itself whenever $t > S_k$, where $0 \leq p \leq n$, $1 \leq q \leq n$.*

Proof. We first prove the *a priori* estimates for N_t when t is large. Let ϑ_t be the formal adjoint of $\bar{\partial}$ with respect to the weighted norm $L^2(D, \phi_t)$. Note that for any $f \in C^\infty_{(p,q)}(D)$,

$$\vartheta_t f = e^{\phi_t} \vartheta (e^{-\phi_t} f) = \vartheta f + tA_0 f$$

for some zeroth order operator A_0. Hence, we have that for any $f \in C^\infty_{(p,q)}(D)$ with compact support in D,

$$\begin{aligned} Q^t(f,f) &= \parallel \overline{\partial} f \parallel^2_{(t)} + \parallel \vartheta_t f \parallel^2_{(t)} \\ &\geq \parallel \overline{\partial} f \parallel^2_{(t)} + \frac{1}{2} \parallel \vartheta f \parallel^2_{(t)} - C_t \parallel f \parallel^2_{(t)} \\ &\geq \parallel f \parallel_1 - C_t \parallel f \parallel^2_{(t)}, \end{aligned}$$

where $\parallel f \parallel_k = \sum_{0 \leq |\alpha| \leq k} \parallel D^\alpha f \parallel_{(t)}$.

Thus the Gårding inequality holds for compactly supported forms and the estimates in the interior are the same as before. We only need to estimate the solution near the boundary. In the following, C and C_k will always denote a constant independent of t.

Since the normal differentiation is controlled by $\overline{\partial}$, $\overline{\partial}^*$ and the tangential derivatives, we shall only consider the action of tangential differentiations. Let U be a special boundary chart near the boundary and $w_1, \cdots, w_n = \partial r$ be a special boundary frame as before, where r is a defining function normalized such that $|dr| = 1$ on bD. We let T^k be a tangential operator of order k and $\eta \in C^\infty_0(U)$ as defined in Proposition 5.2.5. We use induction on k to prove the following estimate:

$$\parallel f \parallel^2_k \leq C_{k,t} \parallel \Box_t f \parallel^2_k, \qquad f \in \mathrm{Dom}(\Box_t) \cap C^\infty_{(p,q)}(\overline{D}). \tag{6.1.3}$$

When $k = 0$, (6.1.3) holds for any $t > 0$ by (6.1.2). We assume that (6.1.3) is true for $k - 1$ where $k \geq 1$.

From the same argument as in Lemma 5.2.2, writing f in the special frame, we see that if $f \in \mathrm{Dom}(\bar{\partial}_t^*) \cap C^\infty_{(p,q)}(\overline{D})$, then $\eta T^k f \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}_t^*)$. We obtain from (6.1.1) that

$$t \parallel \eta T^k f \parallel^2_{(t)} \le (\parallel \overline{\partial}(\eta T^k f) \parallel^2_{(t)} + \parallel \vartheta_t(\eta T^k f) \parallel^2_{(t)}). \tag{6.1.4}$$

We note that the commutator $[\vartheta_t, \eta T^k] = A_k + A^t_{k-1}$, where A_k is a kth order differential operator independent of t and A^t_{k-1} is of order $k-1$. Thus, using arguments similar to (5.2.6) and (5.2.7), keeping track of the dependence on t, we have

$$\begin{aligned}
&\parallel \overline{\partial}(\eta T^k f) \parallel^2_{(t)} + \parallel (\vartheta_t \eta T^k f) \parallel^2_{(t)} \\
&\le C_k(\parallel \eta T^k \overline{\partial} f \parallel^2_{(t)} + \parallel \eta T^k \vartheta_t f \parallel^2_{(t)} + \parallel f \parallel^2_k) \\
&\qquad + C_{k,t} \parallel f \parallel^2_{k-1} \\
&\le C_k(|(\eta T^k f,\ \eta T^k \Box_t f)_{(t)}| + \parallel f \parallel^2_k) + C_{k,t} \parallel f \parallel^2_{k-1} \\
&\le C_k(\parallel \eta T^k f \parallel_{(t)} \parallel \eta T^k \Box_t f \parallel_{(t)} + \parallel f \parallel^2_k) + C_{k,t} \parallel f \parallel^2_{k-1} .
\end{aligned} \tag{6.1.5}$$

Combining (6.1.4) and (6.1.5), we get

$$t \parallel \eta T^k f \parallel^2_{(t)} \le C_k(\parallel \Box_t f \parallel^2_k + \parallel f \parallel^2_k) + C_{k,t} \parallel f \parallel^2_{k-1}, \tag{6.1.6}$$

where the constant C_k is independent of t.

Repeating the arguments of Lemma 5.2.4, we observe that

$$\parallel f \parallel^2_k \le C_k(\parallel \Box_t f \parallel^2_{k-1} + |||f|||^2_k) + C_{k,t} \parallel f \parallel^2_{k-1} .$$

Summing up all the tangential derivatives of the form ηT^k in (6.1.6) and using a partition of unity $\{\eta_i\}_{i=1}^N$, such that $\sum_{i=1}^N \eta_i^2 = 1$ on $\overline{D}$, there exists a constant C_k such that

$$\begin{aligned}
t \parallel f \parallel^2_k \le C_k(&\parallel \Box_t f \parallel^2_k + \parallel f \parallel^2_k) \\
&+ C_{k,t} \parallel \Box_t f \parallel^2_{k-1} + C_{k,t} \parallel f \parallel^2_{k-1} .
\end{aligned} \tag{6.1.7}$$

Choosing $t > C_k + 1$, it follows, using the induction hypothesis, that

$$\begin{aligned}
\parallel f \parallel^2_k &\le C_k \parallel \Box_t f \parallel^2_k + C_{k,t} \parallel \Box_t f \parallel^2_{k-1} + C_{k,t} \parallel f \parallel^2_{k-1} \\
&\le C_{k,t} \parallel \Box_t f \parallel^2_k .
\end{aligned}$$

This proves the *a priori* estimates (6.1.3) for the weighted $\overline{\partial}$-Neumann operators N_t when t is sufficiently large. Using the elliptic regularization method as in the proof of Theorem 5.2.1, one can pass from the *a priori* estimates to actual estimates and the theorem is proved.

Arguing as in Theorem 4.4.3, we can prove that the weighted $\bar{\partial}$-Neumann operator $N_{t,(p,0)}$ also exists for $q = 0$. Let $P_{t,(p,0)}$ denote the weighted Bergman projection from $L^2_{(p,0)}(D)$ onto the closed subspace $\mathcal{H}_{(p,0)}(D) = \{f \in L^2_{(p,0)}(D) \mid \bar{\partial} f = 0\}$ with respect to the weighted norm $L^2(D,\phi_t)$. We have $N_{t,(p,0)} : L^2_{(p,0)}(D) \to L^2_{(p,0)}(D)$ such that

$$\Box_{t,(p,0)} N_{t,(p,0)} = I - P_{t(p,0)}$$

and

$$N_{t,(p,0)} = \bar\partial_t^* N^2_{t,(p,1)} \bar\partial.$$

As in the proof of Corollary 4.4.4, the weighted Bergman projection is given by

$$P_{t,(p,0)} = I - \bar\partial_t^* N_{t,(p,1)} \bar\partial.$$

An operator is called exactly regular on $W^k_{(p,q)}(D)$, $k \geq 0$, if it maps the Sobolev space $W^k_{(p,q)}(D)$ continuously into forms with $W^k(D)$ coefficients. The following theorem shows that all the related operators of N_t are also exactly regular if N_t is exactly regular.

Theorem 6.1.4. *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. For every nonnegative integer k, there exists a constant $S_k > 0$ such that for every $t > S_k$ the operators $\overline{\partial} N_t$, $\overline{\partial}_t^* N_t$, $\overline{\partial}\,\overline{\partial}_t^* N_t$ and $\overline{\partial}_t^* \overline{\partial} N_t$ are exactly regular on $W^k_{(p,q)}(D)$, where $0 \leq p \leq n$, $1 \leq q \leq n$. Furthermore, there exists a constant $S'_k > 0$ such that for $t > S'_k$, the weighted Bergman projection $P_{t,(p,0)}$ maps $W^k_{(p,0)}(D)$ boundedly into itself.*

Proof. Let S_k be as in Theorem 6.1.3 and $t > S_k$. From Theorem 6.1.3, N_t is a bounded map from $W^k_{(p,q)}(D)$ into itself. We shall prove that $\overline{\partial} N_t$ and $\overline{\partial}_t^* N_t$ are exactly regular simultaneously. As before, since $\bar\partial \oplus \vartheta_t$ is elliptic, we only need to prove *a priori* estimates for the tangential derivatives of $\bar\partial N_t f$ and $\bar\partial_t^* N_t f$ for any $f \in C^\infty_{(p,q)}(\overline{D})$. Let η and T be as in Theorem 6.1.3 and let $O_t(\| f \|)$ denote terms which can be bounded by $C_t \| f \|$. We have

$$\begin{aligned}
&\| \eta T^k \overline{\partial} N_t f \|^2_{(t)} + \| \eta T^k \overline{\partial}_t^* N_t f \|^2_{(t)} \\
&= (\eta T^k \overline{\partial} N_t f,\ \overline{\partial} \eta T^k N_t f)_{(t)} + (\eta T^k \overline{\partial}_t^* N_t f,\ \overline{\partial}_t^* \eta T^k N_t f)_{(t)} \\
&\quad + O_t((\| \eta T^k \overline{\partial} N_t f \|_{(t)} + \| \eta T^k \overline{\partial}_t^* N_t f \|_{(t)}) \| N_t f \|_k) \\
&= (\eta T^k \overline{\partial}_t^* \overline{\partial} N_t f,\ \eta T^k N_t f)_{(t)} + (\eta T^k \overline{\partial}\,\overline{\partial}_t^* N_t f,\ \eta T^k N_t f)_{(t)} \\
&\quad + O_t((\| \eta T^k \overline{\partial} N_t f \|_{(t)} + \| \eta T^k \overline{\partial}_t^* N_t f \|_{(t)}) \| N_t f \|_k + \| N_t f \|^2_k) \\
&= (\eta T^k \Box_t N_t f,\ \eta T^k N_t f)_{(t)} \\
&\quad + O_t((\| \eta T^k \overline{\partial} N_t f \|_{(t)} + \| \eta T^k \overline{\partial}_t^* N_t f \|_k) \| N_t f \|_k + \| N_t f \|^2_k) \\
&\leq C \| f \|_k \| N_t f \|_k \\
&\quad + C_t((\| \eta T^k \overline{\partial} N_t f \|_{(t)} + \| \eta T^k \overline{\partial}_t^* N_t f \|_k) \| N_t f \|_k + \| N_t f \|^2_k).
\end{aligned}$$

Using small and large constants in $ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$, it follows from (6.1.3) that

$$\| \eta T^k \overline{\partial} N_t f \|_{(t)} + \| \eta T^k \overline{\partial}_t^* N_t f \|_{(t)} \leq C_{k,t} \| f \|_k\ .$$

By a partition of unity and arguments as before, we obtain the desired *a priori* estimates

$$\| \overline{\partial} N_t f \|_k + \| \overline{\partial}_t^* N_t f \|_k \leq C_{k,t} \| f \|_k\ .$$

For the operators $\overline{\partial}\overline{\partial}_t^* N_t$ and $\overline{\partial}_t^*\overline{\partial} N_t$, $1 \le q \le n$, we have

$$\begin{aligned}
&\| \eta T^k \overline{\partial}_t^* \overline{\partial} N_t f \|_{(t)}^2 + \| \eta T^k \overline{\partial}\overline{\partial}_t^* N_t f \|_{(t)}^2 \\
&\quad = (\eta T^k \overline{\partial}_t^* \overline{\partial} N_t f,\ \eta T^k \overline{\partial}_t^* \overline{\partial} N_t f)_{(t)} + (\eta T^k \overline{\partial}\overline{\partial}_t^* N_t f,\ \eta T^k \overline{\partial}\overline{\partial}_t^* N_t f)_{(t)} \\
&\quad = (\eta T^k \Box_t N_t f,\ \eta T^k \Box_t N_t f)_{(t)} \\
&\qquad - (\eta T^k \overline{\partial}\overline{\partial}_t^* N_t f,\ \eta T^k \overline{\partial}_t^* \overline{\partial} N_t f)_{(t)} - (\eta T^k \overline{\partial}_t^* \overline{\partial} N_t f,\ \eta T^k \overline{\partial}\overline{\partial}_t^* N_t f)_{(t)} \\
&\quad = (\eta T^k f,\ \eta T^k f)_{(t)} - ([\overline{\partial},\ \eta T^k]\overline{\partial}\overline{\partial}_t^* N_t f,\ \eta T^k \overline{\partial} N_t f)_{(t)} \\
&\qquad - ([\overline{\partial}_t^*,\ \eta T^k]\overline{\partial}_t^* \overline{\partial} N_t f,\ \eta T^k \overline{\partial}_t^* N_t f)_{(t)} + E \\
&\quad = (\eta T^k f,\ \eta T^k f)_{(t)} - ([\overline{\partial},\ \eta T^k]\overline{\partial}_t^* N_t f,\ \eta T^k \overline{\partial}_t^* \overline{\partial} N_t f)_{(t)} \\
&\qquad - ([\overline{\partial}_t^*,\ \eta T^k]\overline{\partial} N_t f,\ \eta T^k \overline{\partial}\overline{\partial}_t^* N_t f)_{(t)} + E,
\end{aligned}$$

where the error term E can be estimated by

$$\begin{aligned}
(\| \eta T^k \overline{\partial}\overline{\partial}_t^* N_t f \|_{(t)} + \| \eta T^k \overline{\partial}_t^* \overline{\partial} N_t f \|_{(t)})(\| \overline{\partial} N_t f \|_k + \| \overline{\partial}_t^* N_t f \|_k) \\
+ \| \overline{\partial} N_t f \|_k^2 + \| \overline{\partial}_t^* N_t f \|_k^2 .
\end{aligned}$$

Since $\overline{\partial} N_t$ and $\overline{\partial}_t^* N_t$ are already known to be exactly regular on $W^k_{(p,q)}(D)$, we obtain using small and large constants that

$$\| \eta T^k \overline{\partial}_t^* \overline{\partial} N_t f \|_{(t)}^2 + \| \eta T^k \overline{\partial}\overline{\partial}_t^* N_t f \|_{(t)}^2 \le C_{k,t} \| f \|_k^2 .$$

Summing over a partition of unity, we have proved

$$\| \overline{\partial}_t^* \overline{\partial} N_t f \|_k^2 + \| \overline{\partial}\overline{\partial}_t^* N_t f \|_k^2 \le C_{k,t} \| f \|_k^2$$

when $t > S_k$.

It remains to prove the exact regularity for $P_{t,(p,0)} = I - \bar{\partial}_t^* N_{t,(p,1)} \bar{\partial}$. We use induction on k to prove the *a priori* estimates for $\overline{\partial}_t^* N_{t,(p,1)} \overline{\partial}$. The case when $k = 0$ is obvious. Denoting $N_{t,(p,1)}$ by N_t, we have

$$\begin{aligned}
&\| \eta T^k \overline{\partial}_t^* N_t \overline{\partial} f \|_{(t)}^2 \\
&= (\eta T^k \overline{\partial}_t^* N_t \overline{\partial} f, \eta T^k \overline{\partial}_t^* N_t \overline{\partial} f)_{(t)} \\
&= (\eta T^k N_t \overline{\partial} f, \overline{\partial} \eta T^k \overline{\partial}_t^* N_t \overline{\partial} f)_{(t)} + O(\| N_t \overline{\partial} f \|_k \| \overline{\partial}_t^* N_t \overline{\partial} f \|_k) \\
&= (\eta T^k N_t \overline{\partial} f, \eta T^k \overline{\partial} f)_{(t)} + O(\| N_t \overline{\partial} f \|_k \| \overline{\partial}_t^* N_t \overline{\partial} f \|_k) \\
&= (\eta T^k \overline{\partial}_t^* N_t \overline{\partial} f, \eta T^k f)_{(t)} + O(\| N_t \overline{\partial} f \|_k\ (\| f \|_k + \| \overline{\partial}_t^* N_t \overline{\partial} f \|_k)).
\end{aligned}$$

Summing over a partition of unity and using the fact that $\bar{\partial} \oplus \vartheta_t$ is elliptic, the small and large constants technique gives

$$\| \overline{\partial}_t^* N_t \overline{\partial} f \|_k^2 \le C_k (\| f \|_k^2 + \| N_t \overline{\partial} f \|_k^2). \tag{6.1.8}$$

Using estimate (6.1.1) to $\eta T^k N_t \overline{\partial} f$, we obtain

$$t \parallel \eta T^k N_t \overline{\partial} f \parallel_{(t)}^2 \leq \parallel \overline{\partial} \eta T^k N_t \overline{\partial} f \parallel_{(t)}^2 + \parallel \overline{\partial}_t^* \eta T^k N_t \overline{\partial} f \parallel_{(t)}^2 . \tag{6.1.9}$$

Since $[\bar{\partial}, \eta T^k] + [\vartheta_t, \eta T^k] = B_k + B_{k-1}^t$ where B_k is a kth order differential operator independent of t and B_{k-1}^t is a differential operator of order $k-1$, it follows that

$$\begin{aligned} &\|\overline{\partial} \eta T^k N_t \overline{\partial} f \parallel_{(t)}^2 + \parallel \overline{\partial}_t^* \eta T^k N_t \overline{\partial} f \parallel_{(t)}^2 \\ &\leq C_k(\parallel \overline{\partial}_t^* N_t \overline{\partial} f \parallel_k^2 + \parallel N_t \overline{\partial} f \parallel_k^2) + C_{k,t} \parallel N_t \bar{\partial} f \parallel_{k-1}^2 . \end{aligned}$$

We see, using induction, that $\bar{\partial}_t^* N_t \bar{\partial}$ is bounded on $W^{k-1}(D)$. The above argument implies that $N_t \bar{\partial}$ is also bounded on $W^{k-1}(D)$. Thus, after summing over a partition of unity, if t is chosen to be sufficiently large in (6.1.9), we obtain that

$$\parallel N_t \overline{\partial} f \parallel_k^2 \leq \epsilon \parallel \overline{\partial}_t^* N_t \overline{\partial} f \parallel_k^2 + C_{k,t} \parallel f \parallel_{k-1}^2, \tag{6.1.10}$$

for some small $\epsilon > 0$. Letting ϵ be sufficiently small, (6.1.8) and (6.1.10) together show

$$\parallel \overline{\partial}_t^* N_t \overline{\partial} f \parallel_k^2 \leq C_{k,t} \parallel f \parallel_k^2 .$$

This proves the *a priori* estimate for $\overline{\partial}_t^* N_t \overline{\partial}$ and Theorem 6.1.4.

We remark that the positive number S_k in Theorem 6.1.4 can be chosen to be the same as in Theorem 6.1.3.

Corollary 6.1.5. *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. If $f \in W_{(p,q)}^k(D)$, $k \geq 0$, such that $\overline{\partial} f = 0$, where $0 \leq p \leq n$ and $1 \leq q \leq n$, then there exists $u \in W_{(p,q-1)}^k(D)$ such that $\overline{\partial} u = f$ on D.*

Proof. Since $\overline{\partial} f = 0$, using Theorem 6.1.2 $u = \overline{\partial}_t^* N_t f$ is a solution to the $\overline{\partial}$ equation for any $t > 0$. If t is sufficiently large, $\bar{\partial}_t^* N_t$ is bounded on $W_{(p,q)}^k(D)$ by Theorem 6.1.4. This proves the corollary.

Proof of Theorem 6.1.1. From Corollary 6.1.5, there is $u_k \in W_{(p,q-1)}^k(D)$ satisfying $\overline{\partial} u_k = f$ for each positive integer k. We shall modify u_k to generate a new sequence that converges to a smooth solution.

We claim that $W_{(p,q)}^m(D) \cap \mathrm{Ker}(\overline{\partial})$ is dense in $W_{(p,q)}^s(D) \cap \mathrm{Ker}(\overline{\partial})$ for any $m > s \geq 0$.

Let $g_n \in C_{(p,q)}^\infty(\overline{D})$ be any sequence such that $g_n \to g$ in $W_{(p,q)}^s(D)$. Using Theorem 6.1.4, for sufficiently large t, the Bergman projection with weight $P_t = P_{t,(p,q)}$ is bounded on $W_{(p,q)}^m(D)$. Since $\overline{\partial} g = 0$, we have $g - P_t g = \bar{\partial}_t^* N_t \bar{\partial} g = 0$. Thus, $P_t g_n = g_n' \in W_{(p,q)}^m(D)$, $\overline{\partial} g_n' = 0$ and $g_n' \to g$ in $W_{(p,q)}^s(D)$ since P_t is also bounded on $W_{(p,q)}^s(D)$. This proves the claim.

Since $u_k - u_{k+1}$ is in $W_{(p,q-1)}^k(D) \cap \mathrm{Ker}(\overline{\partial})$, there exists a $v_{k+1} \in W_{(p,q-1)}^{k+1}(D) \cap \mathrm{Ker}(\overline{\partial})$ such that

$$\parallel u_k - u_{k+1} - v_{k+1} \parallel_k \leq 2^{-k}, \quad k = 1, 2, \cdots.$$

Setting $\tilde{u}_{k+1} = u_{k+1} + v_{k+1}$, then $\tilde{u}_{k+1} \in W^{k+1}_{(p,q-1)}(D)$ and $\overline{\partial}\tilde{u}_{k+1} = f$. Inductively, we can choose a new sequence $\tilde{u}_k \in W^k_{(p,q-1)}(D)$ such that $\overline{\partial}\tilde{u}_k = f$ and

$$\| \tilde{u}_{k+1} - \tilde{u}_k \|_k \le 2^{-k}, \quad k = 1, 2, \cdots.$$

Set

$$u_\infty = \tilde{u}_N + \sum_{k=N}^{\infty} (\tilde{u}_{k+1} - \tilde{u}_k), \quad N \in \mathbb{N}.$$

Then u_∞ is well defined and is in $W^N_{(p,q-1)}(D)$ for every N. Thus $u_\infty \in C^\infty_{(p,q-1)}(\overline{D})$ from the Sobolev embedding theorem and $\overline{\partial}u_\infty = f$. This proves Theorem 6.1.1.

We also obtain the following result in the proof:

Corollary 6.1.6. *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \ge 2$. Then $C^\infty_{(p,q)}(\overline{D}) \cap Ker(\overline{\partial})$ is dense in $W^s_{(p,q)}(D) \cap Ker(\overline{\partial})$ in the $W^s_{(p,q)}(D)$ norm, where $0 \le p \le n$ and $0 \le q \le n$. In particular, $C^\infty(\overline{D}) \cap \mathcal{O}(D)$ is dense in $\mathcal{H}(D)$ in $L^2(D)$, where $\mathcal{H}(D)$ is the space of all square integrable holomorphic functions.*

6.2 Sobolev Estimates for the $\overline{\partial}$-Neumann Operator

In this section, using the vector field method we shall give a sufficient condition for verifying global regularity of the $\overline{\partial}$-Neumann problem on a certain class of smooth bounded pseudoconvex domains. In particular, this method can be applied to convex domains and circular domains with transverse symmetries.

Let $D \subseteq \mathbb{C}^n, n \ge 2$, be a smooth bounded pseudoconvex domain, and let r be a smooth defining function for D. Set

$$L_n = \frac{4}{|\nabla r|^2} \sum_{j=1}^{n} \frac{\partial r}{\partial \overline{z}_j} \frac{\partial}{\partial z_j},$$

if $|\nabla r| \ne 0$, and

$$L_{jk} = \frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_k} - \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_j}, \quad \text{for } 1 \le j < k \le n.$$

We have $L_n r = 1$ in a neighborhood of the boundary and the L_{jk}'s are tangent to the level sets of r. Also, the L_{jk}'s span the space of tangential type $(1,0)$ vector fields at every boundary point of D.

Denote by $X_n = (|\nabla r|/\sqrt{2})L_n$ the globally defined type $(1,0)$ vector field which is transversal to the boundary everywhere. Obviously, we have $\| X_n \| = 1$ in some open neighborhood of the boundary. Thus, near every boundary point $p \in bD$, we may choose tangential type $(1,0)$ vector fields $X_1, \cdots, X_{n-1}$ so that $X_1, \cdots, X_{n-1}$ together with X_n form an orthonormal basis of the space of type (1,0) vector fields in some open neighborhood of p. We shall also denote by $\omega_1, \cdots, \omega_n$ the orthonormal frame of (1,0)-forms dual to $X_1, \cdots, X_n$ near p. Note that $\omega_n = (\sqrt{2}/|\nabla r|)\sum_{j=1}^n (\partial r/\partial z_j)dz_j$ is a globally defined (1,0)-form in some open neighborhood of the boundary.

The main idea of this method is to construct a real tangential vector field T on some open neighborhood of the boundary such that the commutators of T with $X_1, \cdots, X_n, \overline{X}_1, \cdots, \overline{X}_n$ have small modulus in X_n direction on the boundary. We formulate the required properties of T in the following condition:

Condition (T). *For any given $\epsilon > 0$ there exists a smooth real vector field $T = T_\epsilon$, depending on ϵ, defined in some open neighborhood of $\overline{D}$ and tangent to the boundary with the following properties:*

(1) *On the boundary, T can be expressed as*

$$T = a_\epsilon(z)(L_n - \overline{L}_n), \quad mod\ (T^{1,0}(bD) \oplus T^{0,1}(bD)),$$

for some smooth function $a_\epsilon(z)$ with $|a_\epsilon(z)| \geq \delta > 0$ for all $z \in bD$, where δ is a positive constant independent of ϵ.

(2) *If S is any one of the vector fields L_n, $\overline{L}_n$, L_{jk} and $\overline{L}_{jk}$, $1 \leq j < k \leq n$, then*

$$[T, S]|_{bD} = A_S(z)L_n, \quad mod\ (T^{1,0}(bD) \oplus T^{0,1}(bD), \overline{L}_n),$$

for some smooth function $A_S(z)$ with $\sup\limits_{bD} |A_S(z)| < \epsilon$.

Here is a simple observation: Near a boundary point p, we have, say, $\partial r/\partial z_n(p) \neq 0$. Thus, for each $j = 1, \cdots, n-1$, we may write

$$X_j = \sum_{k=1}^{n-1} c_{jk} L_{kn},$$

for some smooth functions c_{jk}. It follows that if condition (T) holds on D, then property (2) of condition (T) is still valid with S being taken to be X_j's or $\overline{X}_j$'s for $j = 1, \cdots, n-1$, where X_j's are defined as above in some small open neighborhood of p.

Now we are in a position to prove the main theorem of this section.

Theorem 6.2.1. *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with a smooth defining function r. Suppose that condition (T) holds on D. Then the $\overline{\partial}$-Neumann operator N maps $W^s_{(p,q)}(D)$, $0 \leq p \leq n$, $1 \leq q \leq n$, boundedly into itself for each nonnegative real s.*

Proof. We shall prove the theorem only for nonnegative integers. For any nonnegative real s, the assertion will follow immediately from interpolation (see Theorem B.3 in the Appendix).

In view of the elliptic regularization method employed in Chapter 5, it suffices to prove *a priori* estimates for the $\overline{\partial}$-Neumann operator. The proof will be done by induction on the order of differentiation. Let us assume that the given (p, q)-form $f \in C^\infty_{(p,q)}(\overline{D})$ and the solution $u = N_q f$ to the equation $\Box u = (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})u = f$ is also in $C^\infty_{(p,q)}(\overline{D})$.

The initial step $s = 0$ is obvious, since N is a bounded operator by Theorem 4.4.1. To illustrate the idea, we prove the case $s = 1$ in detail. First, we choose boundary coordinate charts $\{U_\alpha\}_{\alpha=1}^m$ such that $\{U_\alpha\}_{\alpha=1}^m$ and $U_0 = D$ form an open cover of $\overline{D}$. We shall assume that $|\nabla r| > c > 0$ on $\cup_{\alpha=1}^m U_\alpha$ for some positive constant c. Let $\{\zeta_\alpha\}_{\alpha=0}^m$ be a fixed partition of unity subordinate to $\{U_\alpha\}_{\alpha=0}^m$. On each U_α, $1 \leq \alpha \leq m$, let $\omega_{\alpha k}$, $k = 1, \cdots, n$ be an orthonormal frame of (1,0)-forms dual to $X_{\alpha k}$, $k = 1, \cdots, n$. We note that $\omega_{\alpha n} = \omega_n$, $\alpha = 1, \cdots, m$, is a globally defined (1,0)-form dual to $X_n = (|\nabla r|/\sqrt{2})L_n$. Similarly, $X_{\alpha n} = X_n$, $\alpha = 1, \cdots, m$, is also a globally defined type (1,0) vector field.

On each boundary chart U_α, $\alpha = 1, \cdots, m$, we may write

$$u = u^\alpha = \sum{}' u^\alpha_{I,J}\ \omega^{\alpha,I} \wedge \overline{\omega}^{\alpha,J},$$

where $\omega^{\alpha,I} = \omega_{\alpha i_1} \wedge \cdots \wedge \omega_{\alpha i_p}$ and $\overline{\omega}^{\alpha,J} = \overline{\omega}_{\alpha j_1} \wedge \cdots \wedge \overline{\omega}_{\alpha j_q}$. Let T be the smooth real vector field on $\overline{D}$ satisfying the hypothesis of condition (T). For each $s \in \mathbb{N}$, we define

$$T^s u = T^s(\zeta_0 u) + \sum_{\alpha=1}^{m} \sum_{I,J}{}' T^s(\zeta_\alpha u^\alpha_{I,J})\ \omega^{\alpha,I} \wedge \overline{\omega}^{\alpha,J}. \tag{6.2.1}$$

Thus, $T^s u \in \mathcal{D}_{(p,q)}$ using Lemma 5.2.2.

Set

$$\| \overline{X}u \|^2 = \sum_{\alpha=1}^{m} \sum_{I,J}{}' \sum_{k=1}^{n} \| \overline{X}_{\alpha k}(\zeta_\alpha u^\alpha_{I,J}) \|^2,$$

and

$$\| X'u \|^2 = \sum_{\alpha=1}^{m} \sum_{I,J}{}' \sum_{k=1}^{n-1} \| X_{\alpha k}(\zeta_\alpha u^\alpha_{I,J}) \|^2 .$$

From estimate (4.3.1), with $\phi = 0$, and (4.4.6) we obtain, using Theorem 4.4.1,

$$\begin{aligned} \| \overline{X}u \|^2 + \| u \|^2 &\lesssim \| \overline{\partial}u \|^2 + \| \overline{\partial}^* u \|^2 \\ &= \| \overline{\partial}Nf \|^2 + \| \overline{\partial}^* Nf \|^2 \lesssim \| f \|^2 . \end{aligned} \tag{6.2.2}$$

Since, by integration by parts, for $1 \le \alpha \le m$ and $1 \le k \le n-1$,

$$\| X_{\alpha k}(\zeta_\alpha u^\alpha_{I,J}) \|^2 = \| \overline{X}_{\alpha k}(\zeta_\alpha u^\alpha_{I,J}) \|^2 + O(\| \overline{X}u \| + \| Tu \| + \| u \|)\, \| u \|,$$

using small and large constants, we have

$$\| X'u \|^2 \lesssim \| \overline{X}u \|^2 + \| u \|^2 + (sc)\, \| Tu \|^2 . \tag{6.2.3}$$

Here (sc) denotes a small constant that can be made as small as we wish. Estimates (6.2.2) and (6.2.3) together with the interior estimate indicate that if one can control $\| Tu \|$, then $\| u \|_1$ can be estimated. For this reason we shall call $X_{\alpha 1}, \cdots, X_{\alpha n-1}, \overline{X}_{\alpha 1}, \cdots, \overline{X}_{\alpha n-1}$ and $\overline{X}_{\alpha n}$, $1 \le \alpha \le m$, "good" directions.

Our aim thus becomes to estimate $\| Tu \|$. First, the basic estimate shows

$$\| Tu \|^2 \lesssim \| \overline{\partial}Tu \|^2 + \| \overline{\partial}^* Tu \|^2 .$$

We estimate the right-hand side as follows.

$$\begin{aligned}
\| \overline{\partial} Tu \|^2 &= (\overline{\partial} Tu, \overline{\partial} Tu) \\
&= (T\overline{\partial} u, \overline{\partial} Tu) + ([\overline{\partial}, T]u, \overline{\partial} Tu) \\
&= (\overline{\partial} u, -\overline{\partial} T^2 u) + (\overline{\partial} u, [\overline{\partial}, T]Tu) + ([\overline{\partial}, T]u, \overline{\partial} Tu) \\
&\quad + \ O(\| \overline{\partial} Tu \| \| \overline{\partial} u \|) \\
&= (\overline{\partial} u, -\overline{\partial} T^2 u) + (\overline{\partial} u, [[\overline{\partial}, T], T]u) + (-T\overline{\partial} u, [\overline{\partial}, T]u) + ([\overline{\partial}, T]u, \overline{\partial} Tu) \\
&\quad + O((\| \overline{\partial} Tu \| + \| u \|_1) \| \overline{\partial} u \|) \\
&= (\overline{\partial}^* \overline{\partial} u, -T^2 u) + (\overline{\partial} u, [[\overline{\partial}, T], T]u) + (-\overline{\partial} Tu, [\overline{\partial}, T]u) \\
&\quad + \| [\overline{\partial}, T]u \|^2 + ([\overline{\partial}, T]u, \overline{\partial} Tu) + O((\| \overline{\partial} Tu \| + \| u \|_1) \| \overline{\partial} u \|).
\end{aligned}$$

Note that

$$\mathrm{Re}\{(-\overline{\partial} Tu, [\overline{\partial}, T]u) + \ ([\overline{\partial}, T]u, \overline{\partial} Tu)\} = 0.$$

With similar estimates for $\| \overline{\partial}^* Tu \|^2$, we obtain

$$\begin{aligned}
\| \overline{\partial} Tu \|^2 + \| \overline{\partial}^* Tu \|^2 \lesssim \ &\| f \|_1^2 + (sc) \| Tu \|^2 + (sc) \| u \|_1^2 \\
&+ \| [\overline{\partial}, T]u \|^2 + \| [\overline{\partial}^*, T]u \|^2 .
\end{aligned}$$

In order to estimate the crucial commutator terms $\| [\overline{\partial}, T]u \|$ and $\| [\overline{\partial}^*, T]u \|$, we shall use the hypothesis on T. First, from our observation right before Theorem 6.2.1, it is easy to see that on each boundary coordinate chart the commutators between T and $X_1, \cdots, X_{n-1}, \overline{X}_1, \cdots, \overline{X}_{n-1}$ can be controlled using the hypothesis on T. Thus, we need to consider the commutator between T and $\overline{X}_n$ (or X_n) which occurs, when commuting T with $\overline{\partial}$ (or $\overline{\partial}^*$), only for those multiindices (I, J) with $n \notin J$ (or $n \in J$). Such terms can be handled as follows:

$$\begin{aligned}
[\overline{X}_n, T](\zeta_\alpha u^\alpha_{I,J}) &= [(|\nabla r|/\sqrt{2})\overline{L}_n, T](\zeta_\alpha u^\alpha_{I,J}) \\
&= (|\nabla r|/\sqrt{2})[\overline{L}_n, T](\zeta_\alpha u^\alpha_{I,J}) - (T(|\nabla r|/\sqrt{2}))\overline{L}_n(\zeta_\alpha u^\alpha_{I,J}) \\
&= (|\nabla r|/\sqrt{2})[\overline{L}_n, T](\zeta_\alpha u^\alpha_{I,J}) - (T(|\nabla r|)/|\nabla r|)\overline{X}_n(\zeta_\alpha u^\alpha_{I,J}),
\end{aligned}$$

for $\alpha = 1, \cdots, m$. Using the basic estimate, we obtain

$$\sum_{\alpha=1}^{m} \sum_{n\notin J}{}' \| (T(|\nabla r|)/|\nabla r|)\overline{X}_n(\zeta_\alpha u^\alpha_{I,J}) \|^2 \lesssim \| \overline{\partial} u \|^2 + \| \overline{\partial}^* u \|^2 \lesssim \| f \|^2 .$$

The remaining commutator terms can be estimated directly using the hypothesis on T. Thus,

$$\begin{aligned}
\| [\overline{\partial}, T]u \|^2 \ &\lesssim \ \sum_{\alpha=1}^{m} \sum{}' \| \frac{A}{a_\epsilon} T(\zeta_\alpha u^\alpha_{I,J}) \|^2 + \| \overline{X} u \|^2 + \| X' u \|^2 + \| f \|_1^2 \\
&\lesssim \left(\frac{\epsilon}{\delta}\right)^2 \| Tu \|^2 + \| \overline{X} u \|^2 + \| X' u \|^2 + \| f \|_1^2 .
\end{aligned}$$

For $\| [\overline{\partial}^*, T]u \|$ we commute T with X_n if $n \in J$. Hence,

$$[X_n, T](\zeta_\alpha u^\alpha_{I,J}) = (|\nabla r|/\sqrt{2})[L_n, T](\zeta_\alpha u^\alpha_{I,J}) - (T(|\nabla r|)/|\nabla r|)X_n(\zeta_\alpha u^\alpha_{I,J}),$$

for $\alpha = 1, \cdots, m$. Observe that $\pm X_n(\zeta_\alpha u^\alpha_{I,J})$ appears in the coefficient of $\omega^{\alpha,I} \wedge \overline{\omega}^{\alpha,H}$ with $\{n\} \cup H = J$ in $\overline{\partial}^* u$. Meanwhile, all the other terms in the coefficient of $\omega^{\alpha,I} \wedge \overline{\omega}^{\alpha,H}$ are differentiated by $X_1, \cdots, X_{n-1}$ only. Thus we have

$$\sum_{\alpha=1}^{m} \sum_{n\in J}{}' \| (T(|\nabla r|)/|\nabla r|)X_n(\zeta_\alpha u^\alpha_{I,J}) \|^2 \lesssim \| \overline{\partial}^* u \|^2 + \| X' u \|^2 + \| f \|_1^2,$$

and get the estimates as before. Here we use $\| f \|_1^2$ to control the interior term. Now we first choose (sc) to be small enough, and then ϵ to be sufficiently small. From (6.2.3) we obtain

$$\| [\overline{\partial}, T]u \|^2 + \| [\overline{\partial}^*, T]u \|^2 \lesssim \| \overline{X} u \|^2 + \| f \|_1^2 + \gamma \| Tu \|^2,$$

where $\gamma > 0$ is a constant that can be made as small as we wish. Combining these estimates, if we let γ be small enough, we get

$$\| Tu \|^2 \lesssim \| \overline{\partial} Tu \|^2 + \| \overline{\partial}^* Tu \|^2 \lesssim \| f \|_1^2 + (sc) \| u \|_1^2 .$$

This implies

$$\| u \|_1 \leq C \| f \|_1,$$

for some constant $C > 0$ independent of f, and the proof for $s = 1$ is thus complete.

Assume that the $\overline{\partial}$-Neumann operator N is bounded on $W^{s-1}_{(p,q)}(D)$ for some integer $s \geq 2$, i.e.,

$$\| u \|_{s-1} \leq C_{s-1} \| f \|_{s-1},$$

where $C_{s-1} > 0$ is a constant independent of f. The strategy here is the same as before. Using basic estimate and the induction hypothesis, we first establish the following *a priori* estimate

$$\begin{aligned} \| T^s u \|^2 &\lesssim \| \overline{\partial} T^s u \|^2 + \| \overline{\partial}^* T^s u \|^2 \\ &\lesssim \| f \|_s^2 + (sc) \| u \|_s^2 . \end{aligned} \tag{6.2.4}$$

The next step is to consider the action of an arbitrary tangential differential operator of order s. Using (6.2.4), it suffices to consider the estimate near a boundary point p. Let U be a boundary coordinate chart near p, and let $\omega_1, \cdots, \omega_n$ be an orthonormal basis for (1,0)-forms on U dual to $X_1, \cdots, X_n$ defined as before. Denote by $\mathrm{Op}(s, j)$, $1 \leq j \leq s$, a tangential differential operator of order s formed out of $X_1, \cdots, X_{n-1}, \overline{X}_1, \cdots, \overline{X}_{n-1}$ and T with precisely $s - j$ factors of T. Let ζ be a cut-off function supported in U such that $\zeta \equiv 1$ in some open neighborhood of p. We claim that

$$\| \mathrm{Op}(s, j)\zeta u \|^2 \lesssim \| f \|_s^2 + (sc) \| u \|_s^2, \tag{6.2.5}$$

for all $0 \leq j \leq s$. Estimate (6.2.5) will be proved by induction on j. The initial step $j = 0$ is done by (6.2.4). Hence we assume Estimate (6.2.5) holds up to $j-1$ for some $1 \leq j \leq s$. We need to show that (6.2.5) is also true for j. Denote by X_t any one of the vector fields $X_1, \cdots, X_{n-1}$. Then, by commuting one X_t or $\overline{X}_t$ to the left and applying integration by parts, it is easily seen that

$$\| \operatorname{Op}(s,j)\zeta u \|^2 \lesssim \| f \|_s^2 + \| \overline{X}_t \operatorname{Op}(s-1, j-1)\zeta u \|^2 + \| \operatorname{Op}(s, j-1)\zeta u \|^2 .$$

Thus, (6.2.5) is proved inductively for all $0 \leq j \leq s$.

Estimate (6.2.5) shows that all the tangential derivatives of order s can be controlled. Finally, using the noncharacteristic nature of the $\overline{\partial}$-Neumann problem, we also control the differentiation in the normal direction. Therefore, a partition of unity argument gives

$$\| u \|_s \lesssim \| f \|_s + (sc) \| u \|_s$$

which implies, by choosing (sc) sufficiently small,

$$\| u \|_s \lesssim \| f \|_s .$$

Hence, by an induction argument the proof of Theorem 6.2.1 is now complete.

Theorem 6.2.1 provides us with a method for verifying the regularity of the $\overline{\partial}$-Neumann operator N. Once the regularity of N is known, we may also obtain the regularity of other operators related to the $\overline{\partial}$-Neumann operator N as shown in the next theorem.

Theorem 6.2.2. *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. If the $\overline{\partial}$-Neumann operator N is exactly regular on $W^s_{(p,q)}(D)$ for $0 \leq p \leq n$, $1 \leq q \leq n$ and $s \geq 0$, then so are the operators $\overline{\partial}N, \overline{\partial}^* N, \overline{\partial}\overline{\partial}^* N$, $\overline{\partial}^*\overline{\partial}N$ and the Bergman projection $P_{(p,0)}$.*

Proof. The exact regularity of the operators $\overline{\partial}N$, $\overline{\partial}^* N$, $\overline{\partial}\overline{\partial}^* N$ and $\overline{\partial}^*\overline{\partial}N$ can be proved as in Theorem 6.1.4.

For the regularity of the Bergman projections $P_{(p,0)}$, we may assume $p = 0$. Denote as before by P the Bergman projection on functions and by P_t the weighted Bergman projection. Let Φ_t be the multiplication operator by the weight $e^{-t|z|^2}$. Then for any square integrable holomorphic function g and any square integrable function f, we have

$$(Pf, g) = (f, g) = (\Phi_{-t} f, g)_t = (P_t \Phi_{-t} f, g)_t = (\Phi_t P_t(\Phi_{-t} f), g).$$

Hence, we get $P = P\Phi_t P_t \Phi_{-t}$. Recall that $P = I - \overline{\partial}^* N \overline{\partial}$ on smooth bounded pseudoconvex domains (Theorem 4.4.5). Thus, the Bergman projection P on functions can be expressed as

$$P = \Phi_t P_t \Phi_{-t} - \overline{\partial}^* N_1((\overline{\partial}\Phi_t) P_t \Phi_{-t}).$$

Since it has been proved that $\overline{\partial}^* N_1$ preserves $W^s_{(0,1)}(D)$ for each nonnegative real s, if we choose t to be sufficiently large, Theorem 6.1.4 implies that P maps $W^s(D)$ boundedly into itself. This proves the theorem.

Now, we will construct the vector field T when the domain D has a plurisubharmonic defining function or transverse circular symmetry. We say that a smooth bounded domain D has a plurisubharmonic defining function if there exists some smooth defining function $r(z)$ for D satisfying

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(z) t_j \bar{t}_k \geq 0 \quad \text{for } z \in bD \text{ and } t \in \mathbb{C}^n. \tag{6.2.6}$$

Note that (6.2.6) is required to hold only on the boundary, and not in a neighborhood of the boundary. Also, (6.2.6) implies D must be pseudoconvex.

Theorem 6.2.3. *Let $D \subseteq \mathbb{C}^n, n \geq 2$, be a smooth bounded pseudoconvex domain admitting a plurisubharmonic defining function $r(z)$. Then the $\overline{\partial}$-Neumann operator N is exactly regular on $W^s_{(p,q)}(D)$ for $0 \leq p \leq n$, $1 \leq q \leq n$ and all real $s \geq 0$.*

Proof. The proof is based on the observation that if $r(z)$ is a defining function for D that is plurisubharmonic on the boundary, then for each j, derivatives of $(\partial r/\partial z_j)$ of type $(0,1)$ in directions that lie in the null space of the Levi form must vanish. For instance, if coordinates are chosen so that $(\partial/\partial z_1)(p)$ lies in the null space of the Levi form at $p \in bD$, then $(\partial^2 r/\partial z_1 \partial \overline{z}_1)(p) = 0$. Since $r(z)$ is plurisubharmonic on the boundary, applying the complex Hessian $((\partial^2 r/\partial z_j \partial \overline{z}_k)(p))_{j,k=1}^n$ to $(\overline{\alpha}, 1, 0, \cdots, 0) \in \mathbb{C}^n$ for any $\alpha \in \mathbb{C}$, we obtain

$$2\mathrm{Re}\left(\alpha \frac{\partial^2 r}{\partial z_2 \partial \overline{z}_1}(p)\right) + \frac{\partial^2 r}{\partial z_2 \partial \overline{z}_2}(p) \geq 0,$$

which forces $(\partial^2 r/\partial z_2 \partial \overline{z}_1)(p) = 0$. Similarly, we get $(\partial^2 r/\partial z_j \partial \overline{z}_1)(p) = 0$ for $1 \leq j \leq n$.

Fix a point p in the boundary, then $(\partial r/\partial z_j)(p) \neq 0$ for some $1 \leq j \leq n$. Choose such a j and set $X_p = (\partial r/\partial z_j)^{-1}(\partial/\partial z_j)$ in a neighborhood of p. Let $L_1, \cdots, L_{n-1}$ be a local basis for the space of type $(1,0)$ tangential vector fields near p. We may assume that $L_1, \cdots, L_{n-1}$ are tangent to the level sets of r and that the Levi form is diagonal at p in this basis, namely, $\langle [\overline{L}_k, L_j], L_n\rangle(p)$ is a diagonal $(n-1) \times (n-1)$ matrix, where $\langle\ ,\ \rangle$ is the standard Hermitian inner product. Now if the kth eigenvalue is zero, then by the above observation $\overline{L}_k$ annihilates the coefficients of X_p at p. It follows that the system of linear equations

$$\sum_{j=1}^{n-1} \langle [\overline{L}_k, L_j], L_n\rangle(p) a_j = \langle [\overline{L}_k, X_p], L_n\rangle(p), \quad 1 \leq k \leq n-1,$$

is solvable for $a_1, \cdots, a_{n-1} \in \mathbb{C}$. This implies that

$$\left\langle \left[X_p - \sum_{j=1}^{n-1} a_j L_j, \overline{L}_k\right], L_n \right\rangle(p) = 0$$

for $k = 1, \cdots, n-1$. On the other hand, since $X_p(r) = 1 = L_n(r)$, there exist scalars $b_1, \cdots, b_{n-1}$ such that

$$L_n - \sum_{j=1}^{n-1} b_j L_j = X_p - \sum_{j=1}^{n-1} a_j L_j.$$

Therefore, given $\epsilon > 0$, it is easily verified by using a partition of unity $\{\zeta_l\}_{l=1}^m$ with small support that one may patch $L_n - \sum_{j=1}^{n-1} b_j L_j$ together to form a globally defined type $(1,0)$ vector field $X = L_n - Y$, where $Y = \sum_{l=1}^m \zeta_l(\sum_{j=1}^{n-1} b_j L_j)$ is a globally defined tangential type $(1,0)$ vector field, such that

$$\sup_{bD} |\langle [X, \overline{L}_k], L_n\rangle| < \epsilon, \tag{6.2.7}$$

for $1 \le k \le n-1$. Since $X - \overline{X}$ is a purely imaginary tangential vector field, the commutator $[X - \overline{X}, \overline{L}_k]$, $1 \le k \le n-1$, is also a tangential vector field. Thus, we may write

$$[X - \overline{X}, \overline{L}_k] = \alpha_k (L_n - \overline{L}_n), \quad \bmod \ (T^{1,0}(bD) \oplus T^{0,1}(bD)),$$

for $1 \le k \le n-1$. By (6.2.7), we have $|\alpha_k| < \epsilon$ on the boundary. It follows that

$$\sup_{bD} |\langle [X - \overline{X}, S], L_n\rangle| < \epsilon, \tag{6.2.8}$$

when S is any one of the tangential vector fields $L_1, \cdots, L_{n-1}, \overline{L}_1, \cdots, \overline{L}_{n-1}$. In particular, we may achieve that

$$\sup_{bD} |\langle [X - \overline{X}, Y - \overline{Y}], L_n\rangle| < \epsilon, \tag{6.2.9}$$

where ϵ might be different from that in (6.2.8). Since $L_n r = 1$ in a neighborhood of the boundary, we have

$$[X - \overline{X}, L_n] r = (X - \overline{X}) L_n(r) - L_n(X - \overline{X}) r = 0,$$

which implies the vector field $[X - \overline{X}, L_n]$ is tangent to the level sets of r. It follows that $[X - \overline{X}, \mathrm{Re} L_n]$ is also a tangential vector field and we can write

$$[X - \overline{X}, \mathrm{Re} L_n] = \beta (L_n - \overline{L}_n), \quad \bmod \ (T^{1,0}(bD) \oplus T^{0,1}(bD)), \tag{6.2.10}$$

where β is a real-valued function defined in some neighborhood of the boundary.

Now, we set

$$T = e^{r\beta}(X - \overline{X}). \tag{6.2.11}$$

Obviously, T depends on ϵ and satisfies property (1) of condition (T) on the boundary. If S is any one of the vector fields $L_1, \cdots, L_{n-1}, \overline{L}_1, \cdots, \overline{L}_{n-1}$, we have

$$\langle [T, S], L_n\rangle|_{bD} = \langle [X - \overline{X}, S], L_n\rangle|_{bD}.$$

For L_n, using (6.2.10) we get

$$\begin{aligned} \langle [T, L_n], L_n\rangle|_{bD} = & \ \langle [e^{r\beta}(X - \overline{X}), \mathrm{Re} L_n], L_n\rangle|_{bD} \\ & + \frac{1}{2} \langle [e^{r\beta}(X - \overline{X}), L_n - \overline{L}_n], L_n\rangle|_{bD} \\ = & \ 0 + \frac{1}{2} \langle [X - \overline{X}, Y - \overline{Y}], L_n\rangle|_{bD}. \end{aligned}$$

Similarly, we obtain

$$\langle [T, \overline{L}_n], L_n\rangle|_{bD} = -\frac{1}{2}\langle [X-\overline{X}, Y-\overline{Y}], L_n\rangle|_{bD}.$$

Thus, by (6.2.8) and (6.2.9) we see that the vector field T satisfies all the hypotheses of condition (T). Hence, by Theorem 6.2.1, we have proved Theorem 6.2.3.

Theorem 6.2.3 gives a sufficient condition for verifying the exact regularity of the $\overline{\partial}$-Neumann operator. However, this condition in general is not satisfied by every smooth bounded pseudoconvex domain. For instance, by Theorem 6.4.2, the worm domain constructed in Section 6.4 does not enjoy this property. Next, we show that this condition indeed holds on any smooth bounded convex domain. Hence, the $\overline{\partial}$-Neumann problem is exactly regular on any convex domain.

Let $D \subseteq \mathbb{R}^N, N \geq 2$, be a smooth bounded convex domain, and let the origin be contained in D. For any $x \in \mathbb{R}^N$, the Minkowski functional $\mu(x)$ is defined by

$$\mu(x) = \inf\{\lambda > 0 |\ x \in \lambda D\}, \tag{6.2.12}$$

where $\lambda D = \{\lambda y |\ y \in D\}$. Since the boundary of D is smooth, $\mu(x)$ is smooth on $\mathbb{R}^N \setminus \{0\}$. If $x \neq 0$, then the ray $\overrightarrow{ox}$ will intersect the boundary bD at exactly one point, named x'. It is easy to see that the Minkowski functional $\mu(x)$ is equal to the ratio between $d(0,x)$ and $d(0,x')$, where $d(p,q) = \mathrm{dist}(p,q)$. Hence, we have $x = \mu(x)x'$.

Lemma 6.2.4. *Let D be a smooth bounded convex domain in $\mathbb{R}^N$ containing the origin, and let the Minkowski functional $\mu(x)$ be defined as in (6.2.12). Then $\mu(x)$ is a smooth, real-valued function on $\mathbb{R}^N \setminus \{0\}$ satisfying the following properties:*

(1) *$\mu(x)$ is a defining function for D, i.e., $\mu(x) = 1$ and $\nabla\mu(x) \neq 0$ for $x \in bD$,*
(2) *$\mu(x+y) \leq \mu(x) + \mu(y)$ for $x, y \in \mathbb{R}^N$,*
(3) *$\mu(ax) = a\mu(x)$ for $x \in \mathbb{R}^N$ and $a > 0$.*

Proof. Obviously, $\mu(x)$ is a smooth, real-valued function on $\mathbb{R}^N \setminus \{0\}$. (1) and (3) are also clear. To prove (2), let $x, y \neq 0$ be two points in $\mathbb{R}^N$, and let x', y' be the intersections with the boundary of the rays $\overrightarrow{ox}$ and $\overrightarrow{oy}$ respectively. Then we have

$$\begin{aligned} \mu(x+y) &= \mu(\mu(x)x' + \mu(y)y') \\ &= (\mu(x)+\mu(y))\mu\left(\frac{\mu(x)}{\mu(x)+\mu(y)}x' + \frac{\mu(y)}{\mu(x)+\mu(y)}y'\right) \\ &\leq \mu(x) + \mu(y). \end{aligned}$$

Here we have used the fact that D is convex so that the point $z = (\mu(x)/(\mu(x) + \mu(y)))x' + (\mu(y)/(\mu(x)+\mu(y)))y'$ lies in the closure of D. Hence, $\mu(z) \leq 1$. This proves the lemma.

It follows from (2) and (3) of Lemma 6.2.4 that $\mu(x)$ is convex. Consequently,

$$\sum_{j,k=1}^{N} \frac{\partial^2\mu}{\partial x_j \partial x_k}(x) a_j a_k \geq 0, \tag{6.2.13}$$

for $x \in \mathbb{R}^N \setminus \{0\}$ and any $a = (a_1, \cdots, a_N) \in \mathbb{R}^N$. In particular, we have proved the following theorem:

Theorem 6.2.5. *Let D be a smooth bounded convex domain in $\mathbb{C}^n, n \geq 2$. Then the $\overline{\partial}$-Neumann problem is exactly regular on $W^s_{(p,q)}(D)$ for $0 \leq p \leq n$, $1 \leq q \leq n$ and $s \geq 0$.*

Another important class of pseudoconvex domains that satisfy the hypotheses of condition (T) are circular domains with transverse symmetries. A domain D in $\mathbb{C}^n$ is called circular if $e^{i\theta} \cdot z = (e^{i\theta} z_1, \cdots, e^{i\theta} z_n) \in D$ for any $z \in D$ and $\theta \in \mathbb{R}$. D is called Reinhardt if $(e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n) \in D$ for any $z \in D$ and $\theta_1, \cdots, \theta_n \in \mathbb{R}$, and D is called complete Reinhardt if $z = (z_1, \cdots, z_n) \in D$ implies $(w_1, \cdots, w_n) \in D$ for all $|w_j| \leq |z_j|, 1 \leq j \leq n$. Thus, a Reinhardt domain is automatically circular.

Let D be a smooth bounded circular domain in $\mathbb{C}^n, n \geq 2$, and let $r(z)$ be defined as follows

$$r(z) = \begin{cases} d(z, bD), & \text{for } z \notin D \\ -d(z, bD), & \text{for } z \in D, \end{cases} \tag{6.2.14}$$

where $d(z, bD)$ denotes the distance from z to the boundary bD. Then it is easy to see that r is a defining function for D such that $r(z) = r(e^{i\theta} \cdot z)$ and that $|\nabla r| = 1$ on the boundary. Denote by Λ the map of the S^1-action on D from $S^1 \times D$ to D defined by

$$\begin{aligned} \Lambda : S^1 \times D &\to D \\ (e^{i\theta}, z) &\mapsto e^{i\theta} \cdot z = (e^{i\theta} z_1, \cdots, e^{i\theta} z_n). \end{aligned}$$

For each fixed θ, Λ is an automorphism of D and Λ can be extended smoothly to a map from $S^1 \times \overline{D}$ to $\overline{D}$. Hence, for each fixed $z \in \overline{D}$, we consider the orbit of z, namely, the map

$$\begin{aligned} \pi_z : S^1 &\to \overline{D} \\ e^{i\theta} &\mapsto e^{i\theta} \cdot z. \end{aligned}$$

Then, π_z induces a vector field T on $\overline{D}$, in fact on $\mathbb{C}^n$, by

$$T_z = \pi_{z,*} \left(\frac{\partial}{\partial \theta} \bigg|_{\theta = 0} \right) = i \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} - i \sum_{j=1}^n \overline{z}_j \frac{\partial}{\partial \overline{z}_j}, \tag{6.2.15}$$

where $\pi_{z,*}$ is the differential map induced by π_z. Note that T is tangent to the level sets of r. In particular, T is tangent to the boundary of D.

Definition 6.2.6. *Let K be a compact subset of the boundary of a smooth bounded circular domain D. D is said to have transverse circular symmetry on K if for each point $z \in K$ the vector field T defined in (6.2.15) is not contained in $T^{1,0}_z(bD) \oplus T^{0,1}_z(bD)$.*

It is obvious from (6.2.15) and Definition 6.2.6 that D has transverse circular symmetry on the whole boundary if and only if $\sum_{j=1}^n z_j (\partial r / \partial z_j)(z) \neq 0$ on bD. Then, we prove

Theorem 6.2.7. *Let $D \subseteq \mathbb{C}^n, n \geq 2$, be a smooth bounded circular pseudoconvex domain and let r be defined by (6.2.14). Suppose that $\sum_{j=1}^n z_j(\partial r/\partial z_j)(z) \neq 0$ on the boundary. Then the $\overline{\partial}$-Neumann problem is exactly regular on $W^s_{(p,q)}(D)$ for $0 \leq p \leq n$, $1 \leq q \leq n$ and $s \geq 0$.*

Proof. Let T be the vector field defined in (6.2.15). By assumption T is transversal to $T^{1,0}(bD) \oplus T^{0,1}(bD)$ everywhere on the boundary. Let $L_n' = (|\nabla r|^2/4)L_n$ and

$$L_{jk} = \frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_k} - \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_j}, \text{ for } 1 \leq j < k \leq n.$$

It is easy to verify that $[T, \partial/\partial \overline{z}_j] = i\partial/\partial\overline{z}_j$ and $[T, \partial/\partial z_j] = -i\partial/\partial z_j$. Then, we have

$$\begin{aligned}
[T, L_{jk}] &= \left[T, \frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_k} - \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_j}\right] \\
&= T\left(\frac{\partial r}{\partial z_j}\right)\frac{\partial}{\partial z_k} + \frac{\partial r}{\partial z_j}\left[T, \frac{\partial}{\partial z_k}\right] - T\left(\frac{\partial r}{\partial z_k}\right)\frac{\partial}{\partial z_j} - \frac{\partial r}{\partial z_k}\left[T, \frac{\partial}{\partial z_j}\right] \\
&= \left(\frac{\partial}{\partial z_j}(Tr)\right)\frac{\partial}{\partial z_k} + \left(\left[T, \frac{\partial}{\partial z_j}\right]r\right)\frac{\partial}{\partial z_k} - i\frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_k} \\
&\quad - \left(\frac{\partial}{\partial z_k}(Tr)\right)\frac{\partial}{\partial z_j} - \left(\left[T, \frac{\partial}{\partial z_k}\right]r\right)\frac{\partial}{\partial z_j} + i\frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_j} \\
&= -2i\left(\frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_k} - \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_j}\right) \\
&= -2iL_{jk},
\end{aligned}$$

for all $1 \leq j < k \leq n$, since $Tr \equiv 0$, and

$$\begin{aligned}
[T, L_n'] &= \left[T, \sum_{j=1}^n \frac{\partial r}{\partial \overline{z}_j}\frac{\partial}{\partial z_j}\right] \\
&= \sum_{j=1}^n \left(T\left(\frac{\partial r}{\partial \overline{z}_j}\right)\frac{\partial}{\partial z_j} + \frac{\partial r}{\partial \overline{z}_j}\left[T, \frac{\partial}{\partial z_j}\right]\right) \\
&= \sum_{j=1}^n \left(i\frac{\partial r}{\partial \overline{z}_j}\frac{\partial}{\partial z_j} - i\frac{\partial r}{\partial \overline{z}_j}\frac{\partial}{\partial z_j}\right) \\
&\equiv 0.
\end{aligned}$$

Similarly, we have $[T, \overline{L}_n'] \equiv 0$. Since $L_n = (4/|\nabla r|^2)L_n'$ and $|\nabla r| = 1$ on bD, it is easily seen that

$$[T, L_n]|_{bD} = [T, \overline{L}_n]|_{bD} = 0.$$

Hence, condition (T) holds on D. By Theorem 6.2.1 this proves Theorem 6.2.7.

The next result shows that a complete Reinhardt domain always enjoys transverse circular symmetry.

Theorem 6.2.8. *Let $D \subseteq \mathbb{C}^n, n \geq 2$, be a smooth bounded complete Reinhardt pseudoconvex domain with a smooth defining function $r(z) = r(e^{i\theta_1}z_1, \cdots, e^{i\theta_n}z_n)$ for all $\theta_1, \cdots, \theta_n \in \mathbb{R}$. Then, we have $\sum_{j=1}^n z_j(\partial r/\partial z_j) \neq 0$ on bD. In particular, the $\overline{\partial}$-Neumann problem is exactly regular on $W^s_{(p,q)}(D)$ for $0 \leq p \leq n$, $1 \leq q \leq n$ and $s \geq 0$.*

Proof. Let T be defined by (6.2.15). Put $L = \sum_{j=1}^n z_j(\partial/\partial z_j)$. Then, by our construction, we have

$$Tr = -2\mathrm{Im}L(r) = 0$$

on the boundary. Hence, it suffices to show that $\mathrm{Re}L(r) \neq 0$ on the boundary. This is, in turn, equivalent to showing that the real vector field $\sum_{j=1}^n (x_j(\partial/\partial x_j) + y_j(\partial/\partial y_j))$ is transversal to the boundary everywhere.

Suppose now that for some point $p \in bD$ we have

$$\sum_{j=1}^n \left(x_j \frac{\partial r}{\partial x_j} + y_j \frac{\partial r}{\partial y_j} \right)(p) = 0.$$

This implies that the point $p = (x_1(p), y_1(p), \cdots, x_n(p), y_n(p))$ is perpendicular to the normal $(\partial r/\partial x_1,\ \partial r/\partial y_1,\ \cdots,\ \partial r/\partial x_n,\ \partial r/\partial y_n)(p)$. We may assume, by rotation, that $x_j(p) > 0$ and $y_j(p) > 0$ for $1 \leq j \leq n$. Hence, by elementary tangent approximation, there exists a point $q \in D$ such that $|x_j(q)| > |x_j(p)|$ and $|y_j(q)| > |y_j(p)|$ for $1 \leq j \leq n$ which in turn shows $|z_j(p)| < |z_j(q)|$ for $1 \leq j \leq n$. Since D is a complete Reinhardt domain, we must have $p \in D$. This contradicts the fact that p is a boundary point. In view of Theorem 6.2.7, the proof of Theorem 6.2.8 is now complete.

6.3 The Bergman Projection and Boundary Regularity of Biholomorphic Maps

As an application of the regularity theorem proved earlier for the $\overline{\partial}$-Neumann operator, we shall investigate the boundary regularity of a biholomorphic map in this section. Recall that a holomorphic map f between two domains D_1 and D_2 is called biholomorphic if f is one-to-one, onto and the inverse map f^{-1} is also holomorphic.

Let D be a domain in $\mathbb{C}^n$. we denote by $\mathcal{H}(D)$ the space of square integrable holomorphic functions on D as before. Obviously, $\mathcal{H}(D)$ is a closed subspace of $L^2(D)$, and hence is itself a Hilbert space. If $D = \mathbb{C}^n$, then $\mathcal{H}(\mathbb{C}^n) = \{0\}$. Thus, we are interested in the case when $\mathcal{H}(D)$ is nontrivial, in particular, when D is bounded. For any $w \in D$, it is easily verified that the point evaluation map

$$\begin{aligned} \Lambda_w : \mathcal{H}(D) &\to \mathbb{C} \\ f \quad &\mapsto f(w), \end{aligned}$$

by Cauchy's estimate, satisfies

(6.3.1) $$|f(w)| \leq cd(w)^{-n} \parallel f \parallel_{L^2(D)},$$

where $d(w)$ is the distance from w to the complement of D, and the constant c depends only on the space dimension n. Hence, by the Riesz representation theorem, there is a unique element, denoted by $K_D(\cdot, w)$, in $\mathcal{H}(D)$ such that

$$f(w) = \Lambda_w(f) = (f, K_D(\cdot, w)) = \int_D f(z)\overline{K_D(z, w)} dV_z,$$

for all $f \in \mathcal{H}(D)$. The function $K_D(z, w)$ thus defined is called the Bergman kernel function for D. By (6.3.1) the Bergman kernel function clearly satisfies

$$\| K_D(\cdot, w) \|_{L^2(D)} \leq c d(w)^{-n}, \tag{6.3.2}$$

for any $w \in D$.

Next we verify a fundamental symmetry property of $K_D(z, w)$. We shall sometimes omit the subscript D if there is no ambiguity.

Lemma 6.3.1. *The Bergman kernel function $K(z, w)$ satisfies*

$$K(z, w) = \overline{K(w, z)}, \quad \textit{for all } z, w \in D,$$

and hence $K(z, w)$ is anti-holomorphic in w.

Proof. For each $w \in D$, $K(\cdot, w) \in \mathcal{H}(D)$. Hence, by the reproducing property of the kernel function, we obtain

$$\begin{aligned} K(z, w) &= (K(\cdot, w), K(\cdot, z)) \\ &= \overline{(K(\cdot, z), K(\cdot, w))} \\ &= \overline{K(w, z)}. \end{aligned}$$

This proves the lemma.

Since $\mathcal{H}(D)$ is a separable Hilbert space, the Bergman kernel function can also be represented in terms of any orthonormal basis for $\mathcal{H}(D)$.

Theorem 6.3.2. *Let $\{\phi_j(z)\}_{j=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}(D)$. Then*

$$K(z, w) = \sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)}, \quad \textit{for } (z, w) \in D \times D, \tag{6.3.3}$$

where the series (6.3.3) converges uniformly on any compact subset of $D \times D$. In particular, $K(z, \overline{w})$ is holomorphic in $(z, w) \in D \times D^*$, where $D^* = \{\overline{w} \mid w \in D\}$, and hence $K(z, w) \in C^\infty(D \times D)$.

Proof. For any fixed $w \in D$, from general Hilbert space theory we have

$$\begin{aligned} K(z, w) &= \sum_{j=1}^{\infty} (K(\cdot, w), \phi_j(\cdot))\phi_j(z) \\ &= \sum_{j=1}^{\infty} \overline{(\phi_j(\cdot), K(\cdot, w))}\phi_j(z) \\ &= \sum_{j=1}^{\infty} \overline{\phi_j(w)}\phi_j(z), \end{aligned}$$

where the series converges in the L^2 norm, and

$$\| K(\cdot, w) \|_{L^2(D)}^2 = \sum_{j=1}^{\infty} |(K(\cdot, w), \phi_j(\cdot))|^2 = \sum_{j=1}^{\infty} |\phi_j(w)|^2. \tag{6.3.4}$$

Since pointwise convergence is dominated by L^2 convergence in $\mathcal{H}(D)$, we obtain the pointwise convergence of (6.3.3). Therefore, to finish the proof, it suffices to show by a normal family argument that $|\sum_{j=1}^m \phi_j(z)\overline{\phi_j(w)}|$, for any $m \in \mathbb{N}$, is uniformly bounded on any compact subset of $D \times D$. Thus, letting M be a compact subset of D, for any $(z, w) \in M \times M$, then (6.3.4) together with (6.3.2) shows

$$\begin{aligned}
\left| \sum_{j=1}^{m} \phi_j(z)\overline{\phi_j(w)} \right| &\leq \sum_{j=1}^{\infty} |\phi_j(z)||\phi_j(w)| \\
&\leq \left(\sum_{j=1}^{\infty} |\phi_j(z)|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} |\phi_j(w)|^2 \right)^{\frac{1}{2}} \\
&\leq C_M,
\end{aligned}$$

for some constant $C_M > 0$ independent of m. This completes the proof of the theorem.

The Bergman kernel function in general is not computable except for special domains. When D is the unit ball B_n in $\mathbb{C}^n$, we shall apply Theorem 6.3.2 to obtain an explicit formula for the Bergman kernel function on B_n. Obviously, $\{z^\alpha\}$ is an orthogonal basis for $\mathcal{H}(B_n)$, where the index $\alpha = (\alpha_1, \cdots, \alpha_n)$ runs over the multiindices. We shall normalize it using the fact, for $s, t \in \mathbb{N}$ and $0 \leq a < 1$,

$$\begin{aligned}
\int_0^{\sqrt{1-a^2}} x^{2s+1} \left(1 - \frac{x^2}{1-a^2}\right)^{t+1} dx &= \frac{1}{2}(1-a^2)^{s+1} \int_0^1 y^s (1-y)^{t+1} dy \\
&= \frac{1}{2}(1-a^2)^{s+1} B(s+1, t+2) \\
&= \frac{1}{2}(1-a^2)^{s+1} \frac{\Gamma(s+1)\Gamma(t+2)}{\Gamma(s+t+3)},
\end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function and $\Gamma(\cdot)$ is the Gamma function. Hence

$$\begin{aligned}
\| z^\alpha \|^2_{L^2(B_n)} &= \int_{B_n} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} dV_{2n} \\
&= \frac{\pi}{(\alpha_n+1)} \int_{B_{n-1}} |z_1|^{2\alpha_1} \cdots |z_{n-1}|^{2\alpha_{n-1}} (1-|z_1|^2 - \cdots - |z_{n-1}|^2)^{\alpha_n+1} dV_{2n-2} \\
&= \frac{\pi}{(\alpha_n+1)} \int_{B_{n-1}} |z_1|^{2\alpha_1} \cdots |z_{n-2}|^{2\alpha_{n-2}} (1-|z_1|^2 - \cdots - |z_{n-2}|^2)^{\alpha_n+1} \\
&\qquad \cdot |z_{n-1}|^{2\alpha_{n-1}} \left(1 - \frac{|z_{n-1}|^2}{1-|z_1|^2-\cdots-|z_{n-2}|^2}\right)^{\alpha_n+1} dV_{2n-2} \\
&= \frac{\pi}{(\alpha_n+1)} \frac{\pi\Gamma(\alpha_{n-1}+1)\Gamma(\alpha_n+2)}{\Gamma(\alpha_n+\alpha_{n-1}+3)} \int_{B_{n-2}} |z_1|^{2\alpha_1} \cdots |z_{n-2}|^{2\alpha_{n-2}} \\
&\qquad \cdot (1-|z_1|^2 - \cdots - |z_{n-2}|^2)^{\alpha_n+\alpha_{n-1}+2} dV_{2n-4} \\
&= \frac{\pi}{(\alpha_n+1)} \cdot \frac{\pi\Gamma(\alpha_{n-1}+1)\Gamma(\alpha_n+2)}{\Gamma(\alpha_n+\alpha_{n-1}+3)} \cdots \frac{\pi\Gamma(\alpha_1+1)\Gamma(\alpha_n+\cdots+\alpha_2+n)}{\Gamma(\alpha_n+\cdots+\alpha_1+n+1)} \\
&= \frac{\pi^n \cdot \alpha_1! \cdots \alpha_n!}{(\alpha_n+\cdots+\alpha_1+n)!}.
\end{aligned}$$

It follows that the Bergman kernel function on the unit ball B_n is given by

$$\begin{aligned}
K(z,w) &= \sum_\alpha \frac{(\alpha_n+\cdots+\alpha_1+n)!}{\pi^n \cdot \alpha_1! \cdots \alpha_n!} \; z^\alpha \overline{w}^\alpha \\
&= \frac{1}{\pi^n} \sum_{k=0}^\infty \sum_{|\alpha|=k} \frac{(\alpha_n+\cdots+\alpha_1+n)!}{\alpha_1! \cdots \alpha_n!} \; z^\alpha \overline{w}^\alpha \\
&= \frac{1}{\pi^n} \sum_{k=0}^\infty (k+n)(k+n-1)\cdots(k+1)(z_1\overline{w}_1+\cdots+z_n\overline{w}_n)^k \\
&= \frac{1}{\pi^n} \frac{d^n}{dt^n} \left(\frac{1}{1-t} \right) \bigg|_{t=z_1\overline{w}_1+\cdots+z_n\overline{w}_n} \\
&= \frac{n!}{\pi^n} \frac{1}{(1-z\cdot\overline{w})^{n+1}},
\end{aligned}$$

where $z \cdot \overline{w} = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$.

Theorem 6.3.3. *The Bergman kernel function on the unit ball B_n is given by*

$$K(z,w) = \frac{n!}{\pi^n} \frac{1}{(1-z\cdot\overline{w})^{n+1}}, \tag{6.3.5}$$

where $z \cdot \overline{w} = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$.

For any $f \in L^2(D)$, one may write $f = f_1 + f_2$, where $f_1 \in \mathcal{H}(D)$ and $f_2 \in \mathcal{H}(D)^\perp$. It follows now from the reproducing property of the Bergman kernel function that one has

$$\begin{aligned}
Pf(z) = f_1(z) &= (f_1(\cdot), K(\cdot,z)) \\
&= (f_1(\cdot), K(\cdot,z)) + (f_2(\cdot), K(\cdot,z)) \\
&= (f(\cdot), K(\cdot,z)).
\end{aligned}$$

This proves the following theorem:

Theorem 6.3.4. *The Bergman projection $P_D : L^2(D) \to \mathcal{H}(D)$ is represented by*

$$P_D f(z) = \int_D K(z,w) f(w) dV_w, \tag{6.3.6}$$

for all $f \in L^2(D)$ and $z \in D$.

The following result shows how the Bergman kernel function behaves under a biholomorphic map:

Theorem 6.3.5. *Let $f : D_1 \to D_2$ be a biholomorphic map between two domains D_1 and D_2 in $\mathbb{C}^n$. Then*

$$K_{D_1}(z,w) = det f'(z) K_{D_2}(f(z), f(w)) \overline{det f'(w)} \tag{6.3.7}$$

for all $z, w \in D_1$, where $f'(z)$ is the complex Jacobian of f.

Proof. From an elementary calculation, we observe that

$$\det J_{\mathbb{R}} f(z) = |\det f'(z)|^2,$$

where $J_{\mathbb{R}} f(z)$ is the real Jacobian of f via the standard identification between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. Hence, a change of variables shows $h \mapsto (h \circ f)\det f'$ is an isometry between $L^2(D_2)$ and $L^2(D_1)$. Thus, for each $w \in D_1$, we have

$$\det f'(\cdot) K_{D_2}(f(\cdot), f(w)) \overline{\det f'(w)} \in \mathcal{H}(D_1),$$

and for any $h \in \mathcal{H}(D_1)$, using the reproducing property of K_{D_2}, we obtain

$$(h, \det f'(\cdot) K_{D_2}(f(\cdot), f(w)) \overline{\det f'(w)})_{D_1} = h(w).$$

Therefore, from the uniqueness of the kernel function, we must have

$$K_{D_1}(z,w) = \det f'(z) K_{D_2}(f(z), f(w)) \overline{\det f'(w)}.$$

This proves the theorem.

Corollary 6.3.6. *Let $f : D_1 \to D_2$ be a biholomorphic map between two domains D_1 and D_2 in $\mathbb{C}^n$, and let P_1, P_2 be the Bergman projection operator on D_1, D_2 respectively. Then*

$$P_1(u \cdot (g \circ f)) = u \cdot (P_2(g) \circ f) \tag{6.3.8}$$

for all $g \in L^2(D_2)$, where $u = det(f'(z))$ is the determinant of the complex Jacobian of f.

Proof. The proof follows directly from the transformation law of the Bergman kernel functions. For $g \in L^2(D_2)$, $u \cdot (g \circ f) \in L^2(D_1)$. Hence, from Theorem 6.3.5,

$$\begin{aligned}
P_1(u \cdot (g \circ f)) &= \int_{D_1} K_{D_1}(z,w) \det(f'(w)) g(f(w))\, dV_w \\
&= \int_{D_1} u(z) K_{D_2}(f(z), f(w)) |u(w)|^2 g(f(w))\, dV_w \\
&= u(z) \int_{D_2} K_{D_2}(f(z), \eta) g(\eta)\, dV_\eta \\
&= u(z) \cdot (P_2(g) \circ f).
\end{aligned}$$

This proves the corollary.

Now we introduce a condition concerning the regularity of the Bergman projection operator which is useful in proving the regularity of a biholomorphic map near the boundary.

Definition 6.3.7. *A smooth bounded domain D in $\mathbb{C}^n$ is said to satisfy condition R if the Bergman projection P associated with D maps $C^\infty(\overline{D})$ into $C^\infty(\overline{D})\cap\mathcal{O}(D)$.*

Denote by $W_0^s(D)$ the closure of $C_0^\infty(D)$ in $W^s(D)$, and by $\mathcal{H}^s(D) = W^s(D) \cap \mathcal{O}(D)$. The next theorem gives various conditions equivalent to condition R.

Theorem 6.3.8. *Let D be a smooth bounded domain in $\mathbb{C}^n$ with Bergman projection P and Bergman kernel function $K(z,w)$. The following conditions are equivalent:*

(1) *D satisfies condition R.*
(2) *For each positive integer s, there is a nonnegative integer $m = m_s$ such that P is bounded from $W_0^{s+m}(D)$ to $\mathcal{H}^s(D)$.*
(3) *For each multiindex α, there are constants $c = c_\alpha$ and $m = m_\alpha$ such that*

$$\sup_{z\in D}\left|\frac{\partial^\alpha}{\partial z^\alpha}K(z,w)\right| \le cd(w)^{-m},$$

where $d(w)$ is the distance from the point w to the boundary bD.

Before proving Theorem 6.3.8, we shall first prove the following lemma:

Lemma 6.3.9. *Let D be a smooth bounded domain in $\mathbb{C}^n$. Then, for each $s \in \mathbb{N}$, there is a linear differential operator Φ^s of order $n_s = s(s+1)/2$ with coefficients in $C^\infty(\overline{D})$ such that Φ^s maps $W^{s+n_s}(D)$ boundedly into $W_0^s(D)$ and that $P\Phi^s = P$.*

In other words, for each $g \in C^\infty(\overline{D})$ and $s \in \mathbb{N}$, Lemma 6.3.9 allows us to construct a $h = g - \Phi^s g \in C^\infty(\overline{D})$ such that $Ph \equiv 0$ and that h agrees with g up to order $s-1$ on the boundary.

Proof. Let ρ be a smooth defining function for D, and let $\delta > 0$ be so small that $\nabla\rho \neq 0$ on $U_\delta = \{z|\ |\rho(z)| < \delta\}$. Choose a partition of unity $\{\phi_i\}_{i=1}^m$ and, for each i, a complex coordinate z_i in some neighborhood of the support of ϕ_i such that

(1) $\sum_{i=1}^m \phi_i \equiv 1$ on U_ϵ for some $\epsilon < \delta$,
(2) $\text{supp}\phi_i \subset U_\delta$ and $\text{supp}\phi_i \cap bD \neq \emptyset$, and
(3) $\partial\rho/\partial z_i \neq 0$ on $\text{supp}\phi_i$.

To define the operator Φ^s inductively on s, we need the fact that if g is in $C^\infty(\overline{D})$ and vanishes up to order $s-1$ on the boundary, then $g \in W_0^s(D)$. For the initial step $s = 1$, if $h \in C^\infty(\overline{D})$, define

$$\Phi^1 h = h - \sum_{i=1}^m \frac{\partial}{\partial z_i}(\theta_0^i\rho),$$

where $\theta_0^i = (\phi_i h)(\partial\rho/\partial z_i)^{-1}$. It is easy to see that $\Phi^1 h = 0$ on bD, and hence $\Phi^1 h \in W_0^1(D)$.

Suppose $\theta_0^i, \cdots, \theta_{s-1}^i$ have been chosen so that

$$\Phi_i^s h = \phi_i h - \frac{\partial}{\partial z_i}\left(\sum_{k=0}^{s-1}\theta_k^i\rho^{k+1}\right)$$

vanishes to order $s-1$ on the boundary. We define

$$\Phi^s h = h - \sum_{i=1}^{m} \frac{\partial}{\partial z_i}\left(\sum_{k=0}^{s-1} \theta_k^i \rho^{k+1}\right).$$

Since ρ vanishes on the boundary, it is easily verified by integration by parts that $\sum_{i=1}^{m} \frac{\partial}{\partial z_i}(\sum_{k=0}^{s-1} \theta_k^i \rho^{k+1})$ is orthogonal to $\mathcal{H}(D)$. Hence $\Phi^s h \in W_0^s(D)$ and $P\Phi^s h = Ph$. Put $\partial/\partial\nu = \nabla\rho \cdot \nabla/|\nabla\rho|^2$, the normal differentiation, such that $\partial\rho/\partial\nu = 1$. Let

$$\theta_s^i = \frac{(\frac{\partial}{\partial\nu})^s \Phi_i^s h}{(s+1)!\frac{\partial\rho}{\partial z_i}}.$$

Then the functions

$$\Phi_i^{s+1} h = \Phi_i^s h - \frac{\partial}{\partial z_i}(\theta_s^i \rho^{s+1})$$

vanish to order s on the boundary, so does the function

$$\begin{aligned}\Phi^{s+1} h &= \Phi^s h - \sum_{i=1}^{m} \frac{\partial}{\partial z_i}(\theta_s^i \rho^{s+1})\\ &= h - \sum_{i=1}^{m} \frac{\partial}{\partial z_i}\left(\sum_{k=0}^{s} \theta_k^i \rho^{k+1}\right).\end{aligned}$$

Hence, $\Phi^{s+1} h \in W_0^{s+1}(D)$ and $P\Phi^{s+1} = P$. This completes the induction.

It is also easily verified by a simple induction argument that $\Phi^s h$ can be written as

$$\Phi^s h = \sum_{|\alpha| \le k \le n_s} b_{\alpha,k} \rho^k D^\alpha h,$$

where $n_s = s(s+1)/2$ and the $b_{\alpha,k}$'s are in $C^\infty(\overline{D})$ and D^α is the real differential operator of order $|\alpha|$ associated to the multiindex $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{2n})$. This proves the lemma.

We also need negative Sobolev norms for holomorphic functions. For $g \in \mathcal{O}(D)$ and s a positive integer, we define

$$\| g \|_{-s} = \sup \left| \int_D g\overline{\phi} \right|, \tag{6.3.9}$$

where the supremum is taken over all $\phi \in C_0^\infty(D)$ with $\| \phi \|_s = 1$.

Lemma 6.3.10. *Let D be a smooth bounded domain in $\mathbb{C}^n$, $n \ge 2$. For $g \in \mathcal{O}(D)$ and any positive integer s, we have*

(1) $\sup_{z\in D} |g(z)| d(z)^{s+n} \le c_1 \| g \|_{-s}$,

(2) $\| g \|_{-s-n-1} \le c_2 \sup_{z\in D} |g(z)| d(z)^s$,

for some constants c_1 *and* c_2 *independent of* g.

Proof. Let χ be a smooth, nonnegative, radially symmetric function supported in the unit ball B_n in $\mathbb{C}^n$ with $\int_{B_n} \chi(w) dV_w = 1$. For $z \in D$, let $\chi_z(w) = \epsilon^{-2n}\chi((z-w)/\epsilon)$, where $\epsilon = d(z)$. Clearly,

$$\| \chi_z \|_k \leq c(k) d(z)^{-(k+n)},$$

for some constant $c(k) > 0$ depending on k. Then, using polar coordinates and the mean value property of $g \in \mathcal{O}(D)$, we obtain, for $l > n$,

$$\begin{aligned} |g(z)| &= \left| \int_D g(w)\chi_z(w)\ dV_w \right| \\ &\leq \| g \|_{-l+n} \| \chi_z \|_{l-n} \\ &\leq c_1 \| g \|_{-l+n}\ d(z)^{-l}. \end{aligned}$$

Setting $l = s+n$, this proves (1).

For (2), notice that if $\phi \in C_0^\infty(D)$, then by Taylor's expansion and the Sobolev embedding theorem, we have

$$|\phi(z)| \leq c \| \phi \|_{s+n+1}\ d(z)^s.$$

Hence,

$$\begin{aligned} \| g \|_{-s-n-1} &= \sup_{\substack{\phi \in C_0^\infty(D) \\ \|\phi\|_{s+n+1}=1}} \left| \int_D g\overline{\phi} dV_z \right| \\ &\leq c_2 \sup_{z \in D} |g(z)| d(z)^s. \end{aligned}$$

This completes the proof of Lemma 6.3.10.

Proof of Theorem 6.3.8. If D satisfies condition R, from the topology on $C^\infty(\overline{D})$, for each positive integer s there is a nonnegative integer $m = m_s$ such that P maps $W^{s+m}(D)$ boundedly into $\mathcal{H}^s(D)$. In particular, (1) implies (2).

Te see that (2) implies (1), by assumption, for each $s \in \mathbb{N}$, there is a nonnegative integer $m = m_s$ such that P is a bounded operator from $W_0^{s+m_s}(D)$ into $\mathcal{H}^s(D)$. For this fixed $s+m_s$, Lemma 6.3.9 shows that there exists a positive integer $n'_s = n_{s+m_s}$ such that Φ^{s+m_s} maps $W^{s+m_s+n'_s}(D)$ boundedly into $W_0^{s+m_s}(D)$. It follows that, for each $g \in W^{s+m_s+n'_s}(D)$, we have

$$\| Pg \|_s = \| P\Phi^{s+m_s} g \|_s \lesssim \| \Phi^{s+m_s} g \|_{s+m_s} \lesssim \| g \|_{s+m_s+n'_s}\ .$$

Hence, (2) implies (1).

Next, we prove the equivalence of (1) and (3). Suppose (3) holds. Then, for each multiindex α and each $z \in D$, we have by Lemma 6.3.10

$$\| \frac{\partial^\alpha}{\partial z^\alpha} K(z,\cdot) \|_{-s} \leq c_2 \sup_{w \in D} \left| \frac{\partial^\alpha}{\partial z^\alpha} K(z,w) \right| d(w)^{m_\alpha} \leq C,$$

where $s = s_\alpha = m_\alpha + n + 1$. Hence, by using the operator Φ^s constructed in Lemma 6.3.9, for $g \in W^{s+n_s}(D)$ and $z \in D$, we have

$$\begin{aligned}\left|\frac{\partial^\alpha}{\partial z^\alpha} Pg(z)\right| &= \left|\frac{\partial^\alpha}{\partial z^\alpha}\int_D K(z,w)\Phi^s g(w) dV_w\right| \\ &= \left|\int_D \frac{\partial^\alpha}{\partial z^\alpha} K(z,w)\Phi^s g(w) dV_w\right| \\ &\le \| \frac{\partial^\alpha}{\partial z^\alpha} K(z,\cdot) \|_{-s} \| \Phi^s g \|_s \\ &\le C \| g \|_{s+n_s} .\end{aligned}$$

The differentiation under the integral sign is justified, since

$$\begin{aligned}\left|\frac{\partial^\alpha}{\partial z^\alpha} K(z,w)\Phi^s g(w)\right| &\le C_1 \left|\frac{\partial^\alpha}{\partial z^\alpha} K(z,w)\right| \| \Phi^s g \|_s \, d(w)^{m_\alpha} \\ &\le C_2 \| \Phi^s g \|_s,\end{aligned}$$

for some constant $C_2 > 0$ independent of z and w. Thus, condition R holds on D.

On the other hand, if condition R holds on D, then by the Sobolev embedding theorem, for each nonnegative integer s there is an integer $k(s)$ such that

$$\sup_{z\in D}\left|\frac{\partial^\alpha}{\partial z^\alpha} Pf(z)\right| \le C \| f \|_{k(s)},$$

for all multiindices α with $|\alpha| \le s$. Therefore,

$$\begin{aligned}\| \frac{\partial^\alpha}{\partial z^\alpha} K(z,\cdot) \|_{-k(s)} &= \sup_{\substack{\phi\in C_0^\infty(D) \\ \|\phi\|_{k(s)}=1}} \left|\int_D \frac{\partial^\alpha}{\partial z^\alpha} K(z,w)\phi(w) dV_w\right| \\ &= \sup_{\substack{\phi\in C_0^\infty(D) \\ \|\phi\|_{k(s)}=1}} \left|\frac{\partial^\alpha}{\partial z^\alpha} P\phi(z)\right| \\ &\le C,\end{aligned}$$

uniformly as z ranges over D. Hence, by (1) of Lemma 6.3.10, condition (3) holds. This completes the proof of Theorem 6.3.8.

Here are some consequences of condition R.

Corollary 6.3.11. *Let D be a smooth bounded domain in $\mathbb{C}^n$, $n \ge 2$. Suppose that condition R holds on D. Then $K(\cdot, w) \in C^\infty(\overline{D})$ for each $w \in D$.*

Proof. For each fixed $w \in D$, let $\phi_w(z) \in C_0^\infty(D)$ be a smooth real-valued function such that $\phi_w(z)$ is radially symmetric with respect to the center w and $\int_D \phi_w(z)\, dV_z = 1$. Since ϕ_w is constant on the sphere centered at w, applying polar coordinates and using the mean value property of $K(z,\cdot)$, we have

$$K(z,w) = \int_D K(z,\eta)\phi_w(\eta)\, dV_\eta.$$

Hence, $K(\cdot, w) \in C^\infty(\overline{D})$ by condition R on D.

Corollary 6.3.12. *Let D be a smooth bounded domain in $\mathbb{C}^n$, $n \geq 2$. Suppose that condition R holds on D. Then the linear span of $\{K(\cdot, w)|\ w \in D\}$ is dense in $\mathcal{H}^\infty(\overline{D})$ in C^∞ topology.*

Proof. First, for each positive integer s and $g \in \mathcal{H}^\infty(\overline{D})$, Lemma 6.3.9 shows that

$$g = Pg = P\Phi^s g.$$

Hence, we have $\mathcal{H}^\infty(\overline{D}) \subset P(W_0^s(D))$ for all real $s \geq 0$.

Let Λ be the set of functions $\phi \in C_0^\infty(D)$ which are radially symmetric about some point in D with $\int_D \phi dV = 1$. Thus, $P\Lambda = \{K(\cdot, w)|\ w \in D\}$. We claim that the linear span of Λ is dense in $W_0^s(D)$ for each $s \geq 0$. Let $f \in C_0^\infty(D)$. Choose a smooth nonnegative function χ from $C_0^\infty(\mathbb{C}^n)$ which is radially symmetric about the origin with support contained in the unit ball and satisfying $\int_{\mathbb{C}^n} \chi(z) dV_z = 1$. For $\epsilon > 0$, set $\chi_\epsilon(z) = \epsilon^{-2n}\chi(z/\epsilon)$. Then $f_\epsilon = f * \chi_\epsilon$ will converge to f in $W_0^s(D)$. Since

$$f_\epsilon(z) = \epsilon^{-2n} \int_{\mathbb{C}^n} f(w)\chi\left(\frac{z-w}{\epsilon}\right) dV_w$$

can be approximated by finite Riemann sums, this yields the density of span$\{\Lambda\}$ in $W_0^s(D)$.

Now, condition R implies the $\mathcal{H}^s(D)$ closure of the span of $\{K(\cdot, w)|\ w \in D\}$ contains $\mathcal{H}^\infty(\overline{D})$ for every $s \geq 0$. Hence, by the Sobolev embedding theorem, span$\{K(\cdot, w)|\ w \in D\}$ is dense in $\mathcal{H}^\infty(\overline{D})$ in the C^∞ topology. This proves the corollary.

To end this section, we prove the following important consequence of condition R concerning the boundary regularity of a biholomorphic map between two smooth bounded pseudoconvex domains in $\mathbb{C}^n$.

Theorem 6.3.13. *Let D_1 and D_2 be two smooth bounded pseudoconvex domains in $\mathbb{C}^n$, $n \geq 2$, and let f be a biholomorphic map from D_1 onto D_2. Suppose that condition R holds on both D_1 and D_2, then f extends smoothly to the boundary.*

We note from Theorem 1.7.1 that an analog of the Riemann mapping theorem in the complex plane does not hold in $\mathbb{C}^n$ for $n \geq 2$. Theorem 6.3.13 provides an important approach to the classification of domains in higher dimensional spaces. Therefore, given a domain D, it is fundamental to verify whether condition R holds on D or not. When D is a smooth bounded pseudoconvex domain, the Bergman projection P can be expressed in terms of the $\overline{\partial}$-Neumann operator N by the formula (4.4.14), $P = I - \overline{\partial}^* N \overline{\partial}$. We know from previous discussions that condition R holds on the following classes of smooth bounded domains:

(1) D is strongly pseudoconvex (Theorem 5.2.1 and Corollary 5.2.7).
(2) D admits a plurisubharmonic defining function. In particular, if D is convex (Theorems 6.2.3 and 6.2.5).
(3) D is a circular pseudoconvex domain with transverse circular symmetry (Theorem 6.2.7).

In fact, the Bergman projection P is exactly regular on all of the above three classes of pseudoconvex domains.

Theorem 6.3.13 will be proved later. We first prove the following theorem:

Theorem 6.3.14. *Let $f : D_1 \to D_2$ be a biholomorphic map between two smooth bounded pseudoconvex domains D_1 and D_2 in $\mathbb{C}^n$. Then there is a positive integer m such that*

$$d(z, bD_1)^m \lesssim d(f(z), bD_2) \lesssim d(z, bD_1)^{\frac{1}{m}}$$

for all $z \in D_1$.

Proof. By Theorem 3.4.12, there are continuous functions $\rho_j : \overline{D}_j \to \mathbb{R}$, $j = 1, 2$, satisfying

(1) ρ_j is smooth and plurisubharmonic on D_j,
(2) $\rho_j < 0$ on D_j and ρ_j vanishes on bD_j,
(3) $(-\rho_j)^m = -r_j e^{-K|z|^2}$ is smooth on $\overline{D}_j$ for some positive integer m, where r_j is a smooth defining function for D_j.

Property (3) immediately implies that

$$|\rho_j(z)| \lesssim d(z, bD_j)^{\frac{1}{m}}, \quad \text{for } z \in D_j.$$

Since $f : D_1 \to D_2$ is a biholomorphic map, both $\rho_2 \circ f$ and $\rho_1 \circ f^{-1}$ satisfy (1) and (2). Thus, an application of the classical Hopf lemma (see [GiTr 1]) shows that

$$d(z, bD_1) \lesssim |\rho_2 \circ f(z)| \lesssim d(f(z), bD_2)^{\frac{1}{m}}$$

and

$$d(w = f(z), bD_2) \lesssim |\rho_1 \circ f^{-1}(w)| \lesssim d(z = f^{-1}(w), bD_1)^{\frac{1}{m}}.$$

This proves the theorem.

Lemma 6.3.15. *Let $f : D_1 \to D_2$ be a biholomorphic map between two smooth bounded pseudoconvex domains D_1 and D_2 in $\mathbb{C}^n$. Let $u(z) = det(f'(z))$ be the determinant of the complex Jacobian of f. Then, for any positive integer s, there is an integer $j = j(s)$ such that the mapping $\phi \mapsto u \cdot (\phi \circ f)$ is bounded from $W_0^{s+j(s)}(D_2)$ to $W_0^s(D_1)$.*

Proof. It suffices to show the estimate

$$\| u \cdot (\phi \circ f) \|_{W_0^s(D_1)} \le C \| \phi \|_{W_0^{s+j(s)}(D_2)}$$

for all $\phi \in C_0^\infty(D_2)$. Write $f = (f_1, \cdots, f_n)$. For any multiindex α with $|\alpha| \le s$, we have

$$D^\alpha(u \cdot (\phi \circ f)) = \sum D^\beta u \cdot D^\gamma \phi(f) \cdot D^{\delta_1} f_{i_1} \cdots D^{\delta_p} f_{i_p},$$

where $1 \le i_1, \cdots, i_p \le n$, and $\beta, \gamma, \delta_1, \cdots, \delta_p$ are multiindices with $|\beta| \le |\alpha|$, $|\gamma| \le |\alpha| - |\beta|$ and $\sum_{j=1}^p |\delta_j| = |\alpha| - |\beta|$.

Since f is a map between two bounded domains D_1 and D_2 in $\mathbb{C}^n$, the Cauchy estimate implies that

$$\left| \frac{\partial^\beta u}{\partial z^\beta}(z) \right| \le C_\beta d_1(z)^{-(|\beta|+1)},$$

and

$$\left| \frac{\partial^{\delta_j} f_{i_j}}{\partial z^{\delta_j}}(z) \right| \le C_j d_1(z)^{-|\delta_j|},$$

where the constant C_β (C_j) depends on D_2 and the multiindex β (δ_j), and $d_1(z) = d(z, bD_1)$. Hence, for $z \in D_1$,

$$|D^\beta u \cdot D^{\delta_1} f_{i_1} \cdots D^{\delta_p} f_{i_p}(z)| \leq C d_1(z)^{-(|\alpha|+1)}.$$

Also, for any $\phi \in C_0^\infty(D_2)$ and every $k \in \mathbb{N}$, it follows from Taylor's expansion and the Sobolev embedding theorem that

$$|D^\gamma \phi(w)| \leq C \parallel \phi \parallel_{k+|\gamma|+n+1} \cdot d_2(w)^k.$$

Thus, combining the preceding inequalities with Theorem 6.3.14, we obtain

$$\begin{aligned} |D^\alpha(u \cdot (\phi \circ f))(z)| &\lesssim d_1(z)^{-(|\alpha|+1)} \cdot \parallel \phi \parallel_{k+|\alpha|+n+1} \cdot d_2(f(z))^k \\ &\lesssim \parallel \phi \parallel_{k+s+n+1} \cdot d_1(z)^{-s-1+\frac{k}{m}}. \end{aligned}$$

It is now clear by taking $k = m(s+1)$ that the mapping $\phi \mapsto u \cdot (\phi \circ f)$ is bounded from $W_0^{s+j(s)}(D_2)$ to $W_0^s(D_1)$ with $j(s) = m(s+1) + n + 1$. This proves the lemma.

We now return to the proof of Theorem 6.3.13.

Proof of Theorem 6.3.13. Let f be a biholomorphic map from D_1 onto D_2. From Corollary 6.3.6 we obtain

$$P_1(u \cdot (g \circ f)) = u \cdot (P_2(g) \circ f), \tag{6.3.10}$$

for all $g \in L^2(D_2)$, where $u = \det(f'(z))$ is the determinant of the complex Jacobian of f and P_ν, $\nu = 1, 2$, is the Bergman projection on D_ν.

Since condition R holds on D_1, for each positive integer s, there is an integer $m(s)$ such that P_1 maps $W_0^{s+m(s)}(D_1)$ boundedly into $\mathcal{H}^s(D_1)$. On the other hand, by condition R on D_2 we may choose a $g \in W_0^{s+m(s)+j(s)}(D_2)$, as in the proof of Corollary 6.3.12, such that $P_2 g \equiv 1$, where $j(s)$ is determined in Lemma 6.3.15 for the integer $s + m(s)$. Now, Lemma 6.3.15 implies $u \cdot (g \circ f) \in W_0^{s+m(s)}(D_1)$. Hence, from (6.3.10) and condition R on D_1,

$$u = P_1(u \cdot (g \circ f))$$

is in $\mathcal{H}^s(D_1)$. This shows that $u \in C^\infty(\overline{D}_1)$.

Similarly, the determinant $U(w)$ of the complex Jacobian of f^{-1} is also in $C^\infty(\overline{D}_2)$. It follows that $u(z)$ is nonvanishing on $\overline{D}_1$.

Repeating the above arguments, for each $s \in \mathbb{N}$, choose $g_k \in W_0^{s+m(s)+j(s)}(D_2)$, for $k = 1, \cdots, n$, such that $P_2 g_k \equiv w_k$, the kth coordinate function on D_2. Hence,

$$u \cdot f_k = P_1(u \cdot (g_k \circ f))$$

is in $\mathcal{H}^s(D_1)$, where $f = (f_1, \cdots, f_n)$. Since u does not vanish on $\overline{D}_1$, this implies $f_k \in C^\infty(\overline{D}_1)$ for $k = 1, \cdots, n$. It follows that $f \in C^\infty(\overline{D}_1)$. Similarly, we have $f^{-1} \in C^\infty(\overline{D}_2)$. The proof of Theorem 6.3.13 is now complete.

6.4 Worm Domains

In this section we shall construct the so-called worm domains. Such domains possess many pathological properties in complex analysis. We shall prove that such domains do not always have plurisubharmonic defining functions on the boundaries nor do they always have pseudoconvex neighborhood bases.

A Hartogs domain in $\mathbb{C}^2$ is a domain which is invariant under rotation in one of the coordinates. Let D_β be the unbounded worm domain defined by

$$D_\beta = \{(z_1, z_2) \in \mathbb{C}^2 |\ \mathrm{Re}(z_1 e^{-i\log|z_2|^2}) > 0,\ |\log|z_2|^2| < \beta - \frac{\pi}{2}\},$$

for $\beta > \pi/2$. Clearly, D_β is a Hartogs domain. Geometrically, if we use $\mathrm{Re}z_1$, $\mathrm{Im}z_1$ and $|z_2|$ as axes, then D_β can be visualized in $\mathbb{R}^3$ as an open half space in z_1 revolving along the $|z_2|$-axis when $|z_2|$ ranges from $\exp(-\beta/2 + \pi/4)$ to $\exp(\beta/2 - \pi/4)$.

To see that D_β is pseudoconvex, we note that locally, we can substitute the inequality $\mathrm{Re}(z_1 e^{-i\log|z_2|^2}) > 0$ by

$$\mathrm{Re}(z_1 e^{-i\log|z_2|^2 + \arg z_2^2}) = \mathrm{Re}(z_1 e^{-i\log z_2^2}) > 0.$$

Since $z_1 e^{-i\log z_2^2}$ is locally holomorphic, its real part is a pluriharmonic function, with vanishing complex Hessian. D_β is the intersection of two pseudoconvex domains. Thus, it is pseudoconvex. As $\log|z_2|$ changes by a length of π, we see that the half plane $\mathrm{Re}(z_1 e^{-i\log|z_2|^2})$ rotates by an angle of 2π.

To construct a bounded worm domain we shall rotate discs instead of half planes. We define

$$\Omega'_\beta = \{(z_1, z_2) \in \mathbb{C}^2 |\ |z_1 + e^{i\log|z_2|^2}|^2 < 1,\ |\log|z_2|^2| < \beta - \frac{\pi}{2}\}.$$

Since Ω'_β is defined by $|z_1|^2 + 2\mathrm{Re}(z_1 e^{-i\log|z_2|^2}) < 0$, locally, we can view Ω'_β as defined by $|z_1|^2 e^{\arg z_2^2} + 2\mathrm{Re}(z_1 e^{-i\log z_2^2}) < 0$. Since the function $|z_1|^2 e^{\arg z_2^2} = e^{\log|z_1|^2 + \arg z_2^2}$ is plurisubharmonic, it is easy to see that Ω'_β is pseudoconvex and bounded. But it is not smooth at $|\log|z_2|^2| = \beta - \frac{\pi}{2}$. For each fixed $|\log|z_2|^2| < \beta - \frac{\pi}{2}$, Ω'_β is a disc of radius 1 centered at $-e^{i\log|z_2|^2}$ and $(0, z_2) \in b\Omega'_\beta$.

To construct a smooth worm domain, we have to modify Ω'_β. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a fixed smooth function with the following properties:

(1) $\eta(x) \geq 0$, η is even and convex.
(2) $\eta^{-1}(0) = I_{\beta-\pi/2}$, where $I_{\beta-\pi/2} = [-\beta + \pi/2, \beta - \pi/2]$.
(3) there exists an $a > 0$ such that $\eta(x) > 1$ if $x < -a$ or $x > a$.
(4) $\eta'(x) \neq 0$ if $\eta(x) = 1$.

We note that (4) follows from (1) and (2). The existence of such a function is obvious. For each $\beta > \pi/2$, define

$$\Omega_\beta = \{(z_1, z_2) \in \mathbb{C}^2 |\ |z_1 + e^{i\log|z_2|^2}|^2 < 1 - \eta(\log|z_2|^2)\}. \tag{6.4.1}$$

Then, we have:

Proposition 6.4.1. *For each fixed $\beta > \pi/2$, Ω_β is a smooth bounded pseudoconvex domain in $\mathbb{C}^2$.*

Proof. Clearly, by (3), Ω_β is bounded. For the smoothness of Ω_β, we need to show that $\nabla\rho(z) \neq 0$ at every boundary point z, where $\rho(z) = |z_1 + e^{i\log|z_2|^2}|^2 - 1 + \eta(\log|z_2|^2)$ is the defining function for Ω_β. If $(\partial\rho/\partial z_1)(z) = 0$ at some boundary point z, we get

$$\frac{\partial\rho}{\partial z_1}(z) = \overline{z}_1 + e^{-i\log|z_2|^2} = 0.$$

Since $\rho(z) = 0$, it follows that $\eta(\log|z_2|^2) = 1$ whenever $(\partial\rho/\partial z_1)(z) = 0$ at a boundary point z. Now it is easy to see that $(\partial\rho/\partial z_2)(z) \neq 0$ by (4) at such points. This proves the smoothness of Ω_β.

To see that Ω_β is pseudoconvex, we write

$$\rho(z) = |z_1|^2 + 2\mathrm{Re}(z_1 e^{-i\log|z_2|^2}) + \eta(\log|z_2|^2).$$

Again locally, Ω_β can be defined by

$$|z_1|^2 e^{\arg z_2^2} + 2\mathrm{Re}(z_1 e^{-i\log z_2^2}) + \eta(\log|z_2|^2)e^{\arg z_2^2} < 0.$$

The first two terms are plurisubharmonic as before. We only need to show that the last term $\eta(\log(|z_2|^2)e^{\arg z_2^2}$ is plurisubharmonic. A direct calculation shows that

$$\triangle(\eta(\log|z_2|^2)e^{\arg z_2^2}) = \big(\triangle\eta(\log|z_2|^2)\big)e^{\arg z_2^2} + \eta(\log|z_2|^2)\triangle e^{\arg z_2^2} \geq 0,$$

since η is convex and nonnegative from (1). Ω_β is defined locally by a plurisubharmonic function. Thus it is pseudoconvex with smooth boundary.

The following result shows that for each fixed $\beta > \pi/2$ there is no C^2 *global* defining function which is plurisubharmonic on the boundary of Ω_β.

Theorem 6.4.2. *For any $\beta > \pi/2$, there is no C^2 defining function $\tilde{\rho}(z)$ for Ω_β such that $\tilde{\rho}(z)$ is plurisubharmonic on the boundary of Ω_β.*

Proof. Let $\tilde{\rho}(z)$ be such a C^2 defining function for Ω_β that is plurisubharmonic on the boundary $b\Omega_\beta$. Then there is a C^1 positive function h defined in some neighborhood of $b\Omega_\beta$ such that $\tilde{\rho}(z) = h\rho$. Let $A = \{(0, z_2) \in \mathbb{C}^2 |\ |\log|z_2|^2| < \beta - \pi/2\}$. A direct calculation shows that the complex Hessian of $\tilde{\rho}(z)$ acting on any $(\alpha, \beta) \in \mathbb{C}^2$ for any point $p \in A \subset b\Omega_\beta$ is given by

$$\begin{aligned}\mathcal{L}_{\tilde{\rho}(z)}(p; (\alpha,\beta)) &= 2\mathrm{Re}\left[\overline{\alpha}\beta\left(\frac{ih}{z_2} + \frac{\partial h}{\partial z_2}\right)e^{i\log|z_2|^2}\right] \\ &\quad + \left[h + 2\mathrm{Re}\left(\frac{\partial h}{\partial z_1}e^{i\log|z_2|^2}\right)\right]|\alpha|^2.\end{aligned} \tag{6.4.2}$$

Since, by assumption, (6.4.2) is always nonnegative, we must have

$$\left(\frac{ih}{z_2} + \frac{\partial h}{\partial z_2}\right)e^{i\log|z_2|^2} \equiv 0$$

on A, or equivalently,

$$\frac{\partial}{\partial \overline{z}_2}(he^{-i\log|z_2|^2}) \equiv 0$$

on A. Consequently,

$$g(z_2) = h(0, z_2)e^{-i\log|z_2|^2}$$

is a holomorphic function on the annulus A. It follows that

$$g(z_2)e^{i\log z_2^2} = h(0, z_2)e^{-2\arg z_2} = c,$$

is also locally a holomorphic function on A, and hence it must be a constant c, since the right hand side is real. This implies that

$$h(0, z_2) = ce^{2\arg z_2}$$

is a well defined, C^1 positive function on A, which is impossible. This proves Theorem 6.4.2.

In particular, Theorem 6.2.3 cannot be applied to worm domains. In fact we will prove in the next section that the Bergman projection is not regular on worm domains.

Another peculiar phenomenon about worm domains is that they do not have pseudoconvex neighborhood bases if β is sufficiently large. To illustrate this, we first examine the Hartogs triangle

$$G = \{(z_1, z_2) \mid |z_1| < |z_2| < 1\}.$$

By Cauchy's integral formula, any function holomorphic in a neighborhood of $\overline{G}$ extends holomorphically to the bidisc $D^2 = \{(z_1, z_2) \mid |z_1| < 1, |z_2| < 1\}$. Thus if Ω is any pseudoconvex domain containing $\overline{G}$, then Ω contains the larger set D^2 since pseudoconvex domains are domains of holomorphy. This implies that we cannot approximate $\overline{G}$ by a sequence of pseudoconvex domains $\{\Omega^k\}$ such that $\overline{G} \subset \Omega^k$ and $\overline{G} = \cap_k \Omega^k$. However, the Hartogs triangle is not smooth.

We next show that Ω_β does not have a pseudoconvex neighborhood base if $\beta \geq 3\pi/2$. When $\beta \geq 3\pi/2$, $\overline{\Omega}_\beta$ contains the set

$$\begin{aligned} K = &\{(0, z_2) \mid -\pi \leq \log|z_2|^2 \leq \pi\} \\ &\cup \{(z_1, z_2) \mid \log|z_2|^2 = \pi \text{ or } -\pi \text{ and } |z_1 - 1| < 1\}. \end{aligned}$$

Any holomorphic function in a neighborhood of K extends holomorphically to the set

$$\hat{K} = \{(z_1, z_2) \mid -\pi \leq \log|z_2|^2 \leq \pi \text{ and } |z_1 - 1| < 1\}.$$

Thus any holomorphic function in a neighborhood of $\overline{\Omega}_\beta$ extends holomorphically to $\overline{\Omega}_\beta \cup \hat{K}$. This implies that any pseudoconvex domain containing $\overline{\Omega}_\beta$ contains $\hat{K}$.

Theorem 6.4.3. *For $\beta \geq 3\pi/2$, there does not exist a sequence $\{\Omega^k\}$ of pseudoconvex domains in $\mathbb{C}^2$ with $\overline{\Omega}_\beta \subset \Omega^k$ and $\overline{\Omega}_\beta = \cap_k \Omega^k$.*

Thus pseudoconvex domains do not always have a pseudoconvex neighborhood base. We note that any pseudoconvex domain can always be exhausted by pseudoconvex domains from inside.

6.5 Irregularity of the Bergman Projection on Worm Domains

The purpose of this section is to prove that the Bergman projection P is irregular on the worm domain Ω_β in the Sobolev spaces. We first study the Bergman kernel function $K(z,w)$ on the unbounded worm domain D_β where D_β is defined in 6.4. For each fixed z_1 variable, the domain is a union of annuli in z_2. Any holomorphic function in D_β admits a Laurent expansion in z_2. Using Fourier expansion, for any $f \in \mathcal{H}(D_\beta)$, we write

$$f(z) = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} \int_0^{2\pi} f(z_1, e^{i\theta} z_2) e^{-ij\theta}\, d\theta.$$

Let $f_j(z_1,z_2) = \frac{1}{2\pi}\int_0^{2\pi} f(z_1, e^{i\theta}z_2)e^{-ij\theta}\, d\theta$. Then f_j is holomorphic and $f_j(z_1, e^{i\theta}z_2)$ $= e^{ij\theta} f_j(z_1,z_2)$. Such f_j are necessarily of the form $f_j(z_1,z_2) = g(z_1,|z_2|)z_2^j$, where $g(z_1,|z_2|)$ is holomorphic in D_β and locally constant in $|z_2|$. The Bergman space $\mathcal{H}(D_\beta)$ admits an orthogonal decomposition

$$\mathcal{H}(D_\beta) = \bigoplus_{j\in\mathbb{Z}} \mathcal{H}_j(D_\beta).$$

Any f in $\mathcal{H}_j(D_\beta)$ satisfies $f(z_1, e^{i\theta}z_2) = e^{ij\theta} f(z_1,z_2)$. Denote by P_j the orthogonal projection from $L^2(D_\beta)$ onto $\mathcal{H}_j(D_\beta)$. It follows that if $f \in \mathcal{H}(D_\beta)$, we have

$$P_j f(z) = f_j(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z_1, e^{i\theta}z_2)e^{-ij\theta}\, d\theta,$$

and the Bergman kernel function $K_{D_\beta}(z,w)$ associated with D_β satisfies

$$K_{D_\beta}(z,w) = \sum_{j\in\mathbb{Z}} K_j(z,w),$$

where $K_j(z,w)$ is the reproducing kernel for $\mathcal{H}_j(D_\beta)$. Each $K_j(z,w)$ is locally of the form $K_j(z,w) = k_j(z_1,w_1)z_2^j\bar{w}_2^j$. It turns out that for the unbounded worm domain, the kernel K_{-1} can be computed explicitly.

To facilitate the calculation, we introduce the following domain:

$$D'_\beta = \{(z_1,z_2)\in\mathbb{C}^2 |\ |\mathrm{Im} z_1 - \log|z_2|^2| < \pi/2,\ |\log|z_2|^2| < \beta - \pi/2\}.$$

For each fixed z_2, D'_β is an infinite strip in z_1. Thus, D'_β is biholomorphically equivalent to D_β via the mapping

$$\begin{aligned} \varphi:\ D'_\beta &\ \rightarrow\ D_\beta \\ (z_1,z_2) &\mapsto (e^{z_1}, z_2). \end{aligned}$$

Also from the transformation formula (6.3.3) for the Bergman kernel functions, we have

$$K_{D_\beta}(z,w) = \frac{1}{z_1\overline{w}_1} K_{D'_\beta}(\varphi^{-1}(z), \varphi^{-1}(w)).$$

Since φ commutes with rotation in the z_2-variable, we have an analogous transformation law on each component

$$K_j(z,w) = \frac{1}{z_1\overline{w}_1} K'_j(\varphi^{-1}(z), \varphi^{-1}(w)), \tag{6.5.1}$$

where K'_j is the reproducing kernel for the square integrable holomorphic functions H on D'_β satisfying $H(z_1, e^{i\theta}z_2) = e^{ij\theta}H(z_1,z_2)$. The kernel K'_{-1} can be calculated explicitly as follows:

For any $H \in \mathcal{H}_j(D'_\beta)$, we may write $H(z_1,z_2) = h(z_1)z_2^j$, where $h(z_1)$ is holomorphic in z_1. For each $\beta > 0$, let S_β be the strip on the complex plane defined by

$$S_\beta = \{z = x + iy \in \mathbb{C} |\ |y| < \beta\}.$$

It follows that

$$\begin{aligned}
&\| H \|^2_{L^2(D'_\beta)} \\
&= \int_{D'_\beta} |h(z_1)|^2 |z_2|^{2j}\ dx_1 dy_1 dx_2 dy_2 \\
&= 2\pi \int_{|2\log r| < \beta - \frac{\pi}{2}} \int_{|y_1 - 2\log r| < \frac{\pi}{2}} |h(z_1)|^2 r^{2j+1}\ dx_1 dy_1 dr \\
&= \pi \int_{|s| < \beta - \frac{\pi}{2}} \int_{|y_1 - s| < \frac{\pi}{2}} |h(z_1)|^2 e^{(j+1)s}\ dx_1 dy_1 ds \\
&= \pi \int_{-\infty}^{\infty} \int_{S_\beta} |h(z_1)|^2 e^{(j+1)s} \chi_{\frac{\pi}{2}}(y_1 - s) \chi_{\beta - \frac{\pi}{2}}(s)\ dx_1 dy_1 ds \\
&= \int_{S_\beta} |h(z)|^2 \lambda_j(y)\ dx dy,
\end{aligned} \tag{6.5.2}$$

where $\lambda_j(y) = \pi(e^{(j+1)s}\chi_{\beta-\frac{\pi}{2}}) * \chi_{\frac{\pi}{2}}(y)$, $\beta > \pi/2$ and χ_α is the characteristic function on $I_\alpha = (-\alpha, \alpha)$. Let $\lambda(y)$ be a continuous positive bounded function on the interval $I_\beta = \{y \in \mathbb{R} \mid |y| < \beta\}$. Denote by $\mathcal{H}(S_\beta, \lambda)$ the weighted Bergman space on S_β defined by

$$\mathcal{H}(S_\beta, \lambda) = \{f \in \mathcal{O}(S_\beta) |\ \| f \|^2_\lambda = \int_{S_\beta} |f(z)|^2 \lambda(y)\ dx dy < \infty\}.$$

To compute the kernel K'_{-1}, it suffices to compute the Bergman kernel in one variable on a strip S_β with weight $\lambda = \pi\chi_{\beta-\frac{\pi}{2}} * \chi_{\frac{\pi}{2}}(y)$ and the kernel $K'_{-1}(z,w)$ is given by $K'_{-1} = K_\lambda(z_1, w_1)/z_2\bar{w}_2$. The next lemma allows us to compute the weighted Bergman kernel function $K_\lambda(z,w)$ on S_β.

Lemma 6.5.1. *For each $\beta > 0$, let $\lambda(y)$ be a continuous positive bounded function on the interval $I_\beta = \{y \in \mathbb{R} \mid |y| < \beta\}$. Then the weighted Bergman kernel function $K_\lambda(z, w)$ on S_β is given by*

$$K_\lambda(z,w) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i(z-\overline{w})\xi}}{\hat{\lambda}(-2\xi i)}\, d\xi, \tag{6.5.3}$$

where $\hat{\lambda}$ is the Fourier transform of λ if $\lambda(y)$ is viewed as a function on $\mathbb{R}$ that vanishes outside I_β.

Proof. For $f \in \mathcal{H}(S_\beta, \lambda)$ we define the partial Fourier transform $\tilde{f}$ of f with respect to x by

$$\tilde{f}(\xi, y) = \int_{\mathbb{R}} f(x+iy)e^{-ix\xi}\, dx.$$

It is easily verified by Cauchy's theorem that $\tilde{f}(\xi, y) = e^{-y\xi}\tilde{f}_0(\xi)$, where $\tilde{f}_0(\xi) = \tilde{f}(\xi, 0)$. Thus from Plancherel's theorem,

$$\begin{aligned}\| f \|_\lambda^2 &= (2\pi)^{-1}\int_{\mathbb{R}\times I_\beta} e^{-2y\xi} \mid \tilde{f}_0(\xi) \mid^2 \lambda(y) d\xi dy \\ &= (2\pi)^{-1}\int_{\mathbb{R}} \mid \tilde{f}_0(\xi) \mid^2 \hat{\lambda}(-2i\xi) d\xi.\end{aligned} \tag{6.5.4}$$

For $f \in \mathcal{H}(S_\beta, \lambda)$ and $z \in S_\beta$, we have

$$\begin{aligned}\int_{\mathbb{R}} \tilde{f}_0(\xi)e^{iz\xi}\, d\xi &= 2\pi f(z) \\ &= 2\pi \int_{S_\beta} f(w)\overline{K_\lambda(w,z)}\lambda(y)\, dxdy \\ &= \int_{-\beta}^{\beta}\int_{\mathbb{R}} e^{-2y\xi}\tilde{f}_0(\xi)\overline{\widetilde{K}_\lambda((\xi,0),z)}\lambda(y)\, d\xi dy,\end{aligned}$$

where $w = x + iy$. It follows that

$$e^{-i\overline{z}\xi} = \widetilde{K}_\lambda((\xi, 0), z)\int_{-\beta}^{\beta} e^{-2y\xi}\lambda(y)\, dy,$$

and

$$\widetilde{K}_\lambda((\xi,0),z) = \frac{e^{-i\overline{z}\xi}}{\hat{\lambda}(-2\xi i)}.$$

Finally, by the Fourier inversion formula, we obtain

$$K_\lambda(z,w) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i(z-\overline{w})\xi}}{\hat{\lambda}(-2\xi i)}\, d\xi.$$

Here we note that $\hat{\lambda}(-2\xi i)$ is real. This proves the lemma.

We next apply Lemma 6.5.1 to the piecewise linear weight

$$\lambda(y) = \pi\chi_{\beta-\frac{\pi}{2}} * \chi_{\frac{\pi}{2}}(y).$$

Lemma 6.5.2. *For $\beta > \frac{\pi}{2}$, if $\lambda(y) = \pi\chi_{\beta-\frac{\pi}{2}} * \chi_{\frac{\pi}{2}}(y)$, then*

$$\hat{\lambda}(-2\xi i) = \frac{\pi sinh\big((2\beta-\pi)\xi\big) sinh(\pi\xi)}{\xi^2},$$

and

$$K_\lambda(z,w) = \frac{1}{2\pi^2}\int_{\mathbb{R}} \frac{\xi^2 e^{i(z-\overline{w})\xi}}{sinh\big((2\beta-\pi)\xi\big) sinh(\pi\xi)}\, d\xi. \tag{6.5.5}$$

Proof. If $\lambda(y) = \chi_\alpha(y)$ for some $\alpha > 0$, then

$$\hat{\lambda}(-2\xi i) = \int_{-\alpha}^{\alpha} e^{-ix(-2\xi i)}dx = \int_{-\alpha}^{\alpha} e^{-2x\xi}dx = \frac{\sinh(2\alpha\xi)}{\xi}.$$

Hence, for the piecewise linear weight $\lambda(y) = \pi\chi_{\beta-\frac{\pi}{2}} * \chi_{\frac{\pi}{2}}(y)$, we have

$$\hat{\lambda}(-2\xi i) = \frac{\pi\sinh\big((2\beta-\pi)\xi\big)\sinh(\pi\xi)}{\xi^2},$$

and (6.5.5) now follows from Lemma 6.5.1.

We observe that $\xi^2/\sinh\big((2\beta-\pi)\xi\big)\sinh\pi\xi$ has poles at nonzero integer multiples of $\pi i/(2\beta-\pi)$ and i. Let us first assume that $\beta > \pi$, and set $\nu_\beta = \pi/(2\beta-\pi)$ so that $\nu_\beta < 1$. Then, via a standard contour integration, one can obtain the asymptotic expansion of the weighted Bergman kernel function $K_\lambda(z,w)$ and see that it is in fact dominated by the residue of $g(\xi) = \xi^2 e^{i(z-\overline{w})\xi}/\sinh\big((2\beta-\pi)\xi\big)\sinh(\pi\xi)$ at the first pole $\nu_\beta i$.

Lemma 6.5.3. *Let $\beta > \pi$ and $\lambda(y) = \pi\chi_{\beta-\frac{\pi}{2}} * \chi_{\frac{\pi}{2}}(y)$. Then*

$$K_\lambda(z,w) = c_\beta e^{-\nu_\beta(z-\overline{w})} + O(e^{-\mu_\beta(z-\overline{w})}) \tag{6.5.6}$$

for $Re(z-\overline{w}) > 0$ and

$$K_\lambda(z,w) = -c_\beta e^{\nu_\beta(z-\overline{w})} + O(e^{\mu_\beta(z-\overline{w})}) \tag{6.5.7}$$

for $Re(z-\overline{w}) < 0$, where $c_\beta = {\nu_\beta}^3/(\pi^2 sin\nu_\beta\pi)$ and $\mu_\beta = min(2\nu_\beta, 1)$. Furthermore, given any small positive ϵ, the expansion in (6.5.6) or (6.5.7) is uniform for any $z, w \in S_{\beta-\epsilon}$.

Proof. Fix $h > 0$ so that hi is the midpoint between the second and third poles of $g(\xi)$. Denote by Γ_N the rectangular contour with vertices $\pm N$ and $\pm N + ih$. Let us first assume that $2\nu_\beta < 1$. Then, we have $2\nu_\beta < h < 1$ and

$$\begin{aligned}\int_{-N}^{N} \frac{\xi^2 e^{i(z-\overline{w})\xi}}{\sinh\big((2\beta-\pi)\xi\big)\sinh(\pi\xi)}\, d\xi \;+\; I_N \;+\; I_{-N} + \; J_N \\ = 2\pi i(\text{Res } g(\xi) \text{ at } \nu_\beta i \text{ and } 2\nu_\beta i),\end{aligned}$$

where Res $g(\xi)$ denotes the residues of $g(\xi)$ and

$$I_N = i\int_0^h \frac{(N+iy)^2 e^{i(z-\overline{w})(N+iy)}}{\sinh\big((2\beta-\pi)(N+iy)\big)\sinh\pi(N+iy)}\ dy,$$

$$I_{-N} = -i\int_0^h \frac{(-N+iy)^2 e^{i(z-\overline{w})(-N+iy)}}{\sinh\big((2\beta-\pi)(-N+iy)\big)\sinh\pi(-N+iy)}\ dy,$$

and

$$J_N = -\int_{-N}^N \frac{(x+ih)^2 e^{i(z-\overline{w})(x+ih)}}{\sinh\big((2\beta-\pi)(x+ih)\big)\sinh\pi(x+ih)}\ dx.$$

A direct calculation shows that

$$\operatorname*{Res}_{\xi=\nu_\beta i} g(\xi) = \frac{{\nu_\beta}^3 e^{-\nu_\beta(z-\overline{w})}}{i\pi\sin(\nu_\beta\pi)},$$

and

$$\operatorname*{Res}_{\xi=2\nu_\beta i} g(\xi) = \frac{-4{\nu_\beta}^3 e^{-2\nu_\beta(z-\overline{w})}}{i\pi\sin(2\nu_\beta\pi)}.$$

For any $z, w \in S_{\beta-\epsilon}$, we write $z-\overline{w} = u+iv$ with $u>0$. Hence, we have $|v| \le 2(\beta-\epsilon)$ and

$$|I_N| \lesssim \int_0^h \frac{(N^2+y^2)e^{-vN-uy}}{e^{2\beta N}}\ dy \ \lesssim\ \frac{N^2+1}{e^{2\epsilon N}}.$$

It follows that I_N converges to zero uniformly for any $z, w \in S_{\beta-\epsilon}$ such that $\mathrm{Re}(z-\overline{w}) > 0$. Similarly, we get the uniform convergence to zero for I_{-N}. For J_N, we have

$$\begin{aligned}|J_N| &\lesssim \int_{-1}^1 (x^2+h^2)e^{-vx-uh}\ dx + \int_1^N \frac{(x^2+h^2)e^{-vx-uh}}{e^{2\beta x}}\ dx \\ &\quad + \int_{-N}^{-1} \frac{(x^2+h^2)e^{-vx-uh}}{e^{-2\beta x}}\ dx \\ &\lesssim e^{-2\nu_\beta u}\left(1+\int_1^N \frac{x^2+h^2}{e^{2\epsilon x}}\ dx\right).\end{aligned}$$

It follows by letting N tend to infinity that

$$\begin{aligned}&\int_{-\infty}^\infty \frac{\xi^2 e^{i(z-\overline{w})\xi}}{\sinh\big((2\beta-\pi)\xi\big)\sinh(\pi\xi)}\ d\xi \\ &\quad= \frac{2{\nu_\beta}^3 e^{-\nu_\beta(z-\overline{w})}}{\sin\nu_\beta\pi} - \frac{8{\nu_\beta}^3 e^{-2\nu_\beta(z-\overline{w})}}{\sin(2\nu_\beta\pi)} \\ &\qquad + \int_{-\infty}^\infty \frac{(x+ih)^2 e^{i(z-\overline{w})(x+ih)}}{\sinh\big((2\beta-\pi)(x+ih)\big)\sinh\pi(x+ih)}\ dx \\ &\quad= \frac{2{\nu_\beta}^3}{\sin\nu_\beta\pi}e^{-\nu_\beta(z-\overline{w})} + O(e^{-2\nu_\beta(z-\overline{w})}).\end{aligned}$$

Clearly, the estimate is uniform for all $z, w \in S_{\beta-\epsilon}$ with $\operatorname{Re}(z-\overline{w}) > 0$. This proves the case for $2\nu_\beta < 1$.

For cases $2\nu_\beta = 1$ and $2\nu_\beta > 1$, a similar argument applies.

If $\operatorname{Re}(z-\overline{w}) < 0$, we take the rectangular contour Γ_N on the lower half space with vertices $\pm N$, and $\pm N - ih$, and (6.5.7) can be proved similarly. This completes the proof of the lemma.

From (6.5.2), the kernel $K'_{-1}(z,w)$ is given by

$$K'_{-1}(z,w) = K_\lambda(z_1,w_1)/z_2\overline{w}_2,$$

where $K_\lambda(z_1,w_1)$ is calculated in Lemma 6.5.3. If $\beta > \pi$, then (6.5.7) shows that

$$K'_{-1}(z,w) = -c_\beta \frac{e^{\nu_\beta(z_1-\overline{w}_1)}}{z_2\overline{w}_2} + O(e^{\mu_\beta(z_1-\overline{w}_1)})$$

for $\operatorname{Re}(z_1-\overline{w}_1) < 0$. Hence

$$K_{-1}(z,w) = -c_\beta z_1^{\nu_\beta-1}\overline{w}_1^{-\nu_\beta-1} z_2^{-1}\overline{w}_2^{-1} + \frac{1}{z_1\overline{w}_1} O\Big(\Big(\frac{z_1}{\overline{w}_1}\Big)^{\mu_\beta}\Big) \tag{6.5.8}$$

for $|z_1| < |w_1|$. The expansion in (6.5.7) is uniform on $S_{\beta-\epsilon}$ for any small positive ϵ. Thus, for fixed w, we have for any $m \in \mathbb{N}$,

$$|\operatorname{Re}(z_1 e^{-i\log|z_2|^2})|^s \left(\frac{\partial}{\partial z_1}\right)^m K_{-1}(z,w) \notin L^2(D_\beta), \text{ for } s \le m - \nu_\beta.$$

It follows that

$$|\operatorname{Re}(z_1 e^{-i\log|z_2|^2})|^s \left(\frac{\partial}{\partial z_1}\right)^m K_{D_\beta}(z,w) \notin L^2(D_\beta), \tag{6.5.9}$$

for $s \le m - \nu_\beta$. Estimate (6.5.9) also holds for $\pi/2 < \beta \le \pi$. When $\pi/2 < \beta < \pi$, (6.5.9) can be obtained by examining higher order terms in the asymptotic expansion of K_λ.

When $\beta = \pi$, we compute the residue at the double pole $-i$ of (6.5.5) to obtain

$$K_\lambda(z_1,w_1) = \pi^{-2}(-z_1 + \bar{w}_1 - 2)e^{(z_1-\bar{w}_1)} + O(e^{2(z_1-\bar{w}_1)})$$

for $\operatorname{Re}(z_1 - \bar{w}_1) < 0$ and

$$K_{-1}(z,w) = \pi^{-2}(-\log(z_1/\bar{w}_1) - 2)\bar{w}_1^{-2} z_2^{-1}\bar{w}_2^{-1} + O((z_1/\bar{w}_1)^2)$$

for $|z_1| < |w_1|$. Thus (6.5.9) holds for $\beta = \pi$ also. Estimate (6.5.9) is crucial in proving the irregularity of the Bergman projection P measured in the Sobolev norm on the worm domain. For our purpose we need the following fact (See Lemma C.4 in the Appendix).

Lemma 6.5.4. *Let D be a smooth bounded domain in $\mathbb{R}^N$ with a smooth defining function $\rho(x)$. Then, for each $s \geq 0$, the W^{-s} norm of a harmonic function f is equivalent to the L^2 norm of $|\rho|^s f$ on D.*

We first observe the following result for the Bergman projection on the unbounded worm domain D_β.

Proposition 6.5.5. *For each $\beta > \pi/2$, condition R does not hold on D_β. Furthermore, the Bergman projection P_∞ on D_β does not map $C_0^\infty(D_\beta)$ into $W^k(D_\beta)$ when $k \geq \pi/(2\beta - \pi)$.*

Proof. Let $w \in D_\beta$. We choose a real-valued function $f \in C_0^\infty(D_\beta)$ such that f depends on $|z-w|$ and $\int_{D_\beta} f = 1$. Using the same argument as in Corollary 6.3.11, we obtain

$$P_\infty f = K_{D_\beta}(\cdot, w).$$

Since $K_{D_\beta}(\cdot, w) \notin C^\infty(\overline{D}_\beta)$, condition R fails on D_β.

Let $\Gamma_A = \{z \in \mathbb{C}^n \mid |z| < A\}$ be a large ball for $A > 0$. From (6.5.9), we have

$$|\rho_\infty|^s \left(\frac{\partial}{\partial z_1}\right)^m P_\infty f \notin L^2(D_\beta \cap \Gamma_A) \tag{6.5.10}$$

for $s \leq m - \nu_\beta$, where $\rho_\infty = \mathrm{Re}(z_1 e^{-i\log|z_2|^2})$. If $P_\infty f \in W^k(D_\beta \cap \Gamma_A)$, choose a positive integer $m > k$ and let $s = m - k \leq m - \nu_\beta$. Using Lemma 6.5.4, we have $|\rho_\infty|^s \nabla^m P_\infty f \in L^2(D_\beta \cap \Gamma_A)$, a contradiction. Thus $P_\infty f \notin W^k(D_\beta \cap \Gamma_A)$ and the proposition is proved.

We prove the main result of this section on the irregularity of the Bergman projection for the smooth worm domain Ω_β.

Theorem 6.5.6. *For each $\beta > \pi/2$, the Bergman projection P on Ω_β does not map $W^k(\Omega_\beta)$ into $W^k(\Omega_\beta)$ when $k \geq \pi/(2\beta - \pi)$.*

Proof. Assume on the contrary that the Bergman projection P maps $W^k(\Omega_\beta)$ into $W^k(\Omega_\beta)$ with the estimate

$$\| Pf \|_{W^k(\Omega_\beta)} \leq C_k \| f \|_{W^k(\Omega_\beta)} \tag{6.5.11}$$

for $f \in W^k(\Omega_\beta)$ and $k \geq \pi/(2\beta - \pi) = \nu_\beta$.

For any $\mu \geq 1$, let τ_μ be the dilation defined by

$$\begin{aligned} \tau_\mu : \quad \mathbb{C}^2 \quad &\to \quad \mathbb{C}^2 \\ (z_1, z_2) &\mapsto (\mu z_1, z_2). \end{aligned}$$

Denote by $\Omega_{\beta,\mu} = \tau_\mu(\Omega_\beta)$, $\Omega'_{\beta,\mu} = \tau_\mu(\Omega'_\beta)$ where Ω'_β is defined in Section 6.4. Then $\Omega'_{\beta,\mu} \subset \Omega_{\beta,\mu}$ and $\Omega'_{\beta,\mu} \nearrow D_\beta$. Let T_μ be the pullback of the L^2 functions on $\Omega_{\beta,\mu}$, i.e.,

$$\begin{aligned} T_\mu : L^2(\Omega_{\beta,\mu}) &\to L^2(\Omega_\beta) \\ f \quad &\mapsto \quad f \circ \tau_\mu. \end{aligned}$$

A direct calculation shows that

$$\| \left(\frac{\partial}{\partial z}\right)^{\alpha}\left(\frac{\partial}{\partial \overline{z}}\right)^{\gamma} T_\mu f \|_{L^2(\Omega_\beta)} = \mu^{\alpha_1+\gamma_1-1} \| \left(\frac{\partial}{\partial z}\right)^{\alpha}\left(\frac{\partial}{\partial \overline{z}}\right)^{\gamma} f \|_{L^2(\Omega_{\beta,\mu})},$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\gamma = (\gamma_1, \gamma_2)$ are multindices. Thus we have

$$\| T_\mu f \|_{W^l(\Omega_\beta)} \leq \mu^{l-1} \| f \|_{W^l(\Omega_{\beta,\mu})}, \tag{6.5.12}$$

when l is a nonnegative integer. Then, by interpolation, it holds for all real $l \geq 0$.

Let P_μ be the Bergman projection associated with $\Omega_{\beta,\mu}$. Then

$$P_\mu = T_\mu^{-1} P T_\mu. \tag{6.5.13}$$

From the definition (6.4.1) of Ω_β we see that the defining function $\rho(z)$ coincides with $|z_1|^2 + 2\mathrm{Re}(z_1 e^{-i\log|z_2|^2})$ when $\log|z_2|^2 \in I_{\beta-\frac{\pi}{2}}$. Let $\rho_\mu(z) = \mu\rho \circ (\tau_\mu)^{-1}$ so that $\rho_\mu \to \rho_\infty = 2\mathrm{Re}(z_1 e^{-i\log|z_2|^2})$ as $\mu \to \infty$, where ρ_∞ is a defining function for D_β. Write $k = m - s$, where m is an integer and $s \geq 0$. For any $f \in C_0^\infty(\Omega'_{\beta,\mu}) \subset C_0^\infty(\Omega_{\beta,\mu})$, we have using (6.5.11)-(6.5.13) and Lemma 6.5.4,

$$\begin{aligned}
&\| |\rho_\mu|^s \left(\frac{\partial}{\partial z_1}\right)^m P_\mu f \|_{L^2(\Omega_{\beta,\mu})} \\
&\quad = \| |\rho_\mu|^s \left(\frac{\partial}{\partial z_1}\right)^m (T_\mu)^{-1} P T_\mu f \|_{L^2(\Omega_{\beta,\mu})} \\
&\quad = \mu^{s-m+1} \| |\rho|^s \left(\frac{\partial}{\partial z_1}\right)^m P T_\mu f \|_{L^2(\Omega_\beta)} \\
&\quad \leq C\mu^{1-k} \| \left(\frac{\partial}{\partial z_1}\right)^m P T_\mu f \|_{W^{-s}(\Omega_\beta)} \\
&\quad \leq C\mu^{1-k} \| P T_\mu f \|_{W^k(\Omega_\beta)} \\
&\quad \leq C\mu^{1-k} \| T_\mu f \|_{W^k(\Omega_\beta)} \\
&\quad \leq C \| f \|_{W^k(\Omega_{\beta,\mu})},
\end{aligned} \tag{6.5.14}$$

where the constant C is independent of μ. We claim that

$$P_\mu f \rightharpoonup P_\infty f \qquad \text{weakly in } L^2(\mathbb{C}^2),$$

where $P_\infty f$ is the Bergman projection of f on D_β, $P_\infty f = 0$ outside D_β and we have set $P_\mu f = 0$ outside $\Omega_{\beta,\mu}$. Assuming the claim, It follows from (6.5.14) that

$$\| |\rho_\infty|^s \left(\frac{\partial}{\partial z_1}\right)^m P_\infty f \|_{L^2(D_\beta)} \leq C \| f \|_{W^k(D_\beta)} \tag{6.5.15}$$

for any $f \in C_0^\infty(D_\beta)$. This contradicts (6.5.10) and the theorem is proved.

Thus, it remains to prove the claim. Since

$$\| P_\mu f \|_{L^2(\mathbb{C}^2)} \leq \| f \|_{L^2(\mathbb{C}^2)},$$

there exists a subsequence of $P_\mu f$ that converges weakly to $h \in L^2(\mathbb{C}^2)$. Since $\Omega'_{\beta,\mu} \nearrow D_\beta$, h is holomorphic in D_β. Also h vanishes outside $\overline{D}_\beta$ since every compact subset outside $\overline{D}_\beta$ is outside $\overline{\Omega}_{\beta,\mu}$ for sufficiently large μ. To prove that $h = P_\infty f$, we need to show that $f - h \perp \mathcal{H}(D_\beta)$. Choose $M > 1$ so that $\Omega_{\beta,\mu} \subset D_{M\beta}$ for all μ. Obviously $f - P_\mu f \perp \mathcal{H}(D_{M\beta})$. Therefore, by passing to the limit, we obtain that $f - h \perp \mathcal{H}(D_{M\beta})$. The claim will be proved by the following density result:

Lemma 6.5.7. *For each $M > 1$, the space $\mathcal{H}(D_{M\beta})$ is dense in $\mathcal{H}(D_\beta)$.*

Proof. It suffices to show that each $\mathcal{H}_j(D_{M\beta})$ is dense in $\mathcal{H}_j(D_\beta)$, or equivalently, $\mathcal{H}_j(D'_{M\beta})$ is dense in $\mathcal{H}_j(D'_\beta)$. From (6.5.2) and (6.5.4) we have for any $f \in \mathcal{H}_j(D'_\beta)$,

$$\| f \|^2_{L^2(D'_\beta)} = (2\pi)^{-1} \int_{\mathbb{R}} | \tilde{f}_0(\xi) |^2 \hat{\lambda}_j(-2i\xi) d\xi,$$

where $\tilde{f}_0(\xi)$ is the partial Fourier transform of f evaluated at $y = 0$ and $\lambda_j(y) = \pi(e^{(j+1)(\cdot)}\chi_{\beta-\frac{\pi}{2}}) * \chi_{\frac{\pi}{2}}(y)$. Thus, the space $\mathcal{H}_j(D'_\beta)$ is isometric via the Fourier transform to the space of functions on $\mathbb{R}$ which are square integrable with respect to the weight

$$\hat{\lambda}_j(-2\xi i) = \frac{\pi \sinh\left[(2\beta - \pi)\left(\xi - (\frac{j+1}{2})\right)\right]\sinh(\pi\xi)}{\xi\left(\xi - (\frac{j+1}{2})\right)}.$$

Since $C_0^\infty(\mathbb{R})$ is dense in the latter space for any value of β, the lemma follows.

Using Theorem 6.2.2 and Theorem 6.5.6, we also obtain that the $\bar{\partial}$-Neumann operator is irregular on the worm domain.

Corollary 6.5.8. *For each $\beta > \pi/2$, the $\overline{\partial}$-Neumann operator on Ω_β does not map $W^k_{(0,1)}(\Omega_\beta)$ into $W^k_{(0,1)}(\Omega_\beta)$ when $k \geq \pi/(2\beta - \pi)$.*

NOTES

The existence of a smooth solution up to the boundary, using the weighted $\overline{\partial}$-Neumann problem, for the $\overline{\partial}$ equation was proved by J. J. Kohn in [Koh 6]. The equivalence between the Bergman projections and the $\overline{\partial}$-Neumann operators was proved by H. P. Boas and E. J. Straube in [BoSt 2]. Theorem 6.2.1 provides a sufficient condition for verifying the exact regularity of the $\overline{\partial}$-Neumann operators, and the idea has been used in [Che 4] and [BoSt 3,4,5]. The use of a smooth plurisubharmonic defining function (Theorem 6.2.3), based on an observation by A. Noell [Noe 1], was originated in [BoSt 3] where they treated directly the exact regularity of the Bergman projections under the existence of such a defining function. For a convex domain in dimension two, a different proof, using related ideas, was obtained independently in [Che 5]. The use of transverse symmetries for verifying the regularity of the Bergman projection was first initiated by D. Barrett [Bar 1]. The regularity of the $\overline{\partial}$-Neumann problem on circular domains with symmetry (Theorems 6.2.7 and 6.2.8) was proved by S.-C. Chen [Che 3]. See also [BCS 1]. Another sufficient condition related to the De Rham cohomology on the set of infinite type points for the regularity of the $\overline{\partial}$-Neumann operators was also introduced by H. P. Boas and E. J. Straube in [BoSt 5].

Another important class of smooth bounded pseudoconvex domains which is beyond the scope of this book is the class of domains of finite type. The concept of finite type on a pseudoconvex domain in $\mathbb{C}^2$, using the Lie brackets of complex tangential vector fields, was first introduced by J. J. Kohn [Koh 4]. Subsequently,

J. J. Kohn introduced subelliptic multipliers and finite ideal type in [Koh 8] and he proved that finite ideal type condition is sufficient for the subelliptic estimates of the $\overline{\partial}$-Neumann operators. By measuring the order of contact of complex varieties with a hypersurface at the reference point, J. D'Angelo [DAn 1,2] proposed a definition of finite type in $\mathbb{C}^n$. The necessity of finite order of contact of complex varieties for the subelliptic estimates was proved by D. Catlin in [Cat 1].

When the boundary is real analytic near a boundary point, Kohn's theory of ideals of subelliptic multipliers [Koh 8], together with a theorem of Diederich and Fornaess [DiFo 3], showed that a subelliptic estimate on (p,q)-forms for the $\overline{\partial}$-Neumann problem is equivalent to the absence of germs of q dimensional complex varieties in the boundary near the point. In particular, subelliptic estimates always hold on any bounded pseudoconvex domain with real analytic boundary.

D. Catlin also defined in [Cat 4] his own notion of finite type. His theory of multitypes developed in [Cat 2] leads to the construction of a family of smooth bounded plurisubharmonic functions with large Hessian on the boundary. This property is now known as property (P) (see [Cat 3]). Property (P) implies the existence of a compactness estimate for the $\overline{\partial}$-Neumann problem. Therefore, together with a theorem of Kohn and Nirenberg [KoNi 1], global regularity of the $\overline{\partial}$-Neumann problem will follow from property (P). See also the papers by N. Sibony [Sib 2,3] and S. Fu and E. J. Straube [FuSt 1] for related results.

When a smooth bounded pseudoconvex domain has real analytic boundary, it is also important to know the real analytic regularity of the $\overline{\partial}$-Neumann operator near the boundary. Real analytic regularity of a holomorphic function near the boundary is equivalent to holomorphic extension of the function across the boundary. For strongly pseudoconvex domains, an affirmative result of global analytic regularity of the $\overline{\partial}$-Neumann problem had been obtained by M. Derridj and D. S. Tartakoff [DeTa 1] and G. Komatsu [Kom 1]. Local analytic hypoellipticity of the $\overline{\partial}$-Neumann problem on strongly pseudoconvex domains was proved by D. S. Tartakoff [Tar 1,2] and F. Treves [Tre 2]. When the domains are weakly pseudoconvex of special type, some positive results concerning global analytic hypoellipticity of the $\overline{\partial}$-Neumann problem are also available by S.-C. Chen [Che 1,2,6] and M. Derridj [Der 1] using the vector field technique. For local analytic regularity of the $\overline{\partial}$-Neumann problem on certain weakly pseudoconvex domains, see [DeTa 2,3].

For introductory materials on the Bergman kernel function, the reader may consult the survey paper by S. Bell [Bel 4] or the texts by S. G. Krantz [Kra 2] and R. M. Range [Ran 6]. See also the papers by S. Bell [Bel 3], H. P. Boas [Boa 3], S.-C. Chen [Che 7] and N. Kerzman [Ker 2] for the differentiability of the Bergman kernel function near the boundaries of the domains. Theorem 6.3.7 on various equivalent statements of condition R can be found in [BeBo 1]. The operator Φ^s in Lemma 6.3.8 was first constructed by S. Bell in [Bel 2]. Corollary 6.3.12 is the density lemma due to S. Bell [Bel 1]. The smooth extension of a biholomorphic mapping between two smooth bounded domains in $\mathbb{C}^n, n \geq 2$, was first achieved by C. Fefferman in his paper [Fef 1] when the domains are strongly pseudoconvex. Later, condition R was proposed by S. Bell and E. Ligocka in [BeLi 1]. They showed using condition R that, near a boundary point, one may choose special holomorphic local coordinates resulting from the Bergman kernel functions so that any biholomorphic map between these two smooth bounded domains becomes linear (Theorem 6.3.13).

Hence, the biholomorphism extends smoothly up to the boundaries. The present proof of Theorem 6.3.13 was adopted from [Bel 2]. A smooth bounded nonpseudoconvex domain in $\mathbb{C}^2$ which does not satisfy condition R was discovered in [Bar 2]. In contrast to Barrett's counterexample, H. P. Boas and E. J. Straube showed in [BoSt 1] that condition R always holds on any smooth bounded complete Hartogs domain in $\mathbb{C}^2$ regardless of whether it is pseudoconvex or not. Theorem 6.3.14 was proved by R. M. Range [Ran 2].

The construction of worm domains is due to K. Diederich and J. E. Fornaess in [DiFo 1] where Theorem 6.4.3 is proved (see also [FoSte 1]). Our exposition follows that of C. O. Kiselman [Kis 1]. Most of the Section 6.5 is based on [Bar 3]. Recently, based on D. Barrett's result, it was proved by M. Christ in [Chr 2] that condition R does not hold for the Bergman projection on the worm domain. For more about the regularity of the $\overline{\partial}$-Neumann problem and its related questions, the reader may consult the survey paper by H. P. Boas and E. J. Straube [BoSt 6]. We also refer the reader to the book by J. E. Fornaess and B. Stensønes [FoSte 1] for counterexamples on pseudoconvex domains. For recent results on the $\overline{\partial}$-Neumann problem on Lipschitz pseudoconvex domains, see the papers by Bonami-Charpentier [BoCh 1], Henkin-Iordan [HeIo 1], Henkin-Iordan-Kohn [HIK 1], Michel-Shaw [MiSh 1] and Straube [Str 2]. Hölder and L^p estimates of the $\overline{\partial}$-Neumann problem on pseudoconvex domains of finite type in $\mathbb{C}^2$ have been discussed in Chang-Nagel-Stein [CNS 1], Fefferman-Kohn [FeKo 1]. Hölder or L^p estimates for the $\overline{\partial}$-Neumann problem on finite type pseudoconvex domains in $\mathbb{C}^n$ for $n \geq 3$ are still unknown.

CHAPTER 7

CAUCHY-RIEMANN MANIFOLDS AND THE TANGENTIAL CAUCHY-RIEMANN COMPLEX

Let M be a smooth hypersurface in a complex manifold. The restriction of the $\overline{\partial}$ complex to M naturally induces a new differential complex. This complex is called the tangential Cauchy-Riemann complex or the $\overline{\partial}_b$ complex. The tangential Cauchy-Riemann complex, unlike the de Rham or the $\overline{\partial}$ complex, is not elliptic. In general, it is an overdetermined system with variable coefficients.

We have seen in Chapter 3 that the tangential Cauchy-Riemann equations are closely related to the holomorphic extension of a CR function on the hypersurface. The $\overline{\partial}_b$ complex is also important in its own right in the theory of partial differential equations. The tangential Cauchy-Riemann equation associated with a strongly pseudoconvex hypersurface in $\mathbb{C}^2$ provides a nonsolvable first order partial differential equation with variable coefficients. It also serves as a prototype of subelliptic operators.

In the next few chapters, we shall study the solvability and regularity of the $\overline{\partial}_b$ complex. First, in Chapters 8 and 9 the subellipticity and the closed range property of $\overline{\partial}_b$ will be investigated using L^2 method. Then, in Chapter 10 we construct an explicit fundamental solution for $\Box_b$ on the Heisenberg group. Next, the integral representation is used to construct a solution operator for the $\overline{\partial}_b$ operator on a strictly convex hypersurface in Chapter 11. The CR embedding problem will be discussed in Chapter 12.

In this chapter, we shall first define Cauchy-Riemann manifolds and the tangential Cauchy-Riemann complex both extrinsically and intrinsically. The Levi form of a Cauchy-Riemann manifold is introduced. In Section 7.3, we present the famous nonsolvable Lewy operator. In contrast with the Lewy operator, we prove that any linear partial differential operator with constant coefficients is always locally solvable.

7.1 CR Manifolds

Let M be a real smooth manifold of dimension $2n-1$ for $n \geq 2$, and let $T(M)$ be the tangent bundle associated with M. Let $\mathbb{C}T(M) = T(M) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle over M. A CR structure on M is defined as follows.

Definition 7.1.1. *Let M be a real smooth manifold of dimension $2n-1$, $n \geq 2$, and let $T^{1,0}(M)$ be a subbundle of $\mathbb{C}T(M)$. We say that $(M, T^{1,0}(M))$ is a Cauchy-Riemann manifold, abbreviated as CR manifold, with the Cauchy-Riemann structure $T^{1,0}(M)$ if the following conditions are satisfied:*

(1) $dim_{\mathbb{C}} T^{1,0}(M) = n-1$,
(2) $T^{1,0}(M) \cap T^{0,1}(M) = \{0\}$, *where* $T^{0,1} = \overline{T^{1,0}(M)}$,
(3) *(Integrability condition) For any $X_1, X_2 \in \Gamma(U, T^{1,0}(M))$, the Lie bracket $[X_1, X_2]$ is still in $\Gamma(U, T^{1,0}(M))$, where U is any open subset of M and $\Gamma(U, T^{1,0}(M))$ denotes the space of all smooth sections of $T^{1,0}(M)$ over U.*

Here $\overline{T^{1,0}(M)}$ in (2) means the complex conjugation of $T^{1,0}(M)$. Note also that condition (3) in Definition 7.1.1 is void when $n = 2$. The most natural CR manifolds are those defined by smooth hypersurfaces in $\mathbb{C}^n$.

Example 7.1.2. Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a real-valued smooth function. Suppose that the differential $d\rho$ does not vanish on the hypersurface $M = \{z \in \mathbb{C}^n \mid \rho(z) = 0\}$. Then M is a smooth manifold with $\dim_{\mathbb{R}} M = 2n-1$. Define a subbundle $T^{1,0}(M)$ of $\mathbb{C}T(M)$ by $T^{1,0}(M) = T^{1,0}(\mathbb{C}^n) \cap \mathbb{C}T(M)$. It is easily seen that $(M, T^{1,0}(M))$ is a CR manifold with the CR structure $T^{1,0}(M)$ induced from the ambient space $\mathbb{C}^n$.

Hence, it is natural to ask whether a given abstract CR structure $(M, T^{1,0}(M))$ on M can be CR embedded into some $\mathbb{C}^N$ so that the given CR structure coincides with the induced CR structure from the ambient space. The embedding problem of an abstract CR structure will make up the main course of Chapter 12.

Let $(M, T^{1,0}(M))$ and $(N, T^{1,0}(N))$ be two CR manifolds. A smooth mapping φ from M to N is called a CR mapping if $\varphi_* L$ is a smooth section of $T^{1,0}(N)$ for any smooth section L in $T^{1,0}(M)$. Furthermore, if φ has a smooth CR inverse mapping φ^{-1}, then we say that $(M, T^{1,0}(M))$ is CR diffeomorphic to $(N, T^{1,0}(N))$.

We have the following lemma:

Lemma 7.1.3. *Let $(M, T^{1,0}(M))$ be a CR manifold, and let N be a manifold. Suppose that M is diffeomorphic to N via a mapping φ. Then φ induces a CR structure on N, namely, $\varphi_* T^{1,0}(M)$, so that φ becomes a CR diffeomorphism from $(M, T^{1,0}(M))$ onto $(N, \varphi_* T^{1,0}\ (M))$, where φ_* is the differential map induced by φ.*

Proof. We need to check the integrability condition on $\varphi_* T^{1,0}(M)$. However, this follows immediately from the integrability condition on $T^{1,0}(M)$ and the fact that $[\varphi_* X_1, \varphi_* X_2] = \varphi_*[X_1, X_2]$ for any smooth vector fields X_1, X_2 defined on M.

Definition 7.1.4. *A smooth function g defined on a CR manifold $(M, T^{1,0}(M))$ is called a CR function if $\overline{L} g = 0$ for any smooth section $\overline{L}$ in $T^{0,1}(M)$.*

When M is the boundary of a smooth domain in $\mathbb{C}^n$, this definition coincides with Definition 3.0.1. If, in Definition 7.1.4, g is just a distribution, then $\overline{L} g$ should be interpreted in the sense of distribution.

7.2 The Tangential Cauchy-Riemann Complex

Let M be a hypersurface in a complex manifold. The $\overline{\partial}$ complex restricted to M induces the tangential Cauchy-Riemann complex, or the $\overline{\partial}_b$ complex. In fact, the tangential Cauchy-Riemann complex can be formulated on any CR manifold. There are two different approaches in this setting. One way is to define the tangential Cauchy-Riemann complex intrinsically on any abstract CR manifold itself without referring to the ambient space. On the other hand, if the CR manifold is sitting in $\mathbb{C}^n$, or more generally, a complex manifold, the tangential Cauchy-Riemann complex can also be defined extrinsically via the ambient complex structure.

First, we assume that M is a smooth hypersurface in $\mathbb{C}^n$, and let r be a defining function for M. In some open neighborhood U of M, let $I^{p,q}$, $0 \leq p, q \leq n$, be the ideal in $\Lambda^{p,q}(\mathbb{C}^n)$ such that at each point $z \in U$ the fiber $I_z^{p,q}$ is generated by r and $\overline{\partial} r$, namely, each element in the fiber $I_z^{p,q}$ can be expressed in the form

$$rH_1 + \overline{\partial} r \wedge H_2,$$

where H_1 is a smooth (p,q)-form and H_2 is a smooth $(p, q-1)$-form. Denote by $\Lambda^{p,q}(\mathbb{C}^n)|_M$ and $I^{p,q}|_M$ the restriction of $\Lambda^{p,q}(\mathbb{C}^n)$ and $I^{p,q}$ respectively to M. Then, we define

$$\Lambda^{p,q}(M) = \{\text{the orthogonal complement of } I^{p,q}|_M \text{ in } \Lambda^{p,q}(\mathbb{C}^n)|_M\}.$$

We denote by $\mathcal{E}^{p,q}$ the space of smooth sections of $\Lambda^{p,q}(M)$ over M, i.e., $\mathcal{E}^{p,q}(M) = \Gamma(M, \Lambda^{p,q}(M))$. Let τ denote the following map

$$\tau : \Lambda^{p,q}(\mathbb{C}^n) \to \Lambda^{p,q}(M), \tag{7.2.1}$$

where τ is obtained by first restricting a (p,q)-form ϕ in $\mathbb{C}^n$ to M, then projecting the restriction to $\Lambda^{p,q}(M)$. One should note that $\Lambda^{p,q}(M)$ is not intrinsic to M, i.e., $\Lambda^{p,q}(M)$ is not a subspace of the exterior algebra generated by the complexified cotangent bundle of M. This is due to the fact that $\overline{\partial} r$ is not orthogonal to the cotangent bundle of M. Note also that $\mathcal{E}^{p,n} = 0$.

The tangential Cauchy-Riemann operator

$$\overline{\partial}_b : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q+1}(M)$$

is now defined as follows: For any $\phi \in \mathcal{E}^{p,q}(M)$, pick a smooth (p,q)-form ϕ_1 in $\mathbb{C}^n$ that satisfies $\tau\phi_1 = \phi$. Then, $\overline{\partial}_b \phi$ is defined to be $\tau \overline{\partial} \phi_1$ in $\mathcal{E}^{p,q+1}(M)$. If ϕ_2 is another (p,q)-form in $\mathbb{C}^n$ such that $\tau\phi_2 = \phi$, then

$$\phi_1 - \phi_2 = rg + \overline{\partial} r \wedge h,$$

for some (p,q)-form g and $(p, q-1)$-form h. It follows that

$$\overline{\partial}(\phi_1 - \phi_2) = r\overline{\partial} g + \overline{\partial} r \wedge g - \overline{\partial} r \wedge \overline{\partial} h,$$

and hence,

$$\tau \overline{\partial}(\phi_1 - \phi_2) = 0.$$

Thus, the definition of $\overline{\partial}_b$ is independent of the choice of ϕ_1. Since $\overline{\partial}^2 = 0$, we have $\overline{\partial}_b^2 = 0$ and the following boundary complex

$$0 \to \mathcal{E}^{p,0}(M) \xrightarrow{\overline{\partial}_b} \mathcal{E}^{p,1}(M) \xrightarrow{\overline{\partial}_b} \cdots \xrightarrow{\overline{\partial}_b} \mathcal{E}^{p,n-1}(M) \to 0.$$

For the intrinsic approach, we will assume that $(M, T^{1,0}(M))$ is an orientable CR manifold of real dimension $2n-1$ with $n \geq 2$. A real smooth manifold M is said to be orientable if there exists a nonvanishing top degree form on M. We shall assume that M is equipped with a Hermitian metric on $\mathbb{C}T(M)$ so that $T^{1,0}(M)$ is orthogonal to $T^{0,1}(M)$. Denote by $\eta(M)$ the orthogonal complement of $T^{1,0}(M) \oplus T^{0,1}(M)$. It is easily seen that $\eta(M)$ is a line bundle over M. Now denote by $T^{*1,0}(M)$ and $T^{*0,1}(M)$ the dual bundles of $T^{1,0}(M)$ and $T^{0,1}(M)$ respectively. By definition it means that forms in $T^{*1,0}(M)$ annihilate vectors in $T^{0,1}(M) \oplus \eta(M)$ and forms in $T^{*0,1}(M)$ annihilate vectors in $T^{1,0}(M) \oplus \eta(M)$. Define the vector bundle $\Lambda^{p,q}(M)$, $0 \leq p, q \leq n-1$, by

$$\Lambda^{p,q}(M) = \Lambda^p T^{*1,0}(M) \otimes \Lambda^q T^{*0,1}(M).$$

This can be identified with a subbundle of $\Lambda^{p+q}\mathbb{C}T^*(M)$. It follows that $\Lambda^{p,q}(M)$ defined in this way is intrinsic to M. Denote by $\mathcal{E}^{p,q}$ the space of smooth sections of $\Lambda^{p,q}(M)$ over M, i.e., $\mathcal{E}^{p,q}(M) = \Gamma(M, \Lambda^{p,q}(M))$. We define the operator

$$\overline{\partial}_b : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q+1}(M)$$

as follows: If $\phi \in \mathcal{E}^{p,0}$, $\overline{\partial}_b \phi$ is defined by

$$\langle \overline{\partial}_b \phi, (V_1 \wedge \cdots \wedge V_p) \otimes \overline{L} \rangle = \overline{L} \langle \phi, V_1 \wedge \cdots \wedge V_p \rangle$$

for all sections $V_1, \cdots, V_p$ of $T^{1,0}(M)$ and $\overline{L}$ of $T^{0,1}(M)$. Then $\overline{\partial}_b$ is extended to $\mathcal{E}^{p,q}(M)$ for $q > 0$ as a derivation. Namely, if $\phi \in \mathcal{E}^{p,q}(M)$, we define

$$\begin{aligned}
&\langle \overline{\partial}_b \phi, (V_1 \wedge \cdots \wedge V_p) \otimes (\overline{L}_1 \wedge \cdots \wedge \overline{L}_{q+1}) \rangle \\
&= \frac{1}{q+1} \Bigg\{ \sum_{j=1}^{q+1} (-1)^{j+1} \overline{L}_j \langle \phi, (V_1 \wedge \cdots \wedge V_p) \otimes (\overline{L}_1 \wedge \cdots \wedge \widehat{\overline{L}}_j \wedge \cdots \wedge \overline{L}_{q+1}) \rangle \\
&\quad + \sum_{i<j} (-1)^{i+j} \langle \phi, (V_1 \wedge \cdots \wedge V_p) \otimes ([\overline{L}_i, \overline{L}_j] \wedge \overline{L}_1 \wedge \cdots \wedge \widehat{\overline{L}}_i \wedge \cdots \wedge \widehat{\overline{L}}_j \wedge \cdots \wedge \overline{L}_{q+1}) \rangle \Bigg\}.
\end{aligned}$$

Here by $\widehat{\overline{L}}$ we mean that the term $\overline{L}$ is omitted from the expression. If we let $\pi_{p,q}$ be the projection from $\Lambda^{p+q}\mathbb{C}T^*(M)$ onto $\Lambda^{p,q}(M)$, then $\overline{\partial}_b = \pi_{p,q+1} \circ d$, where d is the exterior derivative on M.

One should note how the integrability condition of the CR structure $T^{1,0}(M)$ comes into play in the definition of $\overline{\partial}_b$, and it is standard to see that the following sequence

$$0 \to \mathcal{E}^{p,0}(M) \xrightarrow{\overline{\partial}_b} \mathcal{E}^{p,1}(M) \xrightarrow{\overline{\partial}_b} \cdots \xrightarrow{\overline{\partial}_b} \mathcal{E}^{p,n-1}(M) \to 0,$$

forms a complex, i.e., $\overline{\partial}_b^2 = 0$.

Notice that p plays no role in the formulation of the tangential Cauchy-Riemann operators. Thus, it suffices to consider the action of $\overline{\partial}_b$ on type $(0,q)$-forms, $0 \le q \le n-1$. When the CR manifold $(M, T^{1,0}(M))$ is embedded as a smooth hypersurface in $\mathbb{C}^n$ with the CR structure $T^{1,0}(M)$ induced from the ambient space, the tangential Cauchy-Riemann complex on M can be defined either extrinsically or intrinsically. These two complexes are different, but one can easily show that they are isomorphic. Thus, if the CR manifold is embedded, we shall not distinguish the extrinsic or intrinsic definitions of the tangential Cauchy-Riemann complex. The operator $\overline{\partial}_b$ is a first order differential operator, and one may consider the inhomogeneous $\overline{\partial}_b$ equation

$$\overline{\partial}_b u = f, \tag{7.2.2}$$

where f is a $(0,q)$-form on M. Equation (7.2.2) is overdetermined when $0 < q < n-1$. Since $\overline{\partial}_b^2 = 0$, for equation (7.2.2) to be solvable, it is necessary that

$$\overline{\partial}_b f = 0. \tag{7.2.3}$$

Condition (7.2.3) is called the compatibility condition for the $\overline{\partial}_b$ equation. We shall discuss the solvability and regularity of the $\overline{\partial}_b$ operator in detail in the next few chapters.

Let $L_1, \cdots, L_{n-1}$ be a local basis for smooth sections of $T^{1,0}(M)$ over some open subset $U \subset M$, so $\overline{L}_1, \cdots, \overline{L}_{n-1}$ is a local basis for $T^{0,1}(M)$ over U. Next we choose a local section T of $\mathbb{C}T(M)$ such that $L_1, \cdots, L_{n-1}, \overline{L}_1, \cdots, \overline{L}_{n-1}$ and T span $\mathbb{C}T(M)$ over U. We may assume that T is purely imaginary.

Definition 7.2.1. *The Hermitian matrix* $(c_{ij})_{i,j=1}^{n-1}$ *defined by*

$$[L_i, \overline{L}_j] = c_{ij} T, \quad \textit{mod } (T^{1,0}(U) \oplus T^{0,1}(U)) \tag{7.2.4}$$

is called the Levi form associated with the given CR *structure.*

The Levi matrix (c_{ij}) clearly depends on the choices of $L_1, \cdots, L_{n-1}$ and T. However, the number of nonzero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choices of $L_1, \cdots, L_{n-1}$ and T. Hence, after changing T to $-T$, it makes sense to consider positive definiteness of the matrix (c_{ij}).

Definition 7.2.2. *The* CR *structure is called pseudoconvex at* $p \in M$ *if the matrix* $(c_{ij}(p))$ *is positive semidefinite after an appropriate choice of* T. *It is called strictly pseudoconvex at* $p \in M$ *if the matrix* $(c_{ij}(p))$ *is positive definite. If the* CR *structure is (strictly) pseudoconvex at every point of* M, *then* M *is called a (strictly) pseudoconvex* CR *manifold. If the Levi form vanishes completely on an open set* $U \subset M$, *i.e.,* $c_{ij} = 0$ *on* U *for* $1 \le i, j \le n-1$, M *is called Levi flat.*

Theorem 7.2.3. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded domain with C^∞ boundary. Then D is (strictly) pseudoconvex if and only if $M = bD$ is a (strictly) pseudoconvex CR manifold.*

Proof. Let r be a C^∞ defining function for D, and let $p \in bD$. We may assume that $(\partial r/\partial z_n)(p) \neq 0$. Hence,

$$L_k = \frac{\partial r}{\partial z_n}\frac{\partial}{\partial z_k} - \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_n}, \quad \text{for } k = 1, \cdots, n-1,$$

form a local basis for the tangential type $(1,0)$ vector fields near p on the boundary. If $L = \sum_{j=1}^n a_j(\partial/\partial z_j)$ is a tangential type $(1,0)$ vector field near p, then we have $\sum_{j=1}^n a_j(\partial r/\partial z_j) = 0$ on bD and $L = (\partial r/\partial z_n)^{-1} \sum_{j=1}^{n-1} a_j L_j$ on bD. Hence, if we let $\eta = \partial r - \overline{\partial} r$, we obtain

$$\begin{aligned}
\sum_{i,j=1}^{n-1} c_{ij} a_i \overline{a}_j &= \sum_{i,j=1}^{n-1} \langle \eta, [L_i, \overline{L}_j]\rangle a_i \overline{a}_j \\
&= \sum_{i,j=1}^{n-1} (L_i\langle \eta, \overline{L}_j\rangle - \overline{L}_j\langle \eta, L_i\rangle - 2\langle d\eta, L_i \wedge \overline{L}_j\rangle) a_i \overline{a}_j \\
&= \sum_{i,j=1}^{n-1} 4\langle \partial\overline{\partial} r, L_i \wedge \overline{L}_j\rangle a_i \overline{a}_j \\
&= 4\left|\frac{\partial r}{\partial z_n}\right|^2 \langle \partial\overline{\partial} r, L \wedge \overline{L}\rangle \\
&= 4\left|\frac{\partial r}{\partial z_n}\right|^2 \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial \overline{z}_j} a_i \overline{a}_j,
\end{aligned}$$

which gives the desired equivalence between these two definitions. This proves the theorem.

We note that, locally, a CR manifold in $\mathbb{C}^n$ is pseudoconvex if and only if it is the boundary of a smooth pseudoconvex domain from one side.

Lemma 7.2.4. *Any compact strongly pseudoconvex CR manifold $(M, T^{1,0}(M))$ is orientable.*

Proof. Locally, let $\eta, \omega_1, \cdots, \omega_{n-1}$ be the one forms dual to $T, L_1, \cdots, L_{n-1}$ which are defined as above. The vector field T is chosen so that the Levi form is positive definite. Then we consider the following $2n-1$ form

$$\eta \wedge \omega_1 \wedge \overline{\omega}_1 \wedge \cdots \wedge \omega_{n-1} \wedge \overline{\omega}_{n-1}. \tag{7.2.5}$$

It is not hard to see that the $2n-1$ form (7.2.5) generated by other bases will differ from (7.2.5) only by a positive function. Hence, a partition of unity argument will give the desired nowhere vanishing $2n-1$ form on M, and the lemma is proved.

7.3 Lewy's Equation

In this section, we shall present a partial differential operator of order one with variable coefficients that, in general, does not possess a solution for a given smooth function. This discovery destroys all hope for the existence of solutions to a reasonably smooth partial differential operator. Since this operator arises from the tangential Cauchy-Riemann operator on the boundary of a strongly pseudoconvex domain, this discovery also inspires an intensive investigation of the tangential Cauchy-Riemann operator.

Let Ω_n be the Siegel upper half space defined by

$$\Omega_n = \{(z', z_n) \in \mathbb{C}^n |\ \mathrm{Im} z_n > |z'|^2\}, \tag{7.3.1}$$

where $z' = (z_1, \cdots, z_{n-1})$ and $|z'|^2 = |z_1|^2 + \cdots + |z_{n-1}|^2$. When $n = 1$, Ω_1 is reduced to the upper half space of the complex plane which is conformally equivalent to the unit disc. For $n > 1$, the Cayley transform also maps the unit ball B_n biholomorphically onto the Siegel upper half space Ω_n, i.e.,

$$\begin{aligned} \Phi : B_n &\to \Omega_n \\ z\ &\mapsto w = \Phi(z) \\ &= i\left(\frac{e_n + z}{1 - z_n}\right) = \left(\frac{iz_1}{1 - z_n}, \cdots, \frac{iz_{n-1}}{1 - z_n}, i\frac{1 + z_n}{1 - z_n}\right), \end{aligned} \tag{7.3.2}$$

where $e_n = (0, \cdots, 0, 1)$.

For $n \geq 2$, a simple calculation shows

$$L_k = \frac{\partial}{\partial z_k} + 2i\overline{z}_k \frac{\partial}{\partial z_n} \quad \text{for } k = 1, \cdots, n-1, \tag{7.3.3}$$

forms a global basis for the space of tangential $(1, 0)$ vector fields on the boundary $b\Omega_n$ and

$$[L_j, \overline{L}_k] = -2i\delta_{jk}\frac{\partial}{\partial t},$$

where $z_n = t + is$ and δ_{jk} is the Kronecker delta. It follows that if we choose $T = -2i(\partial/\partial t)$, the Levi matrix (c_{ij}) is the identity matrix which implies that Ω_n is a strongly pseudoconvex domain. Furthermore, the boundary $b\Omega_n$ can be identified with $\mathbb{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$ via the map

$$\pi : (z', t + i|z'|^2) \to (z', t).$$

Therefore, a CR structure can be induced via π on $\mathbb{H}_n$ by

$$Z_k = \pi_* L_k = \frac{\partial}{\partial z_k} + i\overline{z}_k \frac{\partial}{\partial t} \tag{7.3.4}$$

for $k = 1, \cdots, n-1$. Thus, if $f = \sum_{j=1}^{n-1} f_j \overline{\omega}_j$ is a $\overline{\partial}_b$-closed $(0, 1)$-form on $\mathbb{H}_n$, where ω_j is the $(1, 0)$-form dual to Z_j, the solvability of the equation $\overline{\partial}_b u = f$ is equivalent to the existence of a function u satisfying the following system of equations:

$$\overline{Z}_k u = f_k, \quad 1 \leq k \leq n - 1.$$

When $n = 2$, the Siegel upper half space is given by

$$\{(z, w) \in \mathbb{C}^2 \mid |z|^2 - \frac{1}{2i}(w - \overline{w}) < 0\}.$$

Hence, the tangential Cauchy-Riemann operator is generated by

$$\overline{L} = \frac{\partial}{\partial \bar{z}} - 2iz\frac{\partial}{\partial \overline{w}}, \tag{7.3.5}$$

with $w = t+is$, and the corresponding operator, denoted by $\overline{Z}$, via the identification on $\mathbb{H}_2$ is

$$\overline{Z} = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial t}, \tag{7.3.6}$$

where (z, t) with $z = x + iy$ are the coordinates on $\mathbb{H}_2 = \mathbb{C} \times \mathbb{R}$. The coefficients of the operator $\overline{Z}$ defined in (7.3.6) are real analytic. Hence, for a given real analytic function f, the equation

$$\overline{Z}u = f \tag{7.3.7}$$

always has a real analytic solution u locally as is guaranteed by the Cauchy-Kowalevski theorem. However, the next theorem shows that equation (7.3.7) does not possess a solution in general even when f is smooth.

Theorem 7.3.1 (Lewy). *Let f be a continuous real-valued function depending only on t. If there is a C^1 solution $u(x, y, t)$ to the equation (7.3.7), then f must be real analytic in some neighborhood of $t = 0$.*

Proof. Locally, near the origin any point can be expressed in terms of the polar coordinates as

$$(x, y, t) = (re^{i\theta}, t)$$

with $r < R$ and $|t| < R$ for some $R > 0$. Set $s = r^2$. Consider the function $V(s, t)$ defined by

$$V(s, t) = \int_{\{|z|=r\}} u(z, t)\ dz.$$

Then, by Stokes' theorem we have

$$\begin{aligned} V(s, t) &= -\iint\limits_{\{|z|<r\}} \frac{\partial u}{\partial \bar{z}}\ dz \wedge d\bar{z} \\ &= -\iint\limits_{\{|z|<r\}} \left(f + iz\frac{\partial u}{\partial t}\right)\ dz \wedge d\bar{z} \\ &= 2\pi i r^2 f(t) - 2\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^r \rho e^{i\theta}u(\rho e^{i\theta}, t)\rho d\rho d\theta. \end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial V}{\partial s}(s,t) &= 2\pi i f(t) - 2\frac{\partial}{\partial t}\left(\frac{1}{2r}\frac{\partial}{\partial r}\int_0^{2\pi}\int_0^r \rho e^{i\theta}u(\rho e^{i\theta},t)\rho d\rho d\theta\right)\\ &= 2\pi i f(t) - \frac{1}{r}\frac{\partial}{\partial t}\int_0^{2\pi} re^{i\theta}u(re^{i\theta},t)rd\theta\\ &= 2\pi i f(t) + i\frac{\partial}{\partial t}V(s,t).\end{aligned}$$

Set

$$F(t) = \int_0^t f(\eta)\ d\eta.$$

Then, we obtain

$$\left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial s}\right)(V(s,t) + 2\pi F(t)) = 0,$$

which implies that the function $U(s,t) = V(s,t) + 2\pi F(t)$ is holomorphic on the set $\{t + is \in \mathbb{C}|\ |t| < R, 0 < s < R^2\}$, and $U(s,t)$ is continuous up to the real axis $\{s = 0\}$ with real-valued boundary value $2\pi F(t)$. Hence, by the Schwarz reflection principle, $U(s,t)$ can be extended holomorphically across the boundary to the domain $\{t + is \in \mathbb{C}||t| < R, |s| < R^2\}$. In particular, $F(t)$, and hence $f(t)$, must be real analytic on $(-R, R)$. This proves Theorem 7.3.1.

In Section 10.3 we shall give a complete characterization of the local solvability of the Lewy operator (7.3.6).

7.4 Linear Partial Differential Operators with Constant Coefficients

In contrast to the nonsolvable operator (7.3.6), we shall present in this section a fundamental positive result in the theory of partial differential equations which asserts the existence of a distribution fundamental solution to any linear partial differential operator with constant coefficients. It follows by convolution that every partial differential operator with constant coefficients is locally solvable.

Theorem 7.4.1 (Malgrange, Ehrenpreis). *Let*

$$L = \sum_{|\alpha|\le k} a_\alpha D_x^\alpha$$

be a partial differential operator with constant coefficients on $\mathbb{R}^n$*, where* $D_x^\alpha = (\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$ *for any multiindex* $\alpha = (\alpha_1, \cdots, \alpha_n)$ *with nonnegative integer components. If* $f \in C_0^\infty(\mathbb{R}^n)$*, then there exists a* C^∞ *function* $h(x)$ *satisfying* $Lh = f$ *on* $\mathbb{R}^n$*.*

Proof. The proof will be done via the Fourier transform. For any $g \in L^1(\mathbb{R}^n)$, the Fourier transform $\hat{g}(\xi)$ of g is defined by

$$\hat{g}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}g(x)\ dx,$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$. Note first that, via a rotation of coordinates and multiplying L by a constant, we may assume that the corresponding Fourier transform $p(\xi)$ of the operator L is

$$p(\xi) = \xi_n^k + \sum_{j=0}^{k-1} a_j(\xi')\xi_n^j,$$

where $\xi = (\xi', \xi_n)$ with $\xi' \in \mathbb{R}^{n-1}$, and $a_j(\xi')$ is a polynomial in ξ' for $0 \le j \le k-1$.

Next, we complexify $p(\xi)$; namely, we view ξ as a variable in $\mathbb{C}^n$. Hence, for each $\xi' \in \mathbb{R}^{n-1}$, $p(\xi', \xi_n)$ is a polynomial of degree k in ξ_n. Let $\lambda_1(\xi'), \cdots, \lambda_k(\xi')$ be its zeros, arranged so that if $i \le j$, $\mathrm{Im}\lambda_i(\xi') \le \mathrm{Im}\lambda_j(\xi')$, and $\mathrm{Re}\lambda_i(\xi') \le \mathrm{Re}\lambda_j(\xi')$ if $\mathrm{Im}\lambda_i(\xi') = \mathrm{Im}\lambda_j(\xi')$. One sees easily that these k functions $\mathrm{Im}\lambda_j(\xi')$ are continuous in ξ'.

We then need to construct a measurable function

$$\phi : \mathbb{R}^{n-1} \to [-k-1, k+1]$$

such that for all $\xi' \in \mathbb{R}^{n-1}$, we have

$$\min\{|\phi(\xi') - \mathrm{Im}\lambda_j(\xi')| : 1 \le j \le k\} \ge 1.$$

Set $u_0(\xi') = -k-1$ and $u_{k+1}(\xi') = k+1$. For $1 \le j \le k$, define

$$u_j(\xi') = \max\{\min\{\mathrm{Im}\lambda_j(\xi'), k+1\}, -k-1\}.$$

The functions $u_j(\xi')$ are continuous in ξ', so the sets

$$V_j = \{\xi' : u_{j+1}(\xi') - u_j(\xi') \ge 2\},$$

for $j = 0, \cdots, k$, are measurable. It is clear that $\cup_{j=0}^k V_j$ is a covering of $\mathbb{R}^{n-1}$. Thus, we can construct disjoint measurable subsets $W_j \subset V_j$ which still cover $\mathbb{R}^{n-1}$. Define

$$\phi(\xi') = \frac{1}{2}(u_{j+1}(\xi') + u_j(\xi')),$$

if $\xi' \in W_j$. This completes the construction of $\phi(\xi')$.

We define $h(x)$ by

$$h(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\{\mathrm{Im}\xi_n = \phi(\xi')\}} e^{ix\cdot\xi} \left(\frac{\hat{f}(\xi)}{p(\xi)} \right) d\xi_n d\xi'.$$

The key is to observe that, as $|\mathrm{Re}\xi| \to \infty$, $\hat{f}(\xi)$ is rapidly decreasing whereas $\mathrm{Im}\xi$ remains bounded, and to see that the line $\mathrm{Im}\xi_n = \phi(\xi')$ in the ξ_n-plane has distance at least one from any zero of $p(\xi)$ and at most $k+1$ from the real axis. Hence, the integrand is bounded and rapidly decreasing at infinity, so the integral is absolutely convergent. The same reasoning shows that we can differentiate under the integral sign as often as we please. It follows that $h(x)$ is smooth. Finally, we apply L to $h(x)$ and get

$$Lh(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{\{\mathrm{Im}\xi_n = \phi(\xi')\}} e^{ix\cdot\xi} \hat{f}(\xi)\ d\xi_n d\xi'.$$

The integrand on the right-hand side is an entire function which is rapidly decreasing as $|\mathrm{Re}\xi_n| \to \infty$. Therefore, by Cauchy's theorem, the contour of integration can be deformed back to the real axis. By invoking the inverse Fourier transform we obtain $Lh = f$. The proof is now complete.

As an easy consequence of Theorem 7.4.1, if L is a linear partial differential operator with constant coefficients, then for any given function $f(x)$ which is smooth near some point x_0, we can find locally a smooth solution $h(x)$ such that $Lh = f$ near x_0.

We now return to the solvability of the $\overline{\partial}_b$ equation in a very special case. Let the CR manifold $(M, T^{1,0}(M))$ of real dimension $2n-1$, $n \geq 2$, be Levi flat in a neighborhood U of the reference point p, then we can apply the Frobenius theorem (Theorem 1.6.1) to $\mathrm{Re}L_1, \mathrm{Im}L_1, \cdots, \mathrm{Re}L_{n-1}, \mathrm{Im}L_{n-1}$, where $L_1, \cdots, L_{n-1}$ is a local basis of $T^{1,0}(M)$ near p. Thus there exist local coordinates $(x_1, \cdots, x_{2n-2}, t)$ such that the vector fields $\mathrm{Re}L_1, \mathrm{Im}L_1, \cdots, \mathrm{Re}L_{n-1}, \mathrm{Im}L_{n-1}$ span the tangent space of each leaf $\{t = c\}$ for some constant c. Therefore, on each leaf we may apply Theorem 2.3.1, for $n = 2$, or the Newlander-Nirenberg theorem (Theorem 5.4.4), for $n \geq 3$, to show that M is locally foliated by complex submanifolds of complex dimension $n-1$. In this case, the local solvability of the tangential Cauchy-Riemann equation, $\overline{\partial}_b u = f$, where f is a $\overline{\partial}_b$-closed $(0,1)$-form, can be reduced to a $\overline{\partial}$ problem with a parameter. In the next few chapters, the global and local solvability of $\overline{\partial}_b$ will be discussed in detail.

NOTES

The tangential Cauchy-Riemann complex was first introduced by J. J. Kohn and H. Rossi [KoRo 1]. See also the books by A. Boggess [Bog 1], G. B. Folland and J. J. Kohn [FoKo 1] and H. Jacobowitz [Jac 1]. The nonsolvability theorem for the operator (7.3.6) was proved by H. Lewy [Lew 2]. A more general theorem due to L. Hörmander [Hör 1,7] states that the tangential Cauchy-Riemann equation on a real three dimensional CR manifold is not locally solvable if it is not Levi flat. For a proof of the Cauchy-Kowalevski theorem, the reader is referred to [Joh 1]. Theorem 7.4.1 was originally proved by B. Malgrange [Mal 1] and L. Ehrenpreis [Ehr 1]. The proof we present here is due to L. Nirenberg [Nir 2].

CHAPTER 8

SUBELLIPTIC ESTIMATES FOR SECOND ORDER DIFFERENTIAL EQUATIONS AND $\Box_b$

In this chapter, we study subelliptic operators which are not elliptic. We analyze two types of operators in detail. One is a real second order differential equation which is a sum of squares of vector fields. The other is the $\overline{\partial}_b$-Laplacian on a CR manifold. We use pseudodifferential operators to study both operators.

For this purpose, we shall briefly review the definitions and basic properties of the simplest pseudodifferential operators. Using pseudodifferential operators, Hörmander's theorem on the hypoellipticity of sums of squares of vector fields will be discussed in Section 8.2.

The $\overline{\partial}_b$-Laplacian, $\Box_b$, is not elliptic. There is a one-dimensional characteristic set. However, under certain conditions, one is able to establish the 1/2-estimate for the $\Box_b$ operator via potential-theoretic methods and pseudodifferential operators. In the last two sections of this chapter, the 1/2-estimate for the $\Box_b$ operator on compact strongly pseudoconvex CR manifolds is proved, which leads to the existence and regularity theorems of the $\overline{\partial}_b$ equation. Global existence theorems for $\overline{\partial}_b$ on the boundary of a pseudoconvex domain in $\mathbb{C}^n$ will be discussed in Chapter 9.

8.1 Pseudodifferential Operators

We first introduce some simple pseudodifferential operators. Let $\mathcal{S}$ be the Schwartz space in $\mathbb{R}^n$. For the definitions of the space $\mathcal{S}$ and the Sobolev space $W^s(\mathbb{R}^n)$, the reader is referred to the Appendix A. We begin this section with the following definition:

Definition 8.1.1. *A linear operator $T : \mathcal{S} \to \mathcal{S}$ is said to be of order m if for each $s \in \mathbb{R}$ we have*

$$\| Tu \|_s \le C_s \| u \|_{s+m}, \quad \textit{for all } u \in \mathcal{S},$$

with the constant C_s independent of u.

We note that, by definition, any linear operator of order m from $\mathcal{S}$ into itself extends to a bounded linear operator from $W^{s+m}(\mathbb{R}^n)$ to $W^s(\mathbb{R}^n)$ for every $s \in \mathbb{R}$. It is obvious that any differential operator D^α with $|\alpha| = m$ is of order m, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$.

For any $s \in \mathbb{R}$, we define $\Lambda^s : \mathcal{S} \to \mathcal{S}$ by

$$\Lambda^s u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi)\, d\xi$$

where $\hat{u}$ is the Fourier transform of u. Here, $\sigma(\Lambda^s) = (1+|\xi|^2)^{\frac{s}{2}}$ is called the symbol of Λ^s. Obviously, the operator Λ^s is of order s. Λ^s is called a pseudodifferential operator of order s. We should think of Λ^s as a generalization of differential operators to fractional and negative order.

The purpose of this section is to prove some basic properties of the commutators between Λ^s and functions in $\mathcal{S}$. We need the following lemma:

Lemma 8.1.2. $\left((1+|x|^2)/(1+|y|^2)\right)^s \le 2^{|s|}(1+|x-y|^2)^{|s|}$ *for all* $x, y \in \mathbb{R}^n$ *and every* $s \in \mathbb{R}$.

Proof. From the triangle inequality $|x| \le |x-y| + |y|$, we obtain $|x|^2 \le 2(|x-y|^2 + |y|^2)$ and hence $1 + |x|^2 \le 2(1+|x-y|^2)(1+|y|^2)$. Thus, if $s \ge 0$, the lemma is proved. For $s < 0$, the same arguments can be applied with x and y reversed and s replaced by $-s$. This proves the lemma.

We employ Plancherel's theorem to study the commutators of Λ^s and functions in $\mathcal{S}$. We first show that multiplication by a function in $\mathcal{S}$ is of order zero.

Lemma 8.1.3. *For any* $g \in \mathcal{S}$ *and* $s \in \mathbb{R}$, $\| gu \|_s \lesssim \| u \|_s$ *uniformly for all* $u \in \mathcal{S}$.

Proof. By the Fourier transform formula for convolution, we obtain

$$(1+|\xi|^2)^{\frac{s}{2}}\widehat{gu}(\xi) = \frac{1}{(2\pi)^n}\int \left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)^{\frac{s}{2}} \hat{g}(\xi-\eta)(1+|\eta|^2)^{\frac{s}{2}}\hat{u}(\eta)\, d\eta.$$

Now we view $\left((1+|\xi|^2)/(1+|\eta|^2)\right)^{s/2}\hat{g}(\xi-\eta)$ as a kernel $K(\xi,\eta)$ and set $f(\eta) = (1+|\eta|^2)^{s/2}\hat{u}(\eta)$. Lemma 8.1.2 shows that

$$|K(\xi,\eta)| \lesssim (1+|\xi-\eta|^2)^{\frac{|s|}{2}}|\hat{g}(\xi-\eta)|.$$

Since $g \in \mathcal{S}$, it follows that $\hat{g} \in \mathcal{S}$ and that the hypotheses of Theorem B.10 in the Appendix are satisfied by this kernel. Therefore, we have

$$\| gu \|_s \lesssim \| f \| = \| u \|_s .$$

This proves the lemma.

Theorem 8.1.4. *If* $g, h \in \mathcal{S}$, *then for any* $r, s \in \mathbb{R}$, *we have*

(1) $[\Lambda^s, g]$ *is of order* $s-1$,
(2) $[\Lambda^r, [\Lambda^s, g]]$ *is of order* $r+s-2$,
(3) $[[\Lambda^s, g], h]$ *is of order* $s-2$.

Proof. The proof of the theorem will proceed exactly as in Lemma 8.1.3. For (1) it suffices to show that $\Lambda^r[\Lambda^s, g]\Lambda^{1-r-s}$ is of order zero for any $r \in \mathbb{R}$. Let $u \in \mathcal{S}$, and set $f = \Lambda^r[\Lambda^s, g]\Lambda^{1-r-s}u$. A direct calculation shows that

$$\hat{f}(\xi) = \frac{1}{(2\pi)^n}\int K(\xi,\eta)\hat{u}(\eta)\, d\eta,$$

where

$$K(\xi,\eta) = \frac{(1+|\xi|^2)^{\frac{r}{2}}}{(1+|\eta|^2)^{\frac{r+s-1}{2}}}\left((1+|\xi|^2)^{\frac{s}{2}} - (1+|\eta|^2)^{\frac{s}{2}}\right)\hat{g}(\xi-\eta).$$

From the mean value theorem, we have

$$|(1+|\xi|^2)^{\frac{s}{2}} - (1+|\eta|^2)^{\frac{s}{2}}| \lesssim |\xi-\eta| \left((1+|\xi|^2)^{\frac{s-1}{2}} + (1+|\eta|^2)^{\frac{s-1}{2}}\right)$$

for all $\xi, \eta \in \mathbb{R}^n$, hence, by Lemma 8.1.2 we obtain

$$\begin{aligned}|K(\xi,\eta)| &\lesssim |\xi-\eta| \left(\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)^{\frac{r+s-1}{2}} + \left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)^{\frac{r}{2}}\right) |\hat{g}(\xi-\eta)| \\ &\lesssim |\xi-\eta| \left((1+|\xi-\eta|^2)^{\frac{|r+s-1|}{2}} + (1+|\xi-\eta|^2)^{\frac{|r|}{2}}\right) |\hat{g}(\xi-\eta)|.\end{aligned}$$

Since $g \in \mathcal{S}$, so is $\hat{g} \in \mathcal{S}$. Therefore, by Theorem B.10 in the Appendix, (1) is proved.

To prove (2), for $u \in \mathcal{S}$, a similar calculation shows

$$([\Lambda^r, [\Lambda^s, g]]u)\hat{}\,(\xi) = \frac{1}{(2\pi)^n} \int K(\xi,\eta)\hat{u}(\eta)\, d\eta,$$

where

$$K(\xi,\eta) = \left((1+|\xi|^2)^{\frac{r}{2}} - (1+|\eta|^2)^{\frac{r}{2}}\right) \left((1+|\xi|^2)^{\frac{s}{2}} - (1+|\eta|^2)^{\frac{s}{2}}\right) \hat{g}(\xi-\eta).$$

Here $([\Lambda^r, [\Lambda^s, g]]u)\hat{}\,(\xi)$ denotes the Fourier transform of $[\Lambda^r, [\Lambda^s, g]]u$. It follows now from the same estimates as in (1) that

$$\begin{aligned}|K(\xi,\eta)| \lesssim |\xi-\eta|^2 &\left((1+|\xi|^2)^{\frac{r-1}{2}} + (1+|\eta|^2)^{\frac{r-1}{2}}\right) \\ &\cdot \left((1+|\xi|^2)^{\frac{s-1}{2}} + (1+|\eta|^2)^{\frac{s-1}{2}}\right) |\hat{g}(\xi-\eta)|,\end{aligned}$$

which implies easily that $[\Lambda^r, [\Lambda^s, g]]$ is of order $r+s-2$. This proves (2).

For (3), let $u \in \mathcal{S}$. It is easy to get from Taylor's expansion that

$$\begin{aligned}([\Lambda^s, g]u)\hat{}\,(\xi) &= \frac{1}{(2\pi)^n} \int \left((1+|\xi|^2)^{\frac{s}{2}} - (1+|\eta|^2)^{\frac{s}{2}}\right) \hat{g}(\xi-\eta)\hat{u}(\eta)\, d\eta \\ &= \sum_{j=1}^n \frac{1}{(2\pi)^n} \int (\xi_j - \eta_j) \frac{\partial}{\partial \xi_j} (1+|\xi|^2)^{\frac{s}{2}} \hat{g}(\xi-\eta)\hat{u}(\eta)\, d\eta \\ &\quad + \frac{1}{(2\pi)^n} \int O(\xi,\eta)\hat{g}(\xi-\eta)\hat{u}(\eta)\, d\eta \\ &= \widehat{T_1 u} + \widehat{T_2 u},\end{aligned}$$

where $O(\xi,\eta)$ can be estimated by

$$|O(\xi,\eta)| \lesssim |\xi-\eta|^2 \left((1+|\xi|^2)^{\frac{s-2}{2}} + (1+|\eta|^2)^{\frac{s-2}{2}}\right).$$

Now, as in (1), it is easily seen that the operator T_2 is of order $s-2$. Thus, to finish the proof of (3), it suffices to show that $[T_1, h]$ is of order $s-2$. Write

$$\begin{aligned} k(\xi,\eta) &= \sum_{j=1}^{n} (\xi_j - \eta_j) \frac{\partial}{\partial \xi_j} (1+|\xi|^2)^{\frac{s}{2}} \hat{g}(\xi - \eta) \\ &= s \sum_{j=1}^{n} \xi_j (\xi_j - \eta_j)(1+|\xi|^2)^{\frac{s-2}{2}} \hat{g}(\xi - \eta). \end{aligned}$$

Thus, from a direct calculation we obtain

$$([T_1, h]u)\hat{}(\xi) = \frac{1}{(2\pi)^n} \int K(\xi,\eta) \hat{u}(\eta)\ d\eta,$$

where

$$\begin{aligned} K(\xi,\eta) &= \frac{1}{(2\pi)^n} \int \Big(k(\xi,\zeta)\hat{h}(\zeta-\eta) - k(\zeta,\eta)\hat{h}(\xi-\zeta) \Big)\ d\zeta \\ &= \frac{1}{(2\pi)^n} \int \hat{h}(\xi-\zeta)\big(k(\xi,\xi+\eta-\zeta) - k(\zeta,\eta)\big)\ d\zeta. \end{aligned}$$

Since

$$\begin{aligned} &|k(\xi,\xi+\eta-\zeta) - k(\zeta,\eta)| \\ &\qquad \lesssim |\xi-\zeta||\zeta-\eta| \left((1+|\xi|^2)^{\frac{s-2}{2}} + (1+|\zeta|^2)^{\frac{s-2}{2}} \right) |\hat{g}(\zeta-\eta)|, \end{aligned}$$

we can estimate $K(\xi,\eta)$ as follows,

$$\begin{aligned} |K(\xi,\eta)| &\lesssim \int |\xi-\zeta||\zeta-\eta|(1+|\xi|^2)^{\frac{s-2}{2}} |\hat{h}(\xi-\zeta)||\hat{g}(\zeta-\eta)|\ d\zeta \\ &\quad + \int |\xi-\zeta||\zeta-\eta|(1+|\zeta|^2)^{\frac{s-2}{2}} |\hat{h}(\xi-\zeta)||\hat{g}(\zeta-\eta)|\ d\zeta \\ &\lesssim (1+|\xi|^2)^{\frac{s-2}{2}} \bigg(\int |\zeta||\hat{h}(\zeta)||\xi-\eta-\zeta||\hat{g}(\xi-\eta-\zeta)|\ d\zeta \\ &\quad + \int |\zeta|(1+|\zeta|^2)^{\frac{|s-2|}{2}} |\hat{h}(\zeta)||\xi-\eta-\zeta||\hat{g}(\xi-\eta-\zeta)|\ d\zeta \bigg) \\ &\leq C_m (1+|\xi|^2)^{\frac{s-2}{2}} (1+|\xi-\eta|^2)^{-m}, \end{aligned}$$

for any $m \in \mathbb{N}$, where C_m is a constant depending on m. Here we have used the fact that both $\hat{h}$ and $\hat{g}$ are in $\mathcal{S}$. Choosing m to be sufficiently large and applying arguments similar to those used in the proof of (1), (3) is proved. This completes the proof of Theorem 8.1.4.

Theorem 8.1.5. *Let P and Q be two differential operators of order k and m respectively with coefficients in $\mathcal{S}$. Then $[\Lambda^s, P]$ is of order $s+k-1$, $[\Lambda^r, [\Lambda^s, P]]$ is of order $r+s+k-2$, and $[[\Lambda^s, P], Q]$ is of order $s+k+m-2$.*

Proof. Write

$$P = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha,$$

with $a_\alpha(x) \in \mathcal{S}$. Since D^α commutes with Λ^s, the commutator of Λ^s with P is reduced to commutators of Λ^s with $a_\alpha(x)$, composed with D^α. This proves the theorem.

We shall use operators of the form generated by Λ^s, D^α and multiplication by functions in $\mathcal{S}$ plus their Lie brackets. All these are pseudodifferential operators and the computation of their orders is similar to that for differential operators.

8.2 Hypoellipticity of Sum of Squares of Vector Fields

Let Ω be an open neighborhood of the origin in $\mathbb{R}^n$. Let $X_i = \sum_{j=1}^n a_{ij}(\partial/\partial x_j)$, $0 \le i \le k$ with $k \le n$, be vector fields with smooth real-valued coefficients $a_{ij}(x)$ on Ω. Define the second order partial differential operator

$$P = \sum_{i=1}^k X_i^2 + X_0 + b(x), \tag{8.2.1}$$

where $b(x)$ is a smooth real-valued function on Ω.

Denote by $\mathcal{L}_1$ the collection of the X_i's, $0 \le i \le k$. Then, inductively for an integer $m \ge 2$ we define $\mathcal{L}_m$ to be the collection of $\mathcal{L}_{m-1}$ and the vector fields of the form $[X, Y]$ with $X \in \mathcal{L}_1$ and $Y \in \mathcal{L}_{m-1}$.

Definition 8.2.1. *The partial differential operator P defined as in (8.2.1) is said to be of finite type at point $p \in \Omega$ if there exists an m such that $\mathcal{L}_m$ spans the whole tangent space at p.*

Definition 8.2.2. *A partial differential operator P is said to be hypoelliptic in Ω if it satisfies the following property: let u and f be distributions satisfying $Pu = f$ in Ω, then u is smooth on U if f is smooth on U for any open subset U of Ω*

The task of this section is to prove the following main theorem:

Theorem 8.2.3. *Let P be the partial differential operator defined as in (8.2.1). Suppose that P is of finite type at every point in Ω. Then P is hypoelliptic in Ω.*

The heat operator on $\mathbb{R}^{n+1}$ defined by

$$P = \left(\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_n}\right)^2 - \frac{\partial}{\partial t}$$

is a typical example of such an operator, where the coordinates in $\mathbb{R}^{n+1}$ are denoted by $(x,t) = (x_1, \cdots, x_n, t)$. Another simple example with variable coefficients is the Grushin operator on $\mathbb{R}^2$ defined by

$$P = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}.$$

According to Theorem 8.2.3, both operators are hypoelliptic.

To prove Theorem 8.2.3, we begin with the following *a priori* estimate. By shrinking the domain Ω, if necessary, we may assume that $a_{ij}(x)$ and $b(x)$ are in $C^\infty(\overline{\Omega})$ for all i, j.

Lemma 8.2.4. *Let P be defined as in (8.2.1). There exists $C>0$ such that*

$$\sum_{i=1}^{k} \| X_i u \|^2 \le C(|(Pu,u)| + \| u \|^2), \quad u \in C_0^\infty(\Omega). \tag{8.2.2}$$

Proof. Let X_i^* be the adjoint operator for X_i. Then $X_i^* = -X_i + h_i$, where $h_i = -\sum_{j=1}^{n} (\partial a_{ij}/\partial x_j)$. Integration by parts shows

$$-(X_i^2 u, u) = \| X_i u \|^2 + O(\| X_i u \| \| u \|)$$

and

$$(X_0 u, u) = -(u, X_0 u) + O(\| u \|^2).$$

It follows that

$$\mathrm{Re}(X_0 u, u) = O(\| u \|^2).$$

Adding up these estimates, we obtain

$$\sum_{i=1}^{k} \| X_i u \|^2 = -\mathrm{Re}(Pu,u) + O\left(\sum_{i=1}^{k} \| X_i u \| \| u \| + \| u \|^2\right).$$

Using small and large constants, this gives the desired estimate (8.2.2), and the proof is complete.

We first prove the following general theorem.

Theorem 8.2.5. *If $\mathcal{L}_m$ spans the tangent space of Ω for some $m \in \mathbb{N}$, then there exist $\epsilon > 0$ and $C > 0$ such that*

$$\| u \|_\epsilon^2 \le C\left(\sum_{i=0}^{k} \| X_i u \|^2 + \| u \|^2\right), \quad u \in C_0^\infty(\Omega). \tag{8.2.3}$$

Here we may take $\epsilon = 2^{1-m}$.

Proof. We shall denote an element in $\mathcal{L}_j$ by Z_j. By the hypotheses of the theorem, we get

$$\| u \|_\epsilon^2 \lesssim \sum_{j=1}^{n} \| D_j u \|_{\epsilon-1}^2 + \| u \|^2 \lesssim \sum_{Z_m \in \mathcal{L}_m} \| Z_m u \|_{\epsilon-1}^2 + \| u \|^2,$$

where the last summation is a finite sum and $D_j = (\partial/\partial x_j)$.

Therefore, to prove the theorem, it suffices to bound each term $\| Z_m u \|_{\epsilon-1}$ by the right hand side of (8.2.3) for some $\epsilon > 0$. If $m=1$, clearly we can take $\epsilon = 1$. For $m \ge 2$, let $\epsilon \le 1/2$ for the time being. We shall make the choice of ϵ later. We may also assume that $Z_m = XZ_{m-1} - Z_{m-1}X$ with $X \in \mathcal{L}_1$. Thus, we see that

$$\begin{aligned} \| Z_m u \|_{\epsilon-1}^2 &= (Z_m u, \Lambda^{2\epsilon-2} Z_m u) \\ &= (XZ_{m-1}u, \Lambda^{2\epsilon-2} Z_m u) - (Z_{m-1}Xu, \Lambda^{2\epsilon-2} Z_m u). \end{aligned} \tag{8.2.4}$$

Using Theorem 8.1.5, we have

$$\begin{aligned}&|(XZ_{m-1}u, \Lambda^{2\epsilon-2}Z_m u)|\\ &\quad = |(Z_{m-1}u, \Lambda^{2\epsilon-2}Z_m Xu)| + O(\| u \| \| Z_{m-1}u \|_{2\epsilon-1})\\ &\quad \le C(\| Xu \|^2 + \| Z_{m-1}u \|^2_{2\epsilon-1} + \| u \|^2).\end{aligned}$$

Also

$$\begin{aligned}|(Z_{m-1}Xu, \Lambda^{2\epsilon-2}Z_m u)| &= |(Xu, Z_{m-1}\Lambda^{2\epsilon-2}Z_m u)| + O(\| u \| \| Xu \|)\\ &\le C(\| Xu \|^2 + \| Z_{m-1}u \|^2_{2\epsilon-1} + \| u \|^2).\end{aligned}$$

Hence, substituting the above into (8.2.4), by induction, we get

$$\begin{aligned}\| Z_m u \|^2_{\epsilon-1} &\le C\left(\sum_{i=0}^{k} \| X_i u \|^2 + \| Z_{m-1}u \|^2_{2\epsilon-1} + \| u \|^2\right)\\ &\le C\left(\sum_{i=0}^{k} (\| X_i u \|^2 + \| X_i u \|^2_{2^{m-1}\epsilon-1}) + \| u \|^2\right).\end{aligned}$$

Now, for $m \ge 2$, if we take $\epsilon \le 2^{1-m}$, we obtain

$$\| Z_m u \|^2_{\epsilon-1} \le C\left(\sum_{i=0}^{k} \| X_i u \|^2 + \| u \|^2\right).$$

This proves the theorem.

For our purpose, we shall modify the proof of Theorem 8.2.5 to obtain:

Theorem 8.2.6. *Under the hypotheses of Theorem 8.2.3, there exist $\epsilon > 0$ and $C > 0$ such that*

$$\| u \|^2_\epsilon \le C(\| Pu \|^2 + \| u \|^2), \quad u \in C_0^\infty(\Omega). \tag{8.2.5}$$

Proof. As in the proof of Theorem 8.2.5, it suffices to control each term

$$\| Z_m u \|^2_{\epsilon-1} = (XZ_{m-1}u, \Lambda^{2\epsilon-2}Z_m u) - (Z_{m-1}Xu, \Lambda^{2\epsilon-2}Z_m u) \tag{8.2.6}$$

by the right-hand side of (8.2.5) for some $\epsilon > 0$.

Let Q^s be some pseudodifferential operator of order s of the type discussed in Section 8.1. To estimate (8.2.6), we shall distinguish X_0 from the other vector fields $X_1, \cdots, X_k$.

Case (i). $X = X_i$ with $1 \le i \le k$. By Lemma 8.2.4 and the proof of Theorem 8.2.5, we get

$$\| Z_m u \|^2_{\epsilon-1} \le C(\| Pu \|^2 + \| Z_{m-1}u \|^2_{2\epsilon-1} + \| u \|^2). \tag{8.2.7}$$

Case (ii). $X = X_0$. We first write $X_0 = -P^* + \sum_{i=1}^k X_i^2 + \sum_{i=1}^k c_i(x)X_i + g(x)$ with $c_i(x), 1 \le i \le k$, and $g(x)$ belonging to $C^\infty(\overline{\Omega})$. Hence, we have

$$
\begin{aligned}
&(X_0 Z_{m-1}u, \Lambda^{2\epsilon-2} Z_m u)\\
&= -(P^* Z_{m-1}u, \Lambda^{2\epsilon-2} Z_m u) + \sum_{i=1}^k (X_i^2 Z_{m-1}u, \Lambda^{2\epsilon-2} Z_m u)\\
&\quad + \sum_{i=1}^k (c_i X_i Z_{m-1}u, \Lambda^{2\epsilon-2} Z_m u) + (g Z_{m-1}u, \Lambda^{2\epsilon-2} Z_m u).
\end{aligned} \tag{8.2.8}
$$

Obviously, the last two terms in (8.2.8) are bounded by the right-hand side of (8.2.7). Since

$$
\begin{aligned}
&(P^* Z_{m-1}u, \Lambda^{2\epsilon-2} Z_m u) = (Z_{m-1}u, PQ^{2\epsilon-1}u)\\
&= (Z_{m-1}u, Q^{2\epsilon-1}Pu) + \sum_{i=1}^k (Z_{m-1}u, Q^{2\epsilon-1}X_i u) + (Z_{m-1}u, Q^{2\epsilon-1}u),
\end{aligned}
$$

we conclude that the first term on the right-hand side in (8.2.8) is also bounded by the right-hand side of (8.2.7). The second term on the right-hand side of (8.2.8) can be estimated as follows:

$$
\begin{aligned}
&(X_i^2 Z_{m-1}u, \Lambda^{2\epsilon-2} Z_m u)\\
&= -(X_i Z_{m-1}u, X_i Q^{2\epsilon-1}u) + O(\| Pu \|^2 + \| Z_{m-1}u \|_{2\epsilon-1}^2 + \| u \|^2)\\
&= -(X_i Q^{2\epsilon-1} Z_{m-1}u, X_i u) + O(\| Pu \|^2 + \| Z_{m-1}u \|_{2\epsilon-1}^2 + \| u \|^2).
\end{aligned}
$$

It follows from Lemma 8.2.4 that

$$
\begin{aligned}
&|(X_i^2 Z_{m-1}u, \Lambda^{2\epsilon-2} Z_m u)|\\
&\quad \le \| X_i Q^{2\epsilon-1} Z_{m-1}u \|^2 + O(\| Pu \|^2 + \| Z_{m-1}u \|_{2\epsilon-1}^2 + \| u \|^2)\\
&\quad \le C|(PQ^{2\epsilon-1} Z_{m-1}u, Q^{2\epsilon-1} Z_{m-1}u)|\\
&\qquad + O(\| Pu \|^2 + \| Z_{m-1}u \|_{2\epsilon-1}^2 + \| u \|^2)\\
&\quad \le C(|(Pu, Q^{4\epsilon-1} Z_{m-1}u)| + \sum_{i=1}^k |(X_i u, Q^{4\epsilon-1} Z_{m-1}u)|\\
&\qquad + |(u, Q^{4\epsilon-1} Z_{m-1}u)|) + O(\| Pu \|^2 + \| Z_{m-1}u \|_{2\epsilon-1}^2 + \| u \|^2)\\
&\quad \le C(\| Pu \|^2 + \| Z_{m-1}u \|_{4\epsilon-1}^2 + \| u \|^2).
\end{aligned} \tag{8.2.9}
$$

This completes the estimate of the first term on the right-hand side of (8.2.6).

For the second term on the right-hand side of (8.2.6), we write

$$
X_0 = P - \sum_{i=1}^k X_i^2 - b(x).
$$

Thus, we have

$$
\begin{aligned}
&(Z_{m-1}X_0u, \Lambda^{2\epsilon-2}Z_mu) \\
&= (Z_{m-1}Pu, \Lambda^{2\epsilon-2}Z_mu) - \sum_{i=1}^{k} (Z_{m-1}X_i^2u, \Lambda^{2\epsilon-2}Z_mu) \\
&\quad - (Z_{m-1}bu, \Lambda^{2\epsilon-2}Z_mu).
\end{aligned}
\tag{8.2.10}
$$

The first term on the right-hand side of (8.2.10) can be written as

$$
\begin{aligned}
&(Z_{m-1}Pu, \Lambda^{2\epsilon-2}Z_mu) \\
&= -(Pu, Z_{m-1}Q^{2\epsilon-1}u) + (Pu, g_mQ^{2\epsilon-1}u) \\
&= -(Pu, Q^{2\epsilon-1}Z_{m-1}u) - (Pu, [Z_{m-1}, Q^{2\epsilon-1}]u) + (Pu, g_mQ^{2\epsilon-1}u),
\end{aligned}
$$

for some $g_m \in C^\infty(\overline{\Omega})$. Also

$$
(Z_{m-1}bu, \Lambda^{2\epsilon-2}Z_mu) = (bZ_{m-1}u, Q^{2\epsilon-1}u) + ([Z_{m-1}, b]u, Q^{2\epsilon-1}u).
$$

Hence, if $\epsilon \le 1/2$, the first and third terms of (8.2.10) can be estimated by the right-hand side of (8.2.7).

To deal with the second term on the right-hand side of (8.2.10), we have

$$
\begin{aligned}
&(Z_{m-1}X_i^2u, \Lambda^{2\epsilon-2}Z_mu) \\
&\quad = (X_i^2Z_{m-1}u, Q^{2\epsilon-1}u) + ([Z_{m-1}, X_i^2]u, Q^{2\epsilon-1}u) \\
&\quad = (X_i^2Z_{m-1}u, Q^{2\epsilon-1}u) + ([Z_{m-1}, X_i]X_iu, Q^{2\epsilon-1}u) \\
&\qquad + (X_i[Z_{m-1}, X_i]u, Q^{2\epsilon-1}u) \\
&\quad = (X_i^2Z_{m-1}u, Q^{2\epsilon-1}u) + (\widehat{Z}_mX_iu, Q^{2\epsilon-1}u) + (X_i\widehat{Z}_mu, Q^{2\epsilon-1}u).
\end{aligned}
$$

Note that $\widehat{Z}_m$ is a commutator of Z_{m-1} with X_i for some $1 \le i \le k$. Thus, one may apply the proof of Case (i) and (8.2.9) to get

$$
\begin{aligned}
&|(Z_{m-1}X_i^2u, \Lambda^{2\epsilon-2}Z_mu)| \\
&\qquad \le C(\| Pu \|^2 + \| Z_{m-1}u \|^2_{4\epsilon-1} + \| u \|^2 + \| \widehat{Z}_mu \|^2_{2\epsilon-1}) \\
&\qquad \le C(\| Pu \|^2 + \| Z_{m-1}u \|^2_{4\epsilon-1} + parallelu \|^2).
\end{aligned}
$$

This completes the estimate of the second term in (8.2.6).

Consequently, by induction, we obtain

$$
\begin{aligned}
\| Z_mu \|^2_{\epsilon-1} &\le C(\| Pu \|^2 + \| Z_{m-1}u \|^2_{4\epsilon-1} + \| u \|^2) \\
&\le C(\| Pu \|^2 + \sum_{i=0}^{k} \| X_iu \|^2_{4^{m-1}\epsilon-1} + \| u \|^2).
\end{aligned}
$$

Thus, if we take $\epsilon \le 2 \cdot 4^{-m}$, we see that

$$
\| Z_mu \|^2_{\epsilon-1} \le C(\| Pu \|^2 + \| X_0u \|^2_{-\frac{1}{2}} + \| u \|^2).
$$

Since

$$\begin{aligned}\| X_0u \|^2_{-\frac{1}{2}} &= (X_0u, \Lambda^{-1}X_0u)\\ &= (Pu, Q^0u) - \sum_{i=1}^{k} (X_i^2u, Q^0u) - (bu, Q^0u)\\ &\le C(\| Pu \|^2 + \| u \|^2),\end{aligned}$$

the proof of Theorem 8.2.6 is now complete.

The next result shows that estimate (8.2.5) is localizable:

Theorem 8.2.7. *Let ζ, $\zeta_1 \in C_0^\infty(\Omega)$ be two real-valued cut-off functions with $\zeta_1 \equiv 1$ on the support of ζ. Then there is a constant $C > 0$ such that*

$$\| \zeta u \|_\epsilon \le C(\| \zeta_1 Pu \| + \| \zeta_1 u \|), \tag{8.2.11}$$

for all $u \in C^\infty(\Omega)$.

Proof. From (8.2.5) it suffices to estimate $\| [P, \zeta]u \|$ by the right-hand side of (8.2.11). Since

$$[P, \zeta]u = 2\sum_{i=1}^{k} [X_i, \zeta]X_iu + \sum_{i=1}^{k} [X_i, [X_i, \zeta]]u + [X_0, \zeta]u,$$

we have

$$\| [P, \zeta]u \|^2 \le C\left(\sum_{i=1}^{k} \| X_i\zeta_1^2u \|^2 + \| \zeta_1u \|^2\right).$$

Now, as in the proof of Lemma 8.2.4, we get

$$\begin{aligned}\sum_{i=1}^{k} \| X_i\zeta_1^2u \|^2 &= -(P\zeta_1^2u, \zeta_1^2u) + \mathrm{Re}(X_0\zeta_1^2u, \zeta_1^2u)\\ &\quad + O\left(\sum_{i=1}^{k} \| X_i\zeta_1^2u \|\| \zeta_1u \| + \| \zeta_1u \|^2\right)\\ &= -(\zeta_1^2Pu, \zeta_1^2u) - 2\sum_{i=1}^{k} ([X_i, \zeta_1^2]X_iu, \zeta_1^2u) + \mathrm{Re}(X_0\zeta_1^2u, \zeta_1^2u)\\ &\quad + O\left(\sum_{i=1}^{k} \| X_i\zeta_1^2u \|\| \zeta_1u \| + \| \zeta_1u \|^2\right)\\ &= -(\zeta_1^2Pu, \zeta_1^2u) + 4\sum_{i=1}^{k} ([X_i, \zeta_1]\zeta_1u, X_i\zeta_1^2u) + \mathrm{Re}(X_0\zeta_1^2u, \zeta_1^2u)\\ &\quad + O\left(\sum_{i=1}^{k} \| X_i\zeta_1^2u \|\| \zeta_1u \| + \| \zeta_1u \|^2\right).\end{aligned}$$

Hence, using small and large constants, we obtain

$$\sum_{i=1}^{k} \| X_i\zeta_1^2 u \|^2 \leq C(\| \zeta_1 Pu \|^2 + \| \zeta_1 u \|^2).$$

This proves the theorem.

The next step is to iterate the estimate (8.2.5) to obtain the following “bootstrap” *a priori* estimate:

Theorem 8.2.8. *For any $s \in \mathbb{R}$ and $m > 0$, there exists a constant $C_{s,m}$ such that*

$$\| u \|_{s+\epsilon} \leq C_{s,m}(\| Pu \|_s + \| u \|_{-m}), \tag{8.2.12}$$

for all $u \in C_0^\infty(\Omega)$.

Proof. Let $u \in C_0^\infty(\Omega)$. We wish to apply (8.2.5) to $\Lambda^s u$. Hence, we shall assume that estimate (8.2.5) holds for smooth functions that are supported in a fixed open neighborhood U of $\overline{\Omega}$. Let $\eta(x)$ and $g(x)$ be two smooth cut-off functions supported in U such that $\eta \equiv 1$ on the support of g and that $g \equiv 1$ on $\overline{\Omega}$. We claim that the operator $(1-\eta)\Lambda^s g$ acting on $\mathcal{S}$, $s \in \mathbb{R}$, is of negative infinite order, namely, it is a smoothing operator and for any $m > 0$ there is a constant $C_{s,m}$ such that

$$\| (1-\eta)\Lambda^s g u \| \leq C_{s,m} \| u \|_{-m}, \tag{8.2.13}$$

for any $u \in \mathcal{S}$. We first prove the claim (8.2.13). Observe first that $(1-\eta)g \equiv 0$. Hence, we have

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (1-\eta(x))e^{ix\cdot\xi}p(\xi)D_\xi^\alpha \hat{g}(\xi)\ d\xi = 0, \tag{8.2.14}$$

for any polynomial $p(\xi)$ and any multiindex α. A direct calculation shows

$$(1-\eta)\Lambda^s g u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} K(x,\xi)\hat{u}(\xi)\ d\xi,$$

where

$$K(x,\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (1-\eta(x))e^{ix\cdot\zeta}(1+|\zeta|^2)^{\frac{s}{2}}\hat{g}(\zeta-\xi)\ d\zeta.$$

Thus, to prove the claim it suffices to show that for any $a, b \in \mathbb{R}$, the kernel

$$\widehat{K}(x,\xi) = \int_{\mathbb{R}^n} (1-\eta(x))e^{ix\cdot\zeta}(1+|\zeta|^2)^a(1+|\xi|^2)^b\hat{g}(\zeta-\xi)\ d\zeta \tag{8.2.15}$$

satisfies the hypotheses of Theorem B.10 in the Appendix. Using (8.2.14), Lemma 8.1.2 and Taylor's expansion of $(1+|\zeta|^2)^a$ at ξ, for any $c \in \mathbb{R}$, we obtain

$$\begin{aligned}
&|\widehat{K}(x,\xi)| \\
&\lesssim \int_{\mathbb{R}^n} (1+|\xi+t(\zeta-\xi)|^2)^{a-c}|\zeta-\xi|^c(1+|\xi|^2)^b|\hat{g}(\zeta-\xi)|\ d\zeta \\
&\lesssim \int_{\mathbb{R}^n} (1+|t(\zeta-\xi)|^2)^{|a-c|}(1+|\xi|^2)^{a-c}|\zeta-\xi|^c(1+|\xi|^2)^b|\hat{g}(\zeta-\xi)|\ d\zeta,
\end{aligned}$$

where $0 < t < 1$. Thus, if we choose c to be large enough so that $a + b - c \le 0$, it is easily verified by integration by parts that the kernel $\widehat{K}(x, \xi)$ defined in (8.2.15) satisfies

$$|x^\delta \widehat{K}(x, \xi)| \le C_{a,b,\delta}$$

uniformly in x and ξ, where δ is a multiindex. It follows that all the hypotheses of Theorem B.10 are satisfied by the kernel $\widehat{K}(x, \xi)$. This proves the claim.

Now we return to the estimate of $\Lambda^s u$ for $u \in C_0^\infty(\Omega)$. Since $\eta\Lambda^s u$ is supported in U, we may apply estimate (8.2.5) to get

$$\begin{aligned}
\| u \|_{s+\epsilon} &= \| \Lambda^s u \|_\epsilon \\
&\le \| \eta\Lambda^s u \|_\epsilon + \| (1-\eta)\Lambda^s g u \|_\epsilon \\
&\le C(\| P\eta\Lambda^s u \| + \| \eta\Lambda^s u \| + \| u \|_{-m}) \\
&\le C(\| [P, \eta\Lambda^s]u \| + \| Pu \|_s + \| u \|_s + \| u \|_{-m}).
\end{aligned}$$

To handle the term $\| [P, \eta\Lambda^s]u \|$ we write

$$[P, \eta\Lambda^s] = \sum_{i=1}^k Q_i^s X_i + Q_{k+1}^s.$$

Thus,

$$\| [P, \eta\Lambda^s]u \| \le C\left(\sum_{i=1}^k \| X_i u \|_s + \| u \|_s\right).$$

Lemma 8.2.4 then shows that

$$\begin{aligned}
\| X_i u \|_s^2 &= \| \Lambda^s X_i u \|^2 \\
&\le C(\| \eta\Lambda^s X_i u \|^2 + \| (1-\eta)\Lambda^s g X_i u \|^2) \\
&\le C(\| X_i \eta\Lambda^s u \|^2 + \| [\eta\Lambda^s, X_i]u \|^2 + \| u \|_{-m}^2) \\
&\le C(|(P\eta\Lambda^s u, \eta\Lambda^s u)| + \| u \|_s^2 + \| u \|_{-m}^2) \\
&\le C(|([P, \eta\Lambda^s]u, \eta\Lambda^s u)| + \| Pu \|_s^2 + \| u \|_s^2 + \| u \|_{-m}^2) \\
&\le C(\| u \|_s \left(\sum_{i=1}^k \| X_i u \|_s) + \| Pu \|_s^2 + \| u \|_s^2 + \| u \|_{-m}^2\right).
\end{aligned}$$

Using small and large constants, we obtain

$$\| u \|_{s+\epsilon} \le C(\| Pu \|_s + \| u \|_s + \| u \|_{-m}).$$

Finally, observe that for any $\delta > 0$, using the interpolation inequality for Sobolev spaces (Theorem B.2 in the Appendix), there is a constant $C_{s,\delta,m}$ such that

$$\| u \|_s \le \delta \| u \|_{s+\epsilon} + C_{s,\delta,m} \| u \|_{-m},$$

for all $u \in C_0^\infty(\Omega)$. Therefore, letting δ be sufficiently small, Theorem 8.2.8 is proved.

Estimate (8.2.12) can be localized as before to get the following theorem.

Theorem 8.2.9. *Let ζ, ζ_1 be two smooth real-valued cut-off functions supported in Ω with $\zeta_1 \equiv 1$ on the support of ζ. For any $s \in \mathbb{R}$ and $m > 0$ there is a constant $C_{s,m}$ such that*

$$\| \zeta u \|_{s+\epsilon} \leq C_{s,m}(\| \zeta_1 Pu \|_s + \| \zeta_1 u \|_{-m}), \tag{8.2.16}$$

for all $u \in C^\infty(\Omega)$.

We are now ready to prove the main result of this section.

Proof of Theorem 8.2.3. Suppose that u is a distribution with $Pu = f$, where P given by (8.2.1) is of finite type, and that f is smooth on Ω. We wish to show that u is also smooth on Ω. Without loss of generality, we shall show that u is smooth in some open neighborhood V of the origin.

Since u is a distribution, we may assume that locally near the origin u is in W^s for some $s = -m$ with $m > 0$. Let ζ and ζ_k, $k \in \mathbb{N}$, be a sequence of smooth real-valued cut-off functions supported in some open neighborhood of $\overline{V}$ such that $\zeta_1 u \in W^{-m}$, $\zeta_k \equiv 1$ on supp ζ_{k+1}, $\zeta_j \equiv 1$ on supp ζ for all j and $\zeta \equiv 1$ on V.

Let φ be a smooth nonnegative real-valued function supported in the unit open ball of $\mathbb{R}^n$ such that $\varphi(x) = \varphi(|x|)$, $0 \leq \varphi \leq 1$ and $\int_{\mathbb{R}^n} \varphi dx = 1$. For any $\delta > 0$, set $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$ and

$$S_\delta \zeta_k u(x) = \zeta_k u * \varphi_\delta(x) = \int \zeta_k u(y)\varphi_\delta(x-y)\; dy.$$

Clearly, $S_\delta\zeta_k u$ is a smooth function supported in some neighborhood of $\overline{V}$. Hence, we obtain from (8.2.16) with $s = -m$ that

$$\| S_\delta\zeta_k u \|_{-m+\epsilon} \leq C_m(\| PS_\delta\zeta_k u \|_{-m} + \| S_\delta\zeta_k u \|_{-m}).$$

To finish the proof we need the following two key observations. For any $s \in \mathbb{R}$,

(1) $\zeta_k u \in W^s$ with $\| \zeta_k u \|_s \leq C_k$ if and only if $\| S_\delta\zeta_k u \|_s \leq C_k$ for all small $\delta > 0$.

(2) If $\zeta_k u$ is in W^s, then

$$\| [S_\delta\zeta_k, X_i]u \|_s = \| [S_\delta\zeta_k, X_i]\zeta_{k-1} u \|_s \leq C_k \| \zeta_{k-1} u \|_s,$$

where the constant C_k is independent of δ.

These two facts can be verified directly, so we omit the proofs.

Now, using (1) and (2), by commuting P with $S_\delta\zeta$ and applying arguments similar to those above, we obtain

$$\| S_\delta\zeta_2 u \|_{-m+\epsilon} \leq C_{1,m}(\| S_\delta\zeta_1 f \|_{-m} + \| S_\delta\zeta_1 u \|_{-m})$$

with $C_{1,m}$ independent of δ. This implies $\zeta u \in W^{-m+\epsilon}$. Inductively, for any $k \in \mathbb{N}$, we obtain

$$\| S_\delta\zeta_{k+1} u \|_{-m+k\epsilon} \leq C_{k,m}(\| S_\delta\zeta_k f \|_{-m+(k-1)\epsilon} + \| S_\delta\zeta_k u \|_{-m+(k-1)\epsilon}).$$

Choosing k to be sufficiently large, we have $\zeta u \in W^s(V)$ for all $s \geq 0$. Hence, $u \in C^\infty(V)$. This completes the proof of Theorem 8.2.3.

8.3 Subelliptic Estimates for the Tangential Cauchy-Riemann Complex

Let $(M, T^{1,0}(M))$ be a compact orientable CR manifold of real dimension $2n-1$ with $n \geq 3$. Let $\Lambda^{p,q}(M)$, $0 \leq p,\ q \leq n-1$, denote the subbundle of $\Lambda^{p+q}\mathbb{C}T^*(M)$ such that $\Lambda^{p,q}(M) = \Lambda^p T^{*1,0}(M) \otimes \Lambda^q T^{*0,1}(M)$. Let $\mathcal{E}^{p,q}(M)$ be the space of smooth sections of $\Lambda^{p,q}(M)$ over M. Then, we define the tangential Cauchy-Riemann operator $\overline{\partial}_b$ as in Section 7.2, and form the tangential Cauchy-Riemann complex

$$0 \to \mathcal{E}^{p,0}(M) \xrightarrow{\overline{\partial}_b} \mathcal{E}^{p,1}(M) \xrightarrow{\overline{\partial}_b} \cdots \xrightarrow{\overline{\partial}_b} \mathcal{E}^{p,n-1}(M) \to 0. \tag{8.3.1}$$

For any $x_0 \in M$, let $L_1, \cdots, L_{n-1}$ be a local basis for (1,0) vector fields near x_0, and choose a globally defined vector field T which may be assumed to be purely imaginary. This is first done locally, then by restriction to those coordinate transformations which preserve $T^{1,0}(M)$ and orientation, a local choice of sign in the direction of T will extend T to a global one. Fix a Hermitian metric on $\mathbb{C}T(M)$ so that $T^{1,0}(M)$, $T^{0,1}(M)$ and T are mutually orthogonal. We may then assume that $L_1, \cdots, L_{n-1}, \overline{L}_1, \cdots, \overline{L}_{n-1}$ and T is an orthonormal basis in some neighborhood of a reference point $x_0 \in M$. Let $\omega^1, \cdots, \omega^{n-1}$ be an orthonormal basis for (1,0)-forms which is dual to the basis $L_1, \cdots, L_{n-1}$.

Denote by $W^s_{(p,q)}(M)$ the Sobolev space of order s, $s \in \mathbb{R}$, for (p,q)-forms on M. Extend $\overline{\partial}_b$ to $L^2_{(p,q)}(M) = W^0_{(p,q)}(M)$ in the sense of distribution. Thus, the domain of $\overline{\partial}_b$, denoted by $\mathrm{Dom}(\overline{\partial}_b)$, will consist of all $\phi \in L^2_{(p,q)}(M)$ such that $\overline{\partial}_b \phi \in L^2_{(p,q+1)}(M)$, and we have the complex

$$0 \to L^2_{(p,0)}(M) \xrightarrow{\overline{\partial}_b} L^2_{(p,1)}(M) \xrightarrow{\overline{\partial}_b} \cdots \xrightarrow{\overline{\partial}_b} L^2_{(p,n-1)}(M) \to 0. \tag{8.3.2}$$

Therefore, $\overline{\partial}_b$ is a linear, closed, densely defined operator on the Hilbert space $L^2_{(p,q)}(M)$.

Now one can define the adjoint operator $\overline{\partial}_b^*$ of $\overline{\partial}_b$ in the standard way. A (p,q)-form ϕ is in $\mathrm{Dom}(\overline{\partial}_b^*)$ if there exists a $(p,q-1)$-form $g \in L^2_{(p,q-1)}(M)$ such that $(\phi, \overline{\partial}_b \psi) = (g, \psi)$ for every $(p,q-1)$-form $\psi \in \mathrm{Dom}(\overline{\partial}_b)$. In this case we define $\overline{\partial}_b^* \phi = g$. Let

$$\Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b, \tag{8.3.3}$$

be the $\overline{\partial}_b$-Laplacian defined on

$$\begin{aligned} \mathrm{Dom}(\Box_b) = \{\alpha \in L^2_{(p,q)}(M) |\ & \alpha \in \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}_b^*), \\ & \overline{\partial}_b \alpha \in \mathrm{Dom}(\overline{\partial}_b^*) \text{ and } \overline{\partial}_b^* \alpha \in \mathrm{Dom}(\overline{\partial}_b)\}. \end{aligned}$$

It follows from the same arguments as in Proposition 4.2.3 that $\Box_b$ is a linear, closed, densely defined self-adjoint operator from $L^2_{(p,q)}(M)$ into itself. Define a Hermitian form Q_b on $\mathcal{E}^{p,q}(M)$ by

$$Q_b(\phi,\psi) = (\overline{\partial}_b\phi, \overline{\partial}_b\psi) + (\overline{\partial}_b^*\phi, \overline{\partial}_b^*\psi) + (\phi,\psi) = ((\Box_b + I)\phi, \psi), \tag{8.3.4}$$

for $\phi,\psi \in \mathcal{E}^{p,q}(M)$.

Locally on a coordinate neighborhood U, we can express a smooth (p,q)-form ϕ as

$$\phi = \sum_{|I|=p,|J|=q}{}' \phi_{I,J}\ \omega^I \wedge \overline{\omega}^J, \tag{8.3.5}$$

where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$ are multiindices, $w^I = w^{i_1} \wedge \cdots \wedge w^{i_p}$, $\bar{w}^J = \bar{w}^{j_1} \wedge \cdots \wedge \bar{w}^{j_q}$ and the prime means that we sum over only increasing multiindices. Here $\phi_{I,J}$'s are defined for arbitrary I and J so that they are antisymmetric. Then, a direct computation and integration by parts yield

$$\overline{\partial}_b\phi = \sum_{I,J}{}' \sum_j\ \overline{L}_j(\phi_{I,J})\ \overline{\omega}^j \wedge \omega^I \wedge \overline{\omega}^K + \text{terms of order zero}, \tag{8.3.6}$$

and

$$\overline{\partial}_b^*\phi = (-1)^{p-1} \sum_{I,K}{}' \sum_j\ L_j(\phi_{I,jK})\ \omega^I \wedge \overline{\omega}^K + \text{terms of order zero}. \tag{8.3.7}$$

We shall use (8.3.6) and (8.3.7) to obtain the desired estimates. We also abbreviate $\sum_{k,I,J} \| L_k\phi_{I,J} \|^2 + \| \phi \|^2$ by $\| \phi \|_L^2$, and $\sum_{k,I,J} \| \overline{L}_k\phi_{I,J} \|^2 + \| \phi \|^2$ by $\| \phi \|_{\overline{L}}^2$.

The main effort of this section is to derive the subelliptic 1/2-estimate of the form Q_b. Before proceeding to do so, we shall digress for the moment to the regularity theorem for the $\overline{\partial}$ operator. Suppose now that M is the boundary of a smooth bounded strongly pseudoconvex domain D in $\mathbb{C}^n, n \geq 2$. In Chapter 5 (Theorem 5.1.2 and Theorem 5.3.7), we prove the following subelliptic estimates

$$\| f \|^2_{1/2(D)} \leq C(\| \overline{\partial} f \|^2 + \| \overline{\partial}^* f \|^2 + \| f \|^2),$$

for $f \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$, on a smooth bounded strongly pseudoconvex domain D. The proof is based on the *a priori* estimate

$$\int_{bD} |f|^2 dS \leq C(\| \overline{\partial} f \|^2 + \| \overline{\partial}^* f \|^2 + \| f \|^2) = CQ(f,f), \tag{8.3.8}$$

for $f \in \mathcal{D}^1_{(p,q)} = C^1_{(p,q)}(\overline{D}) \cap \mathrm{Dom}(\overline{\partial}^*)$.

In fact, to prove (8.3.8) for a fixed q, $1 \leq q \leq n-1$, one actually does not need strong pseudoconvexity of the domain. The main ingredient is the so-called condition $Z(q)$ defined as follows:

Definition 8.3.1. *Let D be a relatively compact subset with C^∞ boundary in a complex Hermitian manifold of complex dimension $n \geq 2$. D is said to satisfy condition $Z(q)$, $1 \leq q \leq n-1$, if the Levi form associated with D has at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues at every boundary point.*

Obviously, condition $Z(q)$ is satisfied for all q with $1 \leq q \leq n-1$ on any strongly pseudoconvex domain.

Let $x_0 \in M$ be a boundary point and let U be an open neighborhood of x_0. For any $f \in \mathcal{D}_{(p,q)}$ with support in U, the proof of Proposition 5.3.3 shows (with $\phi \equiv 0$) that

$$\begin{aligned} Q(f,f) = {\sum_{I,J}}' \sum_k \| \overline{L}_k f_{I,J} \|^2 + {\sum_{I,K}}' \sum_{j,k} \int_{bD\cap U} \rho_{jk} f_{I,jK} \overline{f}_{I,kK} \, dS \\ + O((\| \overline{\partial} f \| + \| \overline{\partial}^* f \|) \| f \| + \| f \|_{\overline{L}} \| f \|). \end{aligned}$$

We may assume that the Levi form is diagonal at x_0, namely, $\rho_{jk}(x) = \lambda_j \delta_{jk} + b_{jk}(x)$ for $1 \leq j,\ k \leq n-1$, where the λ_j's are the eigenvalues of the Levi form at x_0, δ_{jk} denotes the Kronecker delta and $b_{jk}(x_0) = 0$. It follows that

$$\begin{aligned} &{\sum_{I,K}}' \sum_{j,k} \int_{bD\cap U} \rho_{jk} f_{I,jK} \overline{f}_{I,kK} \, dS \\ &\quad = {\sum_{I,J}}' \left(\sum_{k\in J} \lambda_k \right) \int_{bD} |f_{I,J}|^2 \, dS + \delta O\left({\sum_{I,J}}' \int_{bD} |f_{I,J}|^2 \, dS \right), \end{aligned}$$

where $\delta > 0$ can be made arbitrarily small if U is chosen sufficiently small. Integration by parts also shows

$$\begin{aligned} &\| \overline{L}_k f_{I,J} \|^2 \\ &= -([L_k, \overline{L}_k] f_{I,J}, f_{I,J}) + \| L_k f_{I,J} \|^2 + O(\| f \|_{\overline{L}} \| f \|) \\ &\geq -\lambda_k \int_{bD} |f_{I,J}|^2 \, dS - \delta \int_{bD} |f_{I,J}|^2 \, dS + O(\| f \|_{\overline{L}} \| f \| + \| f \|^2). \end{aligned}$$

Hence, if condition $Z(q)$ holds on bD, then for each fixed J either there is a $k_1 \in J$ with $\lambda_{k_1} > 0$ or there is a $k_2 \notin J$ with $\lambda_{k_2} < 0$. For the former case and any $\epsilon > 0$, we have

$$\begin{aligned} Q(f,f) \geq \epsilon {\sum_{I,J}}' \sum_k \| \overline{L}_k f_{I,J} \|^2 + \epsilon {\sum_{I,J}}' \left(\sum_{k\in J, \lambda_k < 0} \lambda_k \right) \int_{bD} |f_{I,J}|^2 \, dS \\ + {\sum_{I,J}}' \left(\lambda_{k_1} \int_{bD} |f_{I,J}|^2 \, dS \right) - \delta \int_{bD} |f_{I,J}|^2 \, dS \\ + O((\| \overline{\partial} f \| + \| \overline{\partial}^* f \|) \| f \| + \| f \|_{\overline{L}} \| f \| + \| f \|^2). \end{aligned}$$

For the latter case, we see that

$$\begin{aligned} Q(f,f) \geq \epsilon {\sum_{I,J}}' \sum_k \| \overline{L}_k f_{I,J} \|^2 + \epsilon {\sum_{I,J}}' \left(\sum_{k\in J, \lambda_k < 0} \lambda_k \right) \int_{bD} |f_{I,J}|^2 \, dS \\ + (1-\epsilon) {\sum_{I,J}}' (-\lambda_{k_2}) \int_{bD} |f_{I,J}|^2 \, dS - \delta \int_{bD} |f_{I,J}|^2 \, dS \\ + O((\| \overline{\partial} f \| + \| \overline{\partial}^* f \|) \| f \| + \| f \|_{\overline{L}} \| f \| + \| f \|^2). \end{aligned}$$

Thus, choosing ϵ, δ to be small enough and using small and large constants, we obtain (8.3.8). Now, by a partition of unity argument, the next theorem follows immediately from Theorem 5.1.2.

Theorem 8.3.2. *Let D be a relatively compact subset with C^∞ boundary in a complex Hermitian manifold of complex dimension $n \geq 2$. Suppose that condition $Z(q)$ holds for some $q, 1 \leq q \leq n-1$. Then we have*

$$\int_{bD} |f|^2 dS \leq C(\| \overline{\partial} f \|^2 + \| \overline{\partial}^* f \|^2 + \| f \|^2),$$

for $f \in \mathcal{D}_{(p,q)}$. Furthermore, we have

$$\| f \|_{\frac{1}{2}} \leq C(\| \overline{\partial} f \| + \| \overline{\partial}^* f \| + \| f \|), \tag{8.3.9}$$

for $f \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*)$, where the constant $C > 0$ is independent of f.

With Theorem 8.3.2 on hand, we now return to the subelliptic estimate for $\Box_b$ on (p, q)-forms on M. If the CR manifold M is embedded as the boundary of a complex manifold D, topologically one can not distinguish whether M is the boundary of D or M is the boundary of the complement of D. Thus, in order to obtain a similar subelliptic estimate for (p, q)-forms on M, we shall assume that condition $Z(q)$ holds on both D and its complement D^c as motivated by Theorem 8.3.2. Note that condition $Z(q)$ on D^c is equivalent to condition $Z(n-q-1)$ on D. M is said to satisfy condition $Y(q)$, $1 \leq q \leq n-1$, if both conditions $Z(q)$ and $Z(n-q-1)$ are satisfied on D.

In terms of the eigenvalues of the Levi form condition $Y(q)$ means that the Levi form has at least either $\max(n-q, q+1)$ eigenvalues of the same sign or $\min(n-q, q+1)$ pairs of eigenvalues of opposite signs at every point on M. Since condition $Y(q)$ will be used extensively in what follows, we now make a formal definition for any CR manifold.

Definition 8.3.3. *Let M be an oriented CR manifold of real dimension $2n-1$ with $n \geq 2$. M is said to satisfy condition $Y(q)$, $1 \leq q \leq n-1$, if the Levi form has at least either max$(n-q, q+1)$ eigenvalues of the same sign or min$(n-q, q+1)$ pairs of eigenvalues of opposite signs at every point on M.*

It follows that condition $Y(q)$ for $1 \leq q \leq n-2$ holds on any strongly pseudoconvex CR manifold M of real dimension $2n-1$ with $n \geq 3$. Also, it should be pointed out that condition $Y(n-1)$ is violated on any pseudoconvex CR manifold M of real dimension $2n-1$ with $n \geq 2$. In particular, condition $Y(1)$ is not satisfied on any strongly pseudoconvex CR manifold of real dimension three. This phenomenon is related to the nonsolvable Lewy operator which we have discussed in Section 7.3. Another example of a noncompact CR manifold satisfying condition $Y(q)$ will be given in Section 10.1.

Theorem 8.3.4. *Suppose that condition $Y(q)$, for some q with $1 \leq q \leq n-1$, holds on a compact, oriented, CR manifold $(M, T^{1,0}(M))$ of real dimension $2n-1$ with $n \geq 3$. Then we have*

$$\| \phi \|^2_{\frac{1}{2}} \lesssim Q_b(\phi, \phi), \tag{8.3.10}$$

uniformly for all $\phi \in \mathcal{E}^{p,q}(M)$.

Proof. Let $L_1, \cdots, L_{n-1}$ be an orthonormal basis for $T^{1,0}(M)$ locally. Since condition $Y(q)$ implies that the vector fields $L_1, \cdots, L_{n-1}, \overline{L}_1, \cdots, \overline{L}_{n-1}$ and their Lie brackets span the whole complex tangent space, using a partition of unity, the proof is an easy consequence of Theorem 8.2.5 when $m = 2$, and the following theorem:

Theorem 8.3.5. *Under the same hypotheses as in Theorem 8.3.4, for any* $x_0 \in M$*, there is a neighborhood* V_{x_0} *of* x_0 *such that*

$$\| \phi \|_L^2 + \| \phi \|_{\overline{L}}^2 + \sum_{IJ} |Re(T\phi_{I,J}, \phi_{I,J})| \lesssim Q_b(\phi, \phi), \tag{8.3.11}$$

uniformly for all $\phi \in \mathcal{E}^{p,q}(M)$ *with support contained in* V_{x_0}.

Proof. We start with (8.3.6) to obtain

$$\| \overline{\partial}_b \phi \|^2 = \sum_{I,J}{}' \sum_{j \notin J} \| \overline{L}_j \phi_{I,J} \|^2 + \sum_{I,J,L}{}' \sum_{j,l} \epsilon^{jJ}_{lL} (\overline{L}_j \phi_{I,J}, \overline{L}_l \phi_{I,L}) + O(\| \phi \|_{\overline{L}} \| \phi \|),$$

where $\epsilon^{jJ}_{lL} = 0$ unless $j \notin J$, $l \notin L$ and $\{j\} \cup J = \{l\} \cup L$, in which case ϵ^{jJ}_{lL} is the sign of permutation $\binom{jJ}{lL}$. Using the fact that $\phi_{I,J}$ is antisymmetric in J, we rearrange the terms in the above estimate to get

$$\| \overline{\partial}_b \phi \|^2 = \sum_{I,J}{}' \sum_{j} \| \overline{L}_j \phi_{I,J} \|^2 - \sum_{I,K}{}' \sum_{j,k} (\overline{L}_j \phi_{I,kK}, \overline{L}_k \phi_{I,jK}) + O(\| \phi \|_{\overline{L}} \| \phi \|).$$

Using integration by parts, we have

$$\begin{aligned} (\overline{L}_j \phi_{I,kK}, \overline{L}_k \phi_{I,jK}) &= (-L_k \overline{L}_j \phi_{I,kK}, \phi_{I,jK}) + O(\| \phi \|_{\overline{L}} \| \phi \|) \\ &= (L_k \phi_{I,kK}, L_j \phi_{I,jK}) + ([\overline{L}_j, L_k] \phi_{I,kK}, \phi_{I,jK}) \\ &\quad + O((\| \phi \|_L + \| \phi \|_{\overline{L}}) \| \phi \|). \end{aligned}$$

Hence, from (8.3.7) we obtain

$$\begin{aligned} \| \overline{\partial}_b \phi \|^2 = \sum_{I,J}{}' \sum_{j} \| \overline{L}_j \phi_{I,J} \|^2 - \| \overline{\partial}_b^* \phi \|^2 + \sum_{I,K}{}' \sum_{j,k} ([L_j, \overline{L}_k] \phi_{I,jK}, \phi_{I,kK}) \\ + O((\| \phi \|_L + \| \phi \|_{\overline{L}}) \| \phi \|). \end{aligned}$$

To handle the commutator term, we may assume that the Levi form is diagonal at x_0 and that $c_{11}(x_0) \neq 0$, since condition $Y(q)$ holds. It follows that $|c_{11}(x)| > 1/C > 0$ for $x \in V_{x_0}$ if V_{x_0} is chosen to be small enough. For any smooth function f with $f(x_0) = 0$ on M, we have

$$\begin{aligned} |\mathrm{Re}(T\phi_{I,J}, f\phi_{I,L})| &\leq \left| \mathrm{Re} \left(\frac{1}{c_{11}} [L_1, \overline{L}_1] \phi_{I,J}, f\phi_{I,L} \right) \right| + O(\| \phi \|_{\overline{L}} \| \phi \|) \\ &\leq C(\sup_{V_{x_0}} |f|)(\| \phi \|_L^2 + \| \phi \|_{\overline{L}}^2) + O(\| \phi \|_{\overline{L}} \| \phi \|). \end{aligned}$$

Thus, if we denote the eigenvalues of the Levi form at x_0 by $\lambda_1, \cdots, \lambda_{n-1}$, we obtain

$$
\begin{aligned}
Q_b(\phi,\phi) = &\sum_{I,J}{}' \sum_{j} \| \overline{L}_j \phi_{I,J} \|^2 + \sum_{I,J}{}' \sum_{j\in J} \lambda_j \mathrm{Re}(T\phi_{I,J}, \phi_{I,J}) \\
&+ \delta O(\| \phi \|_L^2 + \| \phi \|_{\overline{L}}^2) + O(\| \phi \|_{\overline{L}} \| \phi \|),
\end{aligned}
\tag{8.3.12}
$$

where $\delta > 0$ can be made arbitrarily small, if necessary, by shrinking V_{x_0}.

To finish the proof, we need to control the second term on the right-hand side of (8.3.12). This will be done by using a fraction of the first term on the right-hand side of (8.3.12). Thus, we first use integration by parts to get

$$
\begin{aligned}
\| \overline{L}_j \phi_{I,J} \|^2 = &\| L_j \phi_{I,J} \|^2 - \lambda_j \mathrm{Re}(T\phi_{I,J}, \phi_{I,J}) \\
&+ \delta O(\| \phi \|_L^2 + \| \phi \|_{\overline{L}}^2) + O((\| \phi \|_L + \| \phi \|_{\overline{L}}) \| \phi \|).
\end{aligned}
$$

Next, for each multiindex pair (I, J), set

$$
\sigma(I,J) = \{ j |\ \lambda_j < 0 \text{ if } \mathrm{Re}(T\phi_{I,J}, \phi_{I,J}) > 0 \text{ or } \lambda_j < 0 \text{ if } \mathrm{Re}(T\phi_{I,J}, \phi_{I,J}) < 0 \}.
$$

It follows that, for any small $\epsilon > 0$, we have

$$
\begin{aligned}
\| \phi \|_{\overline{L}}^2 &\geq \epsilon \| \phi \|_{\overline{L}}^2 + (1-\epsilon) \sum_{I,J}{}' \sum_{j\in\sigma(I,J)} \| \overline{L}_j \phi_{I,J} \|^2 \\
&\geq \epsilon \| \phi \|_{\overline{L}}^2 - (1-\epsilon) \sum_{I,J}{}' \sum_{j\in\sigma(I,J)} \lambda_j \mathrm{Re}(T\phi_{I,J}, \phi_{I,J}) \\
&\quad - \delta(\| \phi \|_L^2 + \| \phi \|_{\overline{L}}^2) - C \| \phi \|^2 .
\end{aligned}
$$

Substituting the above into (8.3.12) we obtain

$$
\begin{aligned}
Q_b(\phi,\phi) &\geq \epsilon \| \phi \|_{\overline{L}}^2 - (1-\epsilon) \sum_{I,J}{}' \sum_{j\in\sigma(I,J)} \lambda_j \mathrm{Re}(T\phi_{I,J}, \phi_{I,J}) \\
&\quad + \sum_{I,J}{}' \sum_{j\in J} \lambda_j \mathrm{Re}(T\phi_{I,J}, \phi_{I,J}) - \delta(\| \phi \|_L^2 + \| \phi \|_{\overline{L}}^2) \\
&\quad - O(\| \phi \|_{\overline{L}} \| \phi \|) \\
&= \epsilon \| \phi \|_{\overline{L}}^2 + \sum_{I,J}{}' a_{I,J} \mathrm{Re}(T\phi_{I,J}, \phi_{I,J}) - \delta(\| \phi \|_L^2 + \| \phi \|_{\overline{L}}^2) \\
&\quad - O(\| \phi \|_{\overline{L}} \| \phi \|),
\end{aligned}
\tag{8.3.13}
$$

where

$$
a_{I,J} = \sum_{j\in J\setminus\sigma(I,J)} \lambda_j - (1-\epsilon) \left(\sum_{j\in\sigma(I,J)\setminus J} \lambda_j \right) + \epsilon \left(\sum_{j\in J\cap\sigma(I,J)} \lambda_j \right).
$$

Since condition $Y(q)$ holds at x_0, for each multiindex J with $|J| = q$, one of the following three cases must hold:

(1) If the Levi form has $\max(n-q, q+1)$ eigenvalues of the same sign, then there exists a $j \in J$ and a $k \notin J$ so that λ_j and λ_k are of the same sign which may be assumed to be positive, if necessary, by replacing T by $-T$.

(2) If the Levi form has $\min(n-q, q+1)$ pairs of eigenvalues of opposite signs, then there are $j, k \notin J$ so that $\lambda_j > 0$ and $\lambda_k < 0$.
(3) Under the same hypothesis as in (2), there are $j, k \in J$ so that $\lambda_j > 0$ and $\lambda_k < 0$.

Then it is not hard to verify that by choosing $\epsilon > 0$ to be small enough we can achieve $a_{I,J} > 0$ if $\mathrm{Re}(T\phi_{I,J}, \phi_{I,J}) > 0$, and $a_{I,J} < 0$ if $\mathrm{Re}(T\phi_{I,J}, \phi_{I,J}) < 0$. Since the second term on the right-hand side of (8.3.13) is a finite sum, by letting $\delta > 0$ be sufficiently small, we get

$$Q_b(\phi,\phi) \gtrsim \| \phi \|_L^2 + \sum_{I,J} |\mathrm{Re}(T\phi_{I,J}, \phi_{I,J})| - (sc) \| \phi \|_L^2 - (lc) \| \phi \|^2 .$$

Since

$$\begin{aligned} \| L_j \phi_{I,J} \|^2 \lesssim \| \overline{L}_j \phi_{I,J} \|^2 &+ |\mathrm{Re}(T\phi_{I,J}, \phi_{I,J})| \\ &+ \delta_1 O(\| \phi \|_L^2 + \| \phi \|_{\overline{L}}^2) + O(\| \phi \|_{\overline{L}} \| \phi \|), \end{aligned}$$

where $\delta_1 > 0$ can be made arbitrarily small, by choosing δ_1 and (sc) to be sufficiently small, we obtain

$$\| \phi \|_L^2 + \| \phi \|_{\overline{L}}^2 + \sum_{I,J} |\mathrm{Re}(T\phi_{I,J}, \phi_{I,J})| \lesssim Q_b(\phi,\phi).$$

This completes the proof of the theorem.

Corollary 8.3.6. *Q_b is compact with respect to $L^2_{(p,q)}(M)$.*

Proof. Using Friedrichs' lemma (see Appendix D) and Theorem 8.3.4, we obtain

$$Q_b(\phi,\phi) \geq C \| \phi \|_{\frac{1}{2}}^2, \quad \text{for } \phi \in \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}_b^*).$$

In particular, Q_b is compact with respect to $L^2_{(p,q)}(M)$.

It is easy to see that $(\Box_b + I)^{-1}$ is injective on $L^2_{(p,q)}(M)$. Corollary 8.3.6 implies that $(\Box_b + I)^{-1}$ is compact using Rellich's lemma. We will discuss this in detail in the next section.

The *a priori* estimate obtained in Theorem 8.3.4 is the main ingredient for handling the local regularity problem of the operator $\Box_b$ on compact strongly pseudoconvex CR manifolds of real dimension $2n-1$ with $n \geq 3$. It is Estimate (8.3.10) that enables us to deduce the existence and regularity theorems for the $\overline{\partial}_b$ complex.

8.4 Local Regularity and the Hodge Theorem for $\Box_b$

The main task of this section is to prove the local regularity theorem for the operator $\Box_b$ and its related consequences. This will be done by first proving *a priori* estimates for the operator $\Box_b + I$ on a local coordinate neighborhood U.

Lemma 8.4.1. *Under the same hypotheses as in Theorem 8.3.4, let U be a local coordinate neighborhood, and let $\{\zeta_k\}_{k=1}^{\infty}$ be a sequence of real smooth functions supported in U such that $\zeta_k = 1$ on the support of ζ_{k+1} for all k. Then, for each positive integer k, we have*

$$\| \zeta_k \phi \|^2_{\frac{k}{2}} \lesssim \| \zeta_1 (\Box_b + I)\phi \|^2_{\frac{k-2}{2}} + \| (\Box_b + I)\phi \|^2$$

uniformly for all $\phi \in \mathcal{E}^{p,q}(M)$ supported in U.

Proof. The lemma will be proved by induction. For $k = 1$, by Theorem 8.3.4, we have

$$\| \zeta_1 \phi \|^2_{\frac{1}{2}} \lesssim Q_b(\zeta_1\phi, \zeta_1\phi) = \| \overline{\partial}_b \zeta_1 \phi \|^2 + \| \overline{\partial}_b^* \zeta_1 \phi \|^2 + \| \zeta_1 \phi \|^2 .$$

We estimate the right-hand side as follows:

$$\begin{aligned} \| \overline{\partial}_b \zeta_1 \phi \|^2 &= (\overline{\partial}_b \zeta_1 \phi, \overline{\partial}_b \zeta_1 \phi) \\ &= (\zeta_1 \overline{\partial}_b \phi, \overline{\partial}_b \zeta_1 \phi) + ([\overline{\partial}_b, \zeta_1]\phi, \overline{\partial}_b \zeta_1 \phi) \\ &= (\overline{\partial}_b \phi, \overline{\partial}_b \zeta_1^2 \phi) + (\overline{\partial}_b \phi, [\zeta_1, \overline{\partial}_b]\zeta_1 \phi) + ([\overline{\partial}_b, \zeta_1]\phi, \overline{\partial}_b \zeta_1 \phi) \\ &= (\overline{\partial}_b^* \overline{\partial}_b \phi, \zeta_1^2 \phi) + (\overline{\partial}_b \zeta_1 \phi, [\zeta_1, \overline{\partial}_b]\phi) \\ &\quad + ([\zeta_1, \overline{\partial}_b]\phi, [\zeta_1, \overline{\partial}_b]\phi) + ([\overline{\partial}_b, \zeta_1]\phi, \overline{\partial}_b \zeta_1 \phi). \end{aligned}$$

Note that

$$\operatorname{Re}\big((\overline{\partial}_b \zeta_1 \phi, [\zeta_1, \overline{\partial}_b]\phi) + ([\overline{\partial}_b, \zeta_1]\phi, \overline{\partial}_b \zeta_1 \phi) \big) = 0.$$

A similar argument holds for $\| \overline{\partial}_b^* \zeta_1 \phi \|^2$. Thus, we get

$$\begin{aligned} \| \zeta_1 \phi \|^2_{\frac{1}{2}} &\lesssim Q_b(\zeta_1 \phi, \zeta_1 \phi) \\ &\lesssim \operatorname{Re}((\Box_b + I)\phi, \zeta_1^2 \phi) + O(\| \phi \|^2) \\ &\lesssim \| (\Box_b + I)\phi \| \| \phi \| + O(\| \phi \|^2) \\ &\lesssim \| (\Box_b + I)\phi \|^2, \end{aligned}$$

since $\| \phi \| \lesssim \| (\Box_b + I)\phi \|$. This establishes the initial step.

Let us assume that the assertion is true for all integers up to $k-1$ for some $k > 1$, then we prove it for k. For simplicity we write the standard pseudodifferential operator $\Lambda^{\frac{k-1}{2}} = A_k$ for short, and denote $\zeta_1 A_k \zeta_k$ by P_k. Then, we have

$$\begin{aligned} \| \zeta_k \phi \|^2_{\frac{k}{2}} &\lesssim \| A_k \zeta_k \phi \|^2_{\frac{1}{2}} = \| A_k \zeta_1 \zeta_k \phi \|^2_{\frac{1}{2}} \\ &\lesssim \| \zeta_1 A_k \zeta_k \phi \|^2_{\frac{1}{2}} + \| [A_k, \zeta_1]\zeta_k \zeta_{k-1} \phi \|^2_{\frac{1}{2}} \\ &\lesssim Q_b(P_k \phi, P_k \phi) + \| \zeta_{k-1} \phi \|^2_{\frac{k-2}{2}} . \end{aligned}$$

Let P_k^* be the adjoint operator of P_k, then the first term can be estimated as follows:

$$
\begin{aligned}
&Q_b(P_k\phi, P_k\phi)\\
&= \| \overline{\partial}_b P_k \zeta_{k-1}\phi \|^2 + \| \overline{\partial}_b^* P_k \zeta_{k-1}\phi \|^2 + \| P_k \zeta_{k-1}\phi \|^2\\
&= (\overline{\partial}_b \zeta_{k-1}\phi, P_k^* \overline{\partial}_b P_k \zeta_{k-1}\phi) + (\overline{\partial}_b^* \zeta_{k-1}\phi, P_k^* \overline{\partial}_b^* P_k \zeta_{k-1}\phi) + (\zeta_{k-1}\phi, P_k^* P_k \zeta_{k-1}\phi)\\
&\qquad + O(\| \zeta_{k-1}\phi \|_{\frac{k-1}{2}} (\| \overline{\partial}_b P_k \zeta_{k-1}\phi \| + \| \overline{\partial}_b^* P_k \zeta_{k-1}\phi \|))\\
&= (\overline{\partial}_b \zeta_{k-1}\phi, \overline{\partial}_b P_k^* P_k \zeta_{k-1}\phi) + (\overline{\partial}_b^* \zeta_{k-1}\phi, \overline{\partial}_b^* P_k^* P_k \zeta_{k-1}\phi) + (\zeta_{k-1}\phi, P_k^* P_k \zeta_{k-1}\phi)\\
&\qquad + O(\| \zeta_{k-1}\phi \|^2_{\frac{k-1}{2}} + \| \zeta_{k-1}\phi \|_{\frac{k-1}{2}} (\| \overline{\partial}_b P_k \zeta_{k-1}\phi \| + \| \overline{\partial}_b^* P_k \zeta_{k-1}\phi \|))\\
&= (\overline{\partial}_b \phi, \overline{\partial}_b P_k^* P_k \zeta_{k-1}\phi) + (\overline{\partial}_b^* \phi, \overline{\partial}_b^* P_k^* P_k \zeta_{k-1}\phi) + (\phi, P_k^* P_k \zeta_{k-1}\phi) + O(\cdots)\\
&= ((\square_b + I)\phi, P_k^* P_k \phi) + O(\cdots)\\
&= (P_k \zeta_1 (\square_b + I)\phi, P_k \phi) + O(\cdots)\\
&\lesssim \| P_k \zeta_1 (\square_b + I)\phi \|_{-\frac{1}{2}} \| P_k \phi \|_{\frac{1}{2}} + O(\cdots)\\
&\lesssim (lc) \| \zeta_1 (\square_b + I)\phi \|^2_{\frac{k-2}{2}} + (sc) \| \zeta_k \phi \|^2_{\frac{k}{2}} + (lc) \| \zeta_{k-1}\phi \|^2_{\frac{k-1}{2}}\\
&\qquad + (sc)(\| \overline{\partial}_b P_k \zeta_{k-1}\phi \|^2 + \| \overline{\partial}_b^* P_k \zeta_{k-1}\phi \|^2).
\end{aligned}
$$

Hence, by choosing (sc) small enough and using the induction hypothesis, we obtain

$$
\begin{aligned}
\| \zeta_k \phi \|^2_{\frac{k}{2}} &\lesssim \| \zeta_1 (\square_b + I)\phi \|^2_{\frac{k-2}{2}} + \| \zeta_{k-1}\phi \|^2_{\frac{k-1}{2}}\\
&\lesssim \| \zeta_1 (\square_b + I)\phi \|^2_{\frac{k-2}{2}} + \| \zeta_1 (\square_b + I)\phi \|^2_{\frac{k-3}{2}} + \| (\square_b + I)\phi \|^2\\
&\lesssim \| \zeta_1 (\square_b + I)\phi \|^2_{\frac{k-2}{2}} + \| (\square_b + I)\phi \|^2 .
\end{aligned}
$$

This completes the proof of the lemma.

Theorem 8.4.2. *Under the same hypotheses as in Theorem 8.3.4, given $\alpha \in L^2_{(p,q)}(M)$, let $\phi \in Dom(\square_b)$ be the unique solution of $(\square_b + I)\phi = \alpha$. If U is a subregion of M and $\alpha|_U \in \mathcal{E}^{p,q}(U)$, then $\phi|_U \in \mathcal{E}^{p,q}(U)$. Moreover, if ζ and ζ_1 are two cut-off functions supported in U such that $\zeta_1 = 1$ on the support of ζ, then for each integer $s \geq 0$, there is a constant C_s such that*

$$\| \zeta\phi \|^2_{s+1} \leq C_s (\| \zeta_1 \alpha \|^2_s + \| \alpha \|^2). \tag{8.4.1}$$

Proof. If $\phi|_U$ is smooth, then the estimate (8.4.1) follows immediately from Lemma 8.4.1. Therefore, it remains only to show that $\phi|_U \in \mathcal{E}^{p,q}(U)$. Since the Hermitian form Q_b is not elliptic, namely, Gårding's type inequality does not hold for Q_b, we shall apply the technique of elliptic regularization to Q_b.

The elliptic regularization method has been used in the proof of Theorem 5.2.1 to deduce the regularity of the $\overline{\partial}$-Neumann operator on strongly pseudoconvex domains. Thus, we shall only sketch the idea here and omit the details. Let $\{(U_i, \varphi_i)\}_{i=1}^m$ be an open covering of M formed out of the local coordinate neighborhood systems with local coordinates x_j's on U_i, and let $\{\eta_i\}_{i=1}^m$ be a partition of unity subordinate to $\{U_i\}_{i=1}^m$. Define the form Q_b^ϵ, for each ϵ with $0<\epsilon<<1$, by

$$Q_b^\epsilon(\phi,\psi)=Q_b(\phi,\psi)+\epsilon\sum_{i=1}^m\sum_{j=1}^{2n-1}(D_j\eta_i\phi,D_j\eta_i\psi)$$

for all $\phi,\psi\in\mathcal{E}^{p,q}(M)$, where $D_j=\partial/\partial x_j$. Denote by $\widehat{W}^\epsilon_{(p,q)}(M)$ the completion of $\mathcal{E}^{p,q}(M)$ under Q_b^ϵ.

Let $\phi^\epsilon\in\widehat{W}^\epsilon_{(p,q)}(M)$ be the unique solution to the equation

$$Q_b^\epsilon(\phi^\epsilon,\psi)=(\alpha,\psi),\quad \text{for } \psi\in\widehat{W}^\epsilon_{(p,q)}(M).$$

Then, we have

$$\| \zeta\phi^\epsilon \|_{s+1}^2\lesssim \| \zeta_1\alpha \|_s^2+\| \alpha \|^2 \tag{8.4.2}$$

and the estimate is uniform for all ϵ with $0<\epsilon<<1$. Also, as in the proof of Theorem 5.2.1, $\{\phi^\epsilon\}$ converges to ϕ in $L^2_{(p,q)}(M)$.

The sequence $\{\zeta\phi^\epsilon\}$, by (8.4.2), is uniformly bounded in $W^{s+1}_{(p,q)}(M)$ for each s. Hence, by Rellich's lemma we can extract a subsequence $\{\zeta\phi^{\epsilon_j}\}$ that converges in $W^s_{(p,q)}(M)$ as $\epsilon_j\to 0$. Since $\{\phi^\epsilon\}$ converges to ϕ in $L^2_{(p,q)}(M)$, $\{\zeta\phi^{\epsilon_j}\}$ must converge to $\zeta\phi$ in $W^s_{(p,q)}(M)$ for each s. Finally, by invoking the Sobolev embedding theorem, we have $\zeta\phi\in\mathcal{E}^{p,q}(M)$. This completes the proof of the theorem.

A few consequences now follow immediately from Theorem 8.4.2.

Theorem 8.4.3. *Let α,ϕ,U,ζ and ζ_1 be as in Theorem 8.4.2. If $\alpha|_U\in W^s_{(p,q)}(U)$ for some $s\geq 0$, then $\zeta\phi\in W^{s+1}_{(p,q)}(M)$ and*

$$\| \zeta\phi \|_{s+1}^2\lesssim \| \zeta_1\alpha \|_s^2+\| \alpha \|^2 .$$

Proof. Let ζ_0 be a cut-off function supported in U such that $\zeta_0=1$ on the support of ζ_1. Choose sequences of smooth (p,q)-forms $\{\beta_n\}$ and $\{\gamma_n\}$ with $\operatorname{supp}\{\beta_n\}\subset\operatorname{supp}\{\zeta_0\}$ and $\operatorname{supp}\{\gamma_n\}\subset\operatorname{supp}\{(1-\zeta_0)\}$ such that $\beta_n\to\zeta_0\alpha$ in $W^s_{(p,q)}(M)$ and $\gamma_n\to(1-\zeta_0)\alpha$ in $L^2_{(p,q)}(M)$.

Hence, $\alpha_n=\beta_n+\gamma_n\to\alpha$ in $L^2_{(p,q)}(M)$ and $\zeta_1\alpha_n\to\zeta_1\alpha$ in $W^s_{(p,q)}(M)$. Let $\phi_n\in\operatorname{Dom}(\Box_b)$ be the solution to $(\Box_b+I)\phi_n=\alpha_n$, so $\phi_n\to\phi$ in $L^2_{(p,q)}(M)$. Then, Theorem 8.4.2 shows

$$\| \zeta(\phi_n-\phi_m) \|_{s+1}\lesssim \| \zeta_1(\alpha_n-\alpha_m) \|_s+\| \alpha_n-\alpha_m \| .$$

It follows that $\zeta\phi_n$ is Cauchy in $W^{s+1}_{(p,q)}(M)$, and $\lim_{n\to\infty}\zeta\phi_n=\zeta\phi$ is in $W^{s+1}_{(p,q)}(M)$. Hence, we have

$$\| \zeta\phi \|_{s+1}\lesssim \| \zeta_1\alpha \|_s+\| \alpha \| .$$

This proves the theorem.

Theorem 8.4.4. *Let α, ϕ, U, ζ and ζ_1 be as in Theorem 8.4.2. If $\zeta_1\alpha \in W^s_{(p,q)}(M)$ for some $s \geq 0$, and if ϕ satisfies $(\Box_b + \lambda I)\phi = \alpha$ for some constant λ, then $\zeta\phi \in W^{s+1}_{(p,q)}(M)$. In other words, $\Box_b + \lambda I$ is hypoelliptic for every λ. Moreover, all the eigenforms of $\Box_b$ are smooth.*

Proof. Let $\alpha' = \alpha + (1-\lambda)\phi$, then $(\Box_b + I)\phi = \alpha'$. The assertion now follows from Theorem 8.4.3 and an induction argument. This proves the theorem.

If we patch up the local estimates, we obtain the following global estimate:

Theorem 8.4.5. *Let M be a compact, oriented, CR manifold satisfying condition $Y(q)$. Let $\phi \in Dom(\Box_b)$. If $(\Box_b + I)\phi = \alpha$ with $\alpha \in W^s_{(p,q)}(M)$, $s \geq 0$, then $\phi \in W^{s+1}_{(p,q)}(M)$ and*

$$\| \phi \|_{s+1} \leq C \| \alpha \|_s,$$

where the constant C is independent of α.

Here are some important consequences:

Corollary 8.4.6. *Let M be as in Theorem 8.4.5. The operator $(\Box_b + I)^{-1}$ is compact.*

Proof. Since $(\Box_b + I)^{-1}$ is a bounded operator from $L^2_{(p,q)}(M)$ into $W^1_{(p,q)}(M)$, the assertion follows from Rellich's lemma (see Theorem A.8 in the Appendix).

Corollary 8.4.7. *Let M be as in Theorem 8.4.5. The operator $\Box_b + I$ has a discrete spectrum with no finite limit point, and each eigenvalue occurs with finite multiplicity. All eigenfunctions are smooth. In particular, $Ker(\Box_b)$ is of finite dimension and consists of smooth forms.*

Proof. By Corollary 8.4.6 $(\Box_b + I)^{-1}$ is a compact operator from $L^2_{(p,q)}(M)$ into itself. Hence the spectrum of $(\Box_b + I)^{-1}$ is compact and countable with zero as its only possible limit point. Since $(\Box_b + I)^{-1}$ is injective, zero is not an eigenvalue of $(\Box_b + I)^{-1}$. Each eigenvalue of $(\Box_b + I)^{-1}$ has finite multiplicity. Also λ is an eigenvalue of $\Box_b + I$ if and only if λ^{-1} is an eigenvalue of $(\Box_b + I)^{-1}$. This proves the corollary.

Proposition 8.4.8. *Let M be as in Theorem 8.4.5. $\Box_b$ is hypoelliptic. Moreover, if $\Box_b\phi = \alpha$ with $\alpha \in W^s_{(p,q)}(M)$, $s \geq 0$, we have*

$$\| \phi \|^2_{s+1} \leq C(\| \alpha \|^2_s + \| \phi \|^2),$$

where the constant $C > 0$ is independent of α.

Proof. We show the estimate by an induction on s. If $s = 0$, Theorem 8.4.5 implies

$$\| \phi \|^2_1 \lesssim \| (\Box_b + I)\phi \|^2 \lesssim \| \alpha \|^2 + \| \phi \|^2 .$$

In general, if we assume the assertion holds up to step $s-1$, we have $\phi \in W^s_{(p,q)}(M)$. For the case s, we apply Theorem 8.4.5 again and get

$$\begin{aligned} \| \phi \|^2_{s+1} &\lesssim \| (\Box_b + I)\phi \|^2_s \\ &\lesssim \| \Box_b\phi \|^2_s + \| \phi \|^2_s \\ &\lesssim \| \alpha \|^2_s + \| \phi \|^2, \end{aligned}$$

where the final step is accomplished by the induction hypothesis. This proves the proposition.

Let $\mathcal{H}^b_{(p,q)}(M)$ denote the space of harmonic forms on M, i.e., $\mathcal{H}^b_{(p,q)}(M) = \mathrm{Ker}(\Box_b)$. Thus $\mathcal{H}^b_{(p,q)}(M)$ consists of smooth harmonic (p,q)-forms and is of finite dimension. Using Corollary 8.4.7, $\Box_b$ is bounded away from zero on the orthogonal complement $(\mathcal{H}^b_{(p,q)}(M))^\perp$, namely,

$$\| \Box_b \phi \| \geq \lambda_1 \| \phi \| \tag{8.4.3}$$

for all $\phi \in \mathrm{Dom}(\Box_b) \cap (\mathcal{H}^b_{(p,q)}(M))^\perp$, where λ_1 is the smallest positive eigenvalue of $\Box_b$. It follows from Estimate (8.4.3) and Lemma 4.1.1 that the range of $\Box_b$ is closed. Also, the following strong Hodge type decomposition holds on $L^2_{(p,q)}(M)$:

Proposition 8.4.9. *Let M be as in Theorem 8.4.5. $L^2_{(p,q)}(M)$ admits the strong orthogonal decomposition,*

$$\begin{aligned} L^2_{(p,q)}(M) &= \mathcal{R}(\Box_b) \oplus \mathcal{H}^b_{(p,q)}(M) \\ &= \overline{\partial}_b \overline{\partial}_b^* (Dom\Box_b) \oplus \overline{\partial}_b^* \overline{\partial}_b (Dom\Box_b) \oplus \mathcal{H}^b_{(p,q)}(M), \end{aligned}$$

where $\mathcal{R}(\Box_b)$ denotes the range of $\Box_b$.

Proof. Since $\mathcal{R}(\Box_b)= (\mathcal{H}^b_{(p,q)}(M))^\perp$ and $\mathcal{R}(\overline{\partial}_b\overline{\partial}_b^*) \perp \mathcal{R}(\overline{\partial}_b^*\overline{\partial}_b)$, the decomposition follows.

We can thus define the boundary operator, $N_b : L^2_{(p,q)}(M) \to \mathrm{Dom}(\Box_b)$, as follows: If $\alpha \in \mathcal{H}^b_{(p,q)}(M)$, set $N_b\alpha = 0$. If $\alpha \in \mathcal{R}(\Box_b)$, define $N_b\alpha = \phi$, where ϕ is the unique solution of $\Box_b\phi = \alpha$ with $\phi \perp \mathcal{H}^b_{(p,q)}(M)$, and we extend N_b by linearity. It is easily seen that N_b is a bounded operator.

Let $H^b_{(p,q)}$ denote the orthogonal projection from $L^2_{(p,q)}(M)$ onto $\mathcal{H}^b_{(p,q)}(M)$. Next, we prove the main result of this section.

Theorem 8.4.10. *Suppose that condition $Y(q)$, for some q with $1 \leq q \leq n-1$, holds on a compact, oriented, CR manifold $(M, T^{1,0}(M))$ with $n \geq 3$. Then there exists an operator*

$$N_b : L^2_{(p,q)}(M) \to L^2_{(p,q)}(M)$$

such that:

(1) *N_b is a compact operator,*
(2) *for any $\alpha \in L^2_{(p,q)}(M)$, $\alpha = \overline{\partial}_b\overline{\partial}_b^* N_b\alpha + \overline{\partial}_b^*\overline{\partial}_b N_b\alpha + H^b\alpha$,*
(3) *$N_bH^b = H^bN_b = 0$.*
$N_b\Box_b = \Box_bN_b = I - H^b$ on $Dom(\Box_b)$.
(4) *If N_b is also defined on $L^2_{(p,q+1)}(M)$, then $N_b\overline{\partial}_b = \overline{\partial}_bN_b$ on $Dom(\overline{\partial}_b)$.*
If N_b is also defined on $L^2_{(p,q-1)}(M)$, then $N_b\overline{\partial}_b^* = \overline{\partial}_b^*N_b$ on $Dom(\overline{\partial}_b^*)$.
(5) *$N_b(\mathcal{E}^{p,q}(M)) \subset \mathcal{E}^{p,q}(M)$, and for each positive integer s, the estimate*

$$\| N_b\alpha \|_{s+1} \lesssim \| \alpha \|_s \tag{8.4.4}$$

holds uniformly for all $\alpha \in W^s_{(p,q)}(M)$.

Proof. (1) follows from Proposition 8.4.8 and the Rellich lemma. (2) is just a restatement of Proposition 8.4.9. The assertions in (3) follow immediately from the definition of N_b. For (4), if $\alpha \in \mathrm{Dom}(\overline{\partial}_b)$, then

$$\begin{aligned} N_b\overline{\partial}_b\alpha &= N_b\overline{\partial}_b\overline{\partial}_b^*\overline{\partial}_b N_b\alpha \\ &= N_b\Box_b\overline{\partial}_b N_b\alpha \\ &= \overline{\partial}_b N_b\alpha. \end{aligned}$$

A similar equation holds for $\overline{\partial}_b^*$. For (5), if $\alpha \in \mathcal{E}^{p,q}(M)$, then $\alpha - H^b\alpha \in \mathcal{E}^{p,q}(M)$, we have

$$\Box_b N_b\alpha = \alpha - H^b\alpha.$$

Since $\Box_b$ is hypoelliptic by Theorem 8.4.4, $N_b\alpha \in \mathcal{E}^{p,q}(M)$. Estimate (8.4.4) now follows from Proposition 8.4.8 since

$$\begin{aligned} \| N_b\alpha \|_{s+1} &\lesssim \| \Box_b N_b\alpha \|_s + \| N_b\alpha \| \\ &\lesssim \| \alpha \|_s + \| H^b\alpha \|_s + \| \alpha \| \\ &\lesssim \| \alpha \|_s \, . \end{aligned}$$

Here we have used the fact that $\mathcal{H}^b_{(p,q)}(M)$ is of finite dimension to conclude the estimate: $\| H^b\alpha \|_s \le C_s \| H^b\alpha \| \le C_s \| \alpha \|$ for some constant C_s. This proves the theorem.

Corollary 8.4.11. *Let M be as in Theorem 8.4.10. The range of $\overline{\partial}_b$ on Dom($\overline{\partial}_b$)$\cap$ $L^2_{(p,q-1)}(M)$ is closed.*

Proof. Since $\mathcal{R}(\overline{\partial}_b) \perp \mathrm{Ker}(\overline{\partial}_b^*)$, we have $\mathcal{R}(\overline{\partial}_b) = \overline{\partial}_b\overline{\partial}_b^*(\mathrm{Dom}\Box_b)$.

Definition 8.4.12. *Let M be a compact orientable CR manifold. The Szegö projection S on M is defined to be the orthogonal projection $S = H^b_{(0,0)}$ from $L^2(M)$ onto $\mathcal{H}^b(M) = \mathcal{H}^b_{(0,0)}(M)$.*

If condition $Y(1)$ holds on M, according to Theorem 8.4.10, there exists an operator N_b on $L^2_{(0,1)}(M)$. Then it is easy to obtain the following formula for the Szegö projection.

Theorem 8.4.13. *Let M be a compact orientable CR manifold. Suppose that M satisfies condition $Y(1)$. Then the Szegö projection S on M is given by*

$$S = I - \overline{\partial}_b^* N_b\overline{\partial}_b.$$

Theorem 8.4.10 gives the following solvability and regularity theorem for the $\overline{\partial}_b$ equation.

Theorem 8.4.14. *Under the same hypotheses as in Theorem 8.4.10, for any $\alpha \in L^2_{(p,q)}(M)$ with $\overline{\partial}_b\alpha = 0$ and $H^b\alpha = 0$, there is a unique solution ϕ of $\overline{\partial}_b\phi = \alpha$ with $\phi \perp Ker(\overline{\partial}_b)$. If $\alpha \in \mathcal{E}^{p,q}(M)$, then $\phi \in \mathcal{E}^{p,q-1}(M)$. Furthermore, for each $s \geq 0$, if $\alpha \in W^s_{(p,q)}(M)$, then $\phi \in W^{s+\frac{1}{2}}_{(p,q-1)}(M)$ and*

$$\| \phi \|_{s+\frac{1}{2}} \lesssim \| \alpha \|_s \, . \tag{8.4.5}$$

Proof. By (2) of Theorem 8.4.10, clearly we have $\alpha = \overline{\partial}_b\overline{\partial}_b^*N_b\alpha$. We simply take $\phi = \overline{\partial}_b^*N_b\alpha$, and ϕ is unique by the condition $\phi \perp \mathrm{Ker}(\overline{\partial}_b)$. The smoothness of ϕ follows from (5) of Theorem 8.4.10. For the estimate (8.4.5), let $\{U_i\}_{i=1}^m$ be an open cover of M formed by the coordinate charts $\{U_i\}_{i=1}^m$, and let $\{\zeta_i\}_{i=1}^m$ be a partition of unity subordinate to $\{U_i\}_{i=1}^m$. Then we have

$$\begin{aligned}
&\| \phi \|^2_{s+\frac{1}{2}} = \| \overline{\partial}_b^* N_b\alpha \|^2_{s+\frac{1}{2}} \\
&\quad \leq \| \overline{\partial}_b^* N_b\alpha \|^2_{s+\frac{1}{2}} + \| \overline{\partial}_b N_b\alpha \|^2_{s+\frac{1}{2}} \\
&\quad \lesssim \sum_{i=1}^m \left((\overline{\partial}_b^*\zeta_i N_b\alpha, \Lambda^{2s+1}\overline{\partial}_b^*\zeta_i N_b\alpha) + (\overline{\partial}_b\zeta_i N_b\alpha, \Lambda^{2s+1}\overline{\partial}_b\zeta_i N_b\alpha) \right) \\
&\quad = \sum_{i=1}^m \left((\overline{\partial}_b^* N_b\alpha, \zeta_i\Lambda^{2s+1}\overline{\partial}_b^*\zeta_i N_b\alpha) + (\overline{\partial}_b N_b\alpha, \zeta_i\Lambda^{2s+1}\overline{\partial}_b\zeta_i N_b\alpha) \right) + O(\| N_b\alpha \|^2_{s+1}) \\
&\quad = \sum_{i=1}^m \left((\overline{\partial}_b^* N_b\alpha, \overline{\partial}_b^*\zeta_i\Lambda^{2s+1}\zeta_i N_b\alpha) + (\overline{\partial}_b N_b\alpha, \overline{\partial}_b\zeta_i\Lambda^{2s+1}\zeta_i N_b\alpha) \right) + O(\| N_b\alpha \|^2_{s+1}) \\
&\quad = \sum_{i=1}^m (\alpha, \zeta_i\Lambda^{2s+1}\zeta_i N_b\alpha) + O(\| N_b\alpha \|^2_{s+1}) \\
&\quad \lesssim \| \alpha \|_s \| N_b\alpha \|_{s+1} + \| N_b\alpha \|^2_{s+1}.
\end{aligned}$$

By (5) of Theorem 8.4.10, we thus obtain

$$\| \phi \|_{s+\frac{1}{2}} \lesssim \| \alpha \|_s \, .$$

This completes the proof of Theorem 8.4.14.

A final remark is in order. If D is a relatively compact complex manifold with boundary bD satisfying condition $Z(q)$, $1 \leq q \leq n$, analogous results to Theorems 8.4.10 and 8.4.14 can be obtained for the $\overline{\partial}$-Neumann operator. In particular, $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is hypoelliptic on D.

NOTES

Pseudodifferential operators were introduced by J. J. Kohn and L. Nirenberg [KoNi 2] and L. Hörmander [Hör 4] as a generalization of singular integral operators developed by A. P. Calderón and A. Zygmund [CaZy 1]. These operators have played an important role in the study of linear partial differential equations. We refer the reader to the books by Hörmander [Hör 8], Nagel-Stein [NaSt 1] and Treves [Tre 3] for detailed discussions and applications of pseudodifferential operators. The pseudodifferential operators used in section 8.1 are of the simplest kind.

Theorem 8.2.3 and Theorem 8.2.6 were first proved by L. Hörmander in [Hör 6] where the argument is based on careful analysis of the one-parameter groups generated by the given vector fields. Hörmander's original proof gives very precise ϵ in Theorem 8.2.6. He also showed that finite type condition is necessary for subellipticity of the sum of squares operator. L. P. Rothschild and E. M. Stein [RoSt 1] showed that sharp estimates in L^p spaces and Lipschitz spaces can be achieved. The proof of Theorem 8.2.3 using pseudodifferential operators that we present in Section 8.2 follows the paper by J. J. Kohn [Koh 3] (see also Oleinik-Radkevič [OlRa 1]).

However, it is well known that the finite type condition is not necessary for the hypoellipticity of a sum of squares operator. In [OlRa 1], Theorem 2.5.3, Oleinik and Radkevič showed hypoellipticity when the finite type condition fails on a compact set which is contained in a finite union of hypersurfaces. In [KuSt 1] Kusuoka and Strook showed, by using probabilistic methods, that the operator

$$P_a = \left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial x}\right)^2 + \left(a(t)\frac{\partial}{\partial y}\right)^2$$

is hypoelliptic in $\mathbb{R}^3$ if and only if $\lim_{t\searrow 0} t\log a(t) = 0$, where $a \in C^\infty(\mathbb{R})$ is even, real-valued with derivatives of all orders bounded, non-decreasing on $[0, \infty)$, and vanishing to infinite order at $t = 0$. For instance, such an a is given by $a(t) = e^{-(1/|t|^p)}$, $0 < p < 1$. The new interesting phenomenon in this example is that hypoellipticity depends on the exponential order of vanishing of a at zero. V. S. Fedii [Fed 1] has shown that the operator

$$P_b = \left(\frac{\partial}{\partial t}\right)^2 + \left(b(t)\frac{\partial}{\partial x}\right)^2$$

is hypoelliptic in $\mathbb{R}^2$ for any real-valued $b \in C^\infty(\mathbb{R})$ with $b(t) \neq 0$, for $t \neq 0$. Unlike the previous example, b may vanish at 0 to any exponential order and still P_b is hypoelliptic (see also recent related results in [Koh 12]).

It follows from the work of C. Fefferman and D. H. Phong [FePh 1] that both operators P_a and P_b, when a and b are vanishing to infinite order at $t = 0$, do not satisfy local subelliptic estimates near $t = 0$. The proof of their hypoellipticity does not use such estimates in contrast to the classical proof of Hörmander's Theorem which is based on local subelliptic estimates. The formulation of a necessary and sufficient condition for the hypoellipticity of a sum of squares operator is an open problem.

The materials presented in Sections 8.3 and 8.4 are developed by J. J. Kohn [Koh 2] where condition $Y(q)$ was first introduced. See also [FoKo 1]. Theorem 8.3.2 shows that condition $Z(q)$ is sufficient for the subelliptic 1/2-estimate for the $\overline{\partial}$-Neumann operator on (p, q)-forms. The characterization of when the Morrey-Kohn estimate, (8.3.8), holds was proved by L. Hörmander in [Hör 2]. This lead to a new proof of existence results for $\overline{\partial}$ by A. Andreotti and H. Grauert [AnGr 1] where condition $Z(q)$ was first introduced. The techniques from subelliptic ϵ-estimate to regularity of the solution were treated along the lines of Kohn and Nirenberg [KoNi 1].

There is a vast amount of works concerning the $\overline{\partial}_b$ complex or $\Box_b$ on strongly pseudoconvex CR manifolds (or CR manifolds satisfying condition $Y(q)$). We refer the reader to the papers by Folland-Stein [FoSt 1], Rothschild-Stein [RoSt 1], Beals-Greiner-Stanton [BGS 1] and Boutet de Monvel-Sjöstrand [BdSj 1] as well as the books by Beals-Greiner [BeGr 1], Stein [Ste 4] and Treves [Tre 3].

CHAPTER 9

THE TANGENTIAL CAUCHY-RIEMANN COMPLEX ON PSEUDOCONVEX CR MANIFOLDS

We study existence theorems for the tangential Cauchy-Riemann complex on a smooth pseudoconvex CR manifold M in this chapter. When M is a strongly pseudoconvex CR manifold, L^2 existence theorems and subelliptic estimates for $\bar\partial_b$ and $\Box_b$ have been proved in Chapter 8 using pseudodifferential operators. We shall establish here the existence theorems for $\bar\partial_b$ in the C^∞ and L^2 categories when M is the boundary of a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$. One purpose of this chapter is to study the relationship between $\bar\partial$ and $\bar\partial_b$. To solve $\bar\partial_b$, we construct $\bar\partial$-closed extensions of forms from M to Ω and use the solvability for $\bar\partial$ in Ω. The extension problem can be converted into a $\bar\partial$-Cauchy problem, which is to solve $\bar\partial$ with prescribed support.

The Cauchy problem for a bounded pseudoconvex domain in $\mathbb{C}^n$ is formulated and solved in the L^2 sense in Section 9.1. In Section 9.2 we discuss C^∞ extensions of smooth forms from the boundary and obtain the C^∞ solvability for $\bar\partial_b$ on pseudoconvex boundaries. L^2 existence theorems for $\bar\partial_b$ and estimates in Sobolev spaces are proved in Section 9.3. The closed-range property of $\Box_b$ and the strong Hodge decomposition theorem for $\bar\partial_b$ are proved in Section 9.4.

9.1 The L^2 Cauchy Problem for $\bar\partial$

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, not necessarily with smooth boundary. We study the question of solving $\bar\partial$ with prescribed support. The L^2 or C^∞ Cauchy problem is the following question:

Given a $\bar\partial$-closed (p,q)-form f with L^2 (or C^∞) coefficients in $\mathbb{C}^n$ such that f is supported in $\overline{\Omega}$, can one find $u \in L^2_{(p,q-1)}(\mathbb{C}^n)$ (or $u \in C^\infty_{(p,q-1)}(\mathbb{C}^n)$) such that $\bar\partial u = f$ in $\mathbb{C}^n$ and u is supported in $\overline{\Omega}$?

When $q = 1$, it has been proved in Theorem 3.1.1 that one can solve $\bar\partial$ with compact support for smooth $(0,1)$-forms as long as $\mathbb{C}^n \setminus \Omega$ has no compact components. There are no other restrictions on the boundary of Ω. When $q > 1$, we shall use the duality of the $\bar\partial$-Neumann problem to solve the Cauchy problem on bounded pseudoconvex domains.

Let $\star : C^\infty_{(p,q)}(\overline{\Omega}) \to C^\infty_{(n-p,n-q)}(\overline{\Omega})$ be the Hodge star operator defined by

$$\langle \phi, \psi \rangle dV = \phi \wedge \star\psi,$$

where $\phi, \psi \in C^\infty_{(p,q)}(\overline{\Omega})$ and $dV = i^n dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n$ denotes the volume element in $\mathbb{C}^n$ as before. We can extend $\star$ from $L^2_{(p,q)}(\Omega)$ to $L^2_{(n-p,n-q)}(\Omega)$ in a natural way. This is an algebraic operator given explicitly as follows: if we write

$$\psi = {\sum_{I,J}}' \psi_{I,J} dz^I \wedge d\bar z^J,$$

then

$$\star\psi = {\sum_{I,J}}' i^n \epsilon_{IJ} \bar\psi_{I,J} dz^{[\hat I]} \wedge d\bar z^{[\hat J]},$$

where $[\hat I]$ denotes the increasing $(n-p)$-tuple consisting of elements in $\{1, \cdots, n\}\setminus I$ and ϵ_{IJ} is the sign of the permutation from $(I, J, [\hat I], [\hat J])$ to $(1, 1', \cdots, n, n')$.

Lemma 9.1.1. *For every $\phi \in C^\infty_{(p,q)}(\overline{\Omega})$, we have*

$$\star\star\,\phi = (-1)^{p+q}\phi, \tag{9.1.1}$$

and

$$\vartheta = -\star\bar\partial\star, \tag{9.1.2}$$

where ϑ and $\bar\partial$ are viewed as differential operators.

Proof. (9.1.1) is easy to check. To prove (9.1.2), we have, for any $\psi \in C^\infty_{(p,q-1)}(\overline{\Omega})$ and $\eta \in C^\infty_{(p,q)}(\overline{\Omega})$ such that η has compact support in Ω,

$$(\bar\partial\psi, \eta) = \int_\Omega \bar\partial\psi \wedge \star\eta = (-1)^{p+q}\int_\Omega \psi \wedge \bar\partial\star\eta + \int_\Omega d(\psi\wedge\star\eta).$$

Since $\int_\Omega d(\psi\wedge\star\eta) = 0$, using (9.1.1),

$$(\bar\partial\psi, \eta) = -\int_\Omega \psi\wedge\star\star\,\bar\partial\star\eta = -(\psi, \star\bar\partial\star\eta) = (\psi, \vartheta\eta).$$

This proves (9.1.2).

Theorem 9.1.2. *Let Ω be a bounded pseudoconvex domain with C^1 boundary in $\mathbb{C}^n$, $n \ge 2$. Let $\delta = \sup_{z,z'\in\Omega} |z - z'|$ be the diameter of Ω. For every $f \in L^2_{(p,q)}(\mathbb{C}^n)$, where $0 \le p \le n$ and $1 \le q \le n-1$, with $\bar\partial f = 0$ in the distribution sense in $\mathbb{C}^n$ and f supported in $\overline{\Omega}$, one can find $u \in L^2_{(p,q-1)}(\mathbb{C}^n)$ such that $\bar\partial u = f$ in the distribution sense in $\mathbb{C}^n$ with u supported in $\overline{\Omega}$ and*

$$(n-q)\int_\Omega |\,u\,|^2\, dV \le e\delta^2 \int_\Omega |\,f\,|^2\, dV.$$

Proof. From Theorem 4.4.1, the $\bar\partial$-Neumann operator of degree $(n-p, n-q)$ in Ω, denoted by $N_{(n-p,n-q)}$, exists. We define

$$u = -\star\bar\partial N_{(n-p,n-q)}\star f, \tag{9.1.3}$$

then $u \in L^2_{(p,q-1)}(\Omega)$ and $\star u \in \mathrm{Dom}(\bar\partial^*)$.

Extending u to $\mathbb{C}^n$ by defining $u = 0$ in $\mathbb{C}^n \setminus \Omega$, we claim that $\bar\partial u = f$ in the distribution sense in $\mathbb{C}^n$. First we prove that $\bar\partial u = f$ in the distribution sense in Ω.

From (9.1.1) and (9.1.2),

$$\begin{aligned} \bar\partial u &= -\bar\partial \star \bar\partial N_{(n-p,n-q)} \star f \\ &= (-1)^{p+q+1} \star \star \bar\partial \star \bar\partial N_{(n-p,n-q)} \star f \\ &= (-1)^{p+q} \star \vartheta \bar\partial N_{(n-p,n-q)} \star f \\ &= (-1)^{p+q} \star \bar\partial^* \bar\partial N_{(n-p,n-q)} \star f, \end{aligned} \tag{9.1.4}$$

where ϑ acts in the distribution sense in Ω. On the other hand, for any $\phi \in C^\infty_{(n-p,n-q-1)}(\overline{\Omega})$,

$$\begin{aligned} (\bar\partial\phi, \star f)_\Omega &= (-1)^{p+q} \int_\Omega \bar\partial\phi \wedge f = (-1)^{p+q} \int_\Omega f \wedge \star\star\, \bar\partial\phi \\ &= (-1)^{p+q} (f, \star\bar\partial\phi)_\Omega = (f, \vartheta \star \phi)_\Omega \\ &= (\bar\partial f, \star\phi)_{\mathbb{C}^n} \\ &= 0 \end{aligned} \tag{9.1.5}$$

since supp $f \subset \overline{\Omega}$ and $\bar\partial f = 0$ in the distribution sense in $\mathbb{C}^n$. If Ω has C^1 boundary, using the first part of the proof of Lemma 4.3.2, the set $C^\infty_{(n-p,n-q-1)}(\overline{\Omega})$ is dense in $\mathrm{Dom}(\bar\partial)$ in the graph norm. It follows from the definition of $\bar\partial^*$ that $\star f \in \mathrm{Dom}(\bar\partial^*)$ and $\bar\partial^*(\star f) = 0$. Using Theorem 4.4.1 when $1 \le q < n-1$ and Theorem 4.4.3 when $q = n-1$, we have

$$\bar\partial^* N_{(n-p,n-q)} \star f = N_{(n-p,n-q-1)} \bar\partial^*(\star f) = 0. \tag{9.1.6}$$

Combining (9.1.4) and (9.1.6),

$$\begin{aligned} \bar\partial u &= (-1)^{p+q} \star \bar\partial^* \bar\partial N_{(n-p,n-q)} \star f \\ &= (-1)^{p+q} \star (\bar\partial^*\bar\partial + \bar\partial\bar\partial^*) N_{(n-p,n-q)} \star f \\ &= (-1)^{p+q} \star \star f \\ &= f \end{aligned}$$

in the distribution sense in Ω. Furthermore, we note that $\star u \in \mathrm{Dom}(\bar\partial^*)$ and this additional condition implies that $\bar\partial u = f$ in the distribution sense in $\mathbb{C}^n$. Using

$$\bar\partial^*(\star u) = \vartheta \star u = (-1)^{p+q} \star \bar\partial u = (-1)^{p+q} \star f,$$

where $\vartheta \star u$ is taken in the distribution sense in Ω, we have for any $\psi \in C^\infty_{(p,q)}(\mathbb{C}^n)$,

$$\begin{aligned} (u, \vartheta\psi)_{\mathbb{C}^n} &= (\star\vartheta\psi,\ \star u)_\Omega \\ &= (-1)^{p+q} (\bar\partial \star \psi,\ \star u)_\Omega \\ &= (-1)^{p+q} (\star\psi,\ \bar\partial^*(\star u))_\Omega \\ &= (\star\psi,\ \star f)_\Omega \\ &= (f,\ \psi)_{\mathbb{C}^n}, \end{aligned} \tag{9.1.7}$$

where the third equality holds since $\star u \in \text{Dom}(\bar{\partial}^*)$. Thus $\bar{\partial} u = f$ in the distribution sense in $\mathbb{C}^n$. The estimate for u follows from Theorem 4.4.1. Theorem 9.1.2 is proved.

The assumption that Ω has C^1 boundary is used to show that $C^\infty_{(n-p,n-q-1)}(\overline{\Omega})$ is dense in $\text{Dom}(\bar{\partial})$ in the graph norm so that (9.1.6) holds. Using the proof of Lemma 4.3.2, Theorem 9.1.2 also holds when the domain is star-shaped or locally star-shaped.

When $q \leq n$, including the top degree case, we also have the following result.

Theorem 9.1.3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. For any $f \in L^2_{(p,q)}(\mathbb{C}^n)$, $0 \leq p \leq n$, $1 \leq q \leq n$, such that f is supported in $\overline{\Omega}$ and*

$$\int_\Omega f \wedge g = 0 \qquad \textit{for every} \quad g \in L^2_{(n-p,n-q)}(\Omega) \cap Ker(\bar{\partial}), \tag{9.1.8}$$

one can find $u \in L^2_{(p,q-1)}(\mathbb{C}^n)$ such that $\bar{\partial} u = f$ in the distribution sense in $\mathbb{C}^n$ with u supported in $\overline{\Omega}$ and

$$\int_\Omega |\, u \,|^2 \, dV \leq e\delta^2 \int_\Omega |\, f \,|^2 \, dV,$$

where $\delta = \sup_{z,z' \in \Omega} |z - z'|$ is the diameter of Ω.

If Ω is a bounded pseudoconvex domain with smooth boundary, (9.1.8) can be replaced by the condition

$$\int_\Omega f \wedge g = 0 \qquad \textit{for every } g \in C^\infty_{(n-p,n-q)}(\overline{\Omega}) \cap Ker(\bar{\partial}), \tag{9.1.9}$$

and the same conclusion holds.

Proof. Using Theorem 4.4.1 and Theorem 4.4.3, the $\bar{\partial}$-Neumann operator $N_{(p,q)}$ exists for any $0 \leq p \leq n$ and $0 \leq q \leq n$. When $q = 0$, we have

$$N_{(p,0)} = \bar{\partial}^* N^2_{(p,1)} \bar{\partial}. \tag{9.1.10}$$

For every $0 \leq q \leq n$, the Bergman projection operator $P_{(p,q)}$ is given by

$$\bar{\partial}^* \bar{\partial} N_{(p,q)} = I - P_{(p,q)}. \tag{9.1.11}$$

We define u by

$$u = - \star \bar{\partial} N_{(n-p,n-q)} \star f. \tag{9.1.12}$$

Using Lemma 9.1.1, we have from (9.1.11),

$$\begin{aligned} \bar{\partial} u &= (-1)^{p+q} \star \bar{\partial}^* \bar{\partial} N_{(n-p,n-q)} \star f \\ &= f - (-1)^{p+q} \star P_{(n-p,n-q)} \star f. \end{aligned} \tag{9.1.13}$$

From (9.1.8), we get for any $g \in L^2_{(n-p,n-q)}(\Omega) \cap \mathrm{Ker}(\bar\partial)$,

$$(\star f, g) = (-1)^{p+q} \overline{\int_\Omega g \wedge f} = 0. \tag{9.1.14}$$

Thus $P_{(n-p,n-q)}(\star f) = 0$ and $\bar\partial u = f$ in Ω from (9.1.13).

Using $\star u \in \mathrm{Dom}(\bar\partial^*)$ and extending u to be zero outside Ω, we can repeat the arguments of (9.1.7) to show that $\bar\partial u = f$ in $\mathbb{C}^n$ in the distribution sense. The estimate holds from Theorems 4.4.1 and 4.4.3. Thus, the proposition is proved when f satisfies (9.1.8).

When $b\Omega$ is smooth, we have $C^\infty_{(n-p,n-q)}(\overline{\Omega}) \cap \mathrm{Ker}(\bar\partial)$ is dense in $\mathcal{H}_{(n-p,n-q)}(\Omega) = L^2_{(n-p,n-q)}(\Omega) \cap \mathrm{Ker}(\bar\partial)$ in the $L^2(\Omega)$ norm, using Corollary 6.1.6. Thus if f satisfies condition (9.1.9), it also satisfies condition (9.1.8). Theorem 9.1.3 is proved.

Remark: When $q < n$, condition (9.1.9) implies that $\bar\partial f = 0$ in the distribution sense in $\mathbb{C}^n$. To see this, we take $g = \bar\partial \star v$ for some $v \in C^\infty_{(p,q+1)}(\mathbb{C}^n)$ in (9.1.9). Then we have

$$(f, \vartheta v)_{\mathbb{C}^n} = \int_D f \wedge \star\vartheta v = (-1)^{p+q+1} \int_D f \wedge \bar\partial \star v = 0$$

for any $v \in C^\infty_{(p,q+1)}(\mathbb{C}^n)$. This implies that $\bar\partial f = 0$ in the distribution sense in $\mathbb{C}^n$. From the proof of Theorem 9.1.2, the two conditions are equivalent if $C^\infty_{(n-p,n-q-1)}(\overline{\Omega})$ is dense in $\mathrm{Dom}(\bar\partial)$ in the graph norm.

9.2 $\bar\partial$-Closed Extensions of Forms and C^∞ Solvability of $\bar\partial_b$

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary $M = b\Omega$ and ρ be a smooth defining function for Ω such that $|d\rho| = 1$ on M. We use $\mathcal{E}^{(p,q)}(M)$ to denote the smooth (p,q)-forms on M, where $0 \le p \le n,\ 0 \le q \le n-1$. Here the extrinsic definition for $\mathcal{E}^{(p,q)}(M)$ is used.

We consider the following two kinds of $\bar\partial$-closed extension problems:
Given $\alpha \in \mathcal{E}^{(p,q)}(M)$,

(1) can one find an extension $\tilde\alpha$ of α such that $\tau\tilde\alpha = \alpha$ on M and $\bar\partial\tilde\alpha = 0$ in Ω? (We recall that τ is the projection of smooth (p,q)-forms in $\mathbb{C}^n$ to (p,q)-forms on M which are pointwise orthogonal to the ideal generated by $\bar\partial\rho$.)

(2) can one find an extension $\tilde\alpha$ of α such that $\tilde\alpha = \alpha$ on M and $\bar\partial\tilde\alpha = 0$ in Ω?

When $q < n-1$, it is necessary that $\bar\partial_b\alpha = 0$ on M in order to have a $\bar\partial$-closed extension. When α is a function ($p = q = 0$), this is the question of holomorphic extension of CR functions. In this case problems (1) and (2) are the same. It was proved in Theorem 3.2.2 that any CR function of class C^1 on the boundary of a C^1 bounded domain Ω has a holomorphic extension as long as $\mathbb{C}^n \setminus \Omega$ has no bounded component. When α is a (p,q)-form with $q \ge 1$, (2) seems to be a stronger problem than (1). It will be shown in the next two theorems that these two kinds of extension problems are equivalent for smooth forms also.

Theorem 9.2.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary M. Let $\alpha \in \mathcal{E}^{p,q}(M)$, where $0 \le p \le n$ and $1 \le q \le n-1$. Then there exists $\tilde\alpha \in C^\infty_{(p,q)}(\overline{\Omega})$ such that $\tau\tilde\alpha = \alpha$ on M and $\bar\partial\tilde\alpha = 0$ in Ω if and only if*

$$\int_M \alpha \wedge \psi = 0 \quad \textit{for every } \psi \in C^\infty_{(n-p,n-q-1)}(\overline{\Omega}) \cap Ker(\bar\partial). \tag{9.2.1}$$

Furthermore, when $1 \le q < n-1$, (9.2.1) holds if and only if

$$\bar\partial_b \alpha = 0 \quad \textit{on } M. \tag{9.2.2}$$

Theorem 9.2.2. *Let M and α be the same as in Theorem 9.2.1. There exists $\tilde\alpha$ such that $\tilde\alpha \in C^\infty_{(p,q)}(\overline{\Omega})$, $\tilde\alpha = \alpha$ on M and $\bar\partial\tilde\alpha = 0$ in Ω if and only if (9.2.1) (for $1 \le q \le n-1$) or (9.2.2) (for $1 \le q < n-1$) holds.*

Proof of Theorem 9.2.1. It is easy to see that (9.2.1) and (9.2.2) are necessary conditions for the existence of $\bar\partial$-closed extensions. We assume that α satisfies (9.2.1).

Let α' be a smooth extension of α by extending α componentwise from the boundary to $\overline{\Omega}$. Thus $\alpha' \in C^\infty_{(p,q)}(\overline{\Omega})$ and $\alpha' = \alpha$ on M. We set $f = \bar\partial\alpha'$ in Ω, then $f \in C^\infty_{(p,q+1)}(\overline{\Omega})$ and $f \wedge \bar\partial\rho = \bar\partial_b\alpha \wedge \bar\partial\rho = 0$ on M. Using (9.2.1), we have for any $\bar\partial$-closed $\psi \in C^\infty_{(n-p,n-q-1)}(\overline{\Omega})$,

$$\int_\Omega f \wedge \psi = \int_\Omega \bar\partial(\alpha' \wedge \psi) = \int_M \alpha \wedge \psi = 0.$$

Thus, f satisfies condition (9.1.9) in Theorem 9.1.3. We first assume that the $\bar\partial$-Neumann operator $N_{(n-p,n-q-1)}$ for $(n-p,n-q-1)$-forms on Ω is C^∞ regular up to the boundary and define

$$u = -\star\bar\partial N_{(n-p,n-q-1)}\star f. \tag{9.2.3}$$

Then $u \in C^\infty_{(p,q)}(\overline{\Omega})$. As in Theorem 9.1.3, it follows that $\bar\partial u = f$ in Ω and $\star u \in \mathrm{Dom}(\bar\partial^*)$. Since the boundary is smooth and u is smooth up to the boundary, we can write $u = \tau u + \nu u$. Using Lemma 4.2.1, we have

$$\nu(\star u) = 0 \quad \text{on } M.$$

However, this is equivalent to

$$\tau u = 0 \quad \text{on } M.$$

Setting $\tilde\alpha = \alpha' - u$, we have $\bar\partial\tilde\alpha = 0$ in Ω and $\tau\tilde\alpha = \tau\alpha' = \alpha$ on M. This proves the theorem assuming that $N_{(n-p,n-q-1)}$ is C^∞ regular.

In general, instead of (9.2.3), we define

$$u_t = -\star\bar\partial N^t_{(n-p,n-q-1)}\star f,$$

where $N^t_{(n-p,n-q-1)}$ is the weighted $\bar\partial$-Neumann operator introduced in Theorem 6.1.2 with weight function $\phi_t = t|z|^2,\ t > 0$ and $\star$ is taken with respect to the metric $L^2(D, \phi_t)$. Choosing t sufficiently large, from the proof of Theorem 6.1.4 and Corollary 6.1.5, for each large integer $k > n+2$, there exists a solution u_k such that $u_k \in W^k_{(p,q)}(\Omega) \subset C^1_{(p,q)}(\overline{\Omega})$, $\bar\partial u_k = f$ in Ω, $\tau u_k = 0$ on M and

$$\| u_k \|_{k(\Omega)} \le C_k \| f \|_{k(\Omega)} .$$

To construct a solution $u \in C^\infty_{(p,q)}(\overline{\Omega})$, we set $h_k = u_k - u_{k+1}$. Each h_k is $\bar\partial$-closed and $\tau h_k = 0$ on M. This implies that $\bar\partial h_k = 0$ in $\mathbb{C}^n$ in the distribution sense. Using Friedrichs' lemma and the arguments in the proof of Lemma 4.3.2, we can find a sequence $h^k_n \in C^\infty_{(p,q)}(\Omega)$ such that h^k_n has compact support in Ω, $h^k_n \to h_k$ in $W^k_{(p,q)}(\Omega)$ and $\bar\partial h^k_n \to 0$ in $W^k_{(p,q+1)}(\Omega)$. For each arbitrarily large $m \in \mathbb{N}$, one can find $v^k_n \in W^m_{(p,q)}(\Omega)$ with $\tau v^k_n = 0$ on M and $\bar\partial v^k_n = \bar\partial h^k_n$ in Ω. Setting $\tilde h^k_n = h^k_n - v^k_n$, we have $\bar\partial \tilde h^k_n = 0$ in Ω, $\tilde h^k_n \to h_k$ in $W^k_{(p,q)}(\Omega)$ with $\tilde h^k_n \in W^m_{(p,q)}(\Omega)$ and $\tau \tilde h^k_n = 0$ on M. This implies that inductively one can construct a new sequence $u'_k \in W^k_{(p,q)}(\Omega)$ such that $\bar\partial u'_k = f$ in Ω, $\tau u'_k = 0$ on M and

$$\| u'_k - u'_{k+1} \|_{k(\Omega)} \le 1/2^k, \qquad k \in \mathbb{N}.$$

Writing

$$u = u'_N + \sum_{k=N+1}^{\infty} (u'_k - u'_{k-1}),$$

we have $u \in C^\infty_{(p,q)}(\overline{\Omega})$ such that $\bar\partial u = f$ in Ω, $\tau u = 0$ on M. Setting $\tilde\alpha = \alpha' - u$, the first part of the theorem is proved.

When $1 \le q < n-1$, setting $\psi = \bar\partial u$ for some $u \in C^\infty_{(n-p,n-q-2)}(\overline{\Omega})$ in (9.2.1), we have

$$\int_M \alpha \wedge \psi = \int_M \alpha \wedge \bar\partial u = (-1)^{p+q+1} \int_M \bar\partial_b \alpha \wedge u = 0. \tag{9.2.4}$$

Thus, (9.2.1) implies (9.2.2). We see from (9.2.4) that (9.2.2) also implies (9.2.1), since any $\bar\partial$-closed form ψ in $C^\infty_{(n-p,n-q-1)}(\overline{\Omega})$ can be written as $\psi = \bar\partial u$ for some $u \in C^\infty_{(n-p,n-q-2)}(\overline{\Omega})$ using Theorem 6.1.1. Thus, (9.2.1) and (9.2.2) are equivalent when $q < n-1$. The proof of Theorem 9.2.1 is complete.

In order to prove Theorem 9.2.2, we need the following lemma:

Lemma 9.2.3. *Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary M and let ρ be a smooth defining function for Ω. If $\alpha \in \mathcal{E}^{p,q}(M)$ and $\overline{\partial}_b \alpha = 0$ on M, where $0 \le p \le n,\ 0 \le q \le n-1$, then there exists $E_\infty \alpha$ such that $E_\infty \alpha \in C^\infty_{(p,q)}(\mathbb{C}^n)$, $E_\infty \alpha = \alpha$ on M and*

$$\bar\partial E_\infty \alpha = O(\rho^k) \quad \text{at } M \quad \text{for every positive integer } k.$$

Proof. We first extend α componentwise and smoothly from M to $E\alpha$ in $\mathbb{C}^n$. We claim that for every positive integer k, there exist smooth (p,q)-forms $\alpha_1,\cdots,\alpha_k$ and $(p,q+1)$-forms $\gamma_1,\cdots,\gamma_k$ such that

$$E_k\alpha = E\alpha - \rho\alpha_1 - \frac{\rho^2}{2}\alpha_2 - \cdots - \frac{\rho^k}{k}\alpha_k, \tag{9.2.5 a}$$

and

$$\bar\partial E_k\alpha = \rho^k\left(\gamma_k - \frac{1}{k}\bar\partial\alpha_k\right) = O(\rho^k). \tag{9.2.5 b}$$

Since $\bar\partial_b\alpha = 0$ on M, $\bar\partial E\alpha\wedge\bar\partial\rho = 0$ on M. We can find α_1 and γ_1 such that

$$\bar\partial E\alpha = \bar\partial\rho\wedge\alpha_1 + \rho\gamma_1 = \bar\partial(\rho\alpha_1) + \rho(\gamma_1 - \bar\partial\alpha_1).$$

Setting $E_1\alpha = E\alpha - \rho\alpha_1$, it follows that $\bar\partial E_1\alpha = \rho(\gamma_1 - \bar\partial\alpha_1) = O(\rho)$ at M. This proves (9.2.5 a) and (9.2.5 b) for $k=1$. We also note that α_1 is obtained from the first order derivatives of $E\alpha$ and γ_1 is obtained from the second derivatives of $E\alpha$.

Assuming (9.2.5 a) and (9.2.5 b) have been proved for some $k\in\mathbb{N}$, we apply $\bar\partial$ to both sides of (9.2.5 b) to obtain

$$0 = \bar\partial^2 E_k\alpha = k\rho^{k-1}\bar\partial\rho\wedge\left(\gamma_k - \frac{1}{k}\bar\partial\alpha_k\right) + \rho^k\bar\partial\gamma_k.$$

Hence $\bar\partial\rho\wedge\left(\gamma_k - (1/k)\bar\partial\alpha_k\right) = 0$ on M. Thus, we can find a (p,q)-form α_{k+1} and a $(p,q+1)$-form γ_{k+1} such that $\gamma_k - (1/k)\bar\partial\alpha_k = \bar\partial\rho\wedge\alpha_{k+1} + \rho\gamma_{k+1}$. We define

$$\begin{aligned} E_{k+1}\alpha &= E\alpha - \rho\alpha_1 - \frac{\rho^2}{2}\alpha_2 - \cdots - \frac{\rho^k}{k}\alpha_k - \frac{\rho^{k+1}}{k+1}\alpha_{k+1} \\ &= E_k\alpha - \frac{\rho^{k+1}}{k+1}\alpha_{k+1}, \end{aligned}$$

then

$$\begin{aligned} \bar\partial E_{k+1}\alpha &= \rho^k(\bar\partial\rho\wedge\alpha_{k+1} + \rho\gamma_{k+1}) - \bar\partial\left(\frac{\rho^{k+1}}{k+1}\alpha_{k+1}\right) \\ &= \rho^{k+1}\left(\gamma_{k+1} - \frac{1}{k+1}\bar\partial\alpha_{k+1}\right) = O(\rho^{k+1}). \end{aligned}$$

This proves (9.2.5 a) and (9.2.5 b) for $k+1$. Using induction, (9.2.5 a) and (9.2.5 b) hold for any positive integer k.

To find an extension $E_\infty\alpha$ such that $E_\infty\alpha = \alpha$ on M and $\bar\partial E_\infty\alpha = O(\rho^k)$ at M for every positive integer k, we modify the construction as follows: Let $\Omega_\delta = \{z\in\Omega \mid -\delta < \rho(z) < \delta\}$ be a small tubular neighborhood of M and $\eta(z)$ be a cut-off function such that $\eta\in C_0^\infty(\Omega_\delta)$ and $\eta = 1$ on $\Omega_{\delta/2}$. Let $\pi(z)$ denote the projection from Ω_δ onto M along the normal direction and $n(z)$ denote the unit outward normal at $z\in M$. We define $\tilde\eta_{\epsilon_j}(z) = \eta\big(\pi(z) + \frac{\rho(z)}{\epsilon_j}n(\pi(z))\big)$ and

$$E_\infty\alpha = E\alpha - \sum_{j=1}^\infty \tilde\eta_{\epsilon_j}(z)\frac{\rho^j}{j}\alpha_j,$$

where ϵ_j is chosen to be sufficiently small and $\epsilon_j \searrow 0$. One can choose ϵ_j so small (depending on α_j) such that, for each multiindex $m = (m_1, \cdots, m_{2n})$, we have

$$\left| D^m \left(\tilde{\eta}_{\epsilon_j}(z) \frac{\rho^j}{j} \alpha_j \right) \right| \le C_{m,j} \epsilon_j \le \frac{1}{2^j}, \quad \text{for every } m \text{ with } |m| \le j-1.$$

The series converges in C^k for every $k \in \mathbb{N}$ to some element $E_\infty \alpha \in C^\infty_{(p,q)}(\mathbb{C}^n)$ and $E_\infty \alpha = \alpha$ on M. Furthermore, we have

$$\begin{aligned} \bar\partial E_\infty \alpha &= \bar\partial \left(E\alpha - \sum_{j=1}^{k} \tilde{\eta}_{\epsilon_j}(z) \frac{\rho^j}{j} \alpha_j \right) - \bar\partial \left(\sum_{j=k+1}^{\infty} \tilde{\eta}_{\epsilon_j}(z) \frac{\rho^j}{j} \alpha_j \right) \\ &= O(\rho^k) \quad \text{at } M, \end{aligned}$$

for every positive integer k. This proves the lemma.

Proof of Theorem 9.2.2. Let $\alpha' = E_\infty \alpha$ where $E_\infty \alpha$ is as in Lemma 9.2.3. Using the proof of Theorem 9.2.1, there exists $u \in C^\infty_{(p,q)}(\overline{\Omega})$ such that $\bar\partial u = \bar\partial \alpha'$ in Ω and $\tau u = 0$ on M. Setting $F_0 \alpha = u$, we have

$$\bar\partial F_0 \alpha = \bar\partial \alpha' \quad \text{in } \Omega, \tag{9.2.6 a}$$

$$F_0 \alpha \wedge \bar\partial \rho = 0 \quad \text{on } M. \tag{9.2.6 b}$$

We shall prove that for any nonnegative integer k, there exist $(p, q-1)$-forms $\beta_0, \beta_1, \cdots, \beta_k$ and (p, q)-form η_k such that

$$\begin{aligned} F_0 \alpha = {} & \bar\partial(\rho\beta_0) + \frac{1}{2}\bar\partial(\rho^2\beta_1) + \cdots + \frac{1}{k+1}\bar\partial(\rho^{k+1}\beta_k) \\ & + \rho^{k+1}\left(\eta_k - \frac{1}{k+1}\bar\partial\beta_k \right). \end{aligned} \tag{9.2.7}$$

From (9.2.6 a) and (9.2.6 b), we can write $F_0\alpha = \bar\partial\rho \wedge \beta_0 + \rho\eta_0 = \bar\partial(\rho\beta_0) + \rho(\eta_0 - \bar\partial\beta_0)$ for some $(p, q-1)$-form β_0 and (p, q)-form η_0. This proves (9.2.7) for $k = 0$.

Assuming (9.2.7) is proved for $k \ge 0$, from (9.2.6 a),

$$\bar\partial F_0 \alpha = (k+1)\rho^k \bar\partial\rho \wedge \left(\eta_k - \frac{1}{k+1}\bar\partial\beta_k \right) + \rho^{k+1}\bar\partial\eta_k = \bar\partial\alpha'.$$

Since $\bar\partial\alpha'$ vanishes to arbitrarily high order at the boundary M, we have $\bar\partial\rho \wedge (\eta_k - 1/(k+1)\bar\partial\beta_k) = 0$ on M and there exist a $(p, q-1)$-form β_{k+1} and a (p,q)-form η_{k+1} such that $\eta_k - 1/(k+1)\bar\partial\beta_k = \bar\partial\rho \wedge \beta_{k+1} + \rho\eta_{k+1}$. Substituting this into (9.2.7), we obtain

$$\begin{aligned} F_0 \alpha &= \bar\partial(\rho\beta_0) + \frac{1}{2}\bar\partial(\rho^2\beta_1) + \cdots + \frac{1}{k+1}\bar\partial(\rho^{k+1}\beta_k) \\ &\quad + \rho^{k+1}(\bar\partial\rho \wedge \beta_{k+1} + \rho\eta_{k+1}) \\ &= \bar\partial(\rho\beta_0) + \frac{1}{2}\bar\partial(\rho^2\beta_1) + \cdots + \frac{1}{k+1}\bar\partial(\rho^{k+1}\beta_k) + \frac{1}{k+2}\bar\partial(\rho^{k+2}\beta_{k+1}) \\ &\quad + \rho^{k+2}\left(\eta_{k+1} - \frac{1}{k+2}\bar\partial\beta_{k+1} \right). \end{aligned}$$

Thus, (9.2.7) holds for $k+1$ and by induction, for any nonnegative integer k. Setting

$$F_{k+1}\alpha = F_0\alpha - \sum_{i=0}^{k} \frac{1}{i+1}\bar\partial(\rho^{i+1}\beta_i), \tag{9.2.8}$$

we have

$$\bar\partial F_{k+1}\alpha = \bar\partial\alpha' \quad \text{in } \Omega, \tag{9.2.9a}$$

$$F_{k+1}\alpha = O(\rho^{k+1}) \quad \text{at } M. \tag{9.2.9b}$$

Also each β_i is obtained by taking i-th derivatives of the components of $F_0\alpha$. Thus each β_i is smooth and $F_k\alpha \in C^\infty_{(p,q)}(\overline{\Omega})$.

Let $\eta(z)$ and $\tilde\eta_{\epsilon_j}(z)$ be the same as in Lemma 9.2.3, we define

$$F_\infty\alpha = F_0\alpha - \sum_{i=0}^{\infty} \frac{1}{i+1}\bar\partial\big(\tilde\eta_{\epsilon_j}(z)\rho^{i+1}\beta_i\big). \tag{9.2.10}$$

As in the proof of Lemma 9.2.3, we can choose ϵ_i sufficiently small such that the series converges in every C^k norm to some element $F_\infty\alpha$. $F_\infty\alpha$ satisfies $\bar\partial F_\infty\alpha = \bar\partial\alpha'$ in Ω and $F_\infty\alpha = O(\rho^k)$ at M for every $k = 1, 2, \cdots$. Setting $\tilde\alpha = E_\infty\alpha - F_\infty\alpha$, we have $\tilde\alpha = \alpha$ on M and $\bar\partial\tilde\alpha = 0$ in Ω. This proves the theorem.

The extension result proved in Theorem 9.2.1 can be used to study the global solvability of the equation

$$\bar\partial_b u = \alpha \qquad \text{on } M, \tag{9.2.11}$$

where α is a (p,q)-form with smooth coefficients, $0 \le p \le n$ and $1 \le q \le n-1$. it is easy to see that if (9.2.11) is solvable, then α must satisfy

$$\bar\partial_b\alpha = 0, \qquad \text{when} \quad 1 \le q < n-1. \tag{9.2.12}$$

Also using Stokes' theorem, it is easy to see that if (9.2.11) is solvable for some $u \in \mathcal{E}^{p,q-1}(M)$, then α must satisfy

$$\int_M \alpha\wedge\phi = 0, \quad \phi \in \mathcal{E}^{n-p,n-q-1}(M) \cap \operatorname{Ker}(\bar\partial_b) \tag{9.2.12 a}$$

where $1 \le q \le n-1$. We note that using Theorem 9.2.1, we can substitute ϕ in (9.2.12 a) by $\phi \in C^\infty_{(n-p,n-q-1)}(\overline{\Omega}) \cap \operatorname{Ker}(\bar\partial)$.

When $1 \le q < n-1$, condition (9.2.12 a) always implies condition (9.2.12) (regardless of pseudoconvexity). This can be seen easily if we take ϕ in (9.2.12 a) to be of the form $\bar\partial_b f$, where f is any smooth $(n-p, n-q-2)$-form on M.

Theorem 9.2.4. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary M. For any $\alpha \in \mathcal{E}^{p,q}(M)$, where $0 \le p \le n$ and $1 \le q \le n-1$, there exists $u \in \mathcal{E}^{p,q-1}(M)$ satisfying $\bar\partial_b u = \alpha$ on M if and only if the following conditions hold:*

$$\bar\partial_b \alpha = 0 \qquad \textit{on } M, \quad \textit{when } 1 \le q < n-1,$$

and

$$\int_M \alpha \wedge \psi = 0, \quad \phi \in \mathcal{E}^{n-p,0}(M) \cap Ker(\bar\partial_b), \quad \textit{when } q = n-1.$$

Proof. From Theorem 9.2.1, we can extend α to $\tilde\alpha$ such that $\tilde\alpha \in C^\infty_{(p,q)}(\overline{\Omega})$,

$$\bar\partial\tilde\alpha = 0, \qquad \text{in } \Omega$$

and

$$\tilde\alpha \wedge \bar\partial\rho = \alpha \wedge \bar\partial\rho, \qquad \text{on } M.$$

Using Theorem 6.1.1, we can find a $\tilde u \in C^\infty_{(p,q-1)}(\overline{\Omega})$ such that

$$\bar\partial\tilde u = \tilde\alpha \qquad \text{in } \Omega.$$

Denoting the restriction of $\tilde u$ to M by u, we have $\bar\partial\tilde u \wedge \bar\partial\rho = \tilde\alpha \wedge \bar\partial\rho$ on M, or equivalently $\bar\partial_b u = \alpha$ on M. This proves the theorem.

We conclude this chapter with the following theorem:

Theorem 9.2.5. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary M. For any $\alpha \in \mathcal{E}^{p,q}(M)$, where $0 \le p \le n$ and $1 \le q \le n-1$, the following conditions are equivalent:*

(1) *There exists $u \in \mathcal{E}^{p,q-1}(M)$ satisfying $\bar\partial_b u = \alpha$ on M.*
(2) *There exists $\tilde\alpha \in C^\infty_{(p,q)}(\overline{\Omega})$ with $\tau\tilde\alpha = \alpha$ (or $\tilde\alpha = \alpha$) on M and $\bar\partial\tilde\alpha = 0$ in Ω.*
(3) $\int_M \alpha \wedge \psi = 0, \quad \phi \in \mathcal{E}^{n-p,n-q-1}(M) \cap Ker(\bar\partial_b)$.

When $1 \le q < n-1$, the above conditions are equivalent to

(4) $\bar\partial_b \alpha = 0$ *on M.*

9.3 L^2 Existence Theorems and Sobolev Estimates for $\bar\partial_b$

Let M be the boundary of a smooth domain Ω in $\mathbb{C}^n$. We impose the induced metric from $\mathbb{C}^n$ on M and denote square integrable functions on M by $L^2(M)$. The set of (p,q)-forms on M with L^2 coefficients, denoted by $L^2_{(p,q)}(M)$, is the completion of $\mathcal{E}^{p,q}(M)$ under the sum of L^2 norms of the coefficients. We define the space of (p,q)-forms with $C^k(M)$ coefficients by $C^k_{(p,q)}(M)$. In particular, $\mathcal{E}^{p,q}(M) = C^\infty_{(p,q)}(M)$. By using a partition of unity and the tangential Fourier transform, we can define the Sobolev space $W^s(M)$ for any real number s. Let $W^s_{(p,q)}(M)$ be the subspace of $L^2_{(p,q)}(M)$ with $W^s(M)$ coefficients for $s \ge 0$ and the norm in $W^s_{(p,q)}(M)$ is denoted by $\| \ \|_{s(M)}$. It is clear that $W^0_{(p,q)}(M) = L^2_{(p,q)}(M)$ and $\| \ \|_{0(M)} = \| \ \|_M$.

The L^2 closure of $\bar\partial_b$, still denoted by $\bar\partial_b$, is a linear, closed, densely defined operator such that

$$\bar\partial_b : L^2_{(p,q-1)}(M) \to L^2_{(p,q)}(M). \tag{9.3.1}$$

An element $u \in L^2_{(p,q-1)}(M)$ belongs to $\mathrm{Dom}(\bar\partial_b)$ if and only if $\bar\partial_b u$, defined in the distribution sense, is in $L^2_{(p,q)}(M)$.

Our main result in this section is the following theorem.

Theorem 9.3.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary M. For every $\alpha \in W^s_{(p,q)}(M)$, where $0 \le p \le n,\ 1 \le q \le n-2$ and s is a nonnegative integer, such that*

$$\bar\partial_b \alpha = 0 \qquad \text{on } M, \tag{9.3.2}$$

there exists $u \in W^s_{(p,q-1)}(M)$ satisfying $\bar\partial_b u = \alpha$ on M.

When $q = n-1$, $\alpha \in L^2_{(p,n-1)}(M)$ and α satisfies

$$\int_M \alpha \wedge \phi = 0, \quad \phi \in C^\infty_{(n-p,0)}(M) \cap Ker(\bar\partial_b), \tag{9.3.3}$$

there exists $u \in L^2_{(p,n-2)}(M)$ satisfying $\bar\partial_b u = \alpha$ on M.

Corollary 9.3.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary M. Then $\bar\partial_b : L^2_{(p,q-1)}(M) \to L^2_{(p,q)}(M)$, $0 \le p \le n,\ 1 \le q \le n-1$, has closed range in L^2.*

It is easy to see that (9.3.2) and (9.3.3) are necessary conditions for $\bar\partial_b$ to be solvable in L^2. To prove Theorem 9.3.1, we shall first assume that α is smooth. We then show that there exists a constant C_s independent of α such that

$$\| u \|_{s(M)} \le C_s \| \alpha \|_{s(M)} . \tag{9.3.4}$$

Using Theorem 9.2.4, we can find a solution for any smooth α satisfying (9.3.2) or (9.3.3). However, it is not easy to obtain estimates from this construction. We shall use a different method to solve α by exploiting the relationship between the norms on the boundary M and the tangential Sobolev norms. We also introduce the weighted tangential Sobolev norms.

Let ρ be a defining function for Ω. Let $\tilde\Omega$ be a large ball such that $\Omega \subset\subset \tilde\Omega$. We set

$$\Omega^+ = \tilde\Omega \setminus \overline{\Omega}, \quad \Omega^- = \Omega.$$

For a small $\delta > 0$, we set

$$\begin{aligned} \Omega^-_\delta &= \{z \in \Omega^- \mid\ -\delta < \rho(z) < 0\}, \\ \Omega^+_\delta &= \{z \in \Omega^+ \mid\ 0 < \rho(z) < \delta\}, \\ \Omega_\delta &= \{z \in \tilde\Omega \mid\ -\delta < \rho(z) < \delta\}, \end{aligned}$$

and

$$\Gamma_\epsilon = \{z \in \mathbb{C}^n \mid \rho(z) = \epsilon\}.$$

The special tangential norms in a tubular neighborhood Ω_δ^-, Ω_δ^+ and Ω_δ are defined, as in Section 5.2, by

$$|||f|||^2_{s(\Omega_\delta)} = \int_{-\delta}^{\delta} \|f\|^2_{s(\Gamma_\rho)} d\rho,$$

$$|||f|||^2_{s(\Omega_\delta^-)} = \int_{-\delta}^{0} \|f\|^2_{s(\Gamma_\rho)} d\rho, \quad |||f|||^2_{s(\Omega_\delta^+)} = \int_{0}^{\delta} \|f\|^2_{s(\Gamma_\rho)} d\rho.$$

For each $m \in \mathbb{N}$, $s \in \mathbb{R}$, we set

$$|||D^m f|||_{s(\Omega_\delta)} = \sum_{0 \le k \le m} |||D_\rho^k f|||_{s+m-k(\Omega_\delta)}, \tag{9.3.5}$$

where $D_\rho = \partial/\partial\rho$. We also define the weighted tangential Sobolev norms by

$$\begin{aligned} |||\Theta^m f|||_{s(\Omega_\delta)} &= |||\rho^m D^m f|||_{s(\Omega_\delta)} \\ &= \sum_{0 \le k \le m} |||\rho^m D_\rho^k f|||_{s+m-k(\Omega_\delta)}, \end{aligned} \tag{9.3.6}$$

and similarly

$$|||D\Theta^m f|||_{s(\Omega_\delta)} = \sum_{0 \le k \le m} |||D\rho^m D_\rho^k f|||_{s+m-k(\Omega_\delta)}.$$

Thus Θ can be viewed as a first order differential operator weighted with ρ. Corresponding norms are also defined on Ω_δ^- and Ω_δ^+ similarly. We always assume that $\delta > 0$ to be sufficiently small without specifying so explicitly in the following lemmas.

The next lemma on the extension of smooth functions from the boundary is the key to the proof of (9.3.4).

Lemma 9.3.3. *Let M be the boundary of a smooth domain Ω in $\mathbb{C}^n$. For arbitrary smooth functions u_j on M, $j = 0, 1, \cdots, k_0$, there exists a function $Eu \in C_0^\infty(\Omega_\delta)$ such that $D_\rho^j Eu = u_j$ on M, $j = 0, 1, \cdots, k_0$. Furthermore, for every real number s and nonnegative integer m, there exists a positive constant C depending on m and s but independent of the u_j's such that*

$$|||D^m Eu|||_{s-m+\frac{1}{2}(\Omega_\delta)} \le C \sum_{j=0}^{k_0} \|u_j\|_{s-j(M)}, \tag{9.3.7 i}$$

$$|||\Theta^m Eu|||_{s+\frac{1}{2}(\Omega_\delta)} \le C \sum_{j=0}^{k_0} \|u_j\|_{s-j(M)}. \tag{9.3.7 ii}$$

Proof. Using a partition of unity, it suffices to prove the lemma assuming that u_j is supported in a small neighborhood $U \cap M$, where $U \subset \mathbb{C}^n$ and that there exists a special boundary coordinate chart on U with coordinates $t_1, \cdots, t_{2n-1}, \rho$.

The Fourier transform for u in the special boundary chart is defined by

$$\hat{u}(\tau) = \int_{\mathbb{R}^{2n-1}} e^{-i\langle t,\tau\rangle} u(t)dt,$$

where $\tau = (\tau_1, \cdots, \tau_{2n-1})$ and $\langle t, \tau\rangle = t_1\tau_1 + \cdots + t_{2n-1}\tau_{2n-1}$.

Let ψ be a function in $C_0^\infty(\mathbb{R})$ which is equal to 1 in a neighborhood of 0 and let the partial Fourier transform of Eu be

$$(Eu)\tilde{}\,(\tau, \rho) = \psi(\lambda\rho) \sum_{j=0}^{k_0} \hat{u}_j(\tau) \frac{\rho^j}{j!}, \tag{9.3.8}$$

where $\lambda = (1 + |\tau|^2)^{1/2}$. It is easy to see that $D_\rho^j Eu = u_j$ on M.

To prove (9.3.7 i) and (9.3.7 ii), we note that for every nonnegative integer i, by a change of variables, there exists some $C > 0$ such that

$$\int_{-\infty}^{\infty} |D_\rho^i(\psi(\lambda\rho)\rho^j)|^2 d\rho = \lambda^{2(i-j)-1} \int_{-\infty}^{\infty} |D_\rho^i(\psi(\rho)\rho^j)|^2 d\rho \le C\lambda^{2(i-j)-1}.$$

Thus we have

$$\begin{aligned}
|||D^m Eu|||^2_{s-m+\frac{1}{2}(\Omega_\delta)} &= \sum_{0\le k\le m} |||D_\rho^k Eu|||^2_{s-k+\frac{1}{2}(\Omega_\delta)} \\
&\le C \sum_{j=0}^{k_0} \int_{\mathbb{R}^{2n-1}} \lambda^{2s-2j} |\hat{u}_j(\tau)|^2 d\tau \\
&\le C \sum_{j=0}^{k_0} \|u_j\|^2_{s-j(M)},
\end{aligned}$$

which proves (9.3.7 i). Since

$$\begin{aligned}
\int_{-\infty}^{\infty} \rho^{2m} |D_\rho^i(\psi(\lambda\rho)\rho^j)|^2 d\rho &= \lambda^{2(i-j-m)-1} \int_{-\infty}^{\infty} \rho^{2m} |D_\rho^i(\psi(\rho)\rho^j)|^2 d\rho \\
&\le C\lambda^{2(i-j-m)-1},
\end{aligned}$$

we see that

$$\begin{aligned}
|||\Theta^m Eu|||^2_{s+\frac{1}{2}(\Omega_\delta)} &= \sum_{0\le k\le m} |||\rho^m D_\rho^k Eu|||^2_{s+m-k+\frac{1}{2}(\Omega_\delta)} \\
&\le C \sum_{j=0}^{k_0} \int_{\mathbb{R}^{2n-1}} \lambda^{2s-2j} |\hat{u}_j(\tau)|^2 d\tau \\
&\le C \sum_{j=0}^{k_0} \|u_j\|^2_{s-j(M)}.
\end{aligned}$$

This proves (9.3.7 ii) and the lemma.

Estimate (9.3.7 i) shows that when one extends a function from a smooth boundary, one can have a "gain" of one half derivative. Estimate (9.3.7 ii) shows that the operator ρD for any first order derivative D should be treated as an operator of order zero in view of extension of functions from the boundary. This fact is crucial in the proof of Theorem 9.3.1. We also remark that Lemma 9.3.3 also holds for $k_0 = \infty$ using arguments similar to Lemma 9.2.3.

Lemma 9.3.4. *Let M be the boundary of a smooth bounded domain Ω in $\mathbb{C}^n$. Let $\alpha \in C^\infty_{(p,q)}(M)$, $0 \le p \le n$, $1 \le q \le n-1$, and $\overline{\partial}_b \alpha = 0$ on M. For every positive integer k, there exists a smooth extension $E_k\alpha$ with support in a tubular neighborhood Ω_δ such that $E_k\alpha \in C^\infty_{(p,q)}(\Omega_\delta)$, $E_k\alpha = \alpha$ on M and*

$$\bar\partial E_k \alpha = O(\rho^k) \quad \text{at } M. \tag{9.3.9}$$

Furthermore, for every real number s and nonnegative integer m, there exists a positive constant C_k depending on m and s but independent of α such that

$$|||D^m E_k\alpha|||_{s-m+\frac{1}{2}(\Omega_\delta)} \le C_k \|\alpha\|_{s(M)}, \tag{9.3.10 i}$$

$$|||\Theta^m E_k\alpha|||_{s+\frac{1}{2}(\Omega_\delta)} \le C_k \|\alpha\|_{s(M)}. \tag{9.3.10 ii}$$

Proof. Using Lemma 9.3.3 with $k_0 = 0$, we first extend α componentwise and smoothly from M to $E\alpha$ in $\mathbb{C}^n$ such that $E\alpha$ has compact support in Ω_δ and satisfies the estimates

$$|||D^m E\alpha|||_{s-m+\frac{1}{2}(\Omega_\delta)} \le C \|\alpha\|_{s(M)}, \tag{9.3.11 i}$$

$$|||\Theta^m E\alpha|||_{s+\frac{1}{2}(\Omega_\delta)} \le C \|\alpha\|_{s(M)}, \tag{9.3.11 ii}$$

where C depends on m and s but is independent of α. Using Lemma 9.2.3, for every positive integer k, there exist smooth (p,q)-forms $\alpha_1, \cdots, \alpha_k$ and $(p,q+1)$-forms $\gamma_1, \cdots, \gamma_k$ such that

$$E_k\alpha = E\alpha - \rho\alpha_1 - \frac{\rho^2}{2}\alpha_2 - \cdots - \frac{\rho^k}{k}\alpha_k, \tag{9.3.12 a}$$

and

$$\bar\partial E_k\alpha = \rho^k\left(\gamma_k - \frac{1}{k}\bar\partial\alpha_k\right) = O(\rho^k) \quad \text{at } M. \tag{9.3.12 b}$$

From the proof of Lemma 9.2.3, each component of α_i is a linear combination of the i-th derivatives of $E\alpha$. To show that $E_k\alpha$ satisfies the estimates, it suffices to estimate each $\rho^i\alpha_i$. Using (9.3.11 i) and (9.3.11 ii), we have for any $s \in \mathbb{R}$,

$$\begin{aligned} |||E_k\alpha|||_{s+\frac{1}{2}(\Omega_\delta)} &\le |||E\alpha|||_{s+\frac{1}{2}(\Omega_\delta)} + \sum_{1\le i\le k} |||\rho^i\alpha_i|||_{s+\frac{1}{2}(\Omega_\delta)} \\ &\le C \sum_{0\le i\le k} |||\Theta^i E\alpha|||_{s+\frac{1}{2}(\Omega_\delta)} \\ &\le C\|\alpha\|_{s(M)}. \end{aligned}$$

Again using (9.3.11 i) and (9.3.11 ii), we have for any $s \in \mathbb{R}$, $m \in \mathbb{N}$,

$$|||\Theta^m E_k \alpha|||_{s+\frac{1}{2}(\Omega_\delta)} \le C \sum_{0 \le i \le k} |||\Theta^{m+i} E\alpha|||_{s+\frac{1}{2}(\Omega_\delta)} \le C_k \|\alpha\|_{s(M)},$$

and

$$\begin{aligned} |||D^m E_k \alpha|||_{s-m+\frac{1}{2}(\Omega_\delta)} &\le C \sum_{0 \le i \le k} |||\Theta^i D^m E\alpha|||_{s-m+\frac{1}{2}(\Omega_\delta)} \\ &\le C_k \|\alpha\|_{s(M)}, \end{aligned}$$

where we have used (9.3.7 i) and (9.3.7 ii). This proves Lemma 9.3.4.

The following decomposition of $\overline{\partial}_b$-closed forms on M as the difference between two $\bar{\partial}$-closed forms is an analog of the jump formula for CR functions discussed in Theorem 2.2.3.

Lemma 9.3.5. *Let M be the boundary of a smooth bounded domain Ω in $\mathbb{C}^n$. Let $\alpha \in C^\infty_{(p,q)}(M)$ with $\overline{\partial}_b \alpha = 0$, $0 \le p \le n$, $0 \le q \le n-1$. For each positive integer k, there exist $\alpha^+ \in C^k_{(p,q)}(\overline{\Omega^+})$ and $\alpha^- \in C^k_{(p,q)}(\overline{\Omega})$ such that $\overline{\partial}\alpha^+ = 0$ in Ω^+, $\overline{\partial}\alpha^- = 0$ in Ω and the following decomposition holds:*

$$\alpha^+ - \alpha^- = \alpha \qquad \text{on } M. \tag{9.3.13}$$

Furthermore, we have the following estimates: for every integer $0 \le s \le k-1$, $0 \le m \le s$,

$$|||D^m \alpha^+|||_{s-m+\frac{1}{2}(\Omega^+_\delta)} \le C\|\alpha\|_{s(M)}, \tag{9.3.14 i}$$

$$|||D^m \alpha^-|||_{s-m+\frac{1}{2}(\Omega^-_\delta)} \le C\|\alpha\|_{s(M)}, \tag{9.3.14 ii}$$

where the constant C depends only on m, s, but is independent of α.

Proof. Let k_0 be a positive integer to be determined later. Using Lemma 9.3.4, we extend α from M to $E_{k_0}\alpha$ in $\mathbb{C}^n$ smoothly such that $E_{k_0}\alpha = \alpha$ on M, $E_{k_0}\alpha$ has compact support in Ω_δ and $E_{k_0}\alpha$ satisfies (9.3.9) and (9.3.10) with $k = k_0$.

We define a $(p, q+1)$-form $\tilde{U}_{k_0}$ in $\tilde{\Omega}$ by

$$\tilde{U}_{k_0} = \begin{cases} -\overline{\partial} E_{k_0}\alpha, & \text{if } z \in \Omega^-, \\ 0, & \text{if } z \in M, \\ \overline{\partial} E_{k_0}\alpha, & \text{if } z \in \Omega^+. \end{cases}$$

From (9.3.9), we have $\tilde{U}_{k_0} \in C^{k_0-1}(\tilde{\Omega})$ and $\overline{\partial}\tilde{U}_{k_0} = 0$ in $\tilde{\Omega}$ (in the distribution sense if $k_0 = 1$). It follows from (9.3.10 i) that for any nonnegative integer m, $0 \le m \le k_0 - 1$,

$$\begin{aligned} |||D^m \tilde{U}_{k_0}|||_{s-m-\frac{1}{2}(\Omega_\delta)} &\le C|||D^{m+1} E_{k_0}\alpha|||_{s-m-\frac{1}{2}(\Omega_\delta)} \\ &\le C\|\alpha\|_{s(M)}. \end{aligned} \tag{9.3.15}$$

We define $V_{k_0} = \bar{\partial}^* N^{\tilde{\Omega}}_{(p,q+1)} \tilde{U}_{k_0}$. It follows from Theorem 4.4.1 that $\overline{\partial} V_{k_0} = \tilde{U}_{k_0}$ in $\tilde{\Omega}$. Since $\Box_{(p,q+1)}$ is elliptic in the interior of the domain $\tilde{\Omega}$, $\bar{\partial}^* N^{\tilde{\Omega}}_{(p,q+1)}$ gains one derivative in the interior. Since $\tilde{U}_{k_0}$ has compact support in Ω_δ, we get from (9.3.15) that

$$\begin{aligned} |||D^m V_{k_0}|||_{s-m+\frac{1}{2}(\Omega_\delta)} &\le C|||D^{m+1} E_{k_0}\alpha|||_{s-m-\frac{1}{2}(\Omega_\delta)} \\ &\le C\|\alpha\|_{s(M)}, \end{aligned} \tag{9.3.16}$$

for some $C > 0$ independent of α.

Setting

$$\begin{aligned} \alpha^+ &= \frac{1}{2}(E_{k_0}\alpha - V_{k_0}), \quad z \in \overline{\Omega^+}, \\ \alpha^- &= -\frac{1}{2}(E_{k_0}\alpha + V_{k_0}), \quad z \in \overline{\Omega}, \end{aligned}$$

we see that

$$\alpha = E_{k_0}\alpha = (\alpha^+ - \alpha^-) \qquad \text{on } M.$$

We also have

$$\overline{\partial}\alpha^+ = \frac{1}{2}(\overline{\partial}E_{k_0}\alpha - \overline{\partial}V_{k_0}) = \frac{1}{2}(\overline{\partial}E_{k_0}\alpha - \tilde{U}_{k_0}) = 0 \qquad \text{in } \ \Omega^+,$$

and

$$\overline{\partial}\alpha^- = -\frac{1}{2}(\overline{\partial}E_{k_0}\alpha + \overline{\partial}V_{k_0}) = -\frac{1}{2}(\overline{\partial}E_{k_0}\alpha + \tilde{U}_{k_0}) = 0 \qquad \text{in } \ \Omega.$$

If we choose k_0 sufficiently large ($k_0 \ge n + k + 1$), then $\alpha^+ \in C^k_{(p,q)}(\Omega^+ \cup M)$ and $\alpha^- \in C^k_{(p,q)}(\overline{\Omega})$ by the Sobolev embedding theorem. The estimates (9.3.14 i) and (9.3.14 ii) follow easily from (9.3.16) and (9.3.10). Since this is true for an arbitrarily large ball $\tilde{\Omega}$, the lemma is proved.

Using the weighted $\bar{\partial}$-Neumann operator on Ω^-, we can solve $\bar{\partial}u^- = \alpha^-$ in Ω^- with good estimates up to the boundary. To solve $\bar{\partial}$ for α^+ in Ω^+, we use the following lemma to extend α^+ to be $\bar{\partial}$-closed in $\tilde{\Omega}$ with good estimates.

Lemma 9.3.6. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary M. Let $\alpha \in C^\infty_{(p,q)}(M)$ such that $\overline{\partial}_b\alpha = 0$, where $0 \le p \le n$, $0 \le q < n-1$. For every nonnegative integer k_1, there exists $\tilde{\alpha}^+$ in $C^{k_1}_{(p,q)}(\tilde{\Omega})$ and $\alpha^- \in C^{k_1}_{(p,q)}(\overline{\Omega})$ such that $\overline{\partial}\tilde{\alpha}^+ = 0$ in $\tilde{\Omega}$, $\overline{\partial}\alpha^- = 0$ in Ω^- and $\tilde{\alpha}^+ - \alpha^- = \alpha$ on M. Furthermore, for every $0 \le s \le k_1$, there exists a constant C depending only on s but independent of α such that*

$$\| \tilde{\alpha}^+ \|_{s-\frac{1}{2}(\tilde{\Omega})} \le C\|\alpha\|_{s(M)}. \tag{9.3.17 i}$$

$$\| \alpha^- \|_{s+\frac{1}{2}(\Omega)} \le C\|\alpha\|_{s(M)}. \tag{9.3.17 ii}$$

When $q = n-1$, $\alpha \in C^\infty_{(p,n-1)}(M)$ and α satisfies

$$\int_M \alpha \wedge \phi = 0, \quad \phi \in C^\infty_{(n-p,0)}(M) \cap Ker(\bar{\partial}_b), \tag{9.3.18}$$

the same conclusion holds.

Proof. Let k be an integer with $k > 2(k_1+n)$ and let α^+, α^- be defined as in Lemma 9.3.5. For any $s \ge 0$ and $0 \le m \le s$, using arguments similar to those in the proof of Lemma 5.2.3, the norm $|||D^m \alpha^-|||_{s-m+\frac12(\Omega_\delta^-)}$ is equivalent to $\| \alpha^- \|_{s+\frac12(\Omega_\delta^-)}$. Thus (9.3.17 ii) follows immediately from (9.3.14 ii).

We next extend α^+ to $\tilde\Omega$. By the trace theorem for Sobolev spaces and inequality (9.3.14 i), we have for any integer $0 \le j \le s-1$, $0 \le s \le k$,

$$\|D_\rho^j \alpha^+\|_{s-j(M)} \le C\|D_\rho^{j+1}\alpha^+\|_{s-j-\frac12(\Omega_\delta^+)} \le C\|\alpha\|_{s(M)}.$$

Using the proof of Lemma 9.3.3, we can extend α^+ from Ω^+ to α' in $\tilde\Omega$ such that $\alpha' \in C^k(\tilde\Omega)$, $\overline\partial\alpha' = 0$ in Ω^+ and the following estimates hold: for any integer $0 \le s \le k-1$, $0 \le m \le s$,

$$|||D^m\alpha'|||_{s-m+\frac12(\Omega_\delta^-)} \le C\|\alpha\|_{s(M)}, \tag{9.3.19 i}$$

$$|||\Theta^m\alpha'|||_{s+\frac12(\Omega_\delta^-)} \le C\|\alpha\|_{s(M)}. \tag{9.3.19 ii}$$

We define

$$F\alpha' = -\star\overline\partial N^t_{(n-p,n-q-1)}\star\bar\partial\alpha' \quad \text{in } \Omega,$$

where $N^t_{(n-p,n-q-1)}$ is the weighted $\bar\partial$-Neumann operator on $(n-p, n-q-1)$-forms. Using Theorem 6.1.4, we can choose t sufficiently large such that $F\alpha' \in W^{k-1}_{(p,q)}(\Omega) \subset C^{2k_1+1}_{(p,q)}(\overline\Omega)$. We set $F\alpha' = 0$ outside Ω.

When $q = n-1$, using the definition of α^+, for every $\bar\partial$-closed form $\phi \in C^\infty_{(n-p,0)}(\overline\Omega)$,

$$\begin{aligned}\int_\Omega \bar\partial\alpha' \wedge \phi &= \int_\Omega \bar\partial(\alpha' \wedge \phi) = \int_M \alpha' \wedge \phi = -\frac12\int_M V_{k_0} \wedge \phi \\ &= -\frac12 \int_\Omega \tilde U_{k_0} \wedge \phi = \frac12 \int_M \alpha \wedge \phi = 0,\end{aligned}$$

by (9.3.18). Thus $\bar\partial\alpha'$ satisfies condition (9.1.9). Using Theorem 9.1.2 ($1 \le q \le n-2$) and Theorem 9.1.3 ($q = n-1$) , it follows that

$$\begin{cases} \bar\partial F\alpha' = \bar\partial\alpha' & \text{in } \tilde\Omega, \\ F\alpha' = 0 & \text{in } \Omega^+. \end{cases}$$

We modify $F\alpha'$ to make it smooth at M. Setting $F_0\alpha' = F\alpha'$ and using arguments similar to those in Lemma 9.2.3, one can choose $(p, q-1)$-forms $\beta_0, \cdots, \beta_{k_1}$ and define

$$F_{k_1+1}\alpha' = F_0\alpha' - \sum_{i=0}^{k_1} \frac{1}{i+1}\bar\partial(\rho^{i+1}\beta_i),$$

such that

$$\bar\partial F_{k_1+1}\alpha' = \bar\partial\alpha' \quad \text{in } \Omega,$$

and

$$F_{k_1+1}\alpha' = O(\rho^{k_1+1}) \quad \text{at } M.$$

Each β_i is obtained by taking i-th derivatives of the components of $F_0\alpha'$. Thus $F_{k_1+1}\alpha' \in C^{k_1}_{(p,q)}(\overline{\Omega})$. If we set $F_{k_1+1}\alpha' = 0$ in $\tilde{\Omega} \setminus \Omega$, then $F_{k_1+1}\alpha' \in C^{k_1}_{(p,q)}(\tilde{\Omega})$. We define

$$\tilde{\alpha}^+ = \begin{cases} \alpha' - F_{k_1+1}\alpha', & \text{in } \Omega, \\ \alpha' - F_{k_1+1}\alpha' = \alpha^+, & \text{in } \tilde{\Omega} \setminus \Omega, \end{cases}$$

then $\tilde{\alpha}^+ \in C^{k_1}_{(p,q)}(\tilde{\Omega})$ and $\overline{\partial}\tilde{\alpha}^+ = 0$ in $\tilde{\Omega}$. It remains to show that $\tilde{\alpha}^+$ satisfies (9.3.17 i). Since α^+ satisfies (9.3.14 i) and α' satisfies (9.3.19 i), to estimate $\tilde{\alpha}^+$, we only need to estimate $F_{k_1+1}\alpha'$ in Ω. To prove (9.3.17 i), it suffices to show

$$\| F_{k_1+1}\alpha' \|_{s-\frac{1}{2}(\Omega)} \le C \| \alpha \|_{s(M)} . \tag{9.3.20}$$

From Theorem 6.1.4, we have the estimates

$$\| F\alpha' \|_{s-\frac{1}{2}(\Omega)} \le C \| \bar{\partial}\alpha' \|_{s-\frac{1}{2}(\Omega)} \le C \| \alpha \|_{s(M)} . \tag{9.3.21}$$

We claim that for each positive integer $0 \le m \le k_1 + 1$, $0 \le s \le k_1$,

$$|||\rho^m D^m F\alpha'|||_{s-\frac{1}{2}(\Omega^-_\delta)} \le C \| \alpha \|_{s(M)} . \tag{9.3.22}$$

If the claim is true, then (9.3.20) holds from our construction of $F_{k_1+1}\alpha'$, since $F_{k_1+1}\alpha'$ can be written as combinations of terms in $F\alpha'$ and $\rho^m D^m F\alpha'$. Thus it remains to prove (9.3.22).

(9.3.21) implies that (9.3.22) holds when $m = 0$. To prove the claim for $m > 0$, it suffices to show for each positive integer $0 \le m \le k_1 + 1$, $0 \le s \le k_1$,

$$|||D\Theta^m F\alpha'|||_{s-\frac{3}{2}(\Omega^-_\delta)} \le C \| \alpha \|_{s(M)} \tag{9.3.23}$$

since $F\alpha'$ satisfies an elliptic system $\bar{\partial} \oplus \vartheta_t$. Decompose Ω^-_δ into subdomains Ω_j such that

$$\Omega_j = \{z \in \Omega^-_\delta \mid \delta_{j+1} < -\rho(z) < \delta_j\},$$

where $\delta_j = \delta/2^j$. This is a Whitney type decomposition where the thickness of each Ω_j is comparable to the distance of Ω_j to the boundary. We define $\Omega^*_j = \Omega_{j-1} \cup \overline{\Omega}_j \cup \Omega_{j+1}$. Let ϕ_j be a function in $C^\infty_0(\Omega^*_j)$ such that $0 \le \phi_j \le 1$, $\phi_j = 1$ on Ω_j. Moreover,

$$\sup |\text{grad}\phi_j| \le C\delta_j^{-1}, \tag{9.3.24}$$

where C is independent of j. Since $\bar{\partial} \oplus \vartheta_t$ is elliptic and $\phi_j\Theta^m F\alpha'$ is supported in Ω^*_j, applying Gårding's inequality, we have

$$\begin{aligned} &|||D(\phi_j\Theta^m F\alpha')|||_{s-\frac{3}{2}(\Omega^*_j)} \\ &\le C\big(|||\bar{\partial}(\phi_j\Theta^m F\alpha')|||_{s-\frac{3}{2}(\Omega^*_j)} + |||\vartheta_t(\phi_j\Theta^m F\alpha')|||_{s-\frac{3}{2}(\Omega^*_j)} \\ &\quad + |||\phi_j\Theta^m F\alpha'|||_{s-\frac{3}{2}(\Omega^*_j)}\big), \end{aligned} \tag{9.3.25}$$

where C is independent of j. We also know that

$$
\begin{aligned}
&|||\bar\partial(\phi_j\Theta^m F\alpha')|||_{s-\frac{3}{2}(\Omega_j^*)} \\
&\le |||\bar\partial(\phi_j)\Theta^m F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)} + |||\phi_j\Theta^m\bar\partial F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)} \\
&\quad + |||\phi_j[\bar\partial,\Theta^m]F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)},
\end{aligned}
\tag{9.3.26}
$$

and using (9.3.24),

$$
\begin{aligned}
&|||\bar\partial(\phi_j)\Theta^m F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)} + |||\phi_j[\bar\partial,\Theta^m]F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)} \\
&\qquad \le C|||D\Theta^{m-1}F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)}.
\end{aligned}
\tag{9.3.27}
$$

Substituting (9.3.27) into (9.3.26), we obtain

$$
\begin{aligned}
&|||\bar\partial(\phi_j\Theta^m F\alpha')|||_{s-\frac{3}{2}(\Omega_j^*)} \\
&\le C(|||D\Theta^{m-1}F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)} + |||\Theta^m\bar\partial\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)}) \\
&\le C(|||D\Theta^{m-1}F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)} + |||D\Theta^m\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)}).
\end{aligned}
\tag{9.3.28}
$$

Similarly,

$$
\begin{aligned}
&|||\vartheta_t(\phi_j\Theta^m F\alpha')|||_{s-\frac{3}{2}(\Omega_j^*)} \\
&\le |||\vartheta_t(\phi_j)\Theta^m F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)} + |||\phi_j\Theta^m\vartheta_t F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)} \\
&\qquad + |||\phi_j[\vartheta_t,\Theta^m]F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)} \\
&\le C|||D\Theta^{m-1}F\alpha'|||_{s-\frac{3}{2}(\Omega_j^*)}.
\end{aligned}
\tag{9.3.29}
$$

Substituting (9.3.28) and (9.3.29) into (9.3.25) and summing over j, we have using induction,

$$
\begin{aligned}
&|||D\Theta^m F\alpha'|||_{s-\frac{3}{2}(\Omega^-_{\frac{\delta}{2}})} \\
&\le C(|||D\Theta^{m-1}F\alpha'|||_{s-\frac{3}{2}(\Omega^-_\delta)} + |||\Theta^m\bar\partial\alpha'|||_{s-\frac{3}{2}(\Omega^-_\delta)}) \\
&\le C \parallel \alpha \parallel_{s(M)} .
\end{aligned}
$$

This proves (9.3.23) for a smaller δ. Thus (9.3.17 i) holds. The proof of Lemma 9.3.6 is complete.

We note that both Lemma 9.3.5 and Lemma 9.3.6 hold for $q = 0$, i.e., when α is CR. When $q = 0$ and $n = 1$, Lemma 9.3.5 corresponds to the Plemelj jump formula in one complex variable (see Theorem 2.1.3). In this case, there is no condition on α since $\bar\partial_b\alpha = 0$ on M is always satisfied. In contrast, there is a compatibility condition (9.3.18) for $q = n-1$ in Lemma 9.3.6 for $n > 1$. One should compare this case with Corollary 2.1.4.

Using Lemma 9.3.6, we have the following lemma for smooth $\bar\partial_b$-closed forms:

Lemma 9.3.7. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary M. Let $\alpha \in C^\infty_{(p,q)}(M)$, where $0 \le p \le n$, $1 \le q \le n-2$, such that $\bar\partial_b \alpha = 0$ on M. For every nonnegative integer s, there exists $u_s \in W^s_{(p,q-1)}(M)$ satisfying $\bar\partial_b u_s = \alpha$ on M. Furthermore, there exists a constant C_s independent of α such that*

$$\|u_s\|_{s(M)} \le C_s \|\alpha\|_{s(M)}. \tag{9.3.30}$$

When $q = n-1$, $\alpha \in C^\infty_{(p,n-1)}(M)$ and α satisfies (9.3.3), the same conclusion holds.

Proof. Let s be a fixed nonnegative integer. From Lemma 9.3.6, there exists a decomposition $\alpha = (\tilde\alpha^+ - \alpha^-)$ such that $\tilde\alpha^+ \in C^s_{(p,q)}(\tilde\Omega)$, $\bar\partial\tilde\alpha^+ = 0$ in $\tilde\Omega$, $\alpha^- \in C^s_{(p,q)}(\overline{\Omega})$ and $\bar\partial\alpha^- = 0$ in Ω. We also have the estimates:

$$\|\tilde\alpha^+\|_{s-\frac12(\tilde\Omega)} \le C\|\alpha\|_{s(M)}, \tag{9.3.31 i}$$

$$\|\alpha^-\|_{s+\frac12(\Omega)} \le C\|\alpha\|_{s(M)}. \tag{9.3.31 ii}$$

(9.3.31 i) and (9.3.31 ii) follow from (9.3.17 i) and (9.3.17 ii).

Since Ω is pseudoconvex, we define

$$u^- = \overline{\partial}{}^*_t N^t_{(p,q)}\alpha^-,$$

where $N^t_{(p,q)}$ is the weighted $\bar\partial$-Neumann operator in Ω. If we choose $t > 0$ sufficiently large, using Theorem 6.1.4, it follows that $u^- \in W^{s+\frac12}_{(p,q-1)}(\Omega)$, $\overline{\partial} u^- = \alpha^-$ and

$$|||Du^-|||_{s-\frac12(\Omega^-_\delta)} \le C\|\alpha^-\|_{s+\frac12(\Omega)} \le C\|\alpha\|_{s(M)}$$

for some constant C independent of α. Restricting u^- to the boundary we have $\bar\partial_b u^- = \tau\alpha^-$ on M and using the trace theorem for Sobolev spaces, we obtain

$$\|u^-\|_{s(M)} \le C_s |||Du^-|||_{s-\frac12(\Omega^-_\delta)} \le C_s\|\alpha\|_{s(M)}. \tag{9.3.32}$$

Defining

$$\tilde u^+ = \overline{\partial}{}^* N^{\tilde\Omega}_{(p,q)}\tilde\alpha^+,$$

where $N^{\tilde\Omega}_{(p,q)}$ is the $\bar\partial$-Neumann operator on $\tilde\Omega$, we have $\bar\partial\tilde u^+ = \tilde\alpha^+$ in $\tilde\Omega$ and $\tilde u^+$ is one derivative smoother than $\tilde\alpha^+$ in the interior of $\tilde\Omega$. From (9.3.31 i),

$$|||D\tilde u^+|||_{s-\frac12(\Omega_\delta)} \le C\|\tilde\alpha^+\|_{s-\frac12(\tilde\Omega)} \le C\|\alpha\|_{s(M)}$$

for some constant C independent of α. Restricting $\tilde u^+$ to M, using the trace theorem again,

$$\|\tilde u^+\|_{s(M)} \le C|||D\tilde u^+|||_{s-\frac12(\Omega_\delta)} \le C\|\alpha\|_{s(M)} \tag{9.3.33}$$

for some constant C independent of α. Letting

$$u_s = \tilde u^+ - u^- \qquad \text{on } M,$$

we get $\overline{\partial}_b u_s = \alpha$ on M. We also have from (9.3.32) and (9.3.33),

$$\|u_s\|_{s(M)} \le C_s\|\alpha\|_{s(M)},$$

where C_s is independent of α. This proves (9.3.30) and the lemma.

To prove Theorem 9.3.1, we need the following density lemmas which are of interest independently.

Lemma 9.3.8. *Let M be the boundary of a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$, $n \geq 2$. For each $0 \leq p \leq n$, $0 \leq q < n-1$ and $s \geq 0$, the space $C^\infty_{(p,q)}(M) \cap ker(\bar{\partial}_b)$ is dense in $W^s_{(p,q)}(M) \cap ker(\bar{\partial}_b)$ in the $W^s_{(p,q)}(M)$ norm.*

Proof. We define $\mathcal{Z}^\infty = C^\infty_{(p,q)}(M) \cap \ker(\bar{\partial}_b)$ and $\mathcal{Z}^s = W^s_{(p,q)}(M) \cap \ker(\bar{\partial}_b)$. For any $\alpha \in \mathcal{Z}^s$, using Friedrichs' Lemma (see Appendix D), there exists a sequence of smooth forms $\alpha_m \in C^\infty_{(p,q)}(M)$ such that $\alpha_m \to \alpha$ in $W^s_{(p,q)}(M)$ and $\bar{\partial}_b \alpha_m \to 0$ in $W^s_{(p,q+1)}(M)$. Since $\bar{\partial}_b \alpha_m$ is a smooth form satisfying the compatibility condition (9.3.2) (when $q < n-2$) and (9.3.3) (when $q = n-2$), Lemma 9.3.7 implies that there exists a sufficiently smooth form v_m such that $\bar{\partial}_b v_m = \bar{\partial}_b \alpha_m$ in M with

$$\|v_m\|_{s(M)} \leq C_s \|\bar{\partial}_b \alpha_m\|_{s(M)} \to 0.$$

We set

$$\alpha'_m = \alpha_m - v_m.$$

Then $\bar{\partial}_b \alpha'_m = 0$ and α'_m converges to α in $W^s_{(p,q)}(M)$. Thus, $\mathcal{Z}^k$ is dense in $\mathcal{Z}^s$ in the $W^s_{(p,q)}(M)$ norm where k is an arbitrarily large integer.

For any $\epsilon > 0$ and each positive integer $k > s$, there exists an $\alpha_k \in \mathcal{Z}^k$ such that

$$\|\alpha_k - \alpha\|_{s(M)} < \epsilon.$$

Furthermore, we can require that

$$\|\alpha_k - \alpha_{k+1}\|_{k(M)} < \frac{\epsilon}{2^k}$$

since $\mathcal{Z}^{k+1}$ is dense in $\mathcal{Z}^k$. The series

$$\alpha_k + \sum_{N=k+1}^{\infty} (\alpha_N - \alpha_{N-1})$$

converges in every $W^k_{(p,q)}(M)$ norm to some element α_∞. The Sobolev embedding theorem then assures that α_∞ is in $C^\infty_{(p,q)}(M) \cap \mathrm{Ker}(\bar{\partial}_b)$. We also have

$$\|\alpha_\infty - \alpha\|_{s(M)} < 2\epsilon.$$

This proves the lemma.

When $q = n-1$, we have the density lemma in the L^2 norm.

Lemma 9.3.9. *Let M be the boundary of a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$, $n \geq 2$ and $0 \leq p \leq n$. Let $\mathcal{Z}$ denote the space of all forms in $L^2_{(p,n-1)}(M)$ satisfying (9.3.3) and $\mathcal{Z}^\infty$ be the subspace of all forms in $C^\infty_{(p,n-1)}(M)$ satisfying (9.3.3). Then $\mathcal{Z}^\infty$ is dense in $\mathcal{Z}$ in the L^2 norm.*

Proof. Since the holomorphic degree p plays no role, for simplicity we assume $p = n$. If $\alpha \in L^2_{(n,n-1)}(M)$, we can write $\alpha = f(\star\bar{\partial}\rho)$ for some $f \in L^2(M)$. Using the Hahn-Banach theorem, it suffices to show that any bounded linear functional ℓ on

$L^2_{(n,n-1)}(M)$ that vanishes on $\mathcal{Z}^\infty$ also vanishes on $\mathcal{Z}$. From the Riesz representation theorem, there exists a $g \in L^2(M)$ such that ℓ can be written as

$$\ell(\alpha) = \int_M \alpha \wedge g, \qquad \alpha \in L^2_{(n,n-1)}(M).$$

For any $u \in C^\infty_{(n,n-2)}(M)$, it is easy to see that $\bar\partial_b u \in \mathcal{Z}^\infty$. If ℓ vanishes on $\mathcal{Z}^\infty$, we have

$$\int_M \bar\partial_b u \wedge g = 0, \qquad \text{for any } u \in C^\infty_{(n,n-2)}(M).$$

This implies that $\bar\partial_b g = 0$ in the distribution sense. Using Lemma 9.3.8 when $p = q = 0$, there exists a sequence of smooth functions g_m such that $\bar\partial_b g_m = 0$ and $g_m \to g$ in $L^2(M)$. For any $\alpha \in \mathcal{Z}$, we have

$$\ell(\alpha) = \int_M \alpha \wedge g = \lim_{m\to\infty} \int_M \alpha \wedge g_m = 0.$$

This proves the lemma.

We can now finish the proof of Theorem 9.3.1.

Proof of Theorem 9.3.1. When $1 \le q < n-1$, α can be approximated by smooth $\bar\partial_b$-closed forms α_m in $W^s_{(p,q)}(M)$ according to Lemma 9.3.8. For each α_m, we apply Lemma 9.3.7 to obtain a $(p, q-1)$-form u_m such that $\bar\partial_b u_m = \alpha_m$ and

$$\|u_m\|_{s(M)} \le C_s \|\alpha_m\|_{s(M)}.$$

Thus, u_m converges to some $(p, q-1)$-form u such that $\bar\partial_b u = \alpha$ on M and

$$\|u\|_{s(M)} \le C_s \|\alpha\|_{s(M)}.$$

This proves the theorem when $1 \le q < n-1$.

When $q = n-1$, we approximate α by $\alpha_m \in \mathcal{Z}^\infty$ in the $L^2_{(p,n-1)}(M)$ norm using Lemma 9.3.9. Repeating the arguments above with $s = 0$, we can construct $u \in L^2_{(p,n-2)}(M)$ with $\bar\partial_b u = \alpha$. This completes the proof of Theorem 9.3.1.

Corollary 9.3.2 follows easily from Theorem 9.3.1.

9.4 The Hodge Decomposition Theorem for $\bar\partial_b$

The L^2 existence result proved in Theorem 9.3.1 can be applied to prove the Hodge decomposition theorem for the $\bar\partial_b$ complex on pseudoconvex boundaries. We use the notation $\bar\partial_b^*$ to denote the Hilbert space adjoint of the operator $\bar\partial_b$ with respect to the induced metric on M.

We define $\Box_b$ as in Chapter 8. Let $\Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b$ be defined on $\mathrm{Dom}(\Box_b)$, where

$$\begin{aligned}\mathrm{Dom}(\Box_b) = \{\phi \in L^2_{(p,q)}(M) \mid \phi \in \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}_b^*);\\ \overline{\partial}_b \phi \in \mathrm{Dom}(\overline{\partial}_b^*) \text{ and } \overline{\partial}_b^* \phi \in \mathrm{Dom}(\overline{\partial}_b)\}.\end{aligned}$$

Repeating arguments in the proof of Proposition 4.2.3, we can show that $\Box_b$ is a closed, densely defined self-adjoint operator.

We use $H^b_{(p,q)}$ to denote the projection

$$H^b_{(p,q)} : L^2_{(p,q)}(M) \to \mathrm{Ker}(\Box_b) = \mathrm{Ker}(\bar\partial_b) \cap \mathrm{Ker}(\bar\partial_b^*).$$

When M is the boundary of a smooth bounded pseudoconvex domain, we claim that for $1 \le q \le n-2$,

$$\mathrm{Ker}(\Box_b) = \mathrm{Ker}(\bar\partial_b) \cap \mathrm{Ker}(\bar\partial_b^*) = 0. \tag{9.4.1}$$

To prove (9.4.1), let $\alpha \in \mathrm{Ker}(\Box_b) = \mathrm{Ker}(\bar\partial_b) \cap \mathrm{Ker}(\bar\partial_b^*)$. Then $\alpha = \bar\partial_b u$ for some $u \in L^2_{(p,q-1)}(M)$ by Theorem 9.3.1. Thus

$$(\alpha, \alpha) = (\bar\partial_b u, \alpha) = (u, \bar\partial_b^* \alpha) = 0.$$

We have $H^b_{(p,q)} = 0$ for all $0 \le p \le n,\ 1 \le q \le n-2$. Only $H^b_{(p,0)} \equiv S_{(p,0)}$ (the Szegö projection) and $H^b_{(p,n-1)} \equiv \tilde S_{(p,n-1)}$ are nontrivial where

$$S_{(p,0)} : L^2_{(p,0)}(M) \to \mathrm{Ker}(\bar\partial_b),$$

$$\tilde S_{(p,n-1)} : L^2_{(p,n-1)}(M) \to \mathrm{Ker}(\bar\partial_b^*).$$

We derive some equivalent conditions for (9.3.3) in Theorem 9.3.1. Again one can assume $p = n$. Let $S = S_{(0,0)}$ denote the Szegö projection on functions and $\tilde S$ denote the projection from $L^2_{(n,n-1)}(M)$ onto $L^2_{(n,n-1)} \cap \mathrm{Ker}(\bar\partial_b^*)$. For any $\alpha \in L^2_{(n,n-1)}(M)$, we can write $\alpha = f(\star\bar\partial\rho)$ for some $f \in L^2(M)$, where $\star$ is the Hodge star operator with respect to the standard metric in $\mathbb{C}^n$. The following lemma links condition (9.3.3) to the Szegö projection:

Lemma 9.4.1. *For any $\alpha \in L^2_{(n,n-1)}(M)$, the following conditions are equivalent:*

(1) *α satisfies condition (9.3.3).*
(2) *$S\bar f = 0$, where $\alpha = f(\star\bar\partial\rho)$.*
(3) *$\tilde S\alpha = 0$.*

Proof. Let $\theta = \star(\partial\rho \wedge \bar\partial\rho)$. Since $d\rho = (\partial\rho + \bar\partial\rho)$ vanishes when restricted to M, we have

$$\begin{aligned}\star\partial\rho &= \theta \wedge \bar\partial\rho = \theta \wedge d\rho - \theta \wedge \partial\rho \\ &= \theta \wedge d\rho + \star\bar\partial\rho = \star\bar\partial\rho\end{aligned}$$

on M. If $\psi \in L^2(M)$,

$$\begin{aligned}\int_M \alpha \wedge \psi = \int_M f\psi(\star\bar\partial\rho) &= \int_M f\psi(\star d\rho) - \int_M f\psi(\star\partial\rho) \\ &= \int_M f\psi d\sigma - \int_M f\psi(\star\bar\partial\rho),\end{aligned}$$

where $d\sigma = \star d\rho$ is the surface measure on M. Hence, for any $\psi \in L^2(M)$,

$$\int_M \alpha \wedge \psi = \frac{1}{2}\int_M f\psi d\sigma = \frac{1}{2}(f, \bar{\psi})_M.$$

Since $\mathcal{Z}^\infty$ is dense in $\mathcal{Z}$ from Lemma 9.3.8, (1) and (2) are equivalent.

To prove that (3) and (1) are equivalent, we write $\beta = g(\star\bar{\partial}\rho)$ for any $\beta \in L^2_{(n,n-1)}(M)$, where $g \in L^2(M)$. It is easy to see that $\beta \in \mathrm{Ker}(\bar{\partial}_b^*)$ if and only if $\bar{\partial}_b \bar{g} = 0$ on M. For any $\beta \in L^2_{(n,n-1)}(M) \cap \mathrm{Ker}(\bar{\partial}_b^*)$,

$$(\alpha, \beta)_M = \int_M f\bar{g}d\sigma = 2\int_M \alpha \wedge \bar{g} = 0.$$

This proves that (1) and (3) are equivalent.

We have the following strong Hodge decomposition theorem for $\bar{\partial}_b$.

Theorem 9.4.2. *Let M be the boundary of a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$, $n \geq 2$. Then for any $0 \leq p \leq n$, $0 \leq q \leq n-1$, there exists a linear operator $N_b : L^2_{(p,q)}(M) \to L^2_{(p,q)}(M)$ such that*

(1) *N_b is bounded and $\mathcal{R}(N_b) \subset Dom(\Box_b)$.*

(2) *For any $\alpha \in L^2_{(p,q)}(M)$, we have*

$$\begin{aligned} \alpha &= \overline{\partial}_b \overline{\partial}_b^* N_b \alpha \oplus \overline{\partial}_b^* \overline{\partial}_b N_b \alpha, \quad &&\textit{if} \quad 1 \leq q \leq n-2, \\ \alpha &= \overline{\partial}_b^* \overline{\partial}_b N_b \alpha \oplus S_{(p,0)}\alpha, &&\textit{if} \quad q = 0, \\ \alpha &= \overline{\partial}_b \overline{\partial}_b^* N_b \alpha \oplus \tilde{S}_{(p,n-1)}\alpha, &&\textit{if} \quad q = n-1. \end{aligned}$$

(3) *If $1 \leq q \leq n-2$, we have*

$$\begin{aligned} &N_b \Box_b = \Box_b N_b = I \quad \textit{on } Dom(\Box_b), \\ &\overline{\partial}_b N_b = N_b \overline{\partial}_b \quad \textit{on } Dom(\overline{\partial}_b), \\ &\overline{\partial}_b^* N_b = N_b \overline{\partial}_b^* \quad \textit{on } Dom(\overline{\partial}_b^*). \end{aligned}$$

(4) *If $\alpha \in L^2_{(p,q)}(M)$ with $\overline{\partial}_b \alpha = 0$, where $1 \leq q \leq n-2$ or $\alpha \in L^2_{(p,n-1)}(M)$ with $\tilde{S}_{(p,n-1)}\alpha = 0$, then $\alpha = \overline{\partial}_b \overline{\partial}_b^* N_b \alpha$.*

The solution $u = \overline{\partial}_b^* N_b \alpha$ in (4) is called the canonical solution, i.e., the unique solution orthogonal to $\mathrm{Ker}(\overline{\partial}_b)$.

Proof. From Corollary 9.3.2, the range of $\bar{\partial}_b$, denoted by $\mathcal{R}(\bar{\partial}_b)$, is closed in every degree. If $1 \leq q \leq n-2$, we have from Theorem 9.3.1, $\mathrm{Ker}(\bar{\partial}_b) = \mathcal{R}(\bar{\partial}_b)$ and the following orthogonal decomposition:

$$L^2_{(p,q)}(M) = \mathrm{Ker}(\bar{\partial}_b) \oplus \mathcal{R}(\bar{\partial}_b^*) = \mathcal{R}(\bar{\partial}_b) \oplus \mathcal{R}(\bar{\partial}_b^*). \tag{9.4.2}$$

Repeating the arguments of Theorem 4.4.1, we can prove that for every $\alpha \in \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}_b^*)$,

$$\|\alpha\|^2 \leq c(\|\overline{\partial}_b \alpha\|^2 + \|\overline{\partial}_b^* \alpha\|^2), \tag{9.4.3}$$

and for any $\alpha \in \mathrm{Dom}(\Box_b)$,

$$\|\alpha\|^2 \le c\|\Box_b\alpha,\|^2 \tag{9.4.4}$$

where the constant c is independent of α.

(9.4.4) implies that $\Box_b$ is one-to-one and, from Lemma 4.1.1, that the range of $\Box_b$ is closed. It follows that the strong Hodge decomposition holds:

$$\begin{aligned} L^2_{(p,q)}(M) &= \mathcal{R}(\Box_b) \oplus \mathrm{Ker}(\Box_b) \\ &= \bar\partial_b\bar\partial_b^*(\mathrm{Dom}(\Box_b)) \oplus \bar\partial_b^*\bar\partial_b(\mathrm{Dom}(\Box_b)). \end{aligned}$$

Thus $\Box_b : \mathrm{Dom}(\Box_b) \to L^2_{(p,q)}(M)$ is one-to-one, onto, and it has a unique inverse $N_b : L^2_{(p,q)}(M) \to \mathrm{Dom}(\Box_b)$. Note that N_b is bounded. Following the same argument as in Theorem 4.4.1, we have that N_b satisfies all the conditions (1)-(4) and Theorem 9.4.2 is proved when $1 \le q \le n-2$.

When $q = 0$,

$$L^2_{(p,0)}(M) = \mathcal{R}(\bar\partial_b^*) \oplus \mathrm{Ker}\bar\partial_b.$$

Thus for any $\alpha \perp \mathrm{Ker}(\bar\partial_b)$ and $\alpha \in \mathrm{Dom}(\Box_b)$,

$$\|\alpha\|^2 \le c\|\bar\partial_b\alpha\|^2 \le c\|\Box_b\alpha\|\|\alpha\|. \tag{9.4.5}$$

Thus $\Box_b$ has closed range on $\mathrm{Ker}(\bar\partial_b)^\perp = \mathrm{Ker}(\Box_b)^\perp$ and

$$L^2_{(p,0)}(M) = \mathcal{R}(\Box_b) \oplus \mathrm{Ker}\bar\partial_b.$$

In particular, there exists a bounded operator $N_b : L^2_{(p,0)}(M) \to L^2_{(p,0)}(M)$ satisfying $\Box_b N_b = I - S_{(p,0)}$, $N_b = 0$ on $\mathrm{Ker}(\bar\partial_b)$. This proves (1) and (2) when $q = 0$. Properties (3) and (4) also follow exactly as before. The case for $q = n-1$ can also be proved similarly.

Thus, the strong Hodge decomposition for $\overline{\partial}_b$ holds on the boundary of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ for all (p,q)-forms including $q = 0$ and $q = n-1$.

NOTES

The $\bar\partial$-closed extension of $\bar\partial_b$-closed functions or forms from the boundary of a domain in a complex manifold was studied by J. J. Kohn and H. Rossi [KoRo 1] who first introduced the $\bar\partial_b$ complex. In [KoRo 1], they show that a $\bar\partial$-closed extension exists for any (p,q)-form from the boundary M to the domain Ω in a complex manifold if Ω satisfies condition $Z(n-q-1)$. Formula (9.1.3) was first given there. The L^2 Cauchy problem on any pseudoconvex domain was used by M.-C. Shaw [Sha 6] to study the local solvability for $\bar\partial_b$. Theorem 9.3.3 was first observed by J. P. Rosay in [Rosa 1] where it was pointed out that global smooth solutions can be obtained by combining the results of Kohn [Koh 6] and Kohn-Rossi [KoRo 1]. The $\bar\partial$ Cauchy problem was also discussed by M. Derridj in [Der 1,2]. A. Andreotti and C. D. Hill used reduction to vanishing cohomology arguments to study the Cauchy problem and $\bar{\partial}_b$ in [AnHi 1]. Kernel methods were also used to obtain $\bar{\partial}$-closed extension from boundaries of domains satisfying condition $Z(n-q-1)$ (see Henkin-Leiterer [HeLe 2]). We mention the papers of G. M. Henkin [Hen 3] and H. Skoda [Sko 1] where solutions of $\bar{\partial}_b$, including the top degree case, on strongly pseudoconvex boundaries were studied using integral kernel methods.

Much of the material in Section 9.3 on the L^2 theory of $\bar{\partial}_b$ on weakly pseudoconvex boundaries was based on the work of M.-C. Shaw [Sha 2] and H. P. Boas-M.-C. Shaw [BoSh 1]. In [Sha 1], Kohn's results of the weighted $\bar{\partial}$-Neumann operator on a pseudoconvex domain was extended to an annulus between two pseudoconvex domains. Using the weighted $\bar{\partial}$-Neumann operators constructed in [Koh 6] and [Sha 1], a two-sided $\bar{\partial}$-closed extension for $\bar{\partial}_b$-closed forms away from the top degree was constructed in [Sha 2]. The jump formula proved in Lemma 9.3.5 was derived from the Bochner-Martinelli-Koppelman kernel in [BoSh 1] (c.f. Theorem 11.3.1). Our proof of Lemma 9.3.5 presented here uses an idea of [AnHi 1]. Sobolev estimates for $\bar{\partial}_b$ were also obtained for the top degree case ($q = n-1$) in [BoSh 1]. The proof depends on the regularity of the weighted Szegö projection in Sobolev spaces. For more discussion on the Sobolev estimates for the Szegö projection, see [Boa 1,2,4]. Another proof of Theorem 9.3.1 was given by J. J. Kohn in [Koh 11] using pseudodifferential operators and microlocal analysis.

We point out that all results discussed in this chapter can be generalized to any CR manifolds which are boundaries of domains in complex manifolds, as long as the corresponding $\bar{\partial}$-Neumann operators (or weighted $\bar{\partial}$-Neumann operators) exist and are regular (e.g, pseudoconvex domains in a Stein manifold). However, L^2 existence theorems and the closed range property for $\bar{\partial}_b$ might not be true for abstract CR manifolds. It was observed by D. Burns [Bur 1] that the range of $\bar{\partial}_b$ is not closed in L^2 on a nonembeddable strongly pseudoconvex CR manifold of real dimension three discovered by H. Rossi [Ros 1]. This example along with the interplay between the closed-range property of $\bar{\partial}_b$ and the embedding problem of abstract CR structures will be discussed in Chapter 12.

There are also results on Sobolev estimates for $\Box_b$ on pseudoconvex manifolds. Using subelliptic multipliers combining with microlocal analysis, J. J. Kohn (see [Koh 10,11]) has proved subelliptic estimates for $\Box_b$ when the CR manifold is pseudoconvex and of finite ideal type. If the domain has a plurisubharmonic defining function, Sobolev estimates for $\Box_b$ have been obtained by H. P. Boas and E. J. Straube [BoSt 4]. In particular, the Szegö projections are exactly regular in both cases.

Much less is known for the regularity of $\Box_b$ and the Szegö projection on pseudoconvex manifolds in other function spaces except when the CR manifold is the boundary of a pseudoconvex domain of finite type in $\mathbb{C}^2$. We mention the work of Fefferman-Kohn [FeKo 1] and Nagel-Rosay-Stein-Wainger [NRSW 1] and Christ [Chr 1]. When the CR manifold is of finite type in $\mathbb{C}^n$, $n \geq 3$, and the Levi form is diagonalizable, Hölder estimates for $\bar{\partial}_b$ and $\Box_b$ have been obtained in Fefferman-Kohn-Machedon [FKM 1]. Hölder and L^p estimates for the Szegö projection on convex domains of finite type have been obtained by J. McNeal and E. M. Stein [McSt 2]. It is still an open question whether Hölder or L^p estimates hold for $\Box_b$ and $\bar{\partial}_b$ on general pseudoconvex CR manifolds of finite type.

CHAPTER 10

FUNDAMENTAL SOLUTIONS FOR $\Box_b$ ON THE HEISENBERG GROUP

In Chapters 8 and 9, we have proved the global solvability and regularity for the $\Box_b$ operator on compact pseudoconvex CR manifolds. Under condition $Y(q)$, subelliptic 1/2-estimates for $\Box_b$ were obtained in Chapter 8. On the other hand, it was shown in Section 7.3 that the Lewy operator, which arises from the tangential Cauchy-Riemann operator associated with the Siegel upper half space, does not possess a solution locally in general. The main task of this chapter is to construct a fundamental solution for the $\Box_b$ operator on the Heisenberg group $\mathbb{H}_n$. The Heisenberg group serves as a model for strongly pseudoconvex CR manifolds (or nondegenerate CR manifolds). Using the group structure, we can construct explicitly a solution kernel for $\overline{\partial}_b$ and obtain estimates for the solutions in Hölder spaces. The Cauchy-Szegö kernel on $\mathbb{H}_n$ is discussed in Section 10.2. We construct a relative fundamental solution for $\Box_b$ in the top degree case and deduce from it the necessary and sufficient conditions for the local solvability of the Lewy operator.

10.1 Fundamental Solutions for $\Box_b$ on the Heisenberg Group

Let us recall that the Siegel upper half space Ω_n is defined by

$$\Omega_n = \{(z', z_n) \in \mathbb{C}^n \mid \ \mathrm{Im} z_n > |z'|^2\}, \tag{10.1.1}$$

where $z' = (z_1, \cdots, z_{n-1})$ and $|z'|^2 = |z_1|^2 + \cdots + |z_{n-1}|^2$. Denote by $\mathrm{Aut}(\Omega_n)$ the group of all holomorphic mappings that are one-to-one from Ω_n onto itself. Let $H_n \subset \mathrm{Aut}(\Omega_n)$ be the subgroup defined by

$$H_n = \{h_a; a \in b\Omega_n \mid \ h_a(z) = (a' + z', a_n + z_n + 2i\langle z', a'\rangle)\}, \tag{10.1.2}$$

where $\langle z', a'\rangle$ is the standard inner product in $\mathbb{C}^{n-1}$, i.e., $\langle z', a'\rangle = \sum_{i=1}^{n-1} z_i \bar{a}_i$. To see H_n actually forms a subgroup of $\mathrm{Aut}(\Omega_n)$, put $h_a(z) = (w', w_n)$. Since $\mathrm{Im} a_n = |a'|^2$, we have

$$\mathrm{Im} w_n - |w'|^2 = \mathrm{Im} z_n - |z'|^2.$$

Hence, each h_a maps Ω_n into Ω_n and $b\Omega_n$ into $b\Omega_n$. It is easily verified that if $a, b \in b\Omega_n$, then $h_a \circ h_b = h_c$ with $c = h_a(b)$. It follows that if $b = (-a', -\overline{a}_n)$, then $h_a \circ h_b =$ the identity mapping. Thus, H_n is indeed a subgroup of $\mathrm{Aut}(\Omega_n)$

and it induces a group structure on the boundary $b\Omega_n$. The boundary $b\Omega_n$ can be identified with $\mathbb{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$ via the mapping

$$\pi : (z', t + i|z'|^2) \mapsto (z', t), \tag{10.1.3}$$

where $z_n = t + is$. We shall call $\mathbb{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$ the Heisenberg group of order $n-1$ with the group structure induced from the automorphism subgroup H_n of $\mathrm{Aut}(\Omega_n)$ by

$$(z'_1, t_1) \cdot (z'_2, t_2) = (z'_1 + z'_2, t_1 + t_2 + 2\mathrm{Im}\langle z'_1, z'_2 \rangle). \tag{10.1.4}$$

It is easily verified that

$$\begin{aligned} Z_j &= \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}, \ j = 1, \cdots, n-1, \ \text{and} \\ T &= \frac{\partial}{\partial t}, \end{aligned}$$

are left invariant vector fields with respect to the Lie group structure on $\mathbb{H}_n$ such that

$$[Z_j, \overline{Z}_j] = -2iT, \ \text{for } j = 1, \cdots, n-1, \tag{10.1.5}$$

and that all other commutators vanish. Hence, $\mathbb{H}_n$ is a strongly pseudoconvex CR manifold with type $(1,0)$ vector fields spanned by $Z_1, \cdots, Z_{n-1}$. We fix a left invariant metric on $\mathbb{H}_n$ so that $Z_1, \cdots, Z_{n-1}, \overline{Z}_1, \cdots, \overline{Z}_{n-1}$ and T are orthonormal with respect to this metric. Let the dual basis be $\omega_1, \cdots, \omega_{n-1}, \overline{\omega}_1, \cdots, \overline{\omega}_{n-1}$ and τ, where $\omega_j = dx_j + idy_j$, $j = 1, \cdots, n-1$ and τ is given by

$$\tau = dt + 2\sum_{j=1}^{n-1} (x_j dy_j - y_j dx_j).$$

Hence, $\langle dx_j, dx_j \rangle = \langle dy_j, dy_j \rangle = 1/2$, for $j = 1, \cdots, n-1$, and the volume element is

$$dV = 2^{1-n} dx_1 \cdots dx_{n-1} dy_1 \cdots dy_{n-1} dt.$$

Next we calculate $\Box_b$ on the (p,q)-forms of the Heisenberg group $\mathbb{H}_n$. Since p plays no role in the formulation of the $\bar{\partial}_b$ and $\bar{\partial}_b^*$ operators, we may assume that $p = 0$. Let $f \in C^\infty_{(0,q)}(\mathbb{H}_n)$ be a smooth $(0,q)$-form with compact support on $\mathbb{H}_n$. Write f as

$$f = {\sum_{|J|=q}}' f_J \, \overline{\omega}^J,$$

where $J = (j_1, \cdots, j_q)$ is an increasing multiindex and $\overline{\omega}^J = \overline{\omega}_{j_1} \wedge \cdots \wedge \overline{\omega}_{j_q}$. Then, we have

$$\begin{aligned} \bar{\partial}_b f &= {\sum_{|J|=q}}' \left(\sum_k \overline{Z}_k f_J \, \overline{\omega}_k \right) \wedge \overline{\omega}^J \\ &= {\sum_{|L|=q+1}}' \left({\sum_{k,J}}' \epsilon^L_{kJ} \overline{Z}_k f_J \right) \overline{\omega}^L, \end{aligned}$$

and

$$\overline{\partial}_b^* f = -\sum_{|H|=q-1}{}' \left(\sum_{l,J}{}' \epsilon_{lH}^J Z_l f_J \right) \overline{\omega}^H.$$

It follows that

$$\overline{\partial}_b^* \overline{\partial}_b f = -\sum_{|Q|=q}{}' \left(\sum_{l,L}{}' \epsilon_{lQ}^L Z_l \left(\sum_{k,J}{}' \epsilon_{kJ}^L \overline{Z}_k f_J \right) \right) \overline{\omega}^Q,$$

and

$$\overline{\partial}_b \overline{\partial}_b^* f = -\sum_{|Q|=q}{}' \left(\sum_{k,H}{}' \epsilon_{kH}^Q \overline{Z}_k \left(\sum_{l,J}{}' \epsilon_{lH}^J Z_l f_J \right) \right) \overline{\omega}^Q.$$

For fixed Q and $l \neq k$, it is easily verified that

$$\epsilon_{lQ}^L \epsilon_{kJ}^L = -\epsilon_{kH}^Q \epsilon_{lH}^J,$$

and

$$[Z_l, \overline{Z}_k] = 0.$$

Hence, we obtain

$$\begin{aligned}\Box_b f &= (\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b) \left(\sum_{|J|=q}{}' f_J \, \overline{\omega}^J \right) \\ &= -\sum_{|J|=q}{}' \left(\left(\sum_{k \notin J} Z_k \overline{Z}_k + \sum_{k \in J} \overline{Z}_k Z_k \right) f_J \right) \overline{\omega}^J.\end{aligned}$$

The calculation shows that $\Box_b$ acts on a $(0,q)$-form f diagonally. It is also easily verified that

$$-\left(\sum_{k \notin J} Z_k \overline{Z}_k + \sum_{k \in J} \overline{Z}_k Z_k \right) = -\frac{1}{2} \sum_{k=1}^{n-1} (Z_k \overline{Z}_k + \overline{Z}_k Z_k) + i(n-1-2q)T.$$

Therefore, to invert $\Box_b$ it suffices to invert the operators

$$-\frac{1}{2} \sum_{k=1}^{n-1} (Z_k \overline{Z}_k + \overline{Z}_k Z_k) + i(n-1-2q)T. \tag{10.1.6}$$

In particular, when $n=2$ and $q=0$, $\Box_b$ acts on functions and

$$\begin{aligned}\Box_b &= -Z\overline{Z} \\ &= -\frac{1}{2}(Z\overline{Z} + \overline{Z}Z) + iT,\end{aligned}$$

where $\overline{Z}$ is the Lewy operator. Hence, $\Box_b$ in general is not locally solvable.

However, we shall investigate the solvability and regularity of $\Box_b$ via the following more general operator $\mathcal{L}_\alpha$ defined by

$$\mathcal{L}_\alpha = -\frac{1}{2}\sum_{k=1}^{n-1} (Z_k\overline{Z}_k + \overline{Z}_k Z_k) + i\alpha T, \tag{10.1.7}$$

for $\alpha \in \mathbb{C}$.

The second order term $\mathcal{L}_0 = -\frac{1}{2}\sum_{k=1}^{n-1}(Z_k\overline{Z}_k + \overline{Z}_k Z_k)$ is usually called the sub-Laplacian on a stratified Lie group. By definition, a Lie group is stratified if it is nilpotent and simply connected and its Lie algebra $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$ such that $[V_1, V_j] = V_{j+1}$ for $j < m$ and $[V_1, V_m] = \{0\}$. The Heisenberg group $\mathbb{H}_n$ is a step two nilpotent Lie group, namely, the Lie algebra is stratified with $m = 2$, where V_1 is generated by $Z_1, \cdots, Z_{n-1}, \overline{Z}_1, \cdots, \overline{Z}_{n-1}$ and V_2 is generated by T. By Theorems 8.2.3 and 8.2.5, $\mathcal{L}_0$ satisfies a subelliptic estimate of order 1/2 and is hypoelliptic. We want to construct an explicit fundamental solution φ_0 for $\mathcal{L}_0$. The group structure on the Heisenberg group suggests that one can define a nonisotropic dilation in the following way: for $a > 0$,

$$a(z', t) = (az', a^2 t)$$

which forms an one-parameter subgroup of $\mathrm{Aut}(\mathbb{H}_n)$. We also define a norm on $\mathbb{H}_n$ by $|(z', t)| = (|z'|^4 + t^2)^{\frac{1}{4}}$ to make it homogeneous of degree one with respect to the nonisotropic dilation.

By following the harmonic analysis on real Euclidean spaces, it is reasonable to guess that a fundamental solution φ_0 for $\mathcal{L}_0$ should be given by some negative power of $|(z', t)|$, and that the power should respect the nonisotropic dilation on $\mathbb{H}_n$. In fact, we have the following theorem:

Theorem 10.1.1. *Let $\varphi_0(z', t) = |(z', t)|^{-2(n-1)} = (|z'|^4 + t^2)^{-\frac{n-1}{2}}$. Then $\mathcal{L}_0\varphi_0 = c_0\delta$, where δ is the Dirac function at the origin and c_0 is given by*

$$c_0 = (n-1)^2 \int_{\mathbb{H}_n} ((|z'|^2 + 1)^2 + t^2)^{-\frac{n+1}{2}} dV. \tag{10.1.8}$$

Proof. Define, for $\epsilon > 0$,

$$\varphi_{0,\epsilon}(z', t) = ((|z'|^2 + \epsilon^2)^2 + t^2)^{-\frac{n-1}{2}}.$$

Then, a simple calculation shows that

$$\begin{aligned}
\mathcal{L}_0\varphi_{0,\epsilon}(z', t) &= (n-1)^2\epsilon^2((|z'|^2 + \epsilon^2)^2 + t^2)^{-\frac{n+1}{2}} \\
&= \epsilon^{-2n}(n-1)^2\left(\left(\left|\frac{z'}{\epsilon}\right|^2 + 1\right)^2 + \left(\frac{t}{\epsilon^2}\right)^2\right)^{-\frac{n+1}{2}} \\
&= \epsilon^{-2n}\phi\left(\frac{1}{\epsilon}(z', t)\right),
\end{aligned}$$

where $\phi(z',t) = (n-1)^2((|z'|^2+1)^2+t^2)^{-\frac{n+1}{2}}$. Then, by integration on $\mathbb{H}_n$, we obtain the following:

$$\begin{aligned}\int_{\mathbb{H}_n} \mathcal{L}_0\varphi_{0,\epsilon}(z',t)\ dV &= \int_{\mathbb{H}_n} \epsilon^{-2n}\phi\left(\frac{1}{\epsilon}(z',t)\right)\ dV \\ &= \int_{\mathbb{H}_n} \phi(z',t)\ dV = c_0.\end{aligned}$$

Hence, $\lim_{\epsilon\to 0}\mathcal{L}_0\varphi_{0,\epsilon} = c_0\delta$ in the distribution sense. On the other hand, $\mathcal{L}_0\varphi_{0,\epsilon}$ also tends to $\mathcal{L}_0\varphi_0$ in the distribution sense. This proves the theorem.

It follows that $c_0^{-1}\varphi_0$ is the fundamental solution for $\mathcal{L}_0$. We now proceed to search for a fundamental solution φ_α for $\mathcal{L}_\alpha$ with $\alpha\in\mathbb{C}$. Observe that $\mathcal{L}_\alpha$ has the same homogeneity properties as $\mathcal{L}_0$ with respect to the nonisotropic dilation on $\mathbb{H}_n$, and that $\mathcal{L}_\alpha$ is invariant under unitary transformation in z'-variable. Hence, we can expect that certain φ_α will have the same invariant properties. From these observations we intend to look for a fundamental solution φ_α of the form

$$\varphi_\alpha(z',t) = |(z',t)|^{-2(n-1)}f(t|(z',t)|^{-2}).$$

After a routine, but lengthy, calculation, we see that f must satisfy the following ordinary second order differential equation:

$$\begin{aligned}(1-w^2)^{\frac{3}{2}}f''(w) &- (nw(1-w^2)^{\frac{1}{2}} + i\alpha(1-w^2))f'(w) \\ &+ i(n-1)\alpha w f(w) = 0\end{aligned} \tag{10.1.9}$$

with $w = t|(z',t)|^{-2}$.

By setting $w = \cos\theta, f(w) = g(\theta),\ 0\le\theta\le\pi$, (10.1.9) is reduced to

$$\left(\sin\theta\frac{d}{d\theta} + (n-1)\cos\theta\right)\left(\frac{d}{d\theta} + i\alpha\right)g(\theta) = 0. \tag{10.1.10}$$

Equation (10.1.10) has two linearly independent solutions

$$g_1(\theta) = e^{-i\alpha\theta},$$

and

$$g_2(\theta) = e^{-i\alpha\theta}\int \frac{e^{i\alpha\theta}}{(\sin\theta)^{n-1}}d\theta.$$

Hence, the only bounded solutions for $0\le\theta\le\pi$ are $g(\theta) = ce^{-i\alpha\theta}$. It follows that

$$f(w) = c(w - i\sqrt{1-w^2})^\alpha = c\left(\frac{t-i|z'|^2}{|(z',t)|^2}\right)^\alpha.$$

If we choose $c = i^\alpha$, then

$$\varphi_\alpha(z',t) = (|z'|^2 - it)^{-\frac{(n-1+\alpha)}{2}}(|z'|^2+it)^{-\frac{(n-1-\alpha)}{2}}. \tag{10.1.11}$$

Here we have used the principal branch for the power functions. Then we have the following theorem.

Theorem 10.1.2. *Let $\varphi_\alpha(z', t)$ be defined as in (10.1.11) for $\alpha \in \mathbb{C}$. Then, $\mathcal{L}_\alpha\varphi_\alpha = c_\alpha\delta$, where*

$$c_\alpha = \frac{2^{4-2n}\pi^n}{\Gamma(\frac{n-1+\alpha}{2})\Gamma(\frac{n-1-\alpha}{2})}.$$

Proof. For any $\epsilon > 0$, set

$$\rho_\epsilon(z', t) = |z'|^2 + \epsilon^2 - it,$$

and define

$$\varphi_{\alpha,\epsilon}(z', t) = \rho_\epsilon^{-\frac{(n-1+\alpha)}{2}} \overline{\rho}_\epsilon^{-\frac{(n-1-\alpha)}{2}}. \tag{10.1.12}$$

Hence, $\varphi_{\alpha,\epsilon}$ is smooth and $\varphi_{\alpha,\epsilon}$ tends to φ_α as distributions when ϵ approaches zero. It follows that $\mathcal{L}_\alpha\varphi_{\alpha,\epsilon}$ tends to $\mathcal{L}_\alpha\varphi_\alpha$ as distributions. Thus, it suffices to show that $\mathcal{L}_\alpha\varphi_{\alpha,\epsilon}$ tends to $c_\alpha\delta$ in the distribution sense.

Recall that $Z_k = (\partial/\partial z_k) + i\overline{z}_k(\partial/\partial t)$, for $1 \le k \le n-1$. Then, for a fixed ϵ and any $a \in \mathbb{C}$, a direct computation shows

$$\begin{aligned}
&Z_k\rho_\epsilon^a = 2a\overline{z}_k\rho_\epsilon^{a-1}, \quad \text{and} \quad \overline{Z}_k\overline{\rho}_\epsilon^a = 2az_k\overline{\rho}_\epsilon^{a-1},\\
&Z_k\overline{\rho}_\epsilon^a = \overline{Z}_k\rho_\epsilon^a = 0,\\
&T\rho_\epsilon^a = -ia\rho_\epsilon^{a-1}, \quad \text{and} \quad T\overline{\rho}_\epsilon^a = ia\overline{\rho}_\epsilon^{a-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{L}_\alpha\varphi_{\alpha,\epsilon}(z', t) &= \left(-\frac{1}{2}\sum_{k=1}^{n-1}(Z_k\overline{Z}_k + \overline{Z}_kZ_k) + i\alpha T\right)\left(\rho_\epsilon^{-\frac{(n-1+\alpha)}{2}} \cdot \overline{\rho}_\epsilon^{-\frac{(n-1-\alpha)}{2}}\right)\\
&= \epsilon^2((n-1)^2 - \alpha^2)\rho_\epsilon^{-\frac{(n+1+\alpha)}{2}} \cdot \overline{\rho}_\epsilon^{-\frac{(n+1-\alpha)}{2}}\\
&= \epsilon^{-2n}\mathcal{L}_\alpha\varphi_{\alpha,1}\left(\frac{1}{\epsilon}(z', t)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\mathbb{H}_n} \mathcal{L}_\alpha\varphi_{\alpha,\epsilon}(z', t)\ dV &= \int_{\mathbb{H}_n} \epsilon^{-2n}\mathcal{L}_\alpha\varphi_{\alpha,1}\left(\frac{1}{\epsilon}(z', t)\right)\ dV\\
&= \int_{\mathbb{H}_n} \mathcal{L}_\alpha\varphi_{\alpha,1}(z', t)\ dV.
\end{aligned}$$

Since the mass of $\mathcal{L}_\alpha\varphi_{\alpha,\epsilon}$ concentrates at zero as $\epsilon \to 0$, it follows that $\mathcal{L}_\alpha\varphi_{\alpha,\epsilon}$ tends to $c_\alpha\delta$ with $c_\alpha = \int \mathcal{L}_\alpha\varphi_{\alpha,1}dV$ as distributions. Therefore, it remains only to compute the integral c_α.

We set $a = \frac{1}{2}(n+1+\alpha)$ and $b = \frac{1}{2}(n+1-\alpha)$, then

$$\begin{aligned}
c_\alpha &= \frac{1}{2^{n-1}}\int_{\mathbb{H}_n}((n-1)^2 - \alpha^2)(|z'|^2 + 1 - it)^{-a}(|z'|^2 + 1 + it)^{-b}\ dxdydt\\
&= \frac{((n-1)^2 - \alpha^2)}{2^{n-1}}\int_{\mathbb{C}^{n-1}}(|z'|^2 + 1)^{-n}\ dxdy\int_{-\infty}^{\infty}(1 - it)^{-a}(1 + it)^{-b}\ dt.
\end{aligned}$$

Here, if there is no ambiguity, we shall write $dxdy$ for $dx_1 \cdots dx_{n-1} dy_1 \cdots dy_{n-1}$. The first integral is evaluated by

$$\begin{aligned}
\int_{\mathbb{C}^{n-1}} (|z'|^2 + 1)^{-n} \, dxdy &= \frac{2\pi^{n-1}}{\Gamma(n-1)} \int_0^\infty \frac{r^{2n-3}}{(1+r^2)^n} \, dr \\
&= \frac{\pi^{n-1}}{\Gamma(n-1)} \int_1^\infty t^{-n}(t-1)^{n-2} \, dt, \quad t = 1 + r^2 \\
&= \frac{\pi^{n-1}}{\Gamma(n-1)} \int_0^1 (1-s)^{n-2} \, ds, \quad s = t^{-1} \\
&= \frac{\pi^{n-1}}{\Gamma(n)},
\end{aligned}$$

where $\Gamma(\cdot)$ denotes the Gamma function. For the second integral we first assume that $-(n-1) \le \alpha \le n-1$ so that $a \ge 1$ and $b \ge 1$. We start with the formula

$$\int_0^\infty e^{-xs} x^{b-1} \, dx = \Gamma(b) s^{-b}$$

which is valid if the real part of s is positive. Set $s = 1 + it$, then

$$\Gamma(b)(1+it)^{-b} = \int_0^\infty e^{-ixt} e^{-x} x^{b-1} \, dx = \hat{f}(t),$$

where $\hat{f}(t)$ is the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} e^{-x} x^{b-1}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0. \end{cases}$$

Similarly, we obtain

$$\begin{aligned}
\Gamma(a)(1-it)^{-a} &= \int_0^\infty e^{ixt} e^{-x} x^{a-1} \, dx \\
&= \int_{-\infty}^0 e^{-ixt} e^{-|x|} |x|^{a-1} \, dx \\
&= \hat{g}(t),
\end{aligned}$$

where $g(x)$ is defined by

$$g(x) = \begin{cases} 0, & \text{for } x \ge 0 \\ e^{-|x|}|x|^{a-1}, & \text{for } x < 0. \end{cases}$$

Hence, by the Plancherel theorem, we have

$$\begin{aligned}
\Gamma(a)\Gamma(b) \int_{-\infty}^\infty (1-it)^{-a}(1+it)^{-b} \, dt &= \int_{-\infty}^\infty \hat{f}(t)\hat{g}(t) \, dt \\
&= 2\pi \int_{-\infty}^\infty f(x) g(-x) \, dx \\
&= 2\pi \int_0^\infty e^{-2x} x^{a+b-2} \, dx \\
&= \frac{\pi \Gamma(n)}{2^{n-1}}.
\end{aligned}$$

This implies

$$\int_{-\infty}^{\infty}(1-it)^{-a}(1+it)^{-b}\,dt=\frac{2^{-(n-1)}\pi\Gamma(n)}{\Gamma(a)\Gamma(b)}, \tag{10.1.13}$$

for $-(n-1)\le\alpha\le n-1$. In fact, the left-hand side of (10.1.13) defines an entire function of α from the following equality:

$$\int_{-\infty}^{\infty}(1-it)^{-a}(1+it)^{-b}\,dt=\int_{-\infty}^{\infty}(1+t^2)^{-\frac{n+1}{2}}e^{i\alpha\tan^{-1}t}\,dt.$$

Hence, (10.1.13) holds for all $\alpha\in\mathbb{C}$, and we obtain

$$c_\alpha=\frac{((n-1)^2-\alpha^2)}{2^{n-1}}\cdot\frac{\pi^{n-1}}{\Gamma(n)}\cdot\frac{2^{-(n-1)}\pi\Gamma(n)}{\Gamma(\frac{n+1+\alpha}{2})\Gamma(\frac{n+1-\alpha}{2})}=\frac{2^{4-2n}\pi^n}{\Gamma(\frac{n-1+\alpha}{2})\Gamma(\frac{n-1-\alpha}{2})}.$$

This completes the proof of Theorem 10.1.2.

It follows from Theorem 10.1.2 that $c_\alpha=0$ if $\alpha=\pm(2k+n-1)$ for any nonnegative integer k. Hence, if $\alpha\neq\pm(2k+n-1)$ for $k\in\mathbb{N}\cup\{0\}$, then $\Phi_\alpha=c_\alpha^{-1}\varphi_\alpha$ is a fundamental solution for $\mathcal{L}_\alpha$.

We now derive some consequences from Theorem 10.1.2. The convolution of two functions f and g on $\mathbb{H}_n$ is defined by

$$f*g(u)=\int_{\mathbb{H}_n}f(v)g(v^{-1}u)\,dV(v)=\int_{\mathbb{H}_n}f(uv^{-1})g(v)\,dV(v).$$

Set $\check{g}(u)=g(u^{-1})$, then

$$\int_{\mathbb{H}_n}(f*g)(u)h(u)\,dV(u)=\int_{\mathbb{H}_n}f(u)(h*\check{g})(u)\,dV(u),$$

provided that both sides make sense.

If $\alpha\neq\pm(2k+n-1)$ for $k\in\mathbb{N}\cup\{0\}$, then for any $f\in C_0^\infty(H_n)$, define $K_\alpha f=f*\Phi_\alpha$. It is clear that $K_\alpha f\in C^\infty(\mathbb{H}_n)$ since Φ_α has singularity only at zero. For the rest of this section k will always mean a nonnegative integer.

Theorem 10.1.3. *If $f\in C_0^\infty(\mathbb{H}_n)$ and $\alpha\neq\pm(2k+n-1)$, then $\mathcal{L}_\alpha K_\alpha f=K_\alpha\mathcal{L}_\alpha f=f$.*

Proof. Since $\mathcal{L}_\alpha$ is left invariant, clearly we have $\mathcal{L}_\alpha K_\alpha f=f$. For the other equality, let $g\in C_0^\infty(H_n)$. Notice that $-\alpha\neq\pm(2k+n-1)$ whenever $\alpha\neq\pm(2k+n-1)$. Then

$$\begin{aligned}\int g(u)f(u)\,dV(u)&=\int\mathcal{L}_{-\alpha}K_{-\alpha}g(u)f(u)\,dV(u)\\&=\int(g*\Phi_{-\alpha})(u)\mathcal{L}_\alpha f(u)\,dV(u)\\&=\int g(u)(\mathcal{L}_\alpha f*\check{\Phi}_{-\alpha})(u)\,dV(u)\\&=\int g(u)K_\alpha\mathcal{L}_\alpha f(u)\,dV(u).\end{aligned}$$

Hence, $K_\alpha\mathcal{L}_\alpha f=f$. This proves the theorem.

Theorem 10.1.4. *The operator $\mathcal{L}_\alpha$ is hypoelliptic if and only if $\alpha \neq \pm(2k+n-1)$. In particular, $\Box_b$ is hypoelliptic on $\mathbb{H}_n$ for $(0,q)$-forms when $1 \leq q < n-1$.*

Proof. If $\alpha = \pm(2k+n-1)$ for some nonnegative integer k, then the function $\varphi_\alpha(z',t)$ defined in (10.1.11) is a nonsmooth solution to the equation $\mathcal{L}_\alpha \varphi_\alpha = 0$.

Next, let $\alpha \neq \pm(2k+n-1)$, $f \in \mathcal{D}'$ such that $\mathcal{L}_\alpha f = g$ is smooth on some open set U. Let $V \subset\subset U$ be an open set which is relatively compact in U. Choose a cut-off function $\zeta \in C_0^\infty(U)$ with $\zeta = 1$ in some open neighborhood of $\overline{V}$. Then, by Theorem 10.1.3, $K_\alpha(\zeta g)$ is smooth and satisfies $\mathcal{L}_\alpha K_\alpha(\zeta g) = \zeta g$. Hence, to show that f is smooth on V, it suffices to show that $h = \zeta(f - K_\alpha(\zeta g))$ is smooth on V. Since h is a distribution with compact support, a standard argument from functional analysis shows $h = K_\alpha \mathcal{L}_\alpha h$. But on V we have

$$\mathcal{L}_\alpha h = \mathcal{L}_\alpha f - \mathcal{L}_\alpha K_\alpha(\zeta g) = g - \zeta g = 0.$$

The fact that $\Phi_\alpha(z',t)$ is just singular at the origin will then guarantee that $K_\alpha \mathcal{L}_\alpha h$ is smooth on V which in turns shows h, and hence f, is smooth on V.

The hypoellipticity of $\Box_b$ on $(0,q)$-forms when $1 \leq q < n-1$ follows immediately from the expression of $\Box_b$ in (10.1.6). This proves the theorem.

Theorem 10.1.3 can be used to obtain the following existence and regularity theorem for the $\overline{\partial}_b$ equation on $\mathbb{H}_n$:

Theorem 10.1.5. *Let $f \in C_{(0,q)}(\mathbb{H}_n)$, $1 \leq q < n-1$, with compact support. If $\overline{\partial}_b f = 0$ in the distribution sense, then $u = \overline{\partial}_b^* K f$ satisfies $\overline{\partial}_b u = f$ and $u \in \Lambda^{1/2}_{(0,q)}(\mathbb{H}_n, loc)$, where $K = K_\alpha$ with $\alpha = n-1-2q$. Moreover, if $f \in C^k_{(0,q)}(\mathbb{H}_n)$, $k \in \mathbb{N}$, with compact support, then $u \in C^{k+\frac{1}{2}}_{(0,q)}(\mathbb{H}_n, loc)$.*

Proof. Since f is a continuous $(0,q)$-form with compact support on $\mathbb{H}_n$ and $1 \leq q < n-1$, we obtain from Theorem 10.1.3 that

$$\Box_b K f = (\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b) K f = f,$$

where K acts on f componentwise. The hypothesis $\overline{\partial}_b f = 0$ implies

$$\overline{\partial}_b K f = K \overline{\partial}_b f = 0.$$

Hence,

$$\overline{\partial}_b \overline{\partial}_b^* K f = f.$$

For the regularity of $u = \overline{\partial}_b^* K f$, we write $f = \sum'_{|J|=q} f_J \, \overline{\omega}^J$. Then, we obtain from the previous calculation that

$$\overline{\partial}_b^* K f = -\sideset{}{'}\sum_{|H|=q-1} \left(\sideset{}{'}\sum_{l,J} \epsilon^J_{lH} Z_l(K f_J) \right) \overline{\omega}^H.$$

Hence, it suffices to estimate the following integral, for $1 \leq j \leq n-1$:

$$Z_j \int_{\mathbb{H}_n} f(\xi) \Phi(\xi^{-1}\zeta) \, dV(\xi), \tag{10.1.14}$$

where f is a continuous function with compact support on $\mathbb{H}_n$ and $\zeta = (z', t)$, $\xi = (w', u)$ and

$$\Phi(z', t) = (|z'|^2 - it)^{-(n-1-q)}(|z'|^2 + it)^{-q}.$$

We can rewrite (10.1.14) as

$$-2(n-1-q) f * \Psi^j(\zeta) = -2(n-1-q) \int_{\mathbb{H}_n} f(\xi)\Psi^j(\xi^{-1}\zeta)\, dV(\xi), \tag{10.1.15}$$

where

$$\Psi^j(z', t) = \overline{z}_j(|z'|^2 - it)^{-(n-1-q)-1}(|z'|^2 + it)^{-q}.$$

Define a new kernel $\Psi^j_y(z', t, y)$ on $\mathbb{H}_n \times \mathbb{R}_+$, where $\mathbb{R}_+ = \{y \in \mathbb{R} |\ y > 0\}$, by

$$\Psi^j_y(z', t, y) = \overline{z}_j(|z'|^2 - it + y)^{-(n-1-q)-1}(|z'|^2 + it + y)^{-q}.$$

It is easily seen that $f * \Phi^j_y(z', t, y)$ is smooth on $\mathbb{H}_n \times \mathbb{R}_+$ and

$$\lim_{y\to 0^+} f * \Psi^j_y(z', t, y) = f * \Psi^j(z', t).$$

The assertion then follows from the Hardy-Littlewood lemma proved in Theorem C.1 in the Appendix if one can show, for $0 < y \leq 1/2$, that

$$|\nabla(f * \Psi^j_y)| \leq c y^{-\frac{1}{2}} \parallel f \parallel_{L^\infty(\mathbb{H}_n)}, \tag{10.1.16}$$

for some constant $c > 0$ and $1 \leq j \leq n-1$.

Since $\xi^{-1}\zeta = (z - w, t - u - 2\mathrm{Im} w \cdot \overline{z})$, we may introduce new coordinates $\eta_{2j-1} = \mathrm{Re}(z_j - w_j)$, $\eta_{2j} = \mathrm{Im}(z_j - w_j)$ and $\delta = t - u - 2\mathrm{Im}(w \cdot \overline{z})$ for $1 \leq j \leq n-1$. A direct calculation shows that (10.1.16) will be proved if one can show that

$$I_1 = \int_{|(\eta,\delta)|\leq M} \frac{d\eta_1 \cdots d\eta_{2n-2} d\delta}{(|\eta|^2 + |\delta| + y)^n} \ \leq\ c y^{-\frac{1}{2}}, \tag{10.1.17}$$

$$I_2 = \int_{|(\eta,\delta)|\leq M} \frac{|\eta_j| d\eta_1 \cdots d\eta_{2n-2} d\delta}{(|\eta|^2 + |\delta| + y)^{n+1}} \ \leq\ c y^{-\frac{1}{2}}, \tag{10.1.18}$$

and

$$I_3 = \int_{|(\eta,\delta)|\leq M} \frac{|\eta_j \eta_k| d\eta_1 \cdots d\eta_{2n-2} d\delta}{(|\eta|^2 + |\delta| + y)^{n+1}} \ \leq\ c y^{-\frac{1}{2}}, \tag{10.1.19}$$

where $M > 0$ is a positive constant. Notice first that (10.1.19) follows immediately from (10.1.17). For I_1 we have

$$\begin{aligned}
I_1 &\leq c \int_0^M \int_0^M \frac{r^{2n-3}}{(r^2 + \delta + y)^n}\ dr d\delta \\
&\leq c \int_0^M \frac{r^{2n-3}}{(r^2 + y)^{n-1}}\ dr \\
&\leq c \int_0^{\frac{M}{\sqrt{y}}} \frac{x^{2n-3}}{(1 + x^2)^{n-1}}\ dx \\
&\leq c(M)(-\log y) \\
&\leq c(M, a) y^{-a},
\end{aligned}$$

for any $a > 0$. I_2 can be estimated as follows:

$$\begin{aligned}
I_2 &\le c\int_0^M\int_0^M \frac{r^{2n-2}}{(r^2+\delta+y)^{n+1}}\,drd\delta\\
&\le c\int_0^M \frac{r^{2n-2}}{(r^2+y)^n}\,dr\\
&\le cy^{-\frac{1}{2}}\int_0^\infty \frac{x^{2n-2}}{(1+x^2)^n}\,dx\\
&\le cy^{-\frac{1}{2}}.
\end{aligned}$$

This proves the case when f is continuous with compact support.

If $f \in C^k_{(0,q)}(\mathbb{H}_n)$ with compact support, we let X be any one of the left invariant vector fields $Z_1, \cdots, Z_{n-1}, \overline{Z}_1, \cdots, \overline{Z}_{n-1}$ and T. Then, from (10.1.14) and the convolution formula we get

$$\begin{aligned}
XZ_j\int_{\mathbb{H}_n} f(\xi)\Phi(\xi^{-1}\zeta)\,dV(\xi) &= \int_{\mathbb{H}_n} Xf(\xi)Z_j\Phi(\xi^{-1}\zeta)\,dV(\xi)\\
&= Z_j\int_{\mathbb{H}_n} Xf(\xi)\Phi(\xi^{-1}\zeta)\,dV(\xi).
\end{aligned}$$

Since the left invariant vector fields Z_j, $\overline{Z}_j$, where $j = 1, \cdots, n-1$, and T span the tangent space of $\mathbb{H}_n$, the assertion now follows from differentiating k times and the first part of the proof. This proves the theorem.

The construction developed in this section can be extended to the generalized Heisenberg group $\mathbb{H}_{n,k}$ which we now define. For each $1 \le k \le n-1$, let

$$\Omega_{n,k} = \{(z', z_n) \in \mathbb{C}^n |\ \mathrm{Im} z_n > |z_1|^2 + \cdots + |z_k|^2 - |z_{k+1}|^2 - \cdots - |z_{n-1}|^2\}.$$

The boundary of $\Omega_{n,k}$ is identified with the generalized Heisenberg group $\mathbb{H}_{n,k} = \mathbb{C}^{n-1} \times \mathbb{R}$ by

$$\pi : \left(z', t + i\left(\sum_{j=1}^k |z_j|^2 - \sum_{j=k+1}^{n-1} |z_j|^2\right)\right) \mapsto (z', t),$$

where (z', t) is the coordinates on $\mathbb{H}_{n,k}$.

The group structure on $\mathbb{H}_{n,k}$ is defined by

$$(z', t)\cdot(w', u) = \left(z' + w', t + u + 2\mathrm{Im}\left(\sum_{j=1}^k z_j\overline{w}_j - \sum_{j=k+1}^{n-1} z_j\overline{w}_j\right)\right). \tag{10.1.20}$$

One verifies immediately that

$$\begin{aligned}
Z_j &= \frac{\partial}{\partial z_j} + i\overline{z}_j\frac{\partial}{\partial t}, \quad 1 \le j \le k,\\
Z_j &= \frac{\partial}{\partial z_j} - i\overline{z}_j\frac{\partial}{\partial t}, \quad k+1 \le j \le n-1, \text{ and}\\
T &= \frac{\partial}{\partial t},
\end{aligned}$$

are left-invariant vector fields on $\mathbb{H}_{n,k}$ such that

$$[Z_j, \overline{Z}_j] = \begin{cases} -2iT, & \text{for } 1 \le j \le k, \\ 2iT, & \text{for } k+1 \le j \le n-1, \end{cases}$$

and that all other commutators vanish. It follows that the Z_j's define a nondegenerate CR structure on $\mathbb{H}_{n,k}$ such that the Levi matrix has k positive eigenvalues and $n-1-k$ negative eigenvalues. Without loss of generality, k can be assumed to be at least $(n-1)/2$. We shall call such a CR structure k-strongly pseudoconvex.

We fix a left-invariant metric on $\mathbb{H}_{n,k}$ which makes Z_j, $\overline{Z}_j$ and T, $1 \le j \le n-1$, orthonormal. The dual basis is given by $\omega_1, \cdots, \omega_{n-1}$, $\overline{\omega}_1, \cdots, \overline{\omega}_{n-1}$ and τ, where $\omega_j = dx_j + idy_j$ for $1 \le j \le n-1$ and τ is given by

$$\tau = dt + 2\sum_{j=1}^{k} (x_j dy_j - y_j dx_j) - 2\sum_{j=k+1}^{n-1} (x_j dy_j - y_j dx_j).$$

The volume element is

$$dV = 2^{1-n} dx_1 \cdots dx_{n-1} dy_1 \cdots dy_{n-1} dt.$$

Next, we calculate $\Box_b$ on the generalized Heisenberg group $\mathbb{H}_{n,k}$ as before. Let $K = \{1, \cdots, k\}$ and $K' = \{k+1, \cdots, n-1\}$. For each multiindex J with $|J| = q$, we set

$$\alpha_J = |K \setminus J| + |K' \cap J| - |K \cap J| - |K' \setminus J|,$$

where $|\cdot|$ denotes cardinality of the set. Hence, if $f = \sum'_{|J|=q} f_J\, \overline{\omega}^J$ is a smooth $(0,q)$-form with compact support on $\mathbb{H}_{n,k}$, we get

$$\begin{aligned} \Box_b f &= (\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b)\Bigl(\sum_{|J|=q}{}' f_J\, \overline{\omega}^J\Bigr) \\ &= -\sum_{|J|=q}{}' \Bigl(\Bigl(\sum_{m \notin J} Z_m \overline{Z}_m + \sum_{m \in J} \overline{Z}_m Z_m\Bigr) f_J\Bigr)\, \overline{\omega}^J \\ &= \sum_{|J|=q}{}' \Bigl(\Bigl(-\frac{1}{2}\sum_{m=1}^{n-1} (Z_m \overline{Z}_m + \overline{Z}_m Z_m) + i\alpha_J T\Bigr) f_J\Bigr)\, \overline{\omega}^J. \end{aligned}$$

Notice that $-(n-1) \le \alpha_J \le (n-1)$. The extreme case $\alpha_J = n-1$ occurs if and only if $|J| = n-1-k$ and $J = K'$. On the other hand, $\alpha_J = -(n-1)$ occurs if and only if $|J| = k$ and $J = K$. Hence, we have

Theorem 10.1.6. *$\Box_b$ is hypoelliptic for $(0,q)$-forms, $0 \le q \le n-1$, on $\mathbb{H}_{n,k}$ if $q \ne k$ and $q \ne n-1-k$.*

Proof. The assertion follows immediately from Theorem 10.1.4 if we change the coordinates z_j, $k+1 \le j \le n-1$, to $\overline{z}_j$. This proves the theorem.

We note that Theorem 10.1.6 is a variant of Theorem 8.4.4 since condition $Y(q)$ holds on $\mathbb{H}_{n,k}$ when $q \ne k$ and $q \ne n-1-k$. The conclusion of Theorem 10.1.5

also holds on $\mathbb{H}_{n,k}$ when $q \neq k$ and $q \neq n-1-k$. The proof is exactly the same and we omit the details.

10.2 The Cauchy-Szegö Kernel on the Heisenberg Group

In this section we compute the Szegö projection on $\mathbb{H}_n$. Let Ω_n be the Siegel upper half space in $\mathbb{C}^n$. Denote by $H^2(\Omega_n)$ the Hardy space of all holomorphic functions f defined on Ω_n such that

$$\sup_{s>0} \| f_s(z) \|_{L^2(b\Omega_n)} < \infty,$$

where $f_s(z) = f(z', z_n + is)$ for $z = (z', z_n) \in b\Omega_n$ and $s > 0$. It will be clear later that $H^2(\Omega_n)$ forms a Hilbert space under the norm $\| f \|_{H^2(\Omega_n)} = \sup_{s>0} \| f_s \|_{L^2(b\Omega_n)}$.

If $f(z) \in H^2(\Omega_n)$, then by definition $f(z)$ satisfies

$$\int_{\mathbb{C}^{n-1}} \int_{-\infty}^{\infty} |f(z', t + i|z'|^2 + is)|^2 \, dt dx' dy' < C, \tag{10.2.1}$$

where the constant $C > 0$ is independent of $s > 0$ and $dx'dy'$ stands for $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_{n-1} \wedge dy_{n-1}$ with $z_j = x_j + iy_j$ for $1 \leq j \leq n-1$. By using the mean value property of a holomorphic function it is not hard to see from (10.2.1) that for each $z' \in \mathbb{C}^{n-1}$ and $s > 0$ the function $f(z', t + i|z'|^2 + is)$, when viewed as a function in t on $\mathbb{R}$, is L^2 integrable. Thus, we can form the Fourier transform of $f(z', t + i|z'|^2 + is)$ with respect to t which will be denoted by $\tilde{f}_s(z', \lambda)$. The resulting function $\tilde{f}_s(z', \lambda)$ is L^2 integrable with respect to λ and satisfies

$$\frac{1}{2\pi} \int_{\mathbb{C}^{n-1}} \int_{-\infty}^{\infty} |\tilde{f}_s(z', \lambda)|^2 \, d\lambda dx' dy' < C. \tag{10.2.2}$$

Since f is holomorphic on Ω_n, we have by Cauchy's theorem

$$\tilde{f}_{s+s'}(z', \lambda) = e^{-\lambda s'} \tilde{f}_s(z', \lambda), \tag{10.2.3}$$

for $s, s' > 0$. It follows that, for fixed $s > 0$, we get that

$$\begin{aligned} &\int_{\mathbb{C}^{n-1}} \int_{-\infty}^{\infty} |f(z', t + i|z'|^2 + is + is')|^2 \, dt dx' dy' \\ &\quad = \frac{1}{2\pi} \int_{\mathbb{C}^{n-1}} \int_{-\infty}^{\infty} |\tilde{f}_s(z', \lambda)|^2 e^{-2\lambda s'} \, d\lambda dx' dy', \end{aligned} \tag{10.2.4}$$

which implies $\tilde{f}_s(z', \lambda) = 0$ a.e. for $\lambda < 0$. Therefore, we may assume that $\tilde{f}_s(z', \lambda)$ is concentrated on $\mathbb{R}_+ = \{x \in \mathbb{R} | \ x > 0\}$ with respect to λ. It is also clear from (10.2.3) that

$$\tilde{f}_s(z', \lambda) = \tilde{f}_0(z', \lambda) e^{-\lambda s},$$

for some measurable function $\tilde{f}_0(z', \lambda)$. We set

$$\tilde{f}_0(z', \lambda) = \tilde{f}(z', \lambda) e^{-\lambda |z'|^2}.$$

Since $f(z)$ is holomorphic on Ω_n, the homogeneous tangential Cauchy-Riemann equation on each level set $\{z_n = t + i(|z'|^2 + s)\}$ with $s > 0$ must be satisfied by f, namely,

$$\left(\frac{\partial}{\partial \bar{z}_k} - i z_k \frac{\partial}{\partial t} \right) f(z', t + i(|z'|^2 + s)) = 0, \quad 1 \le k \le n-1.$$

It follows that

$$\left(\frac{\partial}{\partial \bar{z}_k} + \lambda z_k \right) \tilde{f}_s(z', \lambda) = 0, \quad 1 \le k \le n-1,$$

for $s > 0$. Hence, for $1 \le k \le n-1$, we have

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial \bar{z}_k} + \lambda z_k \right) \tilde{f}_0(z', \lambda) \\ &= \left(\frac{\partial}{\partial \bar{z}_k} + \lambda z_k \right) (\tilde{f}(z', \lambda) e^{-\lambda |z'|^2}) \\ &= \frac{\partial \tilde{f}}{\partial \bar{z}_k}(z', \lambda) e^{-\lambda |z'|^2}. \end{aligned}$$

This shows that $\tilde{f}(z', \lambda)$ is holomorphic in z' and measurable in λ. By substituting $\tilde{f}(z', \lambda)$ into (10.2.2), we obtain

$$\frac{1}{2\pi} \int_{\mathbb{C}^{n-1}} \int_0^\infty |\tilde{f}(z', \lambda)|^2 e^{-2\lambda |z'|^2} \cdot e^{-2\lambda s} \, d\lambda dx' dy' < C,$$

where the constant $C > 0$ is independent of $s > 0$. Letting s tend to zero, we see that the function $\tilde{f}(z', \lambda)$ satisfies

$$\frac{1}{2\pi} \int_{\mathbb{C}^{n-1}} \int_0^\infty |\tilde{f}(z', \lambda)|^2 e^{-2\lambda |z'|^2} \, d\lambda dx' dy' < C, \tag{10.2.5}$$

and the function $f(z)$ is recovered by

$$f(z) = f(z', z_n) = \frac{1}{2\pi} \int_0^\infty \tilde{f}(z', \lambda) e^{i\lambda z_n} \, d\lambda, \tag{10.2.6}$$

for $z_n = t + i|z'|^2 + is$ with $s > 0$. Moreover, the Plancherel theorem shows that

$$\begin{aligned} &\lim_{s, s' \to 0} \int_{\mathbb{C}^{n-1}} \int_{-\infty}^\infty |f(z', t + i|z'|^2 + is) - f(z', t + i|z'|^2 + is')|^2 \, dt dx' dy' \\ &= \lim_{s, s' \to 0} \int_{\mathbb{C}^{n-1}} \int_0^\infty |\tilde{f}(z', \lambda)|^2 e^{-2\lambda |z'|^2} (e^{-\lambda s} - e^{-\lambda s'})^2 \, d\lambda dx' dy' \\ &= 0. \end{aligned} \tag{10.2.7}$$

This means that $f(z', t + i|z'|^2 + is)$ converges in the L^2 norm to $f(z', t + i|z'|^2)$ as $s \to 0$.

The next theorem shows that the existence of the function $\tilde{f}(z', \lambda)$ with property (10.2.5) is also sufficient for representing a function $f(z)$ in the Hardy space $H^2(\Omega_n)$.

Theorem 10.2.1. *A complex-valued function f defined on Ω_n belongs to $H^2(\Omega_n)$ if and only if there exists a function $\tilde{f}(z', \lambda)$, $(z', \lambda) \in \mathbb{C}^{n-1} \times \mathbb{R}_+$, which is holomorphic in z' and measurable in λ and satisfies*

$$\frac{1}{2\pi} \int_{\mathbb{C}^{n-1}} \int_0^\infty |\tilde{f}(z', \lambda)|^2 e^{-2\lambda |z'|^2} \, d\lambda dx' dy' < \infty,$$

where $dx'dy'$ stands for $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_{n-1} \wedge dy_{n-1}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} |\ x > 0\}$. The integral

$$f(z) = f(z', z_n) = \frac{1}{2\pi} \int_0^\infty \tilde{f}(z', \lambda) e^{i\lambda z_n} \, d\lambda,$$

for $z_n = t + i|z'|^2 + is$, converges absolutely for all $s > 0$ and defines a function $f(z) \in H^2(\Omega_n)$.

Proof. Suppose that there is a function $\tilde{f}(z', \lambda)$ which is holomorphic in z' and measurable in λ and $\tilde{f}(z', \lambda)$ satisfies (10.2.5). For any z' and $z_n = t + i|z'|^2 + is$ with $s > 0$, we may choose a polydisc $D(z'; r)$ in $\mathbb{C}^{n-1}$ centered at z' with small multiradii $r = (r_1, \cdots, r_{n-1})$ such that $|w'|^2 \leq |z'|^2 + (s/2)$ for all $w' \in D(z'; r)$. Since the value of a holomorphic function is dominated by its L^1 norm, we obtain by Hölder's inequality that

$$\begin{aligned}
&\int_0^\infty |\tilde{f}(z', \lambda) e^{i\lambda z_n}| \, d\lambda \\
&\lesssim \int_0^\infty \left(\int_{D(z';r)} |\tilde{f}(w', \lambda)| \, dV(w') \right) e^{-\lambda |z'|^2 - \lambda s} \, d\lambda \\
&\lesssim \int_0^\infty \left(\int_{D(z';r)} |\tilde{f}(w', \lambda)| e^{-\lambda |w'|^2} \, dV(w') \right) e^{-\lambda s/2} \, d\lambda \\
&\lesssim \left(\int_{D(z';r)} \int_0^\infty |\tilde{f}(w', \lambda)|^2 e^{-2\lambda |w'|^2} \, d\lambda dV(w') \right)^{\frac{1}{2}} \left(\int_{D(z';r)} \int_0^\infty e^{-\lambda s} \, d\lambda dV(w') \right)^{\frac{1}{2}} \\
&< \infty.
\end{aligned}$$

This shows that the integral defined by (10.2.6) converges absolutely and defines a holomorphic function on Ω_n. To see $f(z)$ is actually in $H^2(\Omega_n)$, we apply the Plancherel theorem to the λ-variable and get

$$\begin{aligned}
&\int_{\mathbb{C}^{n-1}} \int_{-\infty}^\infty |f(z', t + i|z'|^2 + is)|^2 \, dt dx' dy' \\
&= \frac{1}{2\pi} \int_{\mathbb{C}^{n-1}} \int_0^\infty |\tilde{f}(z', \lambda)|^2 e^{-2\lambda |z'|^2} \cdot e^{-2\lambda s} \, d\lambda dx' dy' \\
&\leq C,
\end{aligned}$$

for all $s > 0$. This completes the proof of Theorem 10.2.1.

It is clear from the proof of Theorem 10.2.1 that $H^2(\Omega_n)$ can be identified with a closed subspace of $L^2(b\Omega_n)$, namely, any $f(z) \in H^2(\Omega_n)$ is identified with its L^2 limiting value $f(z', t + i|z'|^2)$ on $b\Omega_n$ with the norm

$$\| f \|_{H^2(\Omega_n)} = \sup_{s>0} \| f_s \|_{L^2(b\Omega_n)} = \| f(z', t + i|z'|^2) \|_{L^2(b\Omega_n)} .$$

Thus, following the procedure of the Bergman kernel function, we obtain the reproducing kernel, named Cauchy-Szegö kernel, $S(z, w)$ for the Hardy space $H^2(\Omega_n)$. We make the following definition:

Definition 10.2.2. *The Cauchy-Szegö kernel associated with* Ω_n *is the unique function* $S(z,w)$ *which is holomorphic in* z *and antiholomorphic in* w *with respect to* $z \in \Omega_n$ *and* $w \in \overline{\Omega}_n$ *such that*

$$f(z) = \int_{b\Omega_n} S(z,w) f(w)\ d\sigma_w, \tag{10.2.8}$$

for any $f \in H^2(\Omega_n)$ *and any* $z \in \Omega_n$*, where* $d\sigma_w$ *is the surface element on* $b\Omega_n$*.*

For each fixed $z \in \Omega_n$, (10.2.8) defines a bounded linear functional on $H^2(\Omega_n)$. It is also clear from general Hilbert space theory that $S(z,w)$ can be expressed in terms of any orthonormal basis $\{\phi_k(z)\}_{k=1}^{\infty}$ of $H^2(\Omega_n)$, i.e.,

$$S(z,w) = \sum_{k=1}^{\infty} \phi_k(z)\overline{\phi_k(w)}.$$

Now we want to calculate the Cauchy-Szegö kernel $S(z,w)$ on the Siegel upper half space Ω_n. One way to achieve this goal is via the pullback of the Cauchy-Szegö kernel on the unit ball in $\mathbb{C}^n$ by the inverse Cayley transform. Recall that the Cayley transform $w = \Phi(z)$ is a biholomorphic mapping from the unit ball B_n in $\mathbb{C}^n$ onto the Siegel upper half space Ω_n defined by (7.3.2). Thus, the inverse Cayley transform $\phi = \Phi^{-1}$ is given by

$$\begin{aligned} \phi : \Omega_n &\to B_n \\ w \ \mapsto\ z &= \left(\frac{-2iw_1}{1-iw_n}, \cdots, \frac{-2iw_{n-1}}{1-iw_n}, -\frac{1+iw_n}{1-iw_n} \right). \end{aligned}$$

First, by constructing an orthonormal basis for $H^2(B_n)$ directly, the Cauchy-Szegö kernel on the unit ball can be calculated as follows:

Proposition 10.2.3. *The Cauchy-Szegö kernel* $S(\zeta,\eta)$ *on the unit ball in* $\mathbb{C}^n$ *can be expressed explicitly as*

$$S(\zeta,\eta) = \frac{(n-1)!}{2\pi^n} \frac{1}{(1-\zeta\cdot\overline{\eta})^n}, \tag{10.2.9}$$

where $\zeta\cdot\overline{\eta} = \zeta_1\overline{\eta}_1 + \cdots + \zeta_n\overline{\eta}_n$*.*

Proof. It is clear that $\{\zeta^\alpha\}$ forms an orthogonal basis for the Hardy space $H^2(B_n)$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is any multiindex with $\alpha_j \in \mathbb{N} \cup \{0\}$ for $1 \le j \le n$. Therefore, to get the Cauchy-Szegö kernel, we need to normalize $\{\zeta^\alpha\}$.

We proceed as in Section 6.3 for the Bergman kernel function on the unit ball B_n in $\mathbb{C}^n$. Hence, we get

$$\begin{aligned} c_\alpha &= \int_{bB_n} |\zeta^\alpha|^2\ d\sigma_{2n-1} \\ &= 2(|\alpha|+n) \int_{B_n} |\zeta^\alpha|^2\ dV_{2n} \\ &= \frac{2\pi^n \cdot \alpha_n! \cdots \alpha_1!}{(|\alpha|+n-1)!}. \end{aligned}$$

It follows that the Cauchy-Szegö kernel $S(\zeta,\eta)$ on the unit ball is given by

$$\begin{aligned}
S(\zeta,\eta) &= \sum_{\alpha} \frac{1}{c_\alpha}\zeta^\alpha\overline{\eta}^\alpha \\
&= \frac{(n-1)!}{2\pi^n}\left(1+\sum_{\alpha\neq 0}\ \frac{n(n+1)\cdots(n+|\alpha|-1)}{\alpha_1!\cdots\alpha_n!}\ \zeta^\alpha\overline{\eta}^\alpha\right) \\
&= \frac{(n-1)!}{2\pi^n}\left(1+\sum_{k=1}^{\infty}\sum_{|\alpha|=k} \frac{n(n+1)\cdots(n+|\alpha|-1)}{\alpha_1!\cdots\alpha_n!}\ \zeta^\alpha\overline{\eta}^\alpha\right) \\
&= \frac{(n-1)!}{2\pi^n}\left(1+\sum_{k=1}^{\infty}\ \frac{n(n+1)\cdots(n+k-1)}{k!}\ (\zeta\overline{\eta})^k\right) \\
&= \frac{(n-1)!}{2\pi^n}\frac{1}{(1-\zeta\overline{\eta})^n}.
\end{aligned}$$

This proves the proposition.

Our next step is to pull the Cauchy-Szegö kernel $S(\zeta,\eta)$ on the unit ball back to the Siegel upper half space. Denote by $r(z)=\sum_{j=1}^n |z_j|^2-1$ the defining function for the unit ball in $\mathbb{C}^n$, and fix the standard metric on $\mathbb{C}^n$. Then, the surface element ds on the boundary bB_n is given by the interior product of the volume form dV_{2n} with $dr/|dr|$, namely,

$$\begin{aligned}
ds &= \mathrm{Re}\,\iota^*\left(\left(\sum_{j=1}^n \overline{z}_j dz_j\right)\ \vee\ \left(\frac{1}{2i}\right)^n\left(\bigwedge_{k=1}^n d\overline{z}_k\wedge dz_k\right)\right) \\
&= \mathrm{Re}\,\iota^*\left(\sum_{j=1}^n \frac{1}{2^{n-1}i^n}\overline{z}_j dz_j\wedge\left(\bigwedge_{k\neq j} d\overline{z}_k\wedge dz_k\right)\right),
\end{aligned}$$

where $\vee$ denotes the interior product and $\iota: bB_n\to\mathbb{C}^n$ is the inclusion map. Hence, the pullback of ds by the inverse Cayley transform ϕ is

$$\begin{aligned}
\phi^*(ds) = \frac{1}{2^{n-1}i^n}\Bigg(&\frac{2^{2n-1}i}{|1-iw_n|^{2n}}\cdot\frac{1-i\overline{w}_n}{1-iw_n}\ dw_n\wedge\left(\bigwedge_{j=1}^{n-1} d\overline{w}_j\wedge dw_j\right) \\
&+\sum_{j=1}^{n-1}\frac{2^{2n-1}\overline{w}_j(1+i\overline{w}_n)}{|1-iw_n|^{2n+2}}\ dw_j\wedge\left(\bigwedge_{k\neq j} d\overline{w}_k\wedge dw_k\right)\Bigg).
\end{aligned}$$

Since $w_n=t+i|w'|^2$ on $b\Omega_n$, we have

$$dw_n = dt + i\sum_{j=1}^{n-1}\ (w_j d\overline{w}_j+\overline{w}_j dw_j).$$

It follows that

$$dw_n\wedge\left(\bigwedge_{j=1}^{n-1} d\overline{w}_j\wedge dw_j\right) = dt\wedge\left(\bigwedge_{j=1}^{n-1} d\overline{w}_j\wedge dw_j\right),$$

and, for $1 \le j \le n-1$,

$$dw_j \wedge \left(\bigwedge_{k \neq j} d\overline{w}_k \wedge dw_k \right) = 2iw_j dt \wedge \left(\bigwedge_{j=1}^{n-1} d\overline{w}_j \wedge dw_j \right),$$

on the boundary of Ω_n. Thus,

$$\begin{aligned}
&\phi^*(ds) \\
&= (-2i)^n \left(\frac{i}{|1-iw_n|^{2n}} \cdot \frac{1-i\overline{w}_n}{1-iw_n} + \sum_{k=1}^{n-1} \frac{2i|w_k|^2(1+i\overline{w}_n)}{|1-iw_n|^{2n+2}} \right) dt \wedge \left(\bigwedge_{j=1}^{n-1} d\overline{w}_j \wedge dw_j \right) \\
&= (-2i)^n \left(\frac{i(1+\overline{w}_n^2) + (w_n - \overline{w}_n)(1+i\overline{w}_n)}{|1-iw_n|^{2n+2}} \right) dt \wedge \left(\bigwedge_{j=1}^{n-1} d\overline{w}_j \wedge dw_j \right) \\
&= \frac{2^{2n-1}}{|1-iw_n|^{2n}} \, dt \wedge \left(\bigwedge_{j=1}^{n-1} du_j \wedge dv_j \right),
\end{aligned}$$

where $w_j = u_j + iv_j$.

Since the surface element $d\sigma$ on the boundary $b\Omega_n$ is given by $d\sigma = dt \wedge (\wedge_{j=1}^{n-1} du_j \wedge dv_j)$, the above calculation suggests that the Cauchy-Szegö kernel $S(z,w)$ associated with the Siegel upper half space should be given by

$$\begin{aligned}
&S((z', z_n), (w', w_n)) \\
&\quad = \frac{2^{n-\frac{1}{2}}}{(1-iz_n)^n} \cdot \frac{2^{n-\frac{1}{2}}}{(1+i\overline{w}_n)^n} \cdot \frac{(n-1)!}{2\pi^n} \cdot \frac{1}{(1-\phi(z)\overline{\phi(w)})^n} \\
&\quad = \frac{(-1)^n 2^{n-2}(n-1)!}{\pi^n} \cdot \left(i(z_n - \overline{w}_n) + 2\sum_{j=1}^{n-1} z_j \overline{w}_j \right)^{-n}.
\end{aligned} \tag{10.2.10}$$

We must show that the function $S(z,w)$ obtained in (10.2.10) has the required reproducing property for $H^2(\Omega_n)$ as stated in Definition 10.2.2. Theorem 10.2.1 suggests that one should check the Fourier transform of $S(z,w)$.

Notice first that, for $z \in \Omega_n$ and $w \in \overline{\Omega}_n$, we have

$$\begin{aligned}
\mathrm{Re}\left(i(z_n - \overline{w}_n) + 2\sum_{j=1}^{n-1} z_j \overline{w}_j \right) &= -y_n - v_n + \sum_{j=1}^{n-1} (z_j \overline{w}_j + \overline{z}_j w_j) \\
&= -(y_n - |z'|^2) - (v_n - |w'|^2) - |z' - w'|^2,
\end{aligned}$$

which is always negative. Therefore, we can rewrite $S(z,w)$ as

$$S(z,w) = \frac{2^{n-2}}{\pi^n} \int_0^\infty \lambda^{n-1} e^{(i(z_n - \overline{w}_n) + 2z' \cdot \overline{w}')\lambda} \, d\lambda, \tag{10.2.11}$$

and the above integral converges absolutely for $z \in \Omega_n$ and $w \in \overline{\Omega}_n$. Here we use $z' \cdot \overline{w}'$ to denote the inner product $\sum_{j=1}^{n-1} z_j \overline{w}_j$ in $\mathbb{C}^{n-1}$.

Define

$$\tilde{S}(z', \lambda; w) = \left(\frac{2}{\pi}\right)^{n-1} \lambda^{n-1} e^{(-i\overline{w}_n + 2z'\cdot\overline{w}')\lambda}. \tag{10.2.12}$$

We shall show that for each $w \in \Omega_n$, the integral

$$\frac{1}{2\pi}\int_{\mathbb{C}^{n-1}}\int_0^\infty |\tilde{S}(z', \lambda; w)|^2 e^{-2\lambda|z'|^2}\, d\lambda dx' dy' \tag{10.2.13}$$

converges. Since $w \in \Omega_n$, $w_n = u + i|w'|^2 + iv$ with $v > 0$. Hence, (10.2.13) can be rewritten as

$$\begin{aligned}
&\frac{2^{2n-3}}{\pi^{2n-1}}\int_{\mathbb{C}^{n-1}}\int_0^\infty \lambda^{2n-2} e^{-2\lambda|z'-w'|^2}\cdot e^{-2\lambda v}\, d\lambda dx' dy' \\
&\quad = \frac{2^{2n-3}}{\pi^{2n-1}}\int_{\mathbb{C}^{n-1}}\int_0^\infty \lambda^{2n-2} e^{-2\lambda|z'|^2}\cdot e^{-2\lambda v}\, d\lambda dx' dy' \\
&\quad = \frac{2^{2n-2}}{\pi^n (n-2)!}\int_0^\infty\int_0^\infty \lambda^{2n-2} e^{-2\lambda r^2}\cdot e^{-2\lambda v} r^{2n-3}\, dr d\lambda \\
&\quad = \frac{2^{n-2}}{\pi^n}\int_0^\infty \lambda^{n-1} e^{-2\lambda v}\, d\lambda \\
&\quad = \frac{(n-1)!}{4\pi^n v^n}.
\end{aligned}$$

It follows now from Theorem 10.2.1 and (10.2.11) that

$$\begin{aligned}
S(z, w) &= \frac{2^{n-2}}{\pi^n}\int_0^\infty \lambda^{n-1} e^{(i(z_n - \overline{w}_n) + 2z'\cdot\overline{w}')\lambda}\, d\lambda \\
&= \frac{1}{2\pi}\int_0^\infty \tilde{S}(z', \lambda; w) e^{i\lambda z_n}\, d\lambda.
\end{aligned}$$

Hence, for each $w \in \Omega_n$, $S(\cdot, w) \in H^2(\Omega_n)$. Let $w_j = \alpha_j + i\beta_j$ for $1 \le j \le n-1$ and let $d\alpha' d\beta'$ denote $d\alpha_1 \wedge d\beta_1 \wedge \cdots \wedge d\alpha_{n-1} \wedge d\beta_{n-1}$. Since $S(z, w) = \overline{S(w, z)}$, for any $z \in \Omega_n$ and any $f \in H^2(\Omega_n)$, we obtain

$$\begin{aligned}
&\int_{b\Omega_n} S(z, w) f(w)\, du d\alpha' d\beta' \\
&= \frac{1}{2\pi}\int_{\mathbb{C}^{n-1}}\int_0^\infty \overline{\tilde{S}(w', \lambda; z)}\tilde{f}(w', \lambda) e^{-2\lambda|w'|^2}\, d\lambda d\alpha' d\beta' \\
&= \frac{2^{n-2}}{\pi^n}\int_{\mathbb{C}^{n-1}}\int_0^\infty \lambda^{n-1} e^{\lambda(iz_n + 2z'\cdot\overline{w}' - 2|w'|^2)}\tilde{f}(w', \lambda)\, d\lambda d\alpha' d\beta'
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{n-2}}{\pi^n} \int_{\mathbb{C}^{n-1}} \int_0^\infty \lambda^{n-1} e^{\lambda(iz_n + 2|z'|^2 - 2\overline{z}'\cdot w' - 2|w'-z'|^2)} \tilde{f}(w', \lambda)\, d\lambda d\alpha' d\beta' \\
&= \frac{2^{n-2}}{\pi^n} \int_{\mathbb{C}^{n-1}} \int_0^\infty \lambda^{n-1} e^{\lambda(iz_n + 2|z'|^2)} \cdot e^{-2\lambda|\eta'|^2} \cdot e^{-2\lambda\overline{z}'\cdot(z'+\eta')} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \tilde{f}(z'+\eta', \lambda)\, d\lambda d\zeta' d\xi' \\
&= \frac{1}{2\pi} \frac{2^n}{(n-2)!} \int_0^\infty \lambda^{n-1} e^{i\lambda z_n} \tilde{f}(z', \lambda) \left(\int_0^\infty e^{-2\lambda\rho^2} \rho^{2n-3}\, d\rho \right) d\lambda \\
&= \frac{1}{2\pi} \int_0^\infty \tilde{f}(z', \lambda) e^{i\lambda z_n}\, d\lambda \\
&= f(z),
\end{aligned}
$$

where $\eta_j = w_j - z_j = \zeta_j + i\xi_j$ for $1 \le j \le n-1$ and $d\zeta' d\xi'$ denotes $d\zeta_1 \wedge d\xi_1 \wedge \cdots \wedge d\zeta_{n-1} \wedge d\xi_{n-1}$. The last equality is guaranteed by (10.2.6) in Theorem 10.2.1. Thus, we have shown that the kernel function (10.2.10) reproduces the functions belonging to the Hardy space $H^2(\Omega_n)$. Hence, by the uniqueness of the Cauchy-Szegö kernel function, we obtain the following theorem.

Theorem 10.2.4. *The Cauchy-Szegö kernel function $S(z, w)$ associated with the Siegel upper half space Ω_n is given by*

$$
\begin{aligned}
S(z, w) &= S((z', z_n), (w', w_n)) \\
&= \frac{(-1)^n 2^{n-2} (n-1)!}{\pi^n} \cdot \left(i(z_n - \overline{w}_n) + 2\sum_{j=1}^{n-1} z_j \overline{w}_j \right)^{-n}.
\end{aligned}
$$

Hence, for any $f \in L^2(b\Omega_n)$, the integral

$$
Sf(z) = \int_{b\Omega_n} S(z, w) f(w)\, d\sigma(w), \tag{10.2.14}
$$

defines a function $Sf(z)$ in the Hardy space $H^2(\Omega_n)$ which has a well-defined L^2 integrable limiting value on $b\Omega_n$. We recall that the Szegö projection on $b\Omega_n$ is the orthogonal projection from $L^2(b\Omega_n)$ onto the closed subspace consisting of square integrable CR functions, which coincide with the limiting values of functions belonging to the Hardy space $H^2(\Omega_n)$. We shall still use (10.2.14) to denote the Szegö projection on $b\Omega_n$.

Since, for each $f \in L^2(b\Omega_n)$, $Sf \in H^2(\Omega_n)$, Theorem 10.2.1 shows that

$$
\lim_{\epsilon \to 0} Sf(z', t + i|z'|^2 + i\epsilon^2) = Sf(z', t + i|z'|^2)
$$

in the L^2 sense. Denote $Sf(z', t + i|z'|^2 + i\epsilon^2)$ by $(Sf)_\epsilon(z', t + i|z'|^2)$ which can be regarded as an L^2 integrable function on $b\Omega_n$. Let

$$
\rho_\epsilon(z', t) = |z'|^2 + \epsilon^2 - it,
$$

on $b\Omega_n$. Then we have

Proposition 10.2.5. *For any $f \in L^2(b\Omega_n)$ and any $\epsilon > 0$, $(Sf)_\epsilon$ is given by*

$$(Sf)_\epsilon(z', t + i|z'|^2) = \frac{2^{n-2}(n-1)!}{\pi^n} f * \rho_\epsilon^{-n}(z', t), \tag{10.2.15}$$

where the convolution is taken with respect to the group structure on $b\Omega_n$, and the coordinates on $b\Omega_n$ are $z' = (z_1, \cdots, z_{n-1})$ and t.

Proof. Let $\beta = (z', t + i|z'|^2)$ and $\alpha = (w', u + i|w'|^2)$. Hence,

$$\frac{2^{n-2}(n-1)!}{\pi^n} f * \rho_\epsilon^{-n}(z', t) = \frac{2^{n-2}(n-1)!}{\pi^n} \int_{b\Omega_n} f(\alpha)\rho_\epsilon^{-n}(\alpha^{-1}\beta)\, d\sigma(\alpha).$$

A direct calculation shows that

$$\begin{aligned}\alpha^{-1}\beta &= (-w', -u + i|w'|^2) \cdot (z', t + i|z'|^2) \\ &= (z' - w', t - u + i|z'|^2 + i|w'|^2 - 2iz' \cdot \overline{w}').\end{aligned}$$

It follows that we have

$$\begin{aligned}\rho_\epsilon^{-n}(\alpha^{-1}\beta) &= (|z' - w'|^2 + \epsilon^2 - i(t-u) - z' \cdot \overline{w}' + \overline{z}' \cdot w')^{-n} \\ &= (|z'|^2 + |w'|^2 + \epsilon^2 - i(t-u) - 2z' \cdot \overline{w}')^{-n} \\ &= (-1)^n (i(t + i|z'|^2 + i\epsilon^2 - (u - i|w'|^2)) + 2z' \cdot \overline{w}')^{-n} \\ &= (-1)^n (i(z_n + i\epsilon^2 - \overline{w}_n) + 2z' \cdot \overline{w}')^{-n}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\frac{2^{n-2}(n-1)!}{\pi^n} f * \rho_\epsilon^{-n}(z', t) \\ &= \frac{(-1)^n 2^{n-2}(n-1)!}{\pi^n} \int_{b\Omega_n} \frac{f(w', u)}{(i(z_n + i\epsilon^2 - \overline{w}_n) + 2z' \cdot \overline{w}')^n}\, d\sigma(\alpha) \\ &= Sf(z', t + i|z'|^2 + i\epsilon^2).\end{aligned}$$

This proves the proposition.

Finally, we can describe the Szegö projection on the Heisenberg group $\mathbb{H}_n$ as follows:

Theorem 10.2.6. *The Szegö projection Sf for any $f \in L^2(\mathbb{H}_n)$ is given by*

$$Sf(z', t) = \lim_{\epsilon \to 0} \frac{2^{2n-3}(n-1)!}{\pi^n} f * \rho_\epsilon^{-n}(z', t), \tag{10.2.16}$$

where the convolution is taken with respect to the group structure of $\mathbb{H}_n$.

The convergence of (10.2.16) is guaranteed by Theorem 10.2.1. Notice also that the factor 2^{n-1} that appears in the formulation of (10.2.16) is due to the fact that the volume form dV on the Heisenberg group $\mathbb{H}_n$ has been taken to be

$$dV = 2^{1-n} dx_1 dy_1 \cdots dx_{n-1} dy_{n-1} dt.$$

10.3 Local Solvability of the Lewy Operator

We now return to the local solvability of $\mathcal{L}_{n-1} = \Box_b^0$ on the Heisenberg group. When $\Box_b$ acts on functions, $\Box_b^0$ is not hypoelliptic since it annihilates all CR functions. However, we shall show that, modulo the Szegö projection S, there exists a relative fundamental solution for $\Box_b^0$. Rewrite $\Box_b^0$ as

$$\Box_b^0 = \mathcal{L}_\alpha - i(\alpha - n + 1)T, \tag{10.3.1}$$

for $\alpha \in \mathbb{C}$. Recall that

$$\varphi_\alpha(z', t) = (|z'|^2 - it)^{-\frac{(n-1+\alpha)}{2}}(|z'|^2 + it)^{-\frac{(n-1-\alpha)}{2}},$$

and

$$c_\alpha = \frac{2^{4-2n}\pi^n}{\Gamma(\frac{n-1+\alpha}{2})\Gamma(\frac{n-1-\alpha}{2})}.$$

Then, we have

$$\begin{aligned} \Box_b^0 \varphi_\alpha &= \mathcal{L}_\alpha \varphi_\alpha - i(\alpha - n + 1)T\varphi_\alpha \\ &= c_\alpha \delta - i(\alpha - n + 1)T\varphi_\alpha. \end{aligned} \tag{10.3.2}$$

Now with the aid of the identity

$$\Gamma(w)\Gamma(1-w) = \frac{\pi}{\sin \pi w},$$

we formally differentiate (10.3.2) with respect to α and evaluate it at $\alpha = n - 1$ to get

$$\Box_b^0\left(\frac{(n-2)!}{2^{4-2n}\pi^n}\log\left(\frac{|z'|^2 - it}{|z'|^2 + it}\right)(|z'|^2 - it)^{-n+1}\right) = \delta - \frac{2(n-1)!}{2^{4-2n}\pi^n}(|z'|^2 - it)^{-n}. \tag{10.3.3}$$

Here the logarithm of the quotient means the difference of the corresponding logarithm. Set

$$\varPhi = \frac{(n-2)!}{2^{4-2n}\pi^n}\log\left(\frac{|z'|^2 - it}{|z'|^2 + it}\right)(|z'|^2 - it)^{-n+1}, \tag{10.3.4}$$

and define the operator K by

$$Kf = f * \varPhi, \tag{10.3.5}$$

where the convolution is taken with respect to the group structure on $\mathbb{H}_n$.

Theorem 10.3.1. *Let the operator K be defined as in (10.3.5), then we have*

$$\Box_b^0 \cdot K = K \cdot \Box_b^0 = I - S,$$

when acting on distributions with compact support.

Proof. It suffices to show only that $\Box_b^0 \cdot K = I - S$. The other identity then follows immediately by transposition.

Set $\rho_\epsilon = |z'|^2 + \epsilon^2 - it$, and define

$$\Phi_\epsilon(z', t) = \frac{(n-2)!}{2^{4-2n}\pi^n} \log\left(\frac{|z'|^2 + \epsilon^2 - it}{|z'|^2 + \epsilon^2 + it}\right)(|z'|^2 + \epsilon^2 - it)^{-n+1}.$$

Then, by the calculations done in the proof of Theorem 10.1.2, we obtain

$$\begin{aligned}
\Box_b^0 \Phi_\epsilon &= -\sum_{k=1}^{n-1} Z_k \overline{Z}_k \Phi_\epsilon \\
&= \frac{(n-2)!}{2^{4-2n}\pi^n}\left(-4(n-1)\frac{|z'|^2}{\overline{\rho}_\epsilon {\rho_\epsilon}^n} + 2(n-1)\frac{1}{\overline{\rho}_\epsilon {\rho_\epsilon}^{n-1}}\right) \\
&= \frac{(n-1)!}{2^{4-2n}\pi^n}\left(\frac{4\epsilon^2}{\overline{\rho}_\epsilon \rho_\epsilon^n} - \frac{2}{\rho_\epsilon^n}\right).
\end{aligned}$$

Hence, as $\epsilon \to 0$, we get by the integral evaluated in the proof of Theorem 10.1.2, that

$$\begin{aligned}
\Box_b^0 \Phi &= \left(\frac{4(n-1)!}{2^{4-2n}\pi^n}\int_{\mathbb{H}_n} (|z'|^2 + 1 - it)^{-n}(|z'|^2 + 1 + it)^{-1} dV\right)\delta \\
&\quad - \frac{2(n-1)!}{2^{4-2n}\pi^n}\frac{1}{(|z'|^2 - it)^n} \\
&= \delta - \frac{(n-1)!}{2^{3-2n}\pi^n}\frac{1}{(|z'|^2 - it)^n}.
\end{aligned}$$

The assertion now follows from (10.2.16). This proves the theorem.

Theorem 10.3.1 shows that the operator K inverts $\Box_b^0$ on the space of functions that are orthogonal to the L^2 integrable CR functions. It is also clear that $S\Box_b^0 = \Box_b^0 S = 0$. Then, we have the following local solvability theorem for $\Box_b^0$.

Theorem 10.3.2. *Let $f \in L^2(\mathbb{H}_n)$. The equation $\Box_b^0 u = f$ is solvable in the L^2 sense in some neighborhood of $p \in \mathbb{H}_n$ if and only if $S(f)$ is real analytic in a neighborhood of p.*

Proof. We may assume that f is an L^2 integrable function of compact support. Suppose that $S(f)$ is real analytic near p. Then, by the Cauchy-Kowalevski theorem, there is a real analytic solution u_1 locally such that

$$\Box_b^0 u_1 = S(f)$$

in some neighborhood of p. On the other hand, by Theorem 10.3.1, a solution $u_2 = Kf$ exists for

$$\Box_b^0 u_2 = (I - S)f.$$

Hence, $u = u_1 + u_2$ is a local solution of $\Box_b^0 u = f$.

Conversely, let u be a local solution of $\Box_b^0 u = f$. Choose a cut-off function ζ with $\zeta = 1$ in some open neighborhood of p. Set

$$\Box_b^0(\zeta u) = h.$$

Then, $Sh = 0$ and $f - h = 0$ in some neighborhood of p. Now, from the explicit formula (10.2.16) of the Szegö projection S, it is easily seen that $S(f) = S(f - h)$ is real analytic in some neighborhood of p. This completes the proof of the theorem.

If $n = 2$, we can deduce the local solvability of the Lewy operator from Theorem 10.3.2.

Theorem 10.3.3. *Let $\overline{Z} = (\partial/\partial\overline{z}) - iz(\partial/\partial t)$ and $f \in L^2(\mathbb{H}_2)$. The equation $\overline{Z}u = f$ is locally solvable in the L^2 sense in some open neighborhood of $p \in \mathbb{H}_2$ if and only if $S(\overline{f})$ is real analytic in a neighborhood of p.*

Proof. f is still assumed to be an L^2 integrable function with compact support. If $S(\overline{f})$ is real analytic in some neighborhood of p, then Theorem 10.3.2 assures the existence of a solution v of the equation

$$\Box_b^0 v = -Z\overline{Z}v = \overline{f}$$

which , by conjugation, gives a solution $u = -Z\overline{v}$ of the Lewy equation.

On the other hand, if there exists locally a solution u to the equation $\overline{Z}u = f$, we may assume that u is of compact support. Hence, Theorem 10.3.1 guarantees a solution v of the equation $Zv = u - Su$. Now, by Theorem 10.3.2 again, we see that $S(\overline{f})$ is real analytic in some open neighborhood of p. This proves the theorem.

We note that the Lewy's example can be extended to any tangential Cauchy-Riemann equation $\overline{L}$ of a hypersurface in $\mathbb{C}^2$ which is not Levi-flat, i.e., its Levi form $c(x)$, where

$$[L, \overline{L}] = c(x)T, \quad \mathrm{mod}(L, \overline{L}),$$

does not vanish identically in a neighborhood of the reference point. Note also that, from the discussion at the end of Chapter 7, when the Levi form vanishes completely, the $\overline{\partial}_b$-equation is reduced to a $\overline{\partial}$-equation with a parameter.

Since we have established in Chapter 9 that the range of the $\Box_b$ operator on the boundary of any smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with $n \geq 2$ is closed, the arguments for proving Theorems 10.3.2 and 10.3.3 can then be applied verbatim to the boundary of any smooth bounded pseudoconvex domain with real analytic boundary, provided that the following analyticity hypothesis on the Szegö projection is fulfilled:

Analyticity Hypothesis. *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with real analytic boundary bD and $p \in bD$. Let S be the corresponding Szegö projection on bD. If $f \in L^2(bD)$ vanishes on some open neighborhood U of $p \in bD$, then Sf is real analytic on U.*

Now, with this hypothesis, we can state the following theorem:

Theorem 10.3.4. *Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$ with real analytic boundary. Let r be a real analytic defining function for D and $\overline{L} = (\partial r/\partial \overline{z}_2)(\partial/\partial \overline{z}_1) - (\partial r/\partial \overline{z}_1)(\partial/\partial \overline{z}_2)$. Suppose that the Szegö projection S associated with bD satisfies the analyticity hypothesis. Then the tangential Cauchy-Riemann equation $\overline{L}u = f$ is locally solvable for some L^2 function f near $p \in bD$ if and only if $S(\overline{f})$ is real analytic near p.*

Finally, following the arguments of Theorem 10.1.5 we obtain the regularity theorem of the Lewy operator $\overline{Z}$ in the usual Hölder class.

Theorem 10.3.5. *Let f be a continuous function with compact support on $\mathbb{H}_2$, and let $p \in supp f$. Suppose that $S(\overline{f})$ is real analytic in some open neighborhood U of p, then locally there exists a solution $u \in \Lambda^{1/2}(V)$ on some open neighborhood V of p with $V \subset U$ such that $\overline{Z}u = f$. Furthermore, if $f \in C^k(\mathbb{H}_2)$ for $k \in \mathbb{N} \cup \{0\}$ with compact support, then $u \in C^{k+\frac{1}{2}}(V)$.*

NOTES

Most of the materials in Sections 10.1 are essentially taken from G. B. Folland and E. M. Stein [FoSt 1]. Theorem 10.1.1 was proved by G. B. Folland [Fol 1]. The kernel $\Phi_\alpha = c_\alpha^{-1}\varphi_\alpha$ defined in (10.1.11) is homogeneous of order $-2n+2$ with respect to the nonisotropic dilation on $\mathbb{H}_n$. It follows that the regularity property of the operator $K_\alpha f = f * \Phi_\alpha$ in the nonisotropic normed spaces can be drawn from a general theory described in [FoSt 1]. We refer the reader to the book by E. M. Stein [Ste 4] for a systematic treatment on analysis on Heisenberg groups. The proof of Theorem 10.1.5 follows that of M.-C. Shaw [Sha 9]. The characterization via the Fourier transform of the Hardy space $H^2(\Omega_n)$ on the Siegel upper half space was proved by S. G. Gindikin [Gin 1] (Theorem 10.2.1). The Cauchy-Szegö kernel for the ball B_n in $\mathbb{C}^n$, $n \geq 2$, was found by L. K. Hua [Hua 1] (Proposition 10.2.3), and for the Siegel upper half space Ω_n by S. G. Gindikin [Gin 1]. The characterization of the range of the Lewy operator was proved by Greiner, Kohn and Stein [GKS 1]. (See also [GrSt 1]). We also refer the reader to the books by R. Beals and P. C. Greiner [BeGr 1] and F. Treves [Tre 3,6] for more discussions on Heisenberg group and CR manifolds.

The generalization of the nonsolvability of the Lewy operator to any tangential Cauchy-Riemann equation on a hypersurface which is not Levi-flat in $\mathbb{C}^2$ was proved by L. Hörmander [Hör 1,7]. It is known that the analyticity hypothesis holds on any smooth bounded strongly pseudoconvex domain with real analytic boundary. For instance, see [Tar 1,2] and [Tre 2] for $n \geq 3$ and [Gel 1] for $n = 2$. Unfortunately, there are no general theorems which would guarantee that the Szegö projection S on weakly pseudoconvex boundaries satisfies this hypothesis. One should also note that, in general, the analytic pseudolocality of S is false on pseudoconvex boundaries, as shown by M. Christ and D. Geller in [ChGe 1].

CHAPTER 11

INTEGRAL REPRESENTATIONS FOR $\bar\partial$ AND $\bar\partial_b$

In this chapter the method of integral representation in several complex variables is discussed. This method can be viewed as a generalization of the Cauchy integral formula in one variable to several variables. The integral kernel method gives solutions to $\bar\partial$ and $\bar\partial_b$ represented by integral formulas on strongly pseudoconvex domains or boundaries. The representations are especially easy to construct on a strictly convex domain where solution formulas can be written explicitly. It is in this setting that we derive integral formulas for $\bar\partial$ and $\bar\partial_b$ in this chapter.

The L^2 approach is fruitful for solving $\bar\partial$ and $\bar\partial_b$ in the Sobolev spaces on pseudoconvex domains and their boundaries. In Chapters 4-6, the L^2 method to solve $\bar\partial$ was discussed using the $\bar\partial$-Neumann problem. In Chapters 8 and 9, we studied the global solvability and regularity for the tangential Cauchy-Riemann operator in the Sobolev spaces on compact CR manifolds. However, Hölder and L^p estimates for $\bar\partial$ and $\bar\partial_b$ are not easy to obtain by the L^2 method. An explicit kernel was computed in Chapter 10 for $\Box_b$ on the Heisenberg group and Hölder estimates were obtained for solutions of $\bar\partial_b$. Our goal here is to construct integral formulas for solutions of $\bar\partial$ and $\bar\partial_b$ with Hölder and L^p estimates on strictly convex domains.

In Section 11.1, some terminology necessary in developing the kernel formulas is defined. We derive the Bochner-Martinelli-Koppelman formula as a generalization of the Cauchy integral. Unlike the Cauchy kernel in $\mathbb{C}^1$, the Bochner-Martinelli-Koppelman kernel is only harmonic, but not holomorphic. Then we introduce the Leray kernel and derive the homotopy formula for $\bar\partial$ on convex domains in Section 11.2. Hölder estimates for the solutions of $\bar\partial$ on strictly convex domains are obtained. In Section 11.3 the jump formula derived from the Bochner-Martinelli-Koppelman formula is discussed and homotopy formulas for $\bar\partial_b$ on strictly convex compact boundaries are constructed and estimated.

The kernel method is especially suitable for the local solvability of $\bar\partial_b$ on an open subset with smooth boundary in a strictly convex boundary. It allows the derivation of an explicit formula of a solution kernel on a domain with boundary in a strictly pseudoconvex CR manifold. This is discussed in Section 11.4. The L^p estimates for the local solutions for $\bar\partial_b$ are proved in Section 11.5. We discuss the $\bar\partial_b$-Neumann problem in Section 11.6, which is an analogue for $\bar\partial_b$ of the $\bar\partial$-Neumann problem. The L^2 Hodge decomposition theorem for $\bar\partial_b$ on an open set with boundary in a strictly pseudoconvex CR manifold is proved in Theorem 11.6.4.

11.1 Integral Kernels in Several Complex Variables

Our first goal is to find a fundamental solution of $\bar\partial$ for (p,q)-forms in several complex variables. Since p plays no role in the $\bar\partial$ equation, we shall assume that $p = 0$. In $\mathbb{C}$, the Cauchy kernel is a fundamental solution for $\bar\partial$. This can be derived by differentiating the fundamental solution for $\triangle$. Since

$$\frac{1}{2\pi}\triangle \log|z| = \frac{2}{\pi}\frac{\partial^2}{\partial\bar z\partial z}\log|z| = \delta_0,$$

where δ_0 is the Dirac delta function centered at 0, we have

$$\frac{2}{\pi}\frac{\partial}{\partial\bar z}\frac{\partial}{\partial z}\log|z| = \frac{1}{\pi}\frac{\partial}{\partial\bar z}\frac{1}{z} = \delta_0.$$

This implies that $E(z) = 1/\pi z$ is a fundamental solution for $\partial/\partial\bar z$. For any bounded function f on $\mathbb{C}$ with compact support in $\overline{D}$, where D is a bounded domain in $\mathbb{C}$, we define

$$u(z) = f * E(z) = \frac{1}{\pi}\int_D \frac{f(\zeta)}{z-\zeta}dV = \frac{1}{2\pi i}\int_D \frac{f(\zeta)}{\zeta - z}d\zeta\wedge d\bar\zeta.$$

It follows that $\partial u/\partial\bar z = f$ in $\mathbb{C}$ in the distribution sense. This can also be proved directly as in Theorem 2.1.2.

In $\mathbb{C}^n$ when $n > 1$, we can also derive a fundamental solution for the $\bar\partial$ operator in the top degree case similarly. Let $\alpha = f d\bar z_1\wedge\cdots\wedge d\bar z_n$ be a $(0,n)$-form in $\mathbb{C}^n$ where f is a bounded function with compact support in $\mathbb{C}^n$. Since there is no compatibility condition for α to be solvable, we can derive a solution for the equation $\bar\partial u = \alpha$ as follows: Let $e(z)$ be a fundamental solution for $\triangle$ in $\mathbb{C}^n$, $n \ge 2$, defined by

$$e(z) = e(r) = \frac{-(n-2)!}{4\pi^n}\frac{1}{r^{2n-2}}, \quad r = |z|.$$

Both $e(r)$ and all the first order derivatives of $e(r)$ are locally integrable functions. We define a $(0,n)$-form $e_n = -4e(r)d\bar z_1\wedge\cdots\wedge d\bar z_n$. Then

$$\bar\partial\bar\partial^* e_n = 4\sum_{i=1}^n \frac{\partial^2 e(r)}{\partial\bar z_i\partial z_i}d\bar z_1\wedge\cdots\wedge d\bar z_n = \delta_0 d\bar z_1\wedge\cdots\wedge d\bar z_n.$$

We define

$$E(z) = \bar\partial^* e_n = \sum_{j=1}^n (-1)^j\frac{(n-1)!}{\pi^n}\frac{\bar z_j}{r^{2n}}d\bar z_1\wedge\cdots \widehat{d\bar z_j}\cdots\wedge d\bar z_n,$$

where $\widehat{d\bar z_j}$ denotes that the term $d\bar z_j$ is omitted. It follows that

$$\bar\partial E(z) = \delta_0 d\bar z_1\wedge\cdots\wedge d\bar z_n.$$

Define

$$(11.1.1)\qquad \begin{aligned} u(z) &= f * E(z) \\ &= \sum_{j=1}^{n}\left((-1)^j \frac{(n-1)!}{\pi^n}\int_D \frac{\bar{z}_j - \bar{\zeta}_j}{|z-\zeta|^{2n}} f(\zeta)dV\right)[d\overset{\wedge}{\bar{z}_j}], \end{aligned}$$

where $[d\overset{\wedge}{\bar{z}_j}] = d\bar{z}_1 \wedge \cdots d\overset{\wedge}{\bar{z}_j} \cdots \wedge d\bar{z}_n$. Then u satisfies $\bar{\partial}u = \alpha$ and $E(z)$ is a fundamental solution for $\bar{\partial}$ when $q = n$. For general $0 < q < n$, due to the compatibility condition, the fundamental solution for $\bar{\partial}$ is more involved. We introduce some notation first.

Let $(\zeta - z) = (\zeta_1 - z_1, \zeta_2 - z_2, \cdots, \zeta_n - z_n) \in \mathbb{C}^n$ and $d\zeta = (d\zeta_1, \cdots, d\zeta_n)$. Let $A = (a_1, \cdots, a_n)$, $B = (b_1, \cdots, b_n)$ be two vectors in $\mathbb{C}^n$. We define

$$< A, B > = \sum_{i=1}^{n} a_i b_i, \quad < A, d\zeta > = \sum_{i=1}^{n} a_i d\zeta_i.$$

Thus $< \bar{\zeta} - \bar{z}, \zeta - z > = |\zeta - z|^2$ and $< \bar{\zeta} - \bar{z}, d\zeta > = \sum_{i=1}^n (\bar{\zeta}_i - \bar{z}_i)d\zeta_i$. Let V be an open subset of $\mathbb{C}^n \times \mathbb{C}^n$ with coordinates (ζ, z) and let $G(\zeta, z)$ be a C^1 map from V into $\mathbb{C}^n$ such that $G(\zeta, z) = (g_1(\zeta, z), \cdots, g_n(\zeta, z))$. We define the $(1,0)$-form ω^G by

$$\omega^G = \frac{1}{2\pi i}\frac{< G(\zeta,z), d\zeta >}{< G(\zeta,z), \zeta - z >} = \frac{1}{2\pi i}\frac{\sum_{i=1}^n g_i(\zeta,z)d\zeta_i}{\sum_{i=1}^n g_i(\zeta,z)(\zeta_i - z_i)}$$

on the set of $(\zeta, z) \in \mathbb{C}^n \times \mathbb{C}^n$ where $< G, \zeta - z > \neq 0$.

When $n = 1$, ω^G is independent of G and is the Cauchy kernel. We set the Cauchy-Riemann operator on V equal to

$$\bar{\partial}_{\zeta,z} = \bar{\partial}_\zeta + \bar{\partial}_z,$$

and

$$< \bar{\partial}_{\zeta,z} G(\zeta,z), d\zeta > = \sum_{i=1}^{n} \bar{\partial}_{\zeta,z} g_i(\zeta,z) \wedge d\zeta_i.$$

Let $\Omega(G)$ be an $(n, n-1)$-form in (ζ, z) defined by

$$\Omega(G) = \omega^G \wedge (\bar{\partial}_{\zeta,z}\omega^G)^{n-1} = \omega^G \wedge \underbrace{\bar{\partial}_{\zeta,z}\omega^G \wedge \cdots \wedge \bar{\partial}_{\zeta,z}\omega^G}_{n-1 \text{ times}}.$$

Given m maps $G^i : V \to \mathbb{C}^n$, $i = 1, \cdots, m$, we abbreviate ω^{G^i} by ω^i and $\Omega^{1\cdots m}$ is the $(n, n-m)$-form defined by

$$\begin{aligned} \Omega^{1\cdots m} &= \Omega(G^1, \cdots, G^m) \\ &= \omega^1 \wedge \cdots \wedge \omega^m \wedge \sum_{k_1 + \cdots + k_m = n-m} (\bar{\partial}_{\zeta,z}\omega^1)^{k_1} \wedge \cdots \wedge (\bar{\partial}_{\zeta,z}\omega^m)^{k_m} \end{aligned}$$

on the set where all the denominators are nonvanishing. Since

$$(11.1.2)\quad \bar{\partial}_{\zeta,z}\omega^i = \frac{1}{2\pi i}\frac{< \bar{\partial}_{\zeta,z}G^i, d\zeta >}{< G^i(\zeta,z), \zeta - z >} - \frac{1}{2\pi i}\frac{< \bar{\partial}_{\zeta,z}G^i, \zeta - z > \wedge < G^i, d\zeta >}{(< G^i, \zeta - z >)^2},$$

we have for $k \geq 0$ that

$$\omega^i \wedge (\bar{\partial}_{\zeta,z}\omega^i)^k = \left(\frac{1}{2\pi i}\right)^{k+1} \frac{< G^i, d\zeta >}{< G^i, \zeta - z >} \wedge \left(\frac{< \bar{\partial}_{\zeta,z} G^i, d\zeta >}{< G^i, \zeta - z >}\right)^k. \tag{11.1.3}$$

This follows from the fact that ω^i wedge the last term in (11.1.2) vanishes.

The following lemma is essential in the construction of the kernel formulas.

Lemma 11.1.1. *Let $G^i(\zeta, z) : V \subset \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, $i = 1, \cdots, m$, be C^1 maps. We have*

$$\bar{\partial}_{\zeta,z}\Omega^{1\cdots m} = \sum_{j=1}^{m} (-1)^j \Omega^{1\cdots \hat{j} \cdots m} \tag{11.1.4}$$

on the set where $< G^i, \zeta - z > \neq 0$ for every $i = 1, \cdots, m$, where $\hat{j}$ denotes that the term j is omitted. In particular, we have

$$\bar{\partial}_{\zeta,z}\Omega^1 = 0, \tag{11.1.4-i}$$

$$\bar{\partial}_{\zeta,z}\Omega^{12} = \Omega^1 - \Omega^2, \tag{11.1.4-ii}$$

$$\bar{\partial}_{\zeta,z}\Omega^{123} = -\Omega^{23} + \Omega^{13} - \Omega^{12}, \tag{11.1.4-iii}$$

on the set where the denominators are nonvanishing.

Proof. We use the notation

$$(\bar{\partial}_{\zeta,z}\omega)^{K_m} = (\bar{\partial}_{\zeta,z}\omega^1)^{k_1} \wedge \cdots \wedge (\bar{\partial}_{\zeta,z}\omega^m)^{k_m}$$

for each multiindex $K_m = (k_1, \cdots, k_m)$ and $|K_m| = k_1 + \cdots + k_m$. It follows that

$$\begin{aligned}
\bar{\partial}_{\zeta,z}\Omega^{1\cdots m} &= \sum_{j=1}^{m} (-1)^{j-1} \omega^1 \wedge \cdots \wedge \bar{\partial}_{\zeta,z}\omega^j \wedge \cdots \wedge \omega^m \wedge \sum_{|K_m| = n-m} (\bar{\partial}_{\zeta,z}\omega)^{K_m} \\
&= \sum_{j=1}^{m} (-1)^{j-1} \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^m \wedge \sum_{\substack{|K_m| = n-m+1, \\ k_j \geq 1}} (\bar{\partial}_{\zeta,z}\omega)^{K_m} \\
&= \sum_{j=1}^{m} (-1)^j \Omega^{1\cdots \hat{j} \cdots m} + \sum_{j=1}^{m} (-1)^{j-1} \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^m \\
&\quad \wedge \sum_{\substack{|K_m| = n-m+1, \\ k_j \geq 0}} (\bar{\partial}_{\zeta,z}\omega)^{K_m}.
\end{aligned}$$

We claim that for each K_m such that $|K_m| = n - m + 1$,

$$\sum_{j=1}^{m} (-1)^{j-1} \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^m \wedge (\bar{\partial}_{\zeta,z}\omega)^{K_m} = 0.$$

We first observe that

$$\omega^1 \wedge \cdots \wedge \omega^j \wedge \cdots \wedge \omega^m \wedge (\bar{\partial}_{\zeta,z}\omega)^{K_m} = 0, \tag{11.1.5}$$

since there are $n+1$ $d\zeta' s$. Define

$$\Theta = 2\pi i \sum_{j=1}^{n} (\zeta_j - z_j) d\zeta_j.$$

It is easy to see that

$$\Theta \vee \omega^i = 1, \quad i = 1, \cdots, m, \tag{11.1.6}$$

where $\vee$ denotes the interior product.

Also from (11.1.2), we have

$$\Theta \vee \bar{\partial}_{\zeta,z}\omega^i = 0, \quad i = 1, \cdots, m. \tag{11.1.7}$$

Contraction of equation (11.1.5) with Θ, using (11.1.6) and (11.1.7), gives

$$\begin{aligned}
0 &= \Theta \vee (\omega^1 \wedge \cdots \wedge \omega^j \wedge \cdots \wedge \omega^m \wedge (\bar{\partial}_{\zeta,z}\omega)^{K_m}) \\
&= \sum_{j=1}^{m} (-1)^{j-1} \omega^1 \wedge \cdots \wedge (\Theta \vee \omega^j) \wedge \cdots \wedge \omega^m \wedge (\bar{\partial}_{\zeta,z}\omega)^{K_m} \\
&= \sum_{j=1}^{m} (-1)^{j-1} \omega^1 \wedge \cdots \wedge \overset{\wedge}{\omega^j} \wedge \cdots \wedge \omega^m \wedge (\bar{\partial}_{\zeta,z}\omega)^{K_m}.
\end{aligned}$$

This proves the claim and the lemma.

We also write

$$\Omega^{1\cdots m} = \sum_{q=0}^{n-m} \Omega_q^{1\cdots m},$$

where $\Omega_q^{1\cdots m}$ denotes the piece in $\Omega^{1\cdots m}$ which is of degree $(0,q)$ in z and $(n, n-m-q)$ degree in ζ. If f is a $(0,q')$-form in $\mathbb{C}^n$, the form $\Omega_q^{1\cdots m}(\zeta,z)\wedge f(\zeta)$ is a form of degree $(n, n-m-q+q')$ in ζ and of degree $(0,q)$ in z. We write the form $\Omega_q^{1\cdots m}(\zeta,z)\wedge f(\zeta)$ uniquely as $\sum'_{|J|=q} A_J(\zeta,z)\wedge d\bar{z}^J$ where $A_J(\zeta,z)$ is an $(n, n-m-q+q')$-form in ζ only with coefficients depending on z and J is an increasing multiindex. We shall define the integration of the the form $\Omega^{1\cdots m}(\zeta,z)\wedge f(\zeta)$ with respect to the ζ variables on a $(2n-l)$-dimensional real manifold M as follows:

$$\begin{aligned}
\int_M \Omega^{1\cdots m}(\zeta,z) \wedge f(\zeta) &= \int_M \Omega_q^{1\cdots m}(\zeta,z) \wedge f(\zeta) \\
&= \sum_{|J|=q}{}' \left(\int_M A_J(\zeta,z) \right) d\bar{z}^J,
\end{aligned}$$

where $q = l - m + q'$, provided the integral on the right-hand side exists. Note that from this definition, we have

$$\begin{aligned}\bar{\partial}_z \int_M \Omega^{1\cdots m}(\zeta, z) \wedge f(\zeta) &= (-1)^{2n-l} \int_M \bar{\partial}_z \Omega^{1\cdots m}(\zeta, z) \wedge f(\zeta)\\ &= (-1)^l \int_M \bar{\partial}_z \Omega^{1\cdots m}(\zeta, z) \wedge f(\zeta),\end{aligned}$$

provided that the differentiation under the integral sign is allowed.

Let

$$G^0(\zeta, z) = (\bar{\zeta} - \bar{z}) = (\bar{\zeta}_1 - \bar{z}_1, \cdots, \bar{\zeta}_n - \bar{z}_n).$$

The Bochner-Martinelli-Koppelman kernel $B(\zeta, z)$ is defined by

$$\begin{aligned}B(\zeta, z) &= \Omega(G^0) = \Omega^0\\ &= \frac{1}{(2\pi i)^n} \frac{< \bar{\zeta} - \bar{z}, d\zeta >}{|\zeta - z|^2} \wedge \left(\frac{< d\bar{\zeta} - d\bar{z}, d\zeta >}{|\zeta - z|^2} \right)^{n-1}\\ &= \sum_{q=0}^{n-1} B_q(\zeta, z),\end{aligned} \tag{11.1.8}$$

where B_q is the summand which is of degree $(0, q)$ in z and of degree $(n, n-q-1)$ in ζ. Using (11.1.4-i), we have

$$\bar{\partial}_\zeta B(\zeta, z) + \bar{\partial}_z B(\zeta, z) = 0 \qquad \text{for} \quad \zeta \neq z, \tag{11.1.9}$$

or equivalently, for each $0 \le q \le n$,

$$\bar{\partial}_\zeta B_q(\zeta, z) + \bar{\partial}_z B_{q-1}(\zeta, z) = 0 \qquad \text{for} \quad \zeta \neq z, \tag{11.1.9-q}$$

if we set $B_{-1}(\zeta, z) = B_n(\zeta, z) = 0$. In particular, B_0 is the Bochner-Martinelli kernel defined by (2.2.1) and (11.1.9-q) was proved directly in (3.2.2) when $q = 1$.

When $n = 1$, $B(\zeta, z) = (2\pi i)^{-1} d\zeta / (\zeta - z)$ is the Cauchy kernel. The following theorem shows that the Bochner-Martinelli-Koppelman kernel is indeed a generalization of the Cauchy integral formula to several variables.

Theorem 11.1.2 (Bochner-Martinelli-Koppelman). *Let D be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. For $f \in C^1_{(0,q)}(\overline{D})$, $0 \le q \le n$, the following formula holds:*

$$\begin{aligned}f(z) = \int_{bD} B_q(\cdot, z) \wedge f &+ \int_D B_q(\cdot, z) \wedge \bar{\partial}_\zeta f\\ &+ \bar{\partial}_z \int_D B_{q-1}(\cdot, z) \wedge f, \qquad z \in D,\end{aligned} \tag{11.1.10}$$

where $B(\zeta, z)$ is defined in (11.1.8).

Proof. For $q = 0$, the Bochner-Martinelli formula was proved in Theorem 2.2.1. We first assume that $1 \le q < n$.

Let $z_0 \in D$ and β_ϵ be a small ball of radius ϵ centered at z_0 such that $\overline{\beta}_\epsilon \subset D$. We shall prove the theorem at $z = z_0$. Applying Stokes' theorem to the form $d_\zeta(B_q(\zeta, z) \wedge f(\zeta))$ on $D_\epsilon \equiv D \setminus \overline{\beta}_\epsilon$, we have, using (11.1.9), that

$$\begin{aligned}
&\int_{bD} B_q(\zeta, z) \wedge f - \int_{b\beta_\epsilon} B_q(\zeta, z) \wedge f \\
&= \int_{D_\epsilon} \bar{\partial}_\zeta B_q(\zeta, z) \wedge f - \int_{D_\epsilon} B_q(\zeta, z) \wedge \bar{\partial}_\zeta f \qquad (11.1.11)\\
&= -\int_{D_\epsilon} \bar{\partial}_z B_{q-1}(\zeta, z) \wedge f - \int_{D_\epsilon} B_q(\zeta, z) \wedge \bar{\partial}_\zeta f.
\end{aligned}$$

Since $B(\zeta, z) = O(|\zeta - z|^{-2n+1})$, $B(\zeta, z)$ is an integrable function for each fixed z. We see from the dominated convergence theorem that

$$\int_{D_\epsilon} B_q(\zeta, z) \wedge \bar{\partial}_\zeta f \to \int_D B_q(\zeta, z) \wedge \bar{\partial}_\zeta f. \qquad (11.1.12)$$

Note that

$$\begin{aligned}
\int_{b\beta_\epsilon} B_0(\zeta, z) &= \frac{1}{(2\pi i)^n} \frac{1}{\epsilon^{2n}} \int_{b\beta_\epsilon} < \bar{\zeta} - \bar{z}, d\zeta > \wedge < d\bar{\zeta}, d\zeta >^{n-1} \\
&= \frac{1}{(2\pi i)^n} \frac{1}{\epsilon^{2n}} \int_{\beta_\epsilon} < d\bar{\zeta}, d\zeta >^n = \frac{n!}{\pi^n} \int_{\beta_1} dV = 1.
\end{aligned}$$

For any increasing multiindex $J = (j_1, \cdots, j_q)$, we get that

$$\begin{aligned}
\int_{b\beta_\epsilon} B_q(\zeta, z) \wedge d\bar{\zeta}^J &= \frac{1}{(2\pi i)^n} \frac{1}{\epsilon^{2n}} \int_{b\beta_\epsilon} < \bar{\zeta} - \bar{z}, d\zeta > \wedge < d\bar{\zeta} - d\bar{z}, d\zeta >^{n-1} \wedge d\bar{\zeta}^J \\
&= \frac{1}{(2\pi i)^n} \frac{1}{\epsilon^{2n}} \int_{\beta_\epsilon} < d\bar{\zeta}, d\zeta > \wedge < d\bar{\zeta} - d\bar{z}, d\zeta >^{n-1} \wedge d\bar{\zeta}^J \\
&= \frac{(n-q)(n-1)!}{\pi^n} \left(\int_{\beta_1} dV \right) \wedge d\bar{z}^J \\
&= \frac{n-q}{n} d\bar{z}^J.
\end{aligned}$$

Let $f(z_0)$ denote the $(0, q)$-form whose coefficients are equal to the values of the coefficients of f at z_0. It follows from the above calculation that

$$\begin{aligned}
\int_{b\beta_\epsilon} B_q(\zeta, z) \wedge f &= \frac{n-q}{n} f(z_0) + \int_{b\beta_\epsilon} B_q(\zeta, z) \wedge \big(f - f(z_0)\big) \qquad (11.1.13)\\
&\longrightarrow \frac{n-q}{n} f(z_0), \quad \text{as } \epsilon \to 0.
\end{aligned}$$

The kernel $\bar{\partial}_z B_{q-1}(\zeta, z) = O(|\zeta - z|^{2n})$ is not integrable but the components are the classical singular integrals (see e.g. Stein [Ste 2]). The Principal-value limit defined by

$$\begin{aligned}
\text{P.V.} \int_D \bar{\partial}_z B_{q-1}(\zeta, z_0) \wedge f &= \lim_{\epsilon \to 0} \int_{D_\epsilon} \bar{\partial}_z B_{q-1}(\zeta, z_0) \wedge f \\
&= \lim_{\epsilon \to 0} \left(\bar{\partial}_z \int_{D_\epsilon} B_{q-1}(\zeta, z) \wedge f \right) \Big|_{z = z_0}
\end{aligned}$$

exists for each $z_0 \in D$. We claim that

$$\text{(11.1.14)} \qquad \text{P.V.} \int_D \bar{\partial}_z B_{q-1}(\zeta, z_0) \wedge f = \left(\bar{\partial}_z \int_D B_{q-1}(\zeta, z) \wedge f \right) \Big|_{z=z_0} - \frac{q}{n} f(z_0).$$

We use the notation $d\bar{\zeta} \wedge d\zeta = d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n$ and $[\widehat{d\zeta_j}] = d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar{\zeta}_j \wedge \widehat{d\zeta_j} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n$, where $\widehat{d\zeta_j}$ denotes that the term $d\zeta_j$ is omitted. Let $f(\zeta) = f_J d\bar{\zeta}^J$, where $J = (1, \cdots, q)$. Using Stokes' theorem, we obtain

$$\begin{aligned}
&\int_{\beta_\epsilon} B_{q-1}(\zeta, z) \wedge f(\zeta) \\
&= \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{q} (-1)^j \left(\int_{\beta_\epsilon} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} f_J(\zeta) d\bar{\zeta} \wedge d\zeta \right) \wedge d\bar{z}^{1 \cdots \hat{j} \cdots q} \\
&= \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{q} (-1)^j \int_{\beta_\epsilon} \left(\frac{-1}{n-1} \right) \frac{\partial}{\partial \zeta_j} \left(\frac{1}{|\zeta - z|^{2n-2}} \right) f_J(\zeta) \; d\bar{\zeta} \wedge d\zeta \wedge d\bar{z}^{1 \cdots \hat{j} \cdots q} \\
&= \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{q} \frac{(-1)^j}{n-1} \Bigg(\int_{b\beta_\epsilon} \frac{1}{|\zeta - z|^{2n-2}} f_J(\zeta) [\widehat{d\zeta_j}] \\
&\qquad\qquad + \int_{\beta_\epsilon} \frac{1}{|\zeta - z|^{2n-2}} \frac{\partial f_J}{\partial \zeta_j}(\zeta) d\bar{\zeta} \wedge d\zeta \Bigg) \wedge d\bar{z}^{1 \cdots \hat{j} \cdots q}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\bar{\partial}_z \int_{\beta_\epsilon} B_{q-1}(\zeta, z) \wedge f(\zeta) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{q} (-1)^j \sum_{k=1}^{n} \Bigg(&\int_{b\beta_\epsilon} \frac{\zeta_k - z_k}{|\zeta - z|^{2n}} f_J(\zeta) [\widehat{d\zeta_j}] \\
&+ \int_{\beta_\epsilon} \frac{\zeta_k - z_k}{|\zeta - z|^{2n}} \frac{\partial f_J}{\partial \zeta_j}(\zeta) d\bar{\zeta} \wedge d\zeta \Bigg) d\bar{z}_k \wedge d\bar{z}^{1 \cdots \hat{j} \cdots q}.
\end{aligned}$$

Since

$$\frac{(n-1)!}{(2\pi i)^n} (-1)^j \int_{b\beta_\epsilon} \frac{\zeta_k - z_k}{|\zeta - z|^{2n}} f_J(\zeta) [\widehat{d\zeta_j}] \longrightarrow \delta_{jk} \frac{(-1)^{j-1}}{n} f_J(z_0),$$

and

$$\int_{\beta_\epsilon} \frac{\zeta_k - z_k}{|\zeta - z|^{2n}} \frac{\partial f_J}{\partial \zeta_j}(\zeta) d\bar{\zeta} \wedge d\zeta \longrightarrow 0, \quad \text{as } \epsilon \to 0,$$

we have

$$\bar{\partial}_z \int_{\beta_\epsilon} B_{q-1}(\zeta, z) \wedge f(\zeta) \longrightarrow \frac{q}{n} f_J(z_0) d\bar{z}^J.$$

Thus

$$
\begin{aligned}
&\bar{\partial}_z \left(\int_D B_{q-1}(\zeta, z) \wedge f(\zeta) \right) \Big|_{z=z_0} \\
&= \lim_{\epsilon \to 0} \left(\left[\bar{\partial}_z \int_{D_\epsilon} B_{q-1}(\zeta, z) \wedge f(\zeta) + \bar{\partial}_z \int_{\beta_\epsilon} B_{q-1}(\zeta, z) \wedge f(\zeta) \right] \Big|_{z=z_0} \right) \\
&= \lim_{\epsilon \to 0} \left(\int_{D_\epsilon} \bar{\partial}_z B_{q-1}(\zeta, z_0) \wedge f(\zeta) + \bar{\partial}_z \int_{\beta_\epsilon} B_{q-1}(\zeta, z) \wedge f(\zeta) \Big|_{z=z_0} \right) \\
&= \text{P.V.} \int_D \bar{\partial}_z B_{q-1}(\zeta, z_0) \wedge f(\zeta) + \frac{q}{n} f_J(z_0) d\bar{z}^J.
\end{aligned}
$$

This proves the claim (11.1.14) for the special case of f. The proof for a general $(0,q)$-form f is the same. Combining (11.1.11)-(11.1.14), we have proved the theorem for $0 \le q < n$. When $q = n$, it follows from (11.1.1) that

$$
f(z) = \bar{\partial}_z \int_D B_{n-1}(\cdot, z) \wedge f.
$$

Thus Theorem 11.1.2 holds for all $0 \le q \le n$.

Corollary 11.1.3. *Let D be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. For any $f \in C_{(0,q)}(\overline{D})$, $1 \le q \le n$, such that $f = 0$ on bD and $\bar{\partial} f = 0$ in D in the distribution sense, there exists $u \in C^\alpha_{(0,q-1)}(D)$ with $\bar{\partial} u = f$ in the distribution sense, where $0 < \alpha < 1$. Furthermore, there exists a $C > 0$ such that*

$$
\|u\|_{C^\alpha(D)} \le C \|f\|_{L^\infty(D)}. \tag{11.1.15}
$$

Proof. For $z \in D$, define

$$
u(z) = \int_D B_{q-1}(\cdot, z) \wedge f.
$$

We first prove (11.1.15). Since

$$
\begin{aligned}
&\left| \int_D B_{q-1}(\cdot, z) \wedge f - \int_D B_{q-1}(\cdot, z') \wedge f \right| \\
&\qquad\qquad \le C \left(\sum_{j=1}^n \int_D \left| \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} - \frac{\bar{\zeta}_j - \bar{z}'_j}{|\zeta - z'|^{2n}} \right| dV \right) \|f\|_{L^\infty(D)},
\end{aligned}
$$

it suffices to show that for each $1 \le j \le n$,

$$
\int_D \left| \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} - \frac{\bar{\zeta}_j - \bar{z}'_j}{|\zeta - z'|^{2n}} \right| dV \le C |z - z'| \big| \log |z - z'| \big|. \tag{11.1.16}
$$

Let $|z - z'| = 2\epsilon$. We divide D into three regions: $\beta_\epsilon(z)$, $\beta_\epsilon(z')$ and $D_\epsilon = D \setminus (\beta_\epsilon(z) \cup \beta_\epsilon(z'))$ where $\beta_\epsilon(z)$ is a ball of radius ϵ centered at z. On $\beta_\epsilon(z)$, we have

$$
\int_{\beta_\epsilon(z)} \left| \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} - \frac{\bar{\zeta}_j - \bar{z}'_j}{|\zeta - z'|^{2n}} \right| dV \le 2 \int_{\beta_\epsilon(z)} \frac{1}{|\zeta - z|^{2n-1}} dV \le C |z - z'|.
$$

Similarly, we have the estimate on $\beta_\epsilon(z')$. To estimate the integral on D_ϵ, we note that $\frac{1}{3}|\zeta - z'| \le |\zeta - z| \le 3|\zeta - z'|$ for $\zeta \in D_\epsilon$, thus there exists an $A > 0$ such that

$$\begin{aligned}\int_{D_\epsilon}\left|\frac{\bar\zeta_j - \bar z_j}{|\zeta - z|^{2n}} - \frac{\bar\zeta_j - \bar z'_j}{|\zeta - z'|^{2n}}\right| dV &\le C\int_{\epsilon \le |\zeta - z| \le A}\frac{|z - z'|}{|\zeta - z|^{2n}} dV\\ &\le C|z - z'|\big|\log|z - z'|\big|.\end{aligned}$$

This proves (11.1.16) and (11.1.15) follows.

If $f \in C^1_{(0,q)}(\overline{D})$, Theorem 11.1.2 implies that $\bar\partial u = f$ since $f = 0$ on bD. For $f \in C_{(0,q)}(\overline{D})$, we use an approximation argument. We first assume that the domain D is star-shaped and $0 \in D$. Let $\phi(z) = \phi(|z|)$ be a function supported in $|z| \le 1$ and $\phi \ge 0, \int\phi = 1$. We set $\phi_{\delta_m} = \delta_m^{-2n}\phi(z/\delta_m)$ for $\delta_m \searrow 0$. Extending f to be 0 outside D, we define

$$f_m(z) = f\left(\frac{z}{1 - \frac{1}{m}}\right) * \phi_{\delta_m}$$

for sufficiently small δ_m. One can easily check that f_m has coefficients in $C_0^\infty(D)$, $\bar\partial f_m = 0$ in D and $f_m \to f$ uniformly in D. When the boundary is C^1, we can use a partition of unity $\{\zeta_i\}_{i=1}^N$, with each ζ_i supported in an open set U_i such that $U_i \cap D$ is star-shaped. We then regularize $\zeta_i f$ in U_i as before. It is easy to see that there exists a sequence $f_m \in C^\infty_{(0,q)}(D)$ with compact support in D such that $f_m \to f$ uniformly in D and $\bar\partial f_m \to 0$ uniformly. Applying Theorem 11.1.2 to each f_m and letting $m \to \infty$, we have proved $\bar\partial u = f$ in the distribution sense.

Corollary 11.1.3 allows us to solve the equation $\bar\partial u = f$ for any $\bar\partial$-closed form f with compact support. Thus the Bochner-Martinelli-Koppelman kernel is a fundamental solution for $\bar\partial$ in $\mathbb{C}^n$. In the next section we introduce new kernels and derive a homotopy formula for $\bar\partial$ for forms which do not necessarily have compact support.

11.2 The Homotopy Formula for $\bar\partial$ on Convex Domains

The Bochner-Martinelli-Koppelman kernel is independent of the domain D. Next we introduce another kernel, the Leray kernel, which in general depends on the domain.

Definition 11.2.1. *A $\mathbb{C}^n$-valued C^1 function $G(\zeta, z) = (g_1(\zeta, z), \cdots, g_n(\zeta, z))$ is called a Leray map for D if it satisfies $< G(\zeta, z), \zeta - z > \ne 0$ for every $(\zeta, z) \in bD \times D$.*

In particular, the $\mathbb{C}^n$-valued function $G^0(\zeta, z) = (\bar\zeta - \bar z) = (\bar\zeta_1 - \bar z_1, \cdots, \bar\zeta_n - \bar z_n)$ is a Leray map for any domain D. We use the same notation $\Omega^0 = \Omega(G^0) = B(\zeta, z)$ to denote the Bochner-Martinelli-Koppelman kernel. If $G^1(\zeta, z)$ is another Leray map, we set

$$\Omega^1 = \Omega(G^1) = \left(\frac{1}{2\pi i}\right)^n \frac{< G^1, d\zeta >}{< G^1, \zeta - z >} \wedge \left(\frac{< \bar\partial_{\zeta,z} G^1, d\zeta >}{< G^1, \zeta - z >}\right)^{n-1} \tag{11.2.1}$$

and

$$\Omega^{01} = \Omega(G^0, G^1) = \left(\frac{1}{2\pi i}\right)^n \frac{< \bar\zeta - \bar z, d\zeta >}{|\zeta - z|^2} \wedge \frac{< G^1, d\zeta >}{< G^1, \zeta - z >}$$

$$\wedge \sum_{k_1+k_2=n-2} \left(\frac{< d\bar\zeta - d\bar z, d\zeta >}{|\zeta - z|^2}\right)^{k_1} \wedge \left(\frac{< \bar\partial_{\zeta,z} G^1, d\zeta >}{< G^1, \zeta - z >}\right)^{k_2}. \tag{11.2.2}$$

Notice that Ω^1 and Ω^{01} are well defined for $\zeta \in bD$ and $z \in D$. Also we use the notation Ω^1_q, Ω^{01}_q to denote the summand of forms with degree $(0, q)$ in z in Ω^1, Ω^{01} respectively.

Theorem 11.2.2 (Leray-Koppelman). *Let D be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. Let $G^0 = (\bar\zeta - \bar z)$ and G^1 be another Leray map for D. For $f \in C^1_{(0,q)}(\overline{D})$, $0 \le q \le n$, we have*

$$f(z) = \int_{bD} \Omega^1_q \wedge f + \bar\partial_z T_q f + T_{q+1} \bar\partial f, \quad z \in D, \tag{11.2.3}$$

where

$$T_q f(z) = \int_D \Omega^0_{q-1}(\zeta, z) \wedge f(\zeta) - \int_{bD} \Omega^{01}_{q-1}(\zeta, z) \wedge f(\zeta).$$

Ω^0, Ω^1 and Ω^{01} are defined in (11.1.8), (11.2.1) and (11.2.2) respectively.

Proof. From (11.1.4-ii), we have

$$\bar\partial_{\zeta,z} \Omega^{01} = \Omega^0 - \Omega^1$$

on the set where $\zeta \in bD$ and $z \in D$. Thus, for $z \in D$,

$$\int_{bD} \Omega^0 \wedge f = \int_{bD} \bar\partial_{\zeta,z} \Omega^{01} \wedge f + \int_{bD} \Omega^1 \wedge f.$$

Applying Stokes' theorem, we have

$$\int_{bD} \bar\partial_\zeta \Omega^{01} \wedge f = \int_{bD} d_\zeta(\Omega^{01} \wedge f) - \int_{bD} \Omega^{01} \wedge \bar\partial_\zeta f = -\int_{bD} \Omega^{01} \wedge \bar\partial_\zeta f.$$

Since $\Omega^{01} \wedge f$ is of degree $(n, n-1)$ in ζ, it follows that

$$\int_{bD} \bar\partial_z \Omega^{01} \wedge f = -\bar\partial_z \int_{bD} \Omega^{01} \wedge f.$$

Substituting the above formulas into (11.1.10), we have for $z \in D$,

$$\begin{aligned} f(z) &= \int_{bD} \Omega^0 \wedge f + \int_D \Omega^0 \wedge \bar\partial_\zeta f + \bar\partial_z \int_D \Omega^0 \wedge f \\ &= \int_{bD} \Omega^1 \wedge f + \bar\partial_z \left(\int_D \Omega^0 \wedge f - \int_{bD} \Omega^{01} \wedge f \right) \\ &\quad + \left(\int_D \Omega^0 \wedge \bar\partial_\zeta f - \int_{bD} \Omega^{01} \wedge \bar\partial_\zeta f \right). \end{aligned}$$

(11.2.3) follows from the degree consideration.

Corollary 11.2.3 (Leray). *Let D be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. Let G^1 be any Leray map for D. For any $f \in C^1(\overline{D}) \cap \mathcal{O}(D)$, we have*

$$f(z) = \int_{bD} \Omega_0^1(\zeta, z) \wedge f(\zeta), \quad z \in D,$$

where Ω^1 is defined in (11.2.1) and Ω_0^1 is the piece in Ω^1 of degree (0,0) in z.

Corollary 11.2.3 shows that a holomorphic function in D is represented by its boundary value through any Leray map for D. So far we have not constructed any Leray map other than the Bochner-Martinelli-Koppelman kernel. Our next goal is to construct a Leray map which is holomorphic in the z variable when the domain is convex. We recall the following definition:

Definition 11.2.4. *Let $D \subset\subset \mathbb{R}^N$ be a domain with C^2 boundary and ρ is any C^2 defining function. D is a convex (or strictly convex) domain with C^2 boundary if*

$$\sum_{i,j=1}^{N} \frac{\partial^2 \rho}{\partial x_i \partial x_j}(x) a_i a_j \geq 0 \ (or > 0) \qquad on\ bD,$$

for every $a = (a_1, \cdots, a_N) \neq 0$ with $\sum_{i=1}^N a_i \frac{\partial \rho}{\partial x_i}(x) = 0$ on bD. Here we use $(x_1, \cdots, x_N)$ to denote the real coordinates for $\mathbb{R}^N$ and $a_i \in \mathbb{R}$.

It is easy to check that the definition of convexity or strict convexity is independent of the choice of the defining function ρ. In fact, for a strictly convex domain D, we can choose a special defining function such that its real Hessian is positive definite without restricting to the tangent plane as the next proposition shows.

Proposition 11.2.5. *Let D be a strictly convex domain with C^2 boundary in $\mathbb{R}^N$. Then there exists a C^2 defining function ρ such that*

$$\sum_{i,j=1}^{N} \frac{\partial^2 \rho}{\partial x_i \partial x_j}(x) a_i a_j \geq c|a|^2, \quad for\ all\ x \in bD and\ a \in \mathbb{R}^n, \tag{11.2.4}$$

where c is a positive constant.

Proof. Let ρ_0 be any C^2 defining function for D. We set $\rho = e^{A\rho_0} - 1$ where A is a positive constant. Then ρ is another C^2 defining function. Arguments similar to those in the proof of Theorem 3.4.4 show that ρ is strictly convex and satisfies (11.2.4) if we choose A sufficiently large.

A defining function ρ satisfying (11.2.4) is called a strictly convex defining function for D. By continuity, ρ satisfies (11.2.4) in a small neighborhood of bD.

Lemma 11.2.6. *Let D be a bounded convex domain in $\mathbb{C}^n$ with C^2 boundary and let ρ be a C^2 defining function for D. Then the map*

$$G^1(\zeta, z) = \left(\frac{\partial \rho}{\partial \zeta}\right) = \left(\frac{\partial \rho}{\partial \zeta_1}, \cdots, \frac{\partial \rho}{\partial \zeta_n}\right) \tag{11.2.5}$$

is a Leray map.

Proof. Using convexity, we have for any $z \in D,\ \zeta \in bD$,

$$\mathrm{Re}\sum_{i=1}^{n}\frac{\partial\rho}{\partial\zeta_i}(\zeta_i - z_i) > 0. \tag{11.2.6}$$

Thus G^1 is a Leray map.

Note that G^1 is a Leray map which is independent of z. The importance of the existence of a Leray map which is holomorphic in z (or independent of z) is shown in the next theorem.

Theorem 11.2.7 (A homotopy formula for $\bar\partial$ on convex domains). *Let D be a bounded convex domain in $\mathbb{C}^n$ with C^2 boundary bD and let ρ be a C^2 defining function for D. Suppose that $G^0 = (\bar\zeta - \bar z)$ and G^1 is defined by (11.2.5). For $f \in C^1_{(0,q)}(\overline{D})$, $0 \le q \le n$, we have*

$$f(z) = \bar\partial_z T_q f + T_{q+1}\bar\partial f, \quad z \in D, \quad \textit{if}\ \ 1 \le q \le n, \tag{11.2.7}$$

$$f(z) = \int_{bD}\Omega_0^1 \wedge f + T_1\bar\partial f, \quad z \in D, \quad \textit{if}\ \ q = 0, \tag{11.2.8}$$

where

$$T_q f(z) = \int_D \Omega^0_{q-1}(\zeta, z) \wedge f(\zeta) - \int_{bD}\Omega^{01}_{q-1}(\zeta, z) \wedge f(\zeta). \tag{11.2.9}$$

Ω^0, Ω^1 and Ω^{01} are defined in (11.1.8), (11.2.1) and (11.2.2) respectively.

Proof. Since G^1 is a Leray map which does not depend on z, the kernel Ω^1 has no $d\bar z$'s. Thus for any $1 \le q \le n$, $\Omega^1_q = 0$ and

$$\int_{bD}\Omega^1_q \wedge f = 0.$$

Thus (11.2.7) and (11.2.8) follow from (11.2.3).

Corollary 11.2.8 (A solution operator for $\bar\partial$ on convex domains). *Let D be a bounded convex domain in $\mathbb{C}^n$ with C^2 boundary bD. Let $f \in C^1_{(0,q)}(\overline{D})$, $1 \le q \le n$, with $\bar\partial f = 0$ in D. Then*

$$u = T_q f(z)$$

is a solution to the equation $\bar\partial u = f$, where T_q is defined in (11.2.9).

Formula (11.2.9) gives an explicit solution operator for $\bar\partial$ when the domain is convex. Next we shall estimate the solution kernel in Hölder spaces when the domain is strictly convex.

Lemma 11.2.9. *Let D be a bounded strictly convex domain in $\mathbb{C}^n$ with C^2 boundary bD and let ρ be a strictly convex defining function for D. There exists a constant $C > 0$ such that for any $\zeta \in bD,\ z \in \overline{D}$,*

$$\text{(11.2.10)} \qquad Re < G^1, \zeta - z > \geq C(\rho(\zeta) - \rho(z) + |\zeta - z|^2),$$

where G^1 is defined by (11.2.5).

Proof. Since ρ is a strictly convex defining function satisfying (11.2.4), we apply Taylor's expansion to $\rho(z)$ at the point $\zeta \in bD$, then

$$\begin{aligned}\rho(z) = \rho(\zeta) - 2\mathrm{Re}\sum_{i=1}^n \frac{\partial\rho}{\partial\zeta_i}(\zeta_i - z_i) + \sum_{i,j=1}^n \frac{\partial^2\rho}{\partial\zeta_i\partial\bar\zeta_j}(\zeta_i - z_i)(\bar\zeta_j - \bar z_j)\\ + \mathrm{Re}\sum_{i,j=1}^n \frac{\partial^2\rho}{\partial\zeta_i\partial\zeta_j}(\zeta_i - z_i)(\zeta_j - z_j) + o(|\zeta - z|^2).\end{aligned}$$

Thus, for $|\zeta - z| \leq \epsilon$, where $\epsilon > 0$ is sufficiently small,

$$\mathrm{Re}\sum_{i=1}^n \frac{\partial\rho}{\partial\zeta_i}(\zeta_i - z_i) \geq \frac{1}{2}\rho(\zeta) - \frac{1}{2}\rho(z) + \frac{c}{2}|\zeta - z|^2,$$

where $c > 0$ is the positive constant in (11.2.4). To show that (11.2.10) holds for $|\zeta - z| > \epsilon$, we set

$$\tilde z = \left(1 - \frac{\epsilon}{|\zeta - z|}\right)\zeta + \frac{\epsilon}{|\zeta - z|}z.$$

Then $|\zeta - \tilde z| = \epsilon$ and $\tilde z \in D$ since D is strictly convex. It follows that

$$\begin{aligned}\mathrm{Re} < G^1, \zeta - z > &= \mathrm{Re}\frac{|\zeta - z|}{\epsilon} < G^1, \zeta - \tilde z >\\ &\geq \frac{|\zeta - z|}{2\epsilon}(\rho(\zeta) - \rho(\tilde z) + \frac{c}{2}|\zeta - \tilde z|^2)\\ &\geq \frac{c\epsilon^2}{4} \geq C(\rho(\zeta) - \rho(z) + |\zeta - z|^2),\end{aligned}$$

since $(-\rho(z) + |\zeta - z|^2) \leq M$ for some constant $M > 0$.

Lemma 11.2.10. *Let D be a bounded strictly convex domain in $\mathbb{C}^n$ with C^2 boundary and let ρ be a strictly convex defining function for D. The kernel $\Omega^{01}_{q-1}(\zeta, z)$, $0 < q < n$, is absolutely integrable on bD for any $z \in D$. Furthermore, there exists a constant C such that for any $z \in D$,*

$$\text{(11.2.11)} \qquad \int_{bD} |\Omega^{01}_{q-1}(\zeta, z)| < C,$$

where C is independent of z.

Proof. Let

$$\begin{cases}\Phi(\zeta, z) = < G^1, \zeta - z >,\\ \Phi_0(\zeta, z) = |\zeta - z|^2.\end{cases}$$

Using (11.2.10), the kernel $\Omega^{01}(\zeta, z)$ has singularities only at $\zeta = z$ on bD.

We choose a special coordinate system in a neighborhood of a fixed z near bD. From the definition of Φ, we have $d_\zeta\Phi|_{\zeta=z} = \partial\rho$ and $d_\zeta(\mathrm{Im}\ \Phi)|_{\zeta=z} = \frac{1}{2i}(\partial\rho - \overline{\partial}\rho)$. Thus $d\rho$ and $d_\zeta(\mathrm{Im}\ \Phi)$ are linearly independent at $\zeta = z$. On a small neighborhood $U_\epsilon = \{\zeta \mid |\zeta - z| < \epsilon\}$ of a fixed $z \in D$, Let $(t, y) = (t_1, \cdots, t_{2n-1}, y)$ where $t = (t', t_{2n-1}) = (t_1, \cdots, t_{2n-1})$ are tangential coordinates for $U_\epsilon \cap bD$, $t_i(z) = 0$ and

$$\begin{cases} y = \rho(\zeta), \\ t_{2n-1} = \mathrm{Im}\Phi(\zeta, z). \end{cases} \tag{11.2.12}$$

From (11.2.10) it follows that there exists a positive constant γ_0 such that

$$\begin{cases} |\Phi(\zeta, z)| \geq \gamma_0(|\rho(z)| + |t'|^2 + |t_{2n-1}|), \\ |\zeta - z| \geq \gamma_0(|\rho(z)| + |t|). \end{cases} \tag{11.2.13}$$

Using (11.2.13), we have for some $A > 0$,

$$\begin{aligned} \int_{bD\cap U_\epsilon} |\Omega^{01}_{q-1}(\zeta, z)| &\leq C\left(\sum_{k=1}^{n-1}\int_{bD\cap U_\epsilon}\frac{|\zeta - z|}{|\Phi|^{n-k}|\Phi_0|^k}dS\right) \\ &\leq C\int_{\zeta\in bD\cap U_\epsilon}\frac{1}{|\Phi||\zeta - z|^{2n-3}}dS \\ &\leq C\int_{|t|\leq A}\frac{dt_1 dt_2\cdots dt_{2n-1}}{(|t_{2n-1}| + |t'|^2)|t|^{2n-3}} \\ &\leq C\int_{|t'|\leq A}\frac{|\log|t'||dt_1\cdots dt_{2n-2}}{|t'|^{2n-3}} \\ &\leq C\int_0^A \frac{r^{2n-3}|\log r|dr}{r^{2n-3}} \\ &\leq C, \end{aligned} \tag{11.2.14}$$

where dS is the surface element on bD. This proves the lemma.

Thus the kernel Ω^{01} is integrable uniformly on bD. We have the following Hölder regularity result for $\bar{\partial}$:

Theorem 11.2.11 ($\frac{1}{2}$-Hölder estimates for $\bar{\partial}$ on strictly convex domains). *Let D be a bounded strictly convex domain in $\mathbb{C}^n$ with C^2 boundary. For any $f \in C_{(0,q)}(\overline{D})$, $1 \leq q \leq n$, such that $\bar{\partial}f = 0$ in D, there exists a $u \in C^{1/2}_{(0,q-1)}(D)$ such that $\bar{\partial}u = f$ in D and*

$$\|u\|_{C^{\frac{1}{2}}(D)} \leq C\|f\|_{L^\infty(D)}, \tag{11.2.15}$$

where C is a constant independent of f.

Proof. We first assume that $f \in C^1_{(0,q)}(\overline{D})$. Let

$$u = T_q f = u_0 + u_1,$$

where

$$u_0 = \int_D \Omega^0_{q-1}(\zeta, z) \wedge f(\zeta)$$

and

$$u_1 = -\int_{bD} \Omega^{01}_{q-1}(\zeta, z) \wedge f(\zeta).$$

It follows from Corollary 11.2.8 that $\bar{\partial} T_q f = f$. From Corollary 11.1.3, for any $z,\ z' \in D$,

$$|u_0(z) - u_0(z')| \le C_\alpha \parallel f \parallel_\infty |z - z'|^\alpha$$

for any $\alpha < 1$. Also u_1 is smooth in D. In order to estimate u_1 near the boundary, we use the assumption of strict convexity on D.

We may assume $1 \le q \le n-1$ since $u_1 = 0$ if $q = n$. From Lemma 11.2.10, $u_1 \in C_{(0,q-1)}(\overline{D})$. Since $u_1 \in C^\infty_{(0,q-1)}(D)$, using the Hardy-Littlewood lemma (see Theorem C.1 in the Appendix), to prove that $u_1 \in C^{1/2}_{(0,q-1)}(D)$, it suffices to show that there exists a C such that

$$|\nabla u_1(z)| \le C|\rho(z)|^{-\frac{1}{2}}, \quad z \in D. \tag{11.2.16}$$

Using the same notation as in Lemma 11.2.10, we have for $1 \le q \le n-1$,

$$\begin{aligned}
&\left| \nabla_z \int_{bD \cap U_\epsilon} \Omega^{01}_{q-1}(\zeta, z) \wedge f(\zeta) \right| \\
&\le C \parallel f \parallel_\infty \sum_{k=1}^{n-1} \left(\int\limits_{bD \cap U_\epsilon} \frac{|\zeta - z|}{|\Phi|^{n-k+1}|\Phi_0|^k} dS + \int\limits_{bD \cap U_\epsilon} \frac{1}{|\Phi|^{n-k}|\zeta - z|^{2k}} dS \right).
\end{aligned}$$

To prove (11.2.16), using the change of coordinates (11.2.12) and estimates (11.2.13), it suffices to show that for some $A > 0$, there exists a $C > 0$ such that for $\delta > 0$, $1 \le q \le n-1$,

$$\int_{|t| \le A} \frac{dt_1 dt_2 \cdots dt_{2n-1}}{(\delta + |t_{2n-1}| + |t'|^2)^{n-q+1}|t|^{2q-1}} < C\delta^{-\frac{1}{2}}, \tag{11.2.17}$$

$$\int_{|t| \le A} \frac{dt_1 dt_2 \cdots dt_{2n-1}}{(\delta + |t_{2n-1}| + |t'|^2)^{n-q}(|t| + \delta)^{2q}} < C_\alpha \delta^{-1+\alpha}, \tag{11.2.18}$$

where $0 < \alpha < 1$ and C, C_α are independent of δ. To prove (11.2.17), integrating with respect to t_{2n-1} and then using polar coordinates $|t'| = r$, we have

$$\begin{aligned}
&\int_{|t| \le A} \frac{dt_1 dt_2 \cdots dt_{2n-1}}{(\delta + |t_{2n-1}| + |t'|^2)^{n-q+1}|t|^{2q-1}} \\
&\qquad \le C \int_{|t'| \le A} \frac{dt_1 dt_2 \cdots dt_{2n-2}}{(\delta + |t'|^2)^{n-q}|t'|^{2q-1}} \\
&\qquad \le C \int_0^A \frac{r^{2n-3} dr}{(\delta + r^2)^{n-q} r^{2q-1}} \le C \int_0^A \frac{dr}{\delta + r^2} \le C\delta^{-\frac{1}{2}}.
\end{aligned}$$

(11.2.18) is proved similarly. Thus $u_1 \in C^{1/2}_{(0,q-1)}(D)$ and (11.2.15) is proved for $f \in C^1_{(0,q)}(\overline{D})$.

When $f \in C_{(0,q)}(\overline{D})$, we can find a sequence $f_\epsilon \in C^\infty_{(0,q)}(\overline{D})$ such that $f_\epsilon \to f$ uniformly in D and $\bar{\partial} f_\epsilon = 0$ in D. The f_ϵ's can be constructed easily by a dilation (assuming that $0 \in D$) followed by regularization. Letting $u_\epsilon = T_q f_\epsilon$, we get

$$\|u_\epsilon\|_{C^{\frac{1}{2}}(D)} \leq C\|f_\epsilon\|_{L^\infty(D)}.$$

It is easy to see that u_ϵ converges $C^{1/2}(D)$ to $u = T_q f \in C^{1/2}_{(0,q-1)}(D)$ and $\bar{\partial} u = f$ in the distribution sense.

Remark. In Chapter 5, we have proved that for any $\bar{\partial}$-closed $(0,q)$-form with $W^s(D)$ coefficients in a strictly pseudoconvex domain D, the canonical solution u given by $\bar{\partial}^* N f$ is in $W^{s+\frac{1}{2}}(D)$ where N is the $\bar{\partial}$-Neumann operator (see Theorem 5.2.6). Theorem 11.2.11 gives a solution operator which has a "gain" of 1/2 derivatives in Hölder spaces on strictly convex domains. Near a boundary point of a strictly pseudoconvex domain, locally there exists a holomorphic change of coordinates such that it is strictly convex (see Corollary 3.4.5). Globally, one can also embed strongly pseudoconvex domains in $\mathbb{C}^n$ into strictly convex domains in $\mathbb{C}^N$ for some large N (see, e.g., Fornaess [For 2]). The Hölder 1/2-estimates proved in Theorem 11.2.11 can be extended to any strictly pseudoconvex domain, but we omit the details here.

11.3 Homotopy Formulas for $\bar{\partial}_b$ on Strictly Convex Boundaries

Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary and let ρ be a C^2 defining function for D, normalized such that $|d\rho| = 1$ on bD. f is a $(0,q)$-form on bD with continuous coefficients, denoted by $f \in C_{(0,q)}(bD)$, if and only if

$$f = \tau g, \tag{11.3.1}$$

where g is a continuous $(0,q)$-form in $\mathbb{C}^n$ and τ is the projection operator from $(0,q)$-forms in $\mathbb{C}^n$ onto $(0,q)$-forms on bD (i.e., $(0,q)$-forms which are pointwise orthogonal to $\bar{\partial}\rho$). (11.3.1) is also equivalent to the following condition: for any continuous $(n, n-q-1)$-form ϕ defined in a neighborhood of bD, we have

$$\int_{bD} f \wedge \phi = \int_{bD} g \wedge \phi. \tag{11.3.2}$$

To see that (11.3.1) and (11.3.2) are equivalent, we note that for any $(0,q-1)$-form h with continuous coefficients in $\mathbb{C}^n$,

$$\int_{bD} \bar{\partial}\rho \wedge h \wedge \phi = \int_{bD} (d\rho - \partial\rho) \wedge h \wedge \phi = 0.$$

The space of $(0,q)$-forms with Hölder or L^p coefficients are denoted by $C^\alpha_{(0,q)}(bD)$ or $L^p_{(0,q)}(bD)$, where $0 < \alpha < 1$ and $1 \leq p \leq \infty$. If $u \in L^p_{(0,q-1)}(bD)$, u satisfies

$\bar{\partial}_b u = f$ for some $f \in L^p_{(0,q)}(bD)$ in the distribution sense if and only if for any $\phi \in C^\infty_{(n,n-1-q)}(\mathbb{C}^n)$,

$$\int_{bD} u \wedge \bar{\partial}\phi = (-1)^q \int_{bD} f \wedge \phi. \tag{11.3.3}$$

Let $D^- = D$ and $D^+ = \mathbb{C}^n \setminus \overline{D}$. We define the Bochner-Martinelli-Koppelman transform for any $f \in C_{(0,q)}(bD)$ as follows:

$$\int_{bD} B_q(\zeta, z) \wedge f(\zeta) = \begin{cases} F^-(z), & \text{if } z \in D^-, \\ F^+(z), & \text{if } z \in D^+. \end{cases} \tag{11.3.4}$$

It is easy to see that $F^- \in C^\infty_{(0,q)}(D^-)$ and $F^+ \in C^\infty_{(0,q)}(D^+)$. In fact, F^- and F^+ are continuous up to the boundary if f is Hölder continuous and we have the following jump formula:

Theorem 11.3.1. (Bochner-Martinelli-Koppelman jump formula). *Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. Let $f \in C^1_{(0,q)}(bD)$, where $0 \le q \le n-1$. Then*

$$F^- \in C^\alpha_{(0,q)}(\overline{D}^-), \quad F^+ \in C^\alpha_{(0,q)}(\overline{D}^+), \tag{11.3.5}$$

for every α with $0 < \alpha < 1$ and

$$f = \tau(F^- - F^+), \qquad z \in bD. \tag{11.3.6}$$

Proof. We first assume that the boundary bD is flat with $bD = \{(z_1, \cdots, z_n) \in \mathbb{C}^n \mid \operatorname{Im} z_n = 0\}$ and f has compact support in bD. The coefficients of $B(\zeta, z)$ are of the form

$$\frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}}, \quad j = 1, \cdots, n.$$

We rename the real coordinates $z_j = x_j + iy_j$ by setting $x_{j+n} = y_j$, $j = 1, \cdots, n-1$, and $y = y_n$. Similarly we set $\xi_{j+n} = \eta_j$, $j = 1, \cdots, n-1$, where $\zeta_j = \xi_j + i\eta_j$. Set $x = (x_1, \cdots, x_{2n-1})$, $\xi = (\xi_1, \cdots, \xi_{2n-1})$ and $z = (x, y)$. We define

$$\begin{cases} p_y(\xi) = \dfrac{y}{(|\xi|^2 + y^2)^n}, & y > 0, \\ q^j_y(\xi) = \dfrac{\xi_j}{(|\xi|^2 + y^2)^n}, & y > 0, \quad j = 1, \cdots, 2n-1. \end{cases}$$

Then p_y is a constant multiple of the Poisson kernel for the upper half space $\mathbb{R}^{2n}_+ = \{z| \ y > 0\}$ and the q^j_y's are the conjugate Poisson kernels. If we write $f = \sum'_{|I|=q} f_I d\bar{z}^I$, then each summand in $\int_{bD} B(\zeta, z) \wedge f$ is a constant multiple of the following form:

$$Pf_I(z) = \int_{\xi \in \mathbb{R}^{2n-1}} \frac{y}{(|\xi - x|^2 + y^2)^n} f_I(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_{2n-1} = p_y * f_I,$$

or

$$Q_j f_I(z) = \int_{\xi\in\mathbb{R}^{2n-1}} \frac{\xi_j - x_j}{(|\xi - x|^2 + y^2)^n} f_I(\xi)\ d\xi_1 \wedge \cdots \wedge d\xi_{2n-1} = q_y^j * f_I,$$

where $j = 1, \cdots, 2n-1$. The Poisson integral P is bounded from $C_0(\mathbb{R}^{2n-1})$ to $C(\mathbb{R}^{2n}_+)$. Since it ts a convolution operator, it is bounded from $C_0^1(\mathbb{R}^{2n-1})$ to $C^1(\mathbb{R}^{2n}_+)$.. The integral $Q_j f_I$ is the Poisson integral of the Riesz transform of f_I. To see that $Q_j f_I$ is bounded from $C_0^1(\mathbb{R}^{2n-1})$ to $C^\alpha(\mathbb{R}^{2n}_+)$, we use integration by parts and arguments similar to those used in Corollary 11.1.3. This proves (11.3.5) when the boundary is flat. For the general case, we note that the Bochner-Martinelli-Koppelman kernel is obtained by differentiation of the fundamental solution $e(z)$ for $\triangle$ (c.f. 11.1). Using integration by parts and arguments in the proof of Corollary 11.1.3, one can also prove similarly that $F^- \in C^\alpha_{(0,q)}(\overline{D}^-)$ and $F^+ \in C^\alpha_{(0,q)}(\overline{D}^+)$.

To prove (11.3.6), we first extend f to $\overline{D}$ such that the extension, still denoted by f, is in $C^1_{(0,q)}(\overline{D})$. From Theorem 11.1.2, we have

$$\text{(11.3.7)}\quad \begin{aligned} &\int_{bD} B_q(\cdot, z)\wedge f + \int_D B_q(\cdot, z)\wedge \bar\partial_\zeta f + \bar\partial_z \int_D B_{q-1}(\cdot, z)\wedge f \\ &\qquad = \begin{cases} f(z), & z\in D, \\ 0, & z\in \mathbb{C}^n\setminus \overline{D}. \end{cases} \end{aligned}$$

When $z \in D$, (11.3.7) was proved in (11.1.10). From the proof of (11.1.10), it is easy to see that (11.3.7) holds for $z \in \mathbb{C}^n \setminus \overline{D}$. Since $B(\zeta, z)$ is an integrable kernel in $\mathbb{C}^n$, the term $\int_D B(\cdot, z) \wedge \bar\partial_\zeta f$ is continuous up to the boundary bD. We denote by ν_z the outward unit normal to bD at z. Then for $z \in bD$,

$$\lim_{\epsilon\to 0^+} \left(\int_D B_q(\cdot, z - \epsilon\nu_z)\wedge \bar\partial_\zeta f - \int_D B_q(\cdot, z + \epsilon\nu_z)\wedge \bar\partial_\zeta f \right) = 0.$$

It remains to see that the term $\bar\partial_z \int_D B(\cdot, z)\wedge f$ when restricted to the boundary has no jump in the complex tangential component. For any $\phi \in C^\infty_{(n,n-q-1)}(\mathbb{C}^n)$, we have

$$\text{(11.3.8)}\quad \begin{aligned} &\lim_{\epsilon\to 0^+} \int_{bD} \bar\partial_z \left[\int_D B_{q-1}(\zeta, z - \epsilon\nu_z)\wedge f(\zeta)\right] \wedge \phi(z) \\ &\quad = (-1)^q \lim_{\epsilon\to 0^+} \int_{bD} \left[\int_D B_{q-1}(\zeta, z - \epsilon\nu_z)\wedge f(\zeta)\right] \wedge \bar\partial_z\phi(z) \\ &\quad = (-1)^q \int_{bD} \left[\int_D B_{q-1}(\zeta, z)\wedge f(\zeta)\right] \wedge \bar\partial_z\phi(z). \end{aligned}$$

Similarly, we obtain

$$\text{(11.3.9)}\quad \begin{aligned} &\lim_{\epsilon\to 0^+} \int_{bD} \bar\partial_z \left[\int_D B_{q-1}(\zeta, z + \epsilon\nu_z)\wedge f(\zeta)\right] \wedge \phi(z) \\ &\quad = (-1)^q \int_{bD} \left[\int_D B_{q-1}(\zeta, z)\wedge f(\zeta)\right] \wedge \bar\partial_z\phi(z). \end{aligned}$$

Thus from (11.3.7)-(11.3.9), we get for any $\phi \in C^\infty_{(n,n-q-1)}(\mathbb{C}^n)$,

$$\begin{aligned}\int_{bD} f(z) \wedge \phi(z) &= \lim_{\epsilon \to 0^+} \int_{bD} \left(\int_{bD} [B_q(\zeta, z - \epsilon \nu_z) - B_q(\zeta, z + \epsilon \nu_z)] \wedge f(\zeta) \right) \wedge \phi(z) \\ &= \int_{bD} [F^-(z) - F^+(z)] \wedge \phi(z).\end{aligned}$$

Using (11.3.2), we have proved (11.3.6). This proves the theorem.

Corollary 11.3.2. *Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. For any $f \in C^1_{(0,q)}(bD)$ with $\bar{\partial}_b f = 0$ on bD, we have*

$$f = \tau(F^- - F^+), \quad z \in bD,$$

and $\bar{\partial} F^- = 0$ in D^-, $\bar{\partial} F^+ = 0$ in D^+. Furthermore, we have $F^- \in C^\alpha_{(0,q)}(\overline{D}^-)$, $F^+ \in C^\alpha_{(0,q)}(\overline{D}^+)$ for any $0 < \alpha < 1$.

Proof. Since $F^- \in C^\infty_{(0,q)}(D)$, differentiation under the integral sign and Stoke's theorem imply that for $z \in D$,

$$\begin{aligned}\bar{\partial}_z F^-(z) &= - \int_{bD} \bar{\partial}_z B_q(\zeta, z) \wedge f(\zeta) \\ &= \int_{bD} \bar{\partial}_\zeta B_{q+1}(\zeta, z) \wedge f(\zeta) \\ &= \int_{bD} d_\zeta \left(B(\zeta, z) \wedge f(\zeta) \right) + \int_{bD} B_{q+1}(\zeta, z) \wedge \bar{\partial}_\zeta f(\zeta) \\ &= 0.\end{aligned}$$

Here we have used (11.1.9). Similarly, $\bar{\partial} F^+ = 0$ in D^+. Using Theorem 11.3.1, the corollary is proved.

One should compare Corollary 11.3.2 with Lemma 9.3.5. When $q = 0$, Corollary 11.3.2 implies that any CR function f can be written as the difference of two holomorphic functions. Thus Corollary 11.3.2 generalizes the Plemelj jump formula in $\mathbb{C}$ proved in Theorem 2.1.3.

From Corollary 11.3.2, every $\bar{\partial}_b$-closed form can be written as the jump of two $\bar{\partial}$-closed forms. Solving $\bar{\partial}_b$ is reduced to solving the $\bar{\partial}$ problem on both D^- and D^+. When D is strictly convex, we have already discussed how to solve $\bar{\partial}$ on D by integral formulas. We shall use Theorem 11.3.1 to derive homotopy formulas for $\bar{\partial}_b$ when D is a strictly convex domain with C^2 boundary.

Let ρ be a strictly convex defining function for D. Define C^1 functions G^- and G^+ in $\mathbb{C}^n \times \mathbb{C}^n$ by

(11.3.10) $$G^-(\zeta, z) = \left(\frac{\partial \rho}{\partial \zeta_1}, \cdots, \frac{\partial \rho}{\partial \zeta_n} \right),$$

(11.3.11) $$G^+(\zeta, z) = \left(-\frac{\partial \rho}{\partial z_1}, \cdots, -\frac{\partial \rho}{\partial z_n} \right).$$

Using Lemma 11.2.6, $G^-(\zeta, z)$ is a Leray map for D. Let

$$G^0(\zeta, z) = (\bar{\zeta}_1 - \bar{z}_1, \cdots, \bar{\zeta}_n - \bar{z}_n)$$

be the same as before.

Lemma 11.3.3. *Let D be a strictly convex domain in $\mathbb{C}^n$ with C^2 boundary and ρ be a C^2 strictly convex defining function for D. For $\zeta, z \in bD$, the kernels*

$$\Omega^- = \Omega(G^-), \qquad \Omega^+ = \Omega(G^+), \tag{11.3.12}$$

$$\Omega^{-0} = \Omega(G^-, G^0), \qquad \Omega^{+0} = \Omega(G^+, G^0) \tag{11.3.13}$$

have singularities only when $\zeta = z$. Furthermore, there exists a constant $C > 0$ independent of z such that

$$\int_{bD} (|\Omega^{-0}(\zeta, z)| + |\Omega^{+0}(\zeta, z)|) < C, \quad z \in bD. \tag{11.3.14}$$

Proof. Set

$$\begin{cases} \Phi(\zeta, z) = < G^-(\zeta, z), \zeta - z >, \\ \Psi(\zeta, z) = < G^+(\zeta, z), \zeta - z > . \end{cases}$$

Note that $\Phi(\zeta, z) = \Psi(z, \zeta)$. Using Lemma 11.2.9, there exists a constant $C > 0$ such that for any $\zeta \in bD, z \in \overline{D^-}$,

$$\mathrm{Re}\Phi(\zeta, z) \geq C(\rho(\zeta) - \rho(z) + |\zeta - z|^2). \tag{11.3.15}$$

Let U be some small tubular neighborhood of bD. Again the proof of Lemma 11.2.9 shows that (11.3.15) holds for $\zeta \in \overline{D}^+ \cap U$ if U is sufficiently small. Reversing the role of ζ and z, we have for any $z \in \overline{D}^+ \cap U$ and $\zeta \in \overline{D}$,

$$\begin{aligned} \mathrm{Re}\Psi(\zeta, z) = \mathrm{Re}\Phi(z, \zeta) &= \mathrm{Re} \sum_{i=1}^{n} -\frac{\partial \rho}{\partial z_i}(\zeta_i - z_i) \\ &\geq C(\rho(z) - \rho(\zeta) + |\zeta - z|^2). \end{aligned} \tag{11.3.16}$$

Inequality (11.3.16) holds for $z \in \overline{D}^+ \cap U$ since ρ is strictly convex in a neighborhood of bD. Thus Ω^+ and Ω^{0+} have singularities only at $\zeta = z \in bD$. Using estimate (11.3.15), we have already proved that Ω^{-0} is absolutely integrable in Lemma 11.2.10. Since Ψ satisfies a similar estimate (11.3.16), the proof for Ω^{+0} follows from the arguments of Lemma 11.2.10. This proves (11.3.14) and the lemma.

For $\zeta \neq z$, we set

$$\Gamma = \Omega^{-0} - \Omega^{+0} = \sum_{q=0}^{n-2} \Gamma_q(\zeta, z),$$

where $\Gamma_q = \Omega_q^{-0} - \Omega_q^{+0}$ is the summand which is of degree $(0, q)$ in z. Using Lemma 11.3.3, Γ is an integrable kernel on bD. If $f \in C_{(0,q)}(bD)$, the form

$$H_q f = \int_{bD} \Gamma_{q-1}(\cdot, z) \wedge f = \int_{bD} (\Omega_{q-1}^{-0} - \Omega_{q-1}^{+0}) \wedge f \tag{11.3.17}$$

is a well defined $(0, q-1)$-form on bD with continuous coefficients. The next theorem shows that $\Gamma(\zeta, z)$ is a fundamental solution for $\bar{\partial}_b$ on strictly convex boundaries.

Theorem 11.3.4 (First homotopy formula for $\bar\partial_b$ on strictly convex boundaries). *Let D be a strictly convex domain in $\mathbb{C}^n$ with C^2 boundary and let ρ be a C^2 defining function for D. Then for any $f \in C^1_{(0,q)}(bD)$, $0 < q < n-1$, we have*

$$f(z) = \bar\partial_b H_q f + \tau H_{q+1}\bar\partial_b f, \qquad z \in bD, \tag{11.3.18}$$

where H_q is defined in (11.3.17).

Proof. For any $f \in C^1_{(0,q)}(bD)$, using Theorem 11.3.1, the Bochner-Martinelli-Koppelman transform F^- and F^+ defined by (11.3.4) are continuous up to the boundary. We denote the boundary value of F^- and F^+ by $(\int_{bD} B(\zeta,z)\wedge f(\zeta))^-$ and $(\int_{bD} B(\zeta,z)\wedge f(\zeta))^+$ respectively. From (11.3.6), we have for any $z \in bD$,

$$\begin{aligned} f(z) &= \tau(F^-(z) - F^+(z)) \\ &= \tau\left(\int_{bD} B(\zeta,z)\wedge f(\zeta)\right)^- - \tau\left(\int_{bD} B(\zeta,z)\wedge f(\zeta)\right)^+. \end{aligned} \tag{11.3.19}$$

Applying (11.1.4-ii), we have for any $\zeta \in bD$,

$$\bar\partial_{\zeta,z}\Omega^{-0} = -\Omega^0 + \Omega^- = -B(\zeta,z) + \Omega^-, \qquad z \in D^-,$$

$$\bar\partial_{\zeta,z}\Omega^{+0} = -\Omega^0 + \Omega^+ = -B(\zeta,z) + \Omega^+, \qquad z \in D^+.$$

Thus, for $z \in D^-$,

$$\begin{aligned} &\int_{bD} B(\cdot,z)\wedge f \\ &\quad = -\int_{bD} \bar\partial_{\zeta,z}\Omega^{-0}(\cdot,z)\wedge f + \int_{bD}\Omega^-(\cdot,z)\wedge f \\ &\quad = \bar\partial_z \int_{bD}\Omega^{-0}(\cdot,z)\wedge f + \int_{bD}\Omega^{-0}(\cdot,z)\wedge\bar\partial_b f + \int_{bD}\Omega^-(\cdot,z)\wedge f. \end{aligned} \tag{11.3.20}$$

Similarly, for $z \in D^+$,

$$\begin{aligned} &\int_{bD} B(\cdot,z)\wedge f \\ &\quad = -\int_{bD} \bar\partial_{\zeta,z}\Omega^{+0}(\cdot,z)\wedge f + \int_{bD}\Omega^+(\cdot,z)\wedge f \\ &\quad = \bar\partial_z \int_{bD}\Omega^{+0}(\cdot,z)\wedge f + \int_{bD}\Omega^{+0}(\cdot,z)\wedge\bar\partial_b f + \int_{bD}\Omega^+(\cdot,z)\wedge f. \end{aligned} \tag{11.3.21}$$

Since G^- is independent of z, $\Omega^-(\zeta,z) = \Omega^-_0(\zeta,z)$. It follows that

$$\int_{bD}\Omega^-(\cdot,z)\wedge f = 0, \qquad \text{when} \quad q \neq 0. \tag{11.3.22}$$

Also since G^+ is independent of ζ, we have $\Omega^+(\zeta,z) = \Omega^+_{n-1}(\zeta,z)$ and

$$\int_{bD}\Omega^+(\cdot,z)\wedge f = 0, \qquad \text{when} \quad 0 \le q < n-1. \tag{11.3.23}$$

From Lemma 11.3.3, Ω^{-0} and Ω^{+0} are absolutely integrable kernels. Substituting (11.3.20)-(11.3.23) into (11.3.19) and letting $z \to bD$, we have proved (11.3.18). This proves the theorem.

Corollary 11.3.5 (A solution operator for $\bar{\partial}_b$ on strictly convex boundaries). *Let D be a bounded strictly convex domain in $\mathbb{C}^n$ with C^2 boundary bD. For $f \in C_{(0,q)}(bD)$, $1 \le q \le n-2$, such that $\bar{\partial}_b f = 0$ on bD, define*

$$u(z) = H_q f = \int_{bD} (\Omega_{q-1}^{-0} - \Omega_{q-1}^{+0}) \wedge f, \quad z \in bD. \tag{11.3.24}$$

Then $u \in C_{(0,q-1)}(bD)$ and u satisfies $\bar{\partial}_b u = f$.

Proof. Using Lemma 11.3.3, we have

$$\| u \|_{L^\infty(bD)} \le C \| f \|_{L^\infty(bD)} .$$

Thus $u \in C_{(0,q-1)}(bD)$. From Theorem 11.3.4, it follows that $\bar{\partial}_b u = f$ in the distribution sense.

Remark. Under the same assumption as in Theorem 11.3.4, we also have the following formula when $q = 0$ (f is a function) and $q = n-1$ (the top degree case):

When $q = n-1$, for any $f \in C^1_{(0,n-1)}(bD)$,

$$f(z) = -\tau \int_{bD} \Omega^+_{n-1}(\cdot, z) \wedge f + \bar{\partial}_b \int_{bD} \Gamma_{n-2}(\cdot, z) \wedge f.$$

The kernel $\Omega^+ = \Omega^+_{n-1}$ is a holomorphic function in ζ. If f is a $(0, n-1)$-form satisfying the compatibility condition (9.2.12 a), then

$$\int_{bD} \Omega^+_{n-1}(\cdot, z) \wedge f = 0, \quad z \in D \text{ and } z \to bD.$$

Thus, we have

$$f(z) = \bar{\partial}_b \int_{bD} \Gamma_{n-2}(\cdot, z) \wedge f, \qquad z \in bD.$$

This gives us an explicit solution formula for the $\bar{\partial}_b$ operator on strictly convex boundaries for $q = n-1$.

On the other hand, for any $f \in C^1(bD)$,

$$f(z) = \int_{bD} \Omega^-_0(\cdot, z) \wedge f + \int_{bD} \Gamma_0(\cdot, z) \wedge \bar{\partial}_b f.$$

If f is a CR function, we have

$$f(z) = \int_{bD} \Omega^-_0(\cdot, z) \wedge f, \qquad z \in bD.$$

Thus Ω^-_0 is another reproducing kernel for holomorphic functions in $\mathcal{O}(D) \cap C^1(\overline{D})$. We have already proved in Corollary 2.2.2 that the Bochner-Martinelli kernel is a reproducing kernel. However, Ω^- can be viewed as a true generalization of the Cauchy kernel to $\mathbb{C}^n$ since Ω^- is holomorphic in z.

We shall derive another homotopy formula for $\bar{\partial}_b$ on strictly convex boundaries. Let Ω^{-+} be defined by

$$\Omega^{-+} = \Omega(G^-, G^+), \tag{11.3.25}$$

where G^- and G^+ are defined by (11.3.10) and (11.3.11). We first show that Ω^{-+} is integrable.

Lemma 11.3.6. *Let D be a bounded strictly convex domain in $\mathbb{C}^n$ with C^2 boundary and let ρ be a strictly convex defining function for D. The kernel $\Omega_{q-1}^{-+}(\zeta, z)$, $1 \le q \le n-1$ has singularities only at $\zeta = z$ for $\zeta,\ z \in bD$. Furthermore, there exists a constant C such that for any $z \in bD$,*

$$\int_{bD} |\Omega_{q-1}^{-+}(\zeta, z)| < C, \tag{11.3.26}$$

where C is independent of z.

Proof. Since

$$\Omega_{q-1}^{-+} = \left(\frac{1}{2\pi i}\right)^n \frac{< G^-, d\zeta >}{< G^-, \zeta - z >} \wedge \frac{< G^+, d\zeta >}{< G^+, \zeta - z >}$$
$$\wedge \left(\frac{< \bar{\partial}_{\zeta,z} G^-, d\zeta >}{< G^-, \zeta - z >}\right)^{n-q-1} \wedge \left(\frac{< \bar{\partial}_{\zeta,z} G^+, d\zeta >}{< G^+, \zeta - z >}\right)^{q-1},$$

it follows from (11.3.15) and (11.3.16) that Ω_{q-1}^{-+} has singularities only at $\zeta = z$. Thus we only need to estimate the kernel when ζ is close to z. For a fixed z, let $U_\epsilon = \{\zeta |\ |\zeta - z| < \epsilon\}$ be a sufficiently small neighborhood of z, $\Phi(\zeta, z) = < G^-, \zeta - z >$ and $\Psi(\zeta, z) = < G^+, \zeta - z >$ the same as before. Using the same change of variables $t = (t_1, \cdots, t_{2n-1}) = (t', t_{2n-1})$ as in Lemma 11.2.10 with $t_{2n-1} = \mathrm{Im}\Phi(\zeta, z)$ and $t_i(z) = 0$ for $i = 1, \cdots, 2n-1$, there exists a constant $\gamma_0 > 0$ such that for $\zeta,\ z \in bD$,

$$\begin{cases} |\Phi(\zeta, z)| \ge \gamma_0(|t'|^2 + |t_{2n-1}|), \\ |\Psi(\zeta, z)| \ge \gamma_0(|t'|^2 + |t_{2n-1}|^2), \\ \gamma_0|t| \le |\zeta - z| \le (1/\gamma_0)|t|. \end{cases} \tag{11.3.27}$$

We note that

$$\begin{aligned} | < G^-, d\zeta > \wedge < G^+, d\zeta > | &= | < G^-, d\zeta > \wedge < G^+ - G^-, d\zeta > | \\ &= O(|\zeta - z|). \end{aligned} \tag{11.3.28}$$

Let dS denote the surface element of bD. Repeating the arguments of (11.2.14), using (11.3.27) and (11.3.28), there exists an $A > 0$ such that

$$\begin{aligned} \int_{bD\cap U_\epsilon} |\Omega_{q-1}^{-+}(\zeta, z)| &\le C\left(\int_{bD\cap U_\epsilon} \frac{|\zeta - z|}{|\Phi|^{n-q}|\Psi|^q} dS\right) \\ &\le C \int_{\zeta \in bD\cap\{|\zeta - z| < \epsilon\}} \frac{1}{|\Phi|^{n-q}|\zeta - z|^{2q-1}} dS \\ &\le C \int_{|t|\le A} \frac{dt_1 dt_2 \cdots dt_{2n-1}}{(|t_{2n-1}| + |t'|^2)^{n-q}|t|^{2q-1}} \\ &\le C. \end{aligned}$$

Thus, the kernel Ω^{-+} is absolutely integrable and (11.3.26) is proved.

Theorem 11.3.7 (Second homotopy formula for $\bar\partial_b$ on strictly convex boundaries). *Let D be a strictly convex domain in $\mathbb{C}^n$ with C^2 boundary and let ρ be a C^2 defining function for D. Then for any $f \in C^1_{(0,q)}(bD)$, where $0 < q < n-1$, we have*

$$f(z) = \bar\partial_b R_q f + R_{q+1}\bar\partial_b f, \qquad z \in bD, \tag{11.3.29}$$

where

$$R_q f = \tau \int_{bD} \Omega^{-+}_{q-1}(\cdot, z) \wedge f$$

and Ω^{-+} is defined by (11.3.25).

Proof. Using Lemma 11.3.6, the kernel Ω^{-+} is absolutely integrable. From Theorem 11.3.4, we have when $0 < q < n-1$,

$$f(z) = \bar\partial_b \int_{\zeta\in bD} \Gamma(\zeta, z) \wedge f + \tau \int_{\zeta\in bD} \Gamma(\zeta, z) \wedge \bar\partial_b f.$$

Using (11.1.4-iii), we have for $\zeta,\ z \in bD$ and $\zeta \neq z$,

$$\begin{aligned} \bar\partial_{\zeta,z}\Omega^{-+0} &= \bar\partial_{\zeta,z}\Omega(G^-, G^+, G^0) \\ &= \Omega^{-0} - \Omega^{+0} - \Omega^{-+} = \Gamma - \Omega^{-+}. \end{aligned} \tag{11.3.30}$$

For each fixed $z \in bD$, we claim that Ω^{-+0} and $\bar\partial_{\zeta,z}\Omega^{-+0}$ are absolutely integrable kernels and

$$\left| \int_{bD} \Omega(G^-, G^+, G^0) \right| < C, \tag{11.3.31}$$

$$\left| \int_{bD} \bar\partial_{\zeta,z}\Omega(G^-, G^+, G^0) \right| < C, \tag{11.3.32}$$

where C is independent of z. Let $\Phi_0 = |\zeta - z|^2$ as before. Using (11.3.27) and (11.3.28), (11.3.31) can be estimated by

$$\begin{aligned} \left| \int_{bD\cap U_\epsilon} \Omega(G^-, G^+, G^0) \right| &\le C \sum_{k_1+k_2+k_3=n-3} \int_{bD\cap U_\epsilon} \frac{|\zeta - z|^2}{|\Phi^{k_1+1}\Psi^{k_2+1}\Phi_0^{k_3+1}|} dS \\ &\le C \sum_{k=2}^{n-2} \int_{|t|\le A} \frac{|t|^2 dt_1 dt_2 \cdots dt_{2n-1}}{(|t_{2n-1}| + |t'|^2)^{n-k}|t|^{2k}} < \infty. \end{aligned}$$

Since $\bar\partial_{\zeta,z}\Phi = \bar\partial_\zeta\Phi = O(|\zeta - z|)$ and $\bar\partial_{\zeta,z}\Psi = \bar\partial_z\Psi = O(|\zeta - z|)$, we can use (11.3.27), (11.3.28) and differentiation term by term to get

$$\begin{aligned}
&\left|\int_{bD\cap U_\epsilon}\bar\partial_{\zeta,z}\Omega(G^-,G^+,G^0)\right|\\
&\qquad\le C\sum_{k_1+k_2+k_3=n-3}\int_{bD\cap U_\epsilon}\frac{|\zeta-z|}{|\Phi^{k_1+1}\Psi^{k_2+1}\Phi_0^{k_3+1}|}dS\\
&\qquad\quad+C\sum_{k_1+k_2+k_3=n-2}\int_{bD\cap U_\epsilon}\frac{|\zeta-z|^3}{|\Phi^{k_1+1}\Psi^{k_2+1}\Phi_0^{k_3+1}|}dS\\
&\qquad\le C\sum_{k=1}^{n-2}\int_{|t|\le A}\frac{|t|dt_1dt_2\cdots dt_{2n-1}}{(|t_{2n-1}|+|t'|^2)^{n-k}|t|^{2k}}\\
&\qquad\quad+C\sum_{k=1}^{n-1}\int_{|t|\le A}\frac{|t|^3dt_1dt_2\cdots dt_{2n-1}}{(|t_{2n-1}|+|t'|^2)^{n-k}|t|^{2k+2}}<\infty,
\end{aligned}$$

where dS is the surface element of bD. This proves (11.3.32). From (11.3.31) and (11.3.32), we can interchange the order of integration and differentiation to obtain

$$\begin{aligned}
\int_{\zeta\in bD}\bar\partial_{\zeta,z}\Omega^{-+0}\wedge f&=\int_{\zeta\in bD}\bar\partial_\zeta\Omega^{-+0}\wedge f-\bar\partial_z\int_{\zeta\in bD}\Omega^{-+0}\wedge f\\
&=\int_{\zeta\in bD}\Omega^{-+0}\wedge\bar\partial_b f-\bar\partial_z\int_{\zeta\in bD}\Omega^{-+0}\wedge f,
\end{aligned}\tag{11.3.33}$$

where the last equality follows from Stokes' theorem. The Stokes' theorem can be used here by first substituting $\Phi^\epsilon=\Phi+\epsilon$, $\Psi^\epsilon=\Psi+\epsilon$ and $\Phi_0^\epsilon=\Phi_0+\epsilon$ for Φ, Ψ and Φ_0 respectively in the kernel Ω^{-+0} and then letting $\epsilon\searrow 0$. Similarly,

$$\begin{aligned}
\int_{\zeta\in bD}\bar\partial_{\zeta,z}\Omega^{-+0}\wedge\bar\partial_b f&=\int_{\zeta\in bD}\bar\partial_z\Omega^{-+0}\wedge\bar\partial_b f\\
&=-\bar\partial_z\int_{\zeta\in bD}\Omega^{-+0}\wedge\bar\partial_b f.
\end{aligned}\tag{11.3.34}$$

From (11.3.33) and (11.3.34), we have

$$\bar\partial_b\tau\int_{\zeta\in bD}\bar\partial_{\zeta,z}\Omega^{-+0}\wedge f+\tau\int_{\zeta\in bD}\bar\partial_{\zeta,z}\Omega^{-+0}\wedge\bar\partial_b f=0,\quad z\in bD.$$

Thus using (11.3.30), we obtain

$$\begin{aligned}
f(z)=&\ \bar\partial_b\int_{\zeta\in bD}\Gamma(\zeta,z)\wedge f+\tau\int_{\zeta\in bD}\Gamma(\zeta,z)\wedge\bar\partial_b f\\
=&\left(\bar\partial_b\int_{\zeta\in bD}\bar\partial_{\zeta,z}\Omega^{-+0}\wedge f+\tau\int_{\zeta\in bD}\bar\partial_{\zeta,z}\Omega^{-+0}\wedge\bar\partial_b f\right)\\
&+\left(\bar\partial_b\int_{\zeta\in bD}\Omega^{-+}\wedge f+\tau\int_{\zeta\in bD}\Omega^{-+}\wedge\bar\partial_b f\right)\\
=&\ \bar\partial_b\int_{\zeta\in bD}\Omega^{-+}\wedge f+\tau\int_{\zeta\in bD}\Omega^{-+}\wedge\bar\partial_b f
\end{aligned}$$

for every $z\in bD$. This proves Theorem 11.3.7.

From Lemma 11.3.6 and Theorem 11.3.7, we have derived another solution formula for $\bar\partial_b$.

Corollary 11.3.8 (Second solution operator for $\bar\partial_b$ on strictly convex boundaries). *Let D be a strictly convex domain in $\mathbb{C}^n$ with C^2 boundary and let ρ be a C^2 defining function for D. Let $f \in C_{(0,q)}(bD)$, where $0 < q < n-1$ such that $\bar\partial_b f = 0$ on bD. Setting $u = R_q f$, then u is in $C_{(0,q-1)}(bD)$ and $\bar\partial_b u = f$ on bD.*

Proof. That u is in $C_{(0,q-1)}(bD)$ follows from Lemma 11.3.6. Using Theorem 11.3.7, we have $\bar\partial_b u = f$ on bD in the distribution sense.

Next we shall estimate $R_q f$ in the Hölder and L^p spaces. We use $\| \ \|_{L^p}$ to denote the $L^p_{(0,q)}(bD)$ norms for $(0,q)$-forms.

Theorem 11.3.9 (Hölder and L^p estimates for $\bar\partial_b$ on strictly convex boundaries). *Let D be a strictly convex domain in $\mathbb{C}^n$ with C^3 boundary and ρ be a C^3 defining function for D. For any $f \in L^p_{(0,q)}(bD)$, $1 \le p \le \infty$ and $1 \le q < n-1$, $R_q f$ satisfies the following estimates:*

(1) $\|R_q f\|_{L^{\frac{2n}{2n-1}-\epsilon}} \le C\|f\|_{L^1}$, *for any small* $\epsilon > 0$.
(2) $\|R_q f\|_{L^{p'}} \le C\|f\|_{L^p}$, *where* $\frac{1}{p'} = \frac{1}{p} - \frac{1}{2n}$ *for* $1 < p < 2n$.
(3) $\|R_q f\|_{L^{p'}} \le C\|f\|_{L^p}$, *where* $p = 2n$ *and* $p < p' < \infty$.
(4) $\|R_q f\|_{C^\alpha} \le C\|f\|_{L^p}$, *where* $2n < p < \infty$ *and* $\alpha = \frac12 - \frac{n}{p}$, C^α *is the Hölder space of exponent* α *on* bD.
(5) $\|R_q f\|_{C^{\frac12}} \le C\|f\|_{L^\infty}$.

Proof. We shall prove that the kernel $\Omega^{-+}(\zeta, z)$ is of weak type $\frac{2n}{2n-1}$ on bD uniformly in ζ and in z. (For definition of weak type, see Definition B.5 in the Appendix). Since Ω^{-+} only has singularities when $\zeta = z$, following the change of coordinates $\zeta \to t$ and (11.3.27), it suffices to show that the function

$$K(t) = \frac{1}{(|t_{2n-1}| + |t'|^2)^{n-q}|t'|^{2q-1}} \tag{11.3.35}$$

is of weak type $\frac{2n}{2n-1}$, where $t = (t_1, \cdots, t_{2n-2}, t_{2n-1}) = (t', t_{2n-1})$ and $|t| < 1$. Let A_λ be the subset

$$A_\lambda = \{t \in \mathbb{R}^{2n-1},\ |t| < 1 \mid K(t) > \lambda\}, \qquad \lambda > 0,$$

and let m be the Lebesgue measure in $\mathbb{R}^{2n-1}$. We shall show that there exists a constant $\tilde c > 0$ such that

$$m(A_\lambda) \le \left(\frac{\tilde c}{\lambda}\right)^{\frac{2n}{2n-1}}, \qquad \text{for all} \quad \lambda > 0. \tag{11.3.36}$$

By a change of variables $t \to \tilde t$ such that $t_i = \lambda^{-\frac{1}{2n-1}}\tilde t_i$, $i = 1, \cdots, 2n-2$ and $t_{2n-1} = \lambda^{-\frac{2}{2n-1}}\tilde t_{2n-1}$, we have for some $c > 0$,

$$m(A_\lambda) = c\lambda^{-\frac{2n}{2n-1}} m(A_1) \le \left(\frac{\tilde c}{\lambda}\right)^{\frac{2n}{2n-1}},$$

since $m(A_1) < \infty$. This proves (11.3.36). It follows from Theorem B.11 in the Appendix that the estimates (1), (2) and (3) hold.

To prove (4) and (5), we define $\tilde\Omega^{-+}$ by

$$\tilde\Omega^{-+} = \left(\frac{1}{2\pi i}\right)^n \frac{< G^-, d\zeta >}{\Phi(\zeta,z)} \wedge \frac{< G^+, d\zeta >}{\Psi(\zeta,z)-\mu\rho(z)} \wedge \sum_{k_1+k_2=n-2} \left(\frac{< \bar\partial_{\zeta,z} G^-, d\zeta >}{\Phi(\zeta,z)}\right)^{k_1} \wedge \left(\frac{< \bar\partial_{\zeta,z} G^+, d\zeta >}{\Psi(\zeta,z)-\mu\rho(z)}\right)^{k_2},$$

where $\mu > 0$ is sufficiently large. We first note that the kernel Ω^{-+} is the same as $\tilde\Omega^{-+}$ when $\zeta,\ z \in bD$. It follows from (11.3.15) and (11.3.16) that there exists a $C > 0$ such that for any $\zeta \in bD$ and $z \in D$,

$$\mathrm{Re}\Phi(\zeta,z) \geq C(|\rho(z)| + |\zeta - z|^2), \tag{11.3.37}$$

and

$$\mathrm{Re}\tilde\Psi(\zeta,z) \equiv \mathrm{Re}\Psi(\zeta,z) - \mu\rho(z) \geq C(|\rho(z)| + |\zeta - z|^2), \tag{11.3.38}$$

if μ is chosen sufficiently large. Let $D_\delta = \{z \in D \mid \rho(z) < -\delta\}$ for some $\delta > 0$. From the Hardy-Littlewood lemma (see Theorem C.1 in the Appendix), to prove (4) and (5), it suffices to show that for some small $\delta_0 > 0$ and all $0 < \delta < \delta_0$,

$$\sup_{z\in bD_\delta} \left| \mathrm{grad}_z \int_{bD} \tilde\Omega^{-+}(\cdot,z) \wedge f \right| \leq C\delta^{-\frac{1}{2}-\frac{n}{p}} \|f\|_{L^p}, \tag{11.3.39}$$

where $2n \leq p \leq \infty$. After the same change of variables in a small neighborhood $|\zeta - z| < \epsilon$ that of (11.2.12), applying (11.3.37) and (11.3.38), (11.3.39) is proved for $p = \infty$ if the following holds:

$$\int_{|t|\leq A} \frac{dt_1 dt_2 \cdots dt_{2n-1}}{(\delta + |t_{2n-1}| + |t'|^2)^{n-q+1}|t'|^{2q-1}} < C\delta^{-\frac{1}{2}}, \tag{11.3.40}$$

$$\int_{|t|\leq A} \frac{dt_1 dt_2 \cdots dt_{2n-1}}{(\delta + |t_{2n-1}| + |t'|^2)^{n-q}|t'|^{2q+1}} < C\delta^{-\frac{1}{2}}, \tag{11.3.41}$$

where C is independent of δ. Inequality (11.3.40) is proved in (11.2.17) and (11.3.41) is proved similarly since $1 \leq q < n-1$.

To prove (11.3.39) when $p = 2n$, it suffices to show that

$$\int_{|t|\leq A} \frac{|f| dt_1 dt_2 \cdots dt_{2n-1}}{(\delta + |t_{2n-1}| + |t'|^2)^{n-q+1}|t'|^{2q-1}} < C\delta^{-1}\|f\|_{L^{2n}}, \tag{11.3.42}$$

$$\int_{|t|\leq A} \frac{|f| dt_1 dt_2 \cdots dt_{2n-1}}{(\delta + |t_{2n-1}| + |t'|^2)^{n-q}|t'|^{2q+1}} < C\delta^{-1}\|f\|_{L^{2n}}, \tag{11.3.43}$$

where C is independent of δ. To prove (11.3.42), we set $n^* = \frac{2n}{2n-1}$ and use Hölder's inequality to obtain

$$\begin{aligned}
&\int_{|t|\le A} \frac{|f| dt_1 dt_2 \cdots dt_{2n-1}}{(\delta + |t_{2n-1}| + |t'|^2)^{n-q+1} |t'|^{2q-1}} \\
&\le \|f\|_{L^{2n}} \left(\int_{|t|\le A} \frac{dt_1 dt_2 \cdots dt_{2n-1}}{(\delta + |t_{2n-1}| + |t'|^2)^{n^*(n-q+1)} |t'|^{n^*(2q-1)}} \right)^{\frac{1}{n^*}} \\
&\le \|f\|_{L^{2n}} \left(\int_{|t'|\le A} \frac{dt_1 dt_2 \cdots dt_{2n-2}}{(\delta + |t'|^2)^{n^*(n-q+1)-1} |t'|^{n^*(2q-1)}} \right)^{\frac{1}{n^*}} \\
&\le C\|f\|_{L^{2n}} \left(\int_0^A \frac{r^{2n-3} dr}{(\delta + r^2)^{n^*(n-q+1)-1} r^{n^*(2q-1)}} \right)^{\frac{1}{n^*}} \\
&\le C\|f\|_{L^{2n}} \left(\int_0^A \frac{dr}{(\delta + r^2)^{\frac{2n+1}{2n-1}} r^{\frac{2n-3}{2n-1}}} \right)^{\frac{1}{n^*}}, \quad (v = r/\sqrt{\delta}) \\
&\le C\delta^{-1}\|f\|_{L^{2n}} \left(\int_0^\infty \frac{dv}{(1 + v^2)^{\frac{2n+1}{2n-1}} v^{\frac{2n-3}{2n-1}}} \right)^{\frac{1}{n^*}} \\
&\le C\delta^{-1}\|f\|_{L^{2n}}.
\end{aligned}$$

This proves (11.3.42) and (11.3.43) can be proved similarly. Inequality (11.3.39) is proved for $p = 2n$ and $p = \infty$. The other cases are proved by interpolation (see Theorem B.6 in the Appendix). This completes the proof of Theorem 11.3.9.

Remark. In Chapter 8, we have proved that when bD is strictly pseudoconvex or more generally, bD satisfies condition $\mathrm{Y}(q)$, the canonical solution gains 1/2-derivatives in the Sobolev spaces (see Theorem 8.4.14). Theorem 11.3.9 gives a solution operator which gains 1/2-derivatives in the Hölder space on strictly convex boundaries. This result again can be generalized to any strictly pseudoconvex boundary by a partition of unity since the boundary can be convexified locally. We note that the solution for $\bar{\partial}_b$ defined by (11.3.24) has the same properties as the solution given by $R_q f$ by a similar proof. It is interesting to note that when bD is the boundary of the Siegel upper half space, $R_q f$ obtained by the integral kernel method agrees with the solution obtained in Theorem 10.1.5 using a completely different method. The reader should compare Theorem 11.3.9 with Theorem 10.1.5.

11.4 Solvability for $\bar{\partial}_b$ on CR Manifolds with Boundaries

Let D be a strictly convex domain in $\mathbb{C}^n$ with C^2 boundary bD and $\omega \subset\subset bD$ be a connected open CR manifold with smooth boundary $b\omega$. We consider the $\bar{\partial}_b$ equation

$$(11.4.1) \qquad \overline{\partial}_b u = \alpha \qquad \text{on} \quad \omega,$$

where α is a $(0,q)$-form on ω, $1 \le q \le n-2$. In order for (11.4.1) to be solvable, it is necessary that α satisfies

$$\overline{\partial}_b \alpha = 0 \qquad \text{on} \quad \omega. \tag{11.4.2}$$

Note that when $q = n-1$, (11.4.2) is void and (11.4.1) is related to the local nonsolvable phenomenon of Lewy's equation. Due to the fact that the compatibility condition (11.4.2) is satisfied only on ω instead of the whole boundary bD, this question cannot be answered from the global solvability results obtained in the previous section. The solvability of (11.4.1) depends on the special geometry of the boundary $b\omega$.

In Chapter 9, we have seen that when $q = n-1$ with an additional compatibility condition (9.2.12 a), one still can solve $\bar{\partial}_b$ globally on bD. In fact, we have proved that $\bar{\partial}_b$ has closed range in $L^2_{(0,q)}(bD)$ on any pseudoconvex boundary bD for any $1 \le q \le n-1$. When we discuss the local solvability of (11.4.1), we must avoid the top degree case (when $q = n-1$) due to the Lewy example.

In this section we study the solvability of (11.4.1) on ω for any α satisfying (11.4.2) on ω. When $q = n-2$, there is another compatibility condition for (11.4.1) to be solvable without shrinking. This compatibility condition can be derived as follows:

Let K be a compact set in $\mathbb{C}^n$ and $\mathcal{O}(K)$ be the set of functions which are defined and holomorphic in some open neighborhood of K. Let α be a form in $C_{(0,n-2)}(\overline{\omega})$ such that there exists $u \in C_{(0,n-3)}(\overline{\omega})$ with $\bar{\partial}_b u = \alpha$ on ω. Then for any $g \in \mathcal{O}(b\omega)$, we have

$$\begin{aligned}\int_{b\omega} \alpha \wedge g \wedge dz_1 \wedge \cdots \wedge dz_n &= \int_{b\omega} \bar{\partial}_b u \wedge g \wedge dz_1 \wedge \cdots \wedge dz_n \\ &= \int_{b\omega} \bar{\partial}(u \wedge g \wedge dz_1 \wedge \cdots \wedge dz_n) \\ &= \int_{b\omega} d(u \wedge g \wedge dz_1 \wedge \cdots \wedge dz_n) = 0.\end{aligned}$$

Thus, another necessary condition for (11.4.1) to be solvable for some $u \in C_{(0,n-3)}$ $(\overline{\omega})$ is that

$$\int_{b\omega} \alpha \wedge g \wedge dz_1 \wedge \cdots \wedge dz_n = 0 \qquad \text{for all } g \in \mathcal{O}(b\omega). \tag{11.4.2 a}$$

It is easy to see that (11.4.2 a) is also necessary for the existence of a solution $u \in C_{(0,n-3)}(\omega)$ by approximation. This additional condition makes it necessary to differentiate between $1 \le q < n-2$ and $q = n-2$ when considering (11.4.1). At the end of this section, we will discuss when condition (11.4.2) implies (11.4.2 a) and give an example to show that (11.4.2 a) is indeed an additional compatibility condition.

We first describe the geometry of the boundary $b\omega$ of ω on which one can construct a solution kernel for $\bar{\partial}_b$ on ω. Let ρ be a strictly convex defining function for D. The set ω is a domain in bD defined by some C^2 function r defined in a neighborhood of bD such that

$$\omega = \{z \in \mathbb{C}^n \mid \rho(z) = 0,\ r(z) < 0\}. \tag{11.4.3}$$

We require that r be a C^2 smooth function depending on only one complex variable. Without loss of generality, we may assume that r depends on z_n only. This implies that the hypersurface $M_0 = \{z \in \mathbb{C}^n \mid r(z_n) = 0\}$ is a Levi-flat hypersurface. The boundary $b\omega$ is defined by

$$b\omega = bD \cap M_0 = \{z \in \mathbb{C}^n \mid \rho(z) = 0,\ r(z_n) = 0\}.$$

On $b\omega$, we assume

$$d\rho \wedge dr \neq 0 \qquad \text{on } b\omega. \tag{11.4.4}$$

Thus, the hypersurfaces bD and M_0 intersect transversally over $\mathbb{R}$. If

$$\partial\rho \wedge \partial r \neq 0, \tag{11.4.5}$$

we say that bD and M_0 intersect transversally over $\mathbb{C}$. The points in

$$\Sigma = \{z \in b\omega \mid \partial\rho \wedge \partial r = 0\}$$

are called *characteristic points*. Any point in $b\omega \setminus \Sigma$ is called a *noncharacteristic point* or a *generic point*.

If p is a characteristic point on $b\omega$, the space $T_p^{1,0}(bD) \cap T_p^{1,0}(M_0)$ has complex dimension $n-1$. Near the noncharacteristic point $p \in b\omega$, the set $T_p^{1,0}(bD) \cap T_p^{1,0}(M_0)$ has complex dimension $n-2$. This jump in the dimension of the tangential (1,0) vector fields at the characteristic points makes it difficult to study (11.4.1) by imitating the L^2 techniques used in Chapter 4. We shall study the solvability of (11.4.1) by integral kernels.

The following example shows that in general, an open CR manifold with smooth boundary has characteristic points.

Example. If $D = \{z \in \mathbb{C}^n \mid |z| < 1\}$ is the unit ball and $r(z_n) = \mathrm{Im}\ z_n$, then the boundary $b\omega$ of the set

$$\omega = \{z \in \mathbb{C}^n \mid |z| = 1,\ \mathrm{Im} z_n < 0\}$$

has two characteristic points at $\Sigma = \{(0, \cdots, +1),\ (0, \cdots, -1)\}$, since $\partial\rho \wedge \partial r = 0$ if and only if $z_1 = \cdots = z_{n-1} = 0$.

If $r(z_n) = |z_n|^2$ and $b\omega$ is the boundary of

$$\omega_1 = \{z \in \mathbb{C}^n \mid |z| = 1,\ |z_n|^2 < \frac{1}{2}\},$$

then $b\omega_1$ has no characteristic points.

Notice that ω is simply connected but ω_1 is not.

To use the integral kernels to solve (11.4.1), our starting point is the homotopy formula derived in Theorem 11.3.7. From (11.3.29), we have that Ω^{-+} is a fundamental solution for $\bar{\partial}_b$ on the compact hypersurface bD. Thus, it gives a solution kernel for (11.4.1) if α has compact support in ω. To solve $\bar{\partial}_b$ for forms which do not vanish on $b\omega$, we introduce new kernels constructed from the special defining function r for ω. Set

$$G^\flat(\zeta, z) = G^\flat(\zeta) = \left(0, \cdots, 0, \frac{\partial r(\zeta_n)}{\partial \zeta_n}\right) \tag{11.4.6}$$

and

$$\eta(\zeta, z) = \frac{\partial r(\zeta_n)}{\partial \zeta_n}(\zeta_n - z_n) = < G^\flat, \zeta - z > . \tag{11.4.7}$$

Let

$$\omega^\flat = \frac{1}{2\pi i} \frac{< G^\flat, d\zeta >}{< G^\flat, \zeta - z >} = \frac{1}{2\pi i} \frac{d\zeta_n}{\zeta_n - z_n}, \tag{11.4.8}$$

where the notation $\flat$ is used to indicate that the hypersurface M_0 defined by r is Levi flat. Note that $\omega^\flat$ is independent of r and

$$\bar{\partial}_{\zeta,z}\omega^\flat = 0, \qquad \zeta_n \neq z_n.$$

In other words, it is holomorphic both in the ζ and z variables away from singularities. Setting

$$\Omega^{\flat-+} = \Omega(G^\flat, G^-, G^+), \tag{11.4.9}$$

we see that $\Omega^{\flat-+}$ is an $(n, n-3)$-form. We write

$$\Omega^{\flat-+}(\zeta, z) = \sum_{q=0}^{n-3} \Omega_q^{\flat-+}(\zeta, z), \tag{11.4.10}$$

where

$$\begin{aligned} \Omega_q^{\flat-+} = \frac{1}{(2\pi i)^n} & \frac{d\zeta_n}{\zeta_n - z_n} \wedge \frac{< G^-(\zeta), d\zeta >}{\Phi(\zeta, z)} \wedge \frac{< G^+(z), d\zeta >}{\Psi(\zeta, z)} \\ & \wedge \left[\frac{< \overline{\partial}_\zeta G^-(\zeta), d\zeta >}{\Phi(\zeta, z)}\right]^{n-3-q} \wedge \left[\frac{< \overline{\partial}_z G^+(z), d\zeta >}{\Psi(\zeta, z)}\right]^q \end{aligned}$$

away from the singularities. Thus $\Omega_q^{\flat-+}$ has exactly q $d\bar{z}$'s.

If $\zeta \in b\omega$ and $z \in \omega$, we have

$$r(\zeta_n) - r(z_n) > 0.$$

It follows that $\zeta_n \neq z_n$ when $\zeta \in b\omega$ and $z \in \omega$. The kernel $\Omega^{\flat-+}$ is well defined and smooth when $\zeta \in b\omega$ and $z \in \omega$.

For $\alpha \in C_{(0,q)}(\overline{\omega})$, $1 \le q \le n-2$, we define

$$S_q\alpha = \int_{\zeta \in \omega} \Omega_{q-1}^{-+}(\zeta, z) \wedge \alpha(\zeta) + \int_{\zeta \in b\omega} \Omega_{q-1}^{\flat-+}(\zeta, z) \wedge \alpha(\zeta), \tag{11.4.11}$$

where $\Omega^{-+}(\zeta, z)$ and $\Omega^{\flat-+}(\zeta, z)$ are kernels defined by (11.3.25) and (11.4.10) respectively. The following two theorems are the main results of this section.

Theorem 11.4.1 (A homotopy formula for $\bar{\partial}_b$ on CR manifolds with boundaries). *Let D be a strictly convex domain in $\mathbb{C}^n$ with C^2 boundary and let ρ be a C^2 strictly convex defining function for D. Let $\omega \subset\subset bD$ be an open connected CR manifold with smooth boundary defined by (11.4.3) where $r(z) = r(z_n)$ is a C^2 function. We assume that $d\rho \wedge dr \neq 0$ on $b\omega$. For any $\alpha \in C^1_{(0,q)}(\overline{\omega})$, $1 \le q < n-2$,*

$$\alpha = \bar{\partial}_b S_q \alpha + S_{q+1}\bar{\partial}_b \alpha, \qquad z \in \omega, \tag{11.4.12}$$

where S_q is the integral operator defined by (11.4.11).

Theorem 11.4.2 (A solution operator for $\bar{\partial}_b$ on CR manifolds with boundaries). *Let ω be as in Theorem 11.4.1. For any $\alpha \in C_{(0,q)}(\overline{\omega})$, $1 \le q < n-2$, with $\overline{\partial}_b \alpha = 0$ on ω, the form $u = S_q\alpha$ is in $C_{(0,q-1)}(\omega)$ and $\bar{\partial}_b u = \alpha$ on ω, where S_q is the integral operator defined by (11.4.11).*

When $q = n-2$, we assume furthermore that α satisfies the additional compatibility condition

$$\int_{b\omega} \alpha \wedge g \wedge dz_1 \wedge \cdots \wedge dz_n = 0 \qquad \text{for all } g \in \mathcal{O}(b\omega),$$

the same conclusion holds.

To prove Theorems 11.4.1 and 11.4.2, we start with the following proposition:

Proposition 11.4.3. *Let bD and ω be as in Theorem 11.4.1. For every $f \in C^1_{(0,q)}(\overline{\omega})$, $1 \le q < n-1$,*

$$f(z) = \bar{\partial}_b \int_\omega \Omega^{-+}_{q-1} \wedge f(\zeta) + \tau \int_\omega \Omega^{-+}_q \wedge \bar{\partial}_b f(\zeta) - \tau \int_{b\omega} \Omega^{-+}_q \wedge f(\zeta)$$

for every $z \in \omega$, where Ω^{-+} is defined by (11.3.25).

Proof. We first extend f to $\tilde{f}$ on an open set $\tilde{\omega} \supset\supset \omega$ such that $\tilde{f} \in C^1_{(0,q)}(\tilde{\omega})$. Let $\chi_\epsilon \in C^\infty_0(\tilde{\omega})$ be cut-off functions such that $\chi_\epsilon \equiv 1$ on $\overline{\omega}$ for every ϵ and χ_ϵ converges to the characteristic function of ω as $\epsilon \to 0$. Applying the homotopy formula proved in Theorem 11.3.7 to $\chi_\epsilon \tilde{f}$, we have for $z \in \omega$,

$$\begin{aligned} f(z) = \chi_\epsilon \tilde{f} &= \bar{\partial}_b \int_{bD} \Omega^{-+}_{q-1} \wedge \chi_\epsilon \tilde{f} + \tau \int_{bD} \Omega^{-+}_q \wedge \bar{\partial}_b(\chi_\epsilon \tilde{f}) \\ &= \bar{\partial}_b \int_{bD} \Omega^{-+}_{q-1} \wedge \chi_\epsilon \tilde{f} + \tau \int_{bD} \Omega^{-+}_q \wedge \chi_\epsilon \bar{\partial}_b \tilde{f} \\ &\quad + \tau \int_{\tilde{\omega}\setminus\omega} \Omega^{-+}_q \wedge (\bar{\partial}_b \chi_\epsilon) \wedge \tilde{f}. \end{aligned}$$

For $z \in \omega$, we note that Ω^{-+} is smooth for $\zeta \in (\overline{\tilde{\omega}\setminus\omega})$ and we can apply Stokes' theorem to the third term on the right-hand side to obtain

$$\begin{aligned} &\lim_{\epsilon\to 0} \int_{\tilde{\omega}\setminus\omega} \Omega^{-+}_q \wedge \bar{\partial}_b \chi_\epsilon \wedge \tilde{f} = \lim_{\epsilon\to 0} \int_{\tilde{\omega}\setminus\omega} \Omega^{-+}_q \wedge d\chi_\epsilon \wedge \tilde{f} \\ &= \lim_{\epsilon\to 0} \int_{\tilde{\omega}\setminus\omega} d(\chi_\epsilon \Omega^{-+}_q \wedge \tilde{f}) - \lim_{\epsilon\to 0} \int_{\tilde{\omega}\setminus\omega} \chi_\epsilon d(\Omega^{-+}_q \wedge \tilde{f}) \\ &= -\int_{b\omega} \Omega^{-+}_q \wedge f. \end{aligned}$$

Thus for any $z \in \omega$, letting $\epsilon \to 0$, since Ω^{-+} is an absolutely integrable kernel by Lemma 11.3.6 , we have

$$\begin{aligned} f(z) &= \lim_{\epsilon\to 0} \left(\bar{\partial}_b \int_{bD} \Omega^{-+}_{q-1} \wedge \chi_\epsilon \tilde{f} + \tau \int_{bD} \Omega^{-+}_q \wedge \chi_\epsilon \bar{\partial}_b \tilde{f} \right) \\ &\quad + \lim_{\epsilon\to 0} \tau \int_{\tilde{\omega}\setminus\omega} \Omega^{-+}_q \wedge (\bar{\partial}_b \chi_\epsilon) \wedge \tilde{f} \\ &= \bar{\partial}_b \int_\omega \Omega^{-+}_{q-1} \wedge f + \tau \int_\omega \Omega^{-+}_q \wedge \bar{\partial}_b f - \tau \int_{b\omega} \Omega^{-+}_q \wedge f. \end{aligned}$$

This proves the proposition.

Proof of Theorem 11.4.1. We define

$$\Omega^{\flat -} = \Omega(G^\flat, G^-) \qquad \text{and} \qquad \Omega^{\flat +} = \Omega(G^\flat, G^+),$$

where G^- and G^+ are defined by (11.3.10) and (11.3.11) respectively. The kernels $\Omega^{\flat -}(\zeta, z)$ and $\Omega^{\flat +}(\zeta, z)$ are smooth for $(\zeta, z) \in b\omega \times \omega$. Using (11.1.4-iii), we have

$$\overline{\partial}_{\zeta,z} \Omega^{\flat -+} = -\Omega^{-+} + \Omega^{\flat +} - \Omega^{\flat -} \tag{11.4.13}$$

for any $\zeta \in b\omega$ and $z \in \omega$. Applying Proposition 11.4.3 and (11.4.13), we obtain for $z \in \omega$,

$$\begin{aligned} \alpha(z) &= \overline{\partial}_b \int_\omega \Omega^{-+}_{q-1}(\zeta, z) \wedge \alpha(\zeta) + \tau \int_\omega \Omega^{-+}_q(\zeta, z) \wedge \bar{\partial}_b \alpha(\zeta) \\ &\quad + \tau \int_{b\omega} \overline{\partial}_\zeta \Omega^{\flat -+}_q(\zeta, z) \wedge \alpha(\zeta) + \overline{\partial}_z \int_{b\omega} \Omega^{\flat -+}_{q-1}(\zeta, z) \wedge \alpha(\zeta) \\ &\quad - \tau \int_{b\omega} \Omega^{\flat +}_q(\zeta, z) \wedge \alpha(\zeta) + \tau \int_{b\omega} \Omega^{\flat -}_q(\zeta, z) \wedge \alpha(\zeta). \end{aligned} \tag{11.4.14}$$

We claim that for any $\alpha \in C^1_{(0,q)}(\overline{\omega})$, the following three equalities hold for $z \in \omega$:

$$\begin{aligned} &\text{(i)} \int_{b\omega} \overline{\partial}_\zeta \Omega^{\flat -+}_q(\zeta, z) \wedge \alpha(\zeta) = \int_{b\omega} \Omega^{\flat -+}_q(\zeta, z) \wedge \bar{\partial}_b \alpha(\zeta), \\ &\text{(ii)} \int_{b\omega} \Omega^{\flat -}_q(\zeta, z) \wedge \alpha(\zeta) = 0, \qquad 1 \le q \le n-2, \\ &\text{(iii)} \int_{b\omega} \Omega^{\flat +}_q(\zeta, z) \wedge \alpha(\zeta) = 0, \qquad 1 \le q < n-2. \end{aligned}$$

Since the kernel $\Omega^{\flat -+}(\zeta, z)$ has the factor $d\zeta_1 \wedge \cdots \wedge d\zeta_n$, applying Stokes' theorem, we have

$$\begin{aligned} \int_{b\omega} \overline{\partial}_\zeta \Omega^{\flat -+}(\zeta, z) \wedge \alpha(\zeta) &= \int_{b\omega} d_\zeta (\Omega^{\flat -+}(\zeta, z) \wedge \alpha(\zeta)) \\ &\quad + \int_{b\omega} \Omega^{\flat -+}(\zeta, z) \wedge \bar{\partial}_b \alpha(\zeta), \end{aligned}$$

which proves (i). Since $w^\flat$ is holomorphic in both the ζ and z variables, for any $\zeta \in b\omega$ and $z \in \omega$, we have

$$\Omega^{\flat -} = \omega^\flat \wedge \omega^- \wedge (\overline{\partial}_\zeta \omega^-)^{n-2}, \tag{11.4.15}$$

$$\Omega^{\flat +} = \omega^\flat \wedge \omega^+ \wedge (\overline{\partial}_z \omega^+)^{n-2}. \tag{11.4.16}$$

(ii) follows from (11.4.15) and the fact that integration of an $(n, n-2+q)$-form on $b\omega$ is zero. Similarly when $1 \leq q < n-2$, (iii) follows from type consideration since each component in (11.4.16) has $(n-2)$ $d\overline{z}$'s and no $d\overline{\zeta}$'s. Substituting (i), (ii) and (iii) into (11.4.14), we have proved Theorem 11.4.1.

Proof of Theorem 11.4.2. When $q < n-2$, Theorem 11.4.1 implies Theorem 11.4.2 if $\alpha \in C^1_{(0,q)}(\overline{\omega})$. If α is only in $C_{(0,q)}(\overline{\omega})$, we approximate α by a sequence $\alpha_n \in C^\infty_{(0,q)}(\overline{\omega})$ such that $\alpha_n \to \alpha$ and $\bar\partial_b \alpha_n \to 0$ uniformly on $\overline{\omega}$. This is possible from the proof of Friedrichs' lemma (see Appendix D). It is easy to see that $S_{q+1}\bar\partial_b \alpha_n \to 0$ in the distribution sense in ω and $S_q \alpha_n \to S_q \alpha$ uniformly on compact subset of ω. Thus $u = S_q \alpha$ is in $C_{(0,q-1)}(\omega)$ and $\overline{\partial}_b u = \alpha$ in the distribution sense in ω.

To show that the theorem holds when $q = n-2$, we use (11.4.14) to obtain

$$\alpha = \bar\partial_b S_{n-2}\alpha - \int_{b\omega} \Omega^{\flat +}_{n-2}(\zeta, z) \wedge \alpha(\zeta), \qquad z \in \omega. \tag{11.4.17}$$

To show that the last integral in (11.4.17) vanishes when $q = n-2$, notice that the kernel $\Omega^{\flat +}(\zeta, z)$ is holomorphic in ζ in a neighborhood of $b\omega$ for each fixed $z \in \omega$. Thus from our assumption on α, we have for $z \in \omega$,

$$\int_{b\omega} \Omega^{\flat +}_{n-2}(\zeta, z) \wedge \alpha(\zeta) = 0.$$

This proves the theorem.

In general, the additional assumption (11.4.2 a) on α when $q = n-2$ is necessary. The next proposition characterizes all domains ω such that condition (11.4.2) will imply condition (11.4.2 a). At the end of this section we shall give an example of a $\bar\partial_b$-closed form which does not satisfy condition (11.4.2 a).

Proposition 11.4.4. *Suppose that $\mathcal{O}(\bar\omega)$ is dense in $\mathcal{O}(b\omega)$ (in the $C(b\omega)$ norm). Then any $(0, n-2)$-form $\alpha \in C_{(0,n-2)}(\overline{\omega})$ satisfying condition (11.4.2) also satisfies condition (11.4.2 a). In particular, if polynomials are dense in $\mathcal{O}(b\omega)$, then condition (11.4.2) implies condition (11.4.2 a).*

Proof. From the assumption, for any $g \in \mathcal{O}(b\omega)$, there exists a sequence of holomorphic functions $g_n \in \mathcal{O}(\bar\omega)$ such that g_n converges to g in $C(b\omega)$. We have, for any $\bar\partial_b$-closed α,

$$\begin{aligned}
\int_{b\omega} \alpha \wedge g \wedge dz_1 \wedge \cdots \wedge dz_n &= \lim_{n\to\infty} \int_{b\omega} \alpha \wedge g_n \wedge dz_1 \wedge \cdots \wedge dz_n \\
&= \lim_{n\to\infty} \int_{\omega} \bar\partial(\alpha \wedge g_n \wedge dz_1 \wedge \cdots \wedge dz_n) \\
&= \lim_{n\to\infty} \int_{\omega} \bar\partial_b \alpha \wedge g_n \wedge dz_1 \wedge \cdots \wedge dz_n \\
&= 0.
\end{aligned}$$

Thus condition (11.4.2) implies condition (11.4.2 a). The proposition is proved.

Using Proposition 11.4.4, we have the following:

Corollary 11.4.5. *Let ω be as in Theorem 11.4.1. We assume that the set $\mathbb{C} \setminus S$, where*

$$S = \{z_n \in \mathbb{C} \mid \ z = (z_1, \cdots, z_n) \in b\omega\},$$

is connected. For any $\alpha \in C_{(0,q)}(\overline{\omega})$, $1 \le q \le n-2$, with $\overline{\partial}_b \alpha = 0$ on ω, the form $u = S_q \alpha$ is in $C_{(0,q-1)}(\omega)$ and $\bar{\partial}_b u = \alpha$ on ω, where S_q is the integral operator defined by (11.4.11).

Proof. Using Theorem 11.4.2, we only need to prove the assertion for $q = n-2$. From (11.4.17), it suffices to show that

$$\int_{b\omega} \Omega^{b+}_{n-2}(\zeta, z) \wedge \alpha(\zeta) = 0, \quad z \in \omega.$$

An approximation argument can be applied using the additional assumption on the set S.

Since the set $\mathbb{C} \setminus S$ is connected by assumption, by the Runge approximation theorem, the function

$$h(\zeta_n) = \frac{1}{\zeta_n - z_n}$$

can be approximated by polynomials $P_\nu(\zeta_n, z_n)$ for each fixed z_n in the sup norm on S. We approximate Ψ by $\Psi_\epsilon(\zeta, z) = \Psi(\zeta, z) + \epsilon$ for some ϵ and let $\epsilon \to 0^+$. Then $\Psi_\epsilon(\zeta, z)$ is smooth when $z \in \omega$ and $\zeta \in \omega$. Also Ψ_ϵ is holomorphic in $\zeta \in \omega$. Define

$$\omega^+_\epsilon = \frac{1}{2\pi i} \frac{< G^+(\zeta), d\zeta >}{< G^+(\zeta), \zeta - z > + \epsilon} = \frac{1}{2\pi i} \frac{< G^+(\zeta), d\zeta >}{\Psi(\zeta, z) + \epsilon}.$$

We can apply Stokes' theorem first to the modified kernel with Ψ substituted by Ψ_ϵ, letting $\epsilon \to 0$, to obtain for $z \in \omega$,

$$\begin{aligned}
&\int_{b\omega} \Omega^{b+}_{n-2}(\zeta, z) \wedge \alpha(\zeta) \\
&\quad = \lim_{\nu \to \infty} \lim_{\epsilon \to 0^+} \int_{b\omega} P_\nu(\zeta_n, z_n) d\zeta_n \wedge \omega^+_\epsilon \wedge (\overline{\partial}_z \omega^+_\epsilon)^{n-2} \wedge \alpha(\zeta) \\
&\quad = \lim_{\nu \to \infty} \lim_{\epsilon \to 0^+} \int_{\omega} \overline{\partial}_\zeta \big(P_\nu(\zeta_n, z_n) d\zeta_n \wedge \omega^+_\epsilon \wedge (\overline{\partial}_z \omega^+_\epsilon)^{n-2} \wedge \alpha(\zeta) \big) \\
&\quad = 0
\end{aligned}$$

since every term in the integrand is $\overline{\partial}_\zeta$-closed. This proves the corollary.

Example. We note that the additional assumption on α or ω when $q = n-2$ cannot be removed. Let $bD = \{z \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$ be the unit sphere in $\mathbb{C}^3$. Let

$$\omega = bD \cap \{z \in \mathbb{C}^3 \mid |z_3|^2 < 1/2\}.$$

Then $S = \{z_3 \in \mathbb{C} \mid |z_3|^2 = 1/2\}$ does not satisfies the hypothesis imposed on S in Corollary 11.4.5 since $\mathbb{C} \setminus S$ is not connected. We shall show that equation $\bar{\partial}_b u = \alpha$ is not solvable for $q = 1$ in ω. Let

$$\alpha = \frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{(|z_1|^2 + |z_2|^2)^2}.$$

Then α is a constant multiple of the Bochner-Martinelli-Koppelman kernel in $\mathbb{C}^2$ and $\bar{\partial}\alpha = 0$ for $|z_1|^2 + |z_2|^2 \neq 0$. Thus $\alpha \in C^\infty_{(0,1)}(\overline{\omega})$ and

$$\bar{\partial}_b \alpha = 0 \qquad \text{on } \omega.$$

If $\alpha = \bar{\partial}_b u$ for some $u \in C(\overline{\omega})$, then α must satisfy

$$\int_{b\omega} \alpha \wedge \frac{1}{iz_3} dz_1 \wedge dz_2 \wedge dz_3 = \int_{b\omega} \alpha \wedge dz_1 \wedge dz_2 \wedge d\theta_3 = 0,$$

where $d\theta_3 = dz_3/(iz_3)$. On the other hand, we have that

$$\begin{aligned}\int_{b\omega} \alpha \wedge dz_1 \wedge dz_2 \wedge d\theta_3 &= 8\pi \int_{\{|z_1|^2+|z_2|^2=\frac{1}{2}\}} (\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1) \wedge dz_1 \wedge dz_2 \\ &= 16\pi \int_{\{|z_1|^2+|z_2|^2<\frac{1}{2}\}} d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_1 \wedge dz_2 \\ &\neq 0.\end{aligned}$$

Thus there does not exist any solution $u \in C(\overline{\omega})$. There does not exist any $u \in C(\omega)$ satisfying $\bar{\partial}_b u = \alpha$ either, by an approximation argument. Thus the assumption on α in Theorem 11.4.2 cannot be removed. We note that $\mathcal{O}(\overline{\omega})$ is not dense in $\mathcal{O}(b\omega)$ here.

On the other hand, if

$$\omega = bD \cap \{z \in \mathbb{C}^n \mid \operatorname{Im} z_3 < 0\},$$

then $S = \{z_3 \in \mathbb{C} \mid -1 < \operatorname{Re} z_3 < 1,\ \operatorname{Im} z_3 = 0\}$ and $\mathbb{C} \setminus S$ is connected. Thus it satisfies the hypothesis imposed in Corollary 11.4.5 and we can solve (11.4.1) for all $\bar{\partial}_b$-closed form $\alpha \in C_{(0,q)}(\overline{\omega})$ when $1 \leq q \leq n-2$.

11.5 L^p Estimates for Local Solutions of $\bar{\partial}_b$

Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary M. we shall study the local solvability of the tangential Cauchy-Riemann equations $\overline{\partial}_b$ near a point z_0 in M. After a quadratic change of coordinates we may assume that $z_0 = 0$ and there exists a strictly plurisubharmonic defining function $\rho(z)$ for M which has the following form near the origin:

$$\rho(z) = -\operatorname{Im} z_n + \sum_{j,k=1}^{n} A_{jk} z_j \overline{z}_k + O(|z|^3), \tag{11.5.1}$$

where (A_{jk}) is a positive definite hermitian matrix (see the proof of Corollary 3.4.5). The function ρ is strictly convex near the origin. Let U be a small neighborhood of 0 and $\delta_0 > 0$ be sufficiently small. We define $\{\omega_\delta\}$ by

$$\omega_\delta = \{z \in M \cap U \mid \mathrm{Im}\ z_n < \delta\}, \quad 0 < \delta < \delta_0. \tag{11.5.2}$$

We can always choose U and $\delta_0 > 0$ sufficiently small such that each ω_δ is an open neighborhood in a connected strictly convex hypersurface whose boundary lies in a flat surface. It is easy to see that $\cap_\delta \omega_\delta = \{0\}$. Thus the $\{\omega_\delta\}$ forms a neighborhood base at 0.

Using Theorem 11.4.2 and Corollary 11.4.5, there is a solution operator $S_q\alpha$ satisfying $\bar\partial_b S_q \alpha = \alpha$ on ω_δ for any $\bar\partial_b$-closed $\alpha \in C_{(0,q)}(\overline{\omega}_\delta)$. Our main goal is to prove that there exists a solution operator satisfying L^p estimates on ω_δ.

Theorem 11.5.1 (L^p existence and estimates for local solutions of $\bar\partial_b$). *Let M be a strongly pseudoconvex hypersurface in $\mathbb{C}^n$ and $z_0 \in M$. There exists a neighborhood base $\{\omega_\delta\}$ of z_0 such that for any $\alpha \in L^p_{(0,q)}(\omega_\delta)$, $1 \le q \le n-2$ and $1 < p < \infty$, satisfying $\overline\partial_b \alpha = 0$, there exists $u \in L^p_{(0,q-1)}(\omega_\delta)$ satisfying $\overline\partial_b u = \alpha$. Furthermore, there exists a positive constant c such that the following estimate holds:*

$$\|u\|_{L^p_{(0,q-1)}(\omega_\delta)} \le c\|\alpha\|_{L^p_{(0,q)}(\omega_\delta)}, \tag{11.5.3}$$

where c depends on p, ω_δ but is independent of α.

Corollary 11.5.2. *Let M and ω_δ be as in Theorem 11.5.1. The range of $\overline\partial_b$ is closed in the $L^p_{(0,q)}(\omega_\delta)$ space, where $1 < p < \infty$ and $1 \le q \le n-2$.*

Corollary 11.5.3. *Let M and ω_δ be as in Theorem 11.5.1. For each $1 \le q < n-2$, there exists a solution operator $\tilde S_q$ given by integral kernels such that for any $\overline\partial_b$-closed $\alpha \in L^p_{(0,q)}(\omega_\delta)$, $1 < p < \infty$, we have $\bar\partial_b \tilde S_q \alpha = \alpha$ and*

$$\|\tilde S_q \alpha\|_{L^p_{(0,q-1)}(\omega_\delta)} \le c\|\alpha\|_{L^p_{(0,q)}(\omega_\delta)},$$

where c depends on p, ω_δ but is independent of α.

The rest of this section is to prove Theorem 11.5.1. Let ω_δ be defined by (11.5.2). To prove Theorem 11.5.1, we first prove the L^p estimates for the solution constructed for $\overline\partial_b$-closed forms with $C(\overline{\omega}_\delta)$ coefficients in Theorem 11.4.2.

Proposition 11.5.4. *Let M be a strongly pseudoconvex CR manifold defined by (11.5.1) and ω_δ be defined by (11.5.2). For any $\alpha \in C_{(0,q)}(\overline{\omega}_\delta)$ such that $\overline\partial_b \alpha = 0$, $1 \le q \le n-2$, there exists a solution $u \in C_{(0,q-1)}(\omega_\delta)$ satisfying $\overline\partial_b u = \alpha$ on ω_δ. Furthermore, for every $1 < p < \infty$, there exists a constant $C_p > 0$ such that*

$$\|u\|_{L^p_{(0,q-1)}(\omega_\delta)} \le C_p\|\alpha\|_{L^p_{(0,q)}(\omega_\delta)}, \tag{11.5.4}$$

where C_p is independent of α and small perturbation of δ.

Proof. Let

$$u(z) \equiv S_q\alpha(z) = I_1(\alpha) + I_2(\alpha), \tag{11.5.5}$$

where

$$I_1(\alpha) = \int_{\omega_\delta} \Omega^{-+}_{q-1}(\zeta, z) \wedge \alpha(\zeta)$$

and

$$I_2(\alpha) = \int_{b\omega_\delta} \Omega^{\flat -+}_{q-1}(\zeta, z) \wedge \alpha(\zeta).$$

Since the set $\mathbb{C}\setminus S$ is connected. it follows from Theorem 11.4.2 and Corollary 11.4.5 that for every $1 \le q \le n-2$, $\bar{\partial}_b u = \alpha$ on ω_δ and $u \in C_{(0,q-1)}(\omega_\delta)$.

To prove Proposition 11.5.4, we only need to prove that u satisfies (11.5.4). Using Theorem 11.3.9, there exists a $C > 0$ such that the integral $I_1(\alpha)$ satisfies

$$\| I_1(\alpha) \|_{L^p(\omega_\delta)} \le C \| \alpha \|_{L^p(\omega_\delta)} . \tag{11.5.6}$$

We only need to estimate $I_2(\alpha)$.

Since $I_2(\alpha)$ is an integral on $b\omega_\delta$, we rewrite $I_2(\alpha)$ to be an integral on ω_δ to facilitate the L^p estimates. Since the kernel $\Omega^{\flat -+}(\zeta, z)$ has singularities at $\zeta_n = z_n$ for any $\zeta, z \in \omega_\delta$, we shall modify the kernel first so that Stokes' theorem can be applied.

Let $r(z) = r(z_n) = \mathrm{Im}\ z_n$. Then for any $\zeta, z \in \overline{\omega}_\delta$,

$$r(z) - r(\zeta) - 2\mathrm{Re}\frac{\partial r(\zeta)}{\partial \zeta_n}(z_n - \zeta_n) = 0. \tag{11.5.7}$$

We set

$$\begin{aligned}\tilde{\eta}(\zeta, z) &= \frac{\partial r(\zeta)}{\partial \zeta_n}(\zeta_n - z_n) - (r(\zeta) - \delta)\\ &= \eta(\zeta, z) - (r(\zeta) - \delta).\end{aligned}$$

It follows from (11.5.7) that

$$\begin{aligned}\mathrm{Re}\ \tilde{\eta}(\zeta, z) &= \frac{1}{2}\Big(-r(\zeta) - r(z)\Big) + \delta\\ &= \frac{1}{2}\Big(-\big(r(\zeta) - \delta\big) - \big(r(z) - \delta\big)\Big)\\ &> 0\end{aligned} \tag{11.5.8}$$

for all $\zeta, z \in \omega_\delta$. Thus Re $\tilde{\eta}(\zeta, z)$ vanishes only when ζ and z are both in $b\omega_\delta$. Also we have

$$\tilde{\eta}(\zeta, z) = \eta(\zeta, z), \quad \text{when } \zeta \in b\omega_\delta \text{ and } z \in \omega_\delta.$$

We define the kernel $\tilde{\Omega}^{\flat -+}(\zeta, z)$ by modifying $\Omega^{\flat -+}$ with $\tilde{\eta}$ substitute for η. Set

$$\tilde{\Omega}^{\flat -+}(\zeta, z) = \sum_{q=0}^{n-3} \tilde{\Omega}^{\flat -+}_q(\zeta, z), \tag{11.5.9}$$

where

$$\begin{aligned}\tilde{\Omega}^{\flat -+}_q(\zeta, z) = \frac{1}{(2\pi i)^n} \frac{\frac{\partial r}{\partial \zeta_n} d\zeta_n}{\tilde{\eta}(\zeta, z)} \wedge \frac{< G^-(\zeta), d\zeta >}{\Phi(\zeta, z)} \wedge \frac{< G^+(z), d\zeta >}{\Psi(\zeta, z)}\\ \wedge \left[\frac{< \overline{\partial}_\zeta G^-(\zeta), d\zeta >}{\Phi(\zeta, z)}\right]^{n-3-q} \wedge \left[\frac{< \overline{\partial}_z G^+(z), d\zeta >}{\Psi(\zeta, z)}\right]^q\end{aligned}$$

away from the singularities. The kernel $\tilde\Omega_q^{\flat-+}$ has exactly q $d\bar z$'s. Since

$$\tilde\Omega^{\flat-+}(\zeta,z)=\Omega^{\flat-+}(\zeta,z),\quad \text{when } \zeta\in b\omega_\delta \text{ and } z\in\omega_\delta, \tag{11.5.10}$$

we shall substitute $\tilde\Omega^{\flat-+}$ in $I_2(\alpha)$ for $\Omega^{\flat-+}$. The advantage is that $\tilde\Omega^{\flat-+}$ is integrable for each fixed $z\in\omega_\delta$ since $\tilde\eta$ satisfies (11.5.8). Thus for any $z\in\omega_\delta$, by Stokes' theorem and a limiting argument (substituting $\Phi_\epsilon=\Phi+\epsilon$ and $\Psi_\epsilon=\Psi+\epsilon$ for Φ and Ψ, approximating α by smooth forms α_ϵ such that $\alpha_\epsilon\to\alpha$ and $\overline\partial_b\alpha_\epsilon\to 0$ uniformly on ω_δ, then letting $\epsilon\searrow 0$), we can write

$$\begin{aligned} I_2(\alpha)(z)&=\int_{\zeta\in b\omega_\delta}\tilde\Omega_{q-1}^{\flat-+}\wedge\alpha(\zeta)=\int_{\zeta\in\omega_\delta}\overline\partial_\zeta\left(\tilde\Omega_{q-1}^{\flat-+}\wedge\alpha(\zeta)\right)\\ &=\int_{\zeta\in\omega_\delta}\overline\partial_\zeta\tilde\Omega_{q-1}^{\flat-+}\wedge\alpha(\zeta).\end{aligned} \tag{11.5.11}$$

From (11.3.28), we have

$$|<G^-(\zeta),d\zeta>\wedge<G^+(z),d\zeta>|=O(|\zeta-z|).$$

Thus for every $1\le q<n-1$,

$$\tilde\Omega_{q-1}^{\flat-+}(\zeta,z)=\frac{O(|\zeta-z|)}{\Phi(\zeta,z)^{n-q-1}\Psi(\zeta,z)^q\tilde\eta(\zeta_n,z_n)}. \tag{11.5.12}$$

We write

$$\begin{aligned}\overline\partial_\zeta\tilde\Omega_{q-1}^{\flat-+}(\zeta,z)=\frac{1}{(2\pi i)^n}\,\overline\partial_\zeta\Bigg[&\frac{\frac{\partial r}{\partial\zeta_n}d\zeta_n}{\tilde\eta(\zeta,z)}\wedge\frac{<G^-,d\zeta>}{\Phi(\zeta,z)}\wedge\frac{<G^+-G^-,d\zeta>}{\Psi(\zeta,z)}\\ &\wedge\left(\frac{<\overline\partial_\zeta G^-,d\zeta>}{\Phi(\zeta,z)}\right)^{n-2-q}\wedge\left(\frac{<\overline\partial_z G^+,d\zeta>}{\Psi(\zeta,z)}\right)^{q-1}\Bigg].\end{aligned}$$

It follows from the definition of Φ and Ψ that

$$\begin{cases}\overline\partial_\zeta\Phi(\zeta,z)=O(|\zeta-z|),\\ \overline\partial_\zeta\Psi(\zeta,z)=0.\end{cases}$$

Using $<G^-,d\zeta>=\partial_\zeta\rho$, $\frac{\partial r}{\partial\zeta_n}d\zeta_n=\partial_\zeta r$ and estimate (11.3.15), after grouping terms of the same form together, we have

$$\begin{aligned}|\overline\partial_\zeta\tilde\Omega_{q-1}^{\flat-+}(\zeta,z)|\le C\sum\Bigg[&\frac{|\zeta-z|}{|\tilde\eta(\zeta,z)||\Phi(\zeta,z)|^{n-1-q}|\Psi(\zeta,z)|^q}\\ &+\frac{|\partial_\zeta\rho\wedge\partial_\zeta r\wedge V_{2n-3}(\zeta)|}{|\tilde\eta(\zeta,z)||\Phi(\zeta,z)|^{n-1-q}|\Psi(\zeta,z)|^q}\\ &+\frac{|\zeta-z||\partial_\zeta\rho\wedge\partial_\zeta r\wedge\overline\partial_\zeta\tilde\eta\wedge V_{2n-4}(\zeta)|}{|\tilde\eta(\zeta,z)|^2|\Phi(\zeta,z)|^{n-1-q}|\Psi(\zeta,z)|^q}\Bigg],\end{aligned}$$

where $\sum$ ranges over all possible monomials $V_{2n-3}(\zeta)$ and $V_{2n-4}(\zeta)$ of degree $2n-3$ and $2n-4$ respectively in $d\zeta_1, d\overline{\zeta}_1, \cdots, d\zeta_n, d\overline{\zeta}_n$. Let

$$K_1(\zeta, z) = \frac{|\zeta - z|}{|\tilde{\eta}(\zeta, z)||\Phi(\zeta, z)|^{n-1-q}|\Psi(\zeta, z)|^q}, \tag{11.5.13}$$

$$K_2(\zeta, z) = \frac{|\partial_\zeta \rho \wedge \partial_\zeta r \wedge V_{2n-3}(\zeta)|}{|\tilde{\eta}(\zeta, z)||\Phi(\zeta, z)|^{n-1-q}|\Psi(\zeta, z)|^q}, \tag{11.5.14}$$

$$K_3(\zeta, z) = \frac{|\zeta - z||\partial_\zeta \rho \wedge \partial_\zeta r \wedge \overline{\partial}_\zeta \tilde{\eta} \wedge V_{2n-4}(\zeta)|}{|\tilde{\eta}(\zeta, z)|^2|\Phi(\zeta, z)|^{n-1-q}|\Psi(\zeta, z)|^q}. \tag{11.5.15}$$

We define

$$J_i(\alpha)(z) = \int_{\zeta \in \omega_\delta} |\alpha(\zeta)| K_i(\zeta, z) dm_{2n-1}(\zeta), \qquad i = 1, 2, 3,$$

where $m_{2n-1}(\zeta)$ is the surface measure of ω_δ. For $z \in \omega_\delta$, using $|\tilde{\eta}(\zeta, z)| > 0$,

$$\int_{\zeta \in \omega_\delta} |K_i(\zeta, z)| dm_{2n-1}(\zeta) \le C_z, \qquad i = 1, 2, 3,$$

where C_z depends on z. Thus the operator J_i is bounded from $L^p(\omega_\delta)$ to $L^p(\omega_\delta, \mathrm{loc})$, $i = 1, 2, 3$. Near the boundary $b\omega_\delta$, the singularities of K_i are not absolutely integrable, but are of Hilbert integral type (see Theorem B.9 in the Appendix). However, we shall show that there exists a constant $c > 0$ such that

$$\|J_i(\alpha)\|_{L^p(\omega_\delta)} \le c\|\alpha\|_{L^p(\omega_\delta)} \qquad i = 1, 2, 3. \tag{11.5.16}$$

Let $r_\delta(\zeta_n) = r(\zeta_n) - \delta$ be the defining function for ω_δ. To prove (11.5.16), we use the following lemma:

Lemma 11.5.5. *If for every $0 < \epsilon < 1$, there exist a constant c_ϵ such that $K_i(\zeta, z)$ satisfies*

$$\int_{\zeta \in \omega_\delta} |r_\delta(\zeta_n)|^{-\epsilon} K_i(\zeta, z) \le c_\epsilon |r_\delta(z_n)|^{-\epsilon} \quad \text{for all } z \in \omega_\delta, \tag{11.5.17}$$

$$\int_{z \in \omega_\delta} |r_\delta(z_n)|^{-\epsilon} K_i(\zeta, z) \le c_\epsilon |r_\delta(\zeta_n)|^{-\epsilon} \quad \text{for all } \zeta \in \omega_\delta, \tag{11.5.18}$$

then for $1 < p < \infty$, there exists $c_p > 0$ such that

$$\|J_i(\alpha)\|_{L^p(\omega_\delta)} \le c_p \|\alpha\|_{L^p(\omega_\delta)}$$

for all $\alpha \in L^p(\omega_\delta)$.

Proof. By Hölder's inequality and (11.5.17), we have

$$\begin{aligned}|J_i(\alpha)(z)|^p &\le \int_{\zeta\in\omega_\delta} K_i(\zeta,z)|\alpha(\zeta)|^p|r_\delta(\zeta_n)|^{\epsilon p/p'}dm_{2n-1}(\zeta)\\ &\quad\cdot\left(\int_{\zeta\in\omega_\delta} K_i(\zeta,z)|r_\delta(\zeta_n)|^{-\epsilon}dm_{2n-1}(\zeta)\right)^{p/p'}\\ &\le (c_\epsilon)^{p/p'}|r_\delta(z_n)|^{-\epsilon p/p'}\int_{\zeta\in\omega_\delta} K_i(\zeta,z)|\alpha(\zeta)|^p|r_\delta(\zeta_n)|^{\epsilon p/p'}dm_{2n-1}(\zeta),\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Integrating with respect to z and interchanging the order of integration we obtain, using (11.5.18),

$$\begin{aligned}&\int_{z\in\omega_\delta}|J_i(\alpha)(z)|^p dm_{2n-1}\\ &\le (c_\epsilon)^{p/p'}\int_{\zeta\in\omega_\delta}\left[\int_{z\in\omega_\delta}|r_\delta(z_n)|^{\frac{-\epsilon p}{p'}}K_i(\zeta,z)dm(z)\right]\cdot|\alpha(\zeta)|^p|r_\delta(\zeta_n)|^{\frac{\epsilon p}{p'}}dm(\zeta)\\ &\le (c_\epsilon)^{p/p'}(c_{\epsilon p/p'})\|\alpha\|^p_{L^p(\omega_\delta)}.\end{aligned}$$

This proves Lemma 11.5.5 with the constant $c_p=(c_\epsilon)^{1/p'}(c_{\epsilon p/p'})^{1/p}$ if one chooses ϵ so small such that $0<\epsilon<1$ and $0<\epsilon p/p'<1$.

Thus, to prove (11.5.16), it suffices to prove (11.5.17) and (11.5.18) for $i=1,2,3$. Using a partition of unity in both ζ and z variables, we can assume that ζ lies in a coordinate patch U and z lies in a coordinate patch W. When $U\cap W=\emptyset$, then $|\Phi(\zeta,z)|>0$ and $|\Psi(\zeta,z)|>0$ for $\zeta\in U$ and $z\in W$ and the estimation will be simpler. We assume U and W are the same coordinate patch and omit the other cases.

Let Σ denote the set of the characteristic points, i.e., points where $\partial\rho\wedge\partial r=0$ on $b\omega_\delta$. We first assume that $U\cap\Sigma=\emptyset$. We shall choose special coordinates for $\omega_\delta\cap U$.

Since $d\rho\wedge dr\ne 0$ on $b\omega_\delta$, we can choose $r(\zeta_n)$ as a coordinate function near $U\cap\omega_\delta$. Since $d_\zeta\Phi(\zeta,z)|_{\zeta=z}=\partial\rho(\zeta)$ and $\partial\rho=-\overline{\partial}\rho$ on ω_δ, it follows that $\partial\rho(\zeta)=\frac{1}{2}(\partial\rho-\overline{\partial}\rho)=id_\zeta\mathrm{Im}\Phi(\zeta,z)|_{\zeta=z}$. Thus,

$$\partial\rho(\zeta)=id_\zeta\mathrm{Im}\Phi(\zeta,z)+O(|\zeta-z|).$$

Similarly for $\zeta\in b\omega_\delta$, we have

$$\partial_\zeta r=id_\zeta\mathrm{Im}\tilde{\eta}(\zeta_n,z_n)+O(|\zeta_n-z_n|).$$

Thus, if $\zeta\in b\omega_\delta\setminus\Sigma$,

$$\begin{aligned}&dr(\zeta_n)\wedge d_\zeta\mathrm{Im}\Phi(\zeta,z)\wedge d_\zeta\mathrm{Im}\tilde{\eta}(\zeta_n,z_n)\wedge d\rho(\zeta)|_{\zeta=z}\\ &\quad=-dr(\zeta_n)\wedge\partial_\zeta\rho(\zeta)\wedge\partial_\zeta r(\zeta_n)\wedge d\rho(\zeta)\\ &\quad=-\overline{\partial}_\zeta r\wedge\partial_\zeta\rho\wedge\partial_\zeta r\wedge\overline{\partial}_\zeta\rho\\ &\quad\ne 0.\end{aligned}$$

Let $\mathrm{Im}\Phi(\zeta,z)=t_1$ and $\mathrm{Im}\tilde{\eta}(\zeta,z)=t_2$. We can choose coordinates $(r(\zeta_n),t_1,\cdots,t_{2n-2})$ with $t_i(z)=0$, $i=1,\cdots,2n-2$. From (11.3.15), (11.3.16) and (11.5.8), there exists $c>0$ with

$$|\Phi(\zeta,z)|\geq c(|t|^2+|t_1|), \tag{11.5.19}$$

$$|\mathrm{Re}\Psi(\zeta,z)|\geq c|t|^2, \tag{11.5.20}$$

$$|\mathrm{Re}\tilde{\eta}(\zeta,z)|\geq c(|r(\zeta)-\delta|+|r(z)-\delta|). \tag{11.5.21}$$

It follows that there exists $C>0$ independent of z such that

$$\begin{aligned}
|K_1(\zeta,z)|&\leq\frac{C}{(|r(\zeta)-\delta|+|r(z)-\delta|)|t|^{2n-3}},\\
|K_2(\zeta,z)|&\leq\frac{C}{(|r(\zeta)-\delta|+|r(z)-\delta|)(|t_1|+|t|^2)^{n-1-q}|t|^{2q}},\\
|K_3(\zeta,z)|&\leq\frac{C}{(|r(\zeta)-\delta|+|r(z)-\delta|+|t_2|)^2(|t_1|+|t|^2)^{n-1-q}|t|^{2q-1}}.
\end{aligned}$$

Estimate (11.5.17) will be proved for $i=1,2,3$ when ζ and z are away from the characteristic points using the following lemma (letting $\mu=|r(\zeta)-\delta|$ and $\sigma=|r(z)-\delta|$).

Lemma 11.5.6. *Let $t=(t_1,\cdots,t_{2n-2})$ and $dt=dt_1dt_2\cdots dt_{2n-2}$. For any $0<\epsilon<1$, $A>0$, there exists $c_\epsilon^i>0$, $i=0,1,2,3$, such that the following inequalities hold: For any $\sigma>0$, $1\leq q\leq n-2$,*

(1) $\displaystyle\int_0^\infty\frac{\mu^{-\epsilon}}{\sigma+\mu}d\mu\leq c_\epsilon^0\sigma^{-\epsilon}$,

(2) $\displaystyle\int_0^\infty\int_{|t|\leq A}\frac{\mu^{-\epsilon}}{(\sigma+\mu)|t|^{2n-3}}dtd\mu\leq c_\epsilon^1\sigma^{-\epsilon}$,

(3) $\displaystyle\int_0^\infty\int_{|t|\leq A}\frac{\mu^{-\epsilon}}{(\sigma+\mu)(|t_1|+|t|^2)^{n-1-q}|t|^{2q}}dtd\mu\leq c_\epsilon^2\sigma^{-\epsilon}$,

(4) $\displaystyle\int_0^\infty\int_{|t|\leq A}\frac{\mu^{-\epsilon}dtd\mu}{(\sigma+\mu+|t_2|)^2(|t_1|+|t|^2)^{n-1-q}|t|^{2q-1}}\leq c_\epsilon^3\sigma^{-\epsilon}$.

Proof. (1) follows from a change of variables to the case when $\sigma=1$. In fact one can show using contour integration that $c_\epsilon^0=\pi/\sin\pi\epsilon$.

Estimate (2) follows from (1) by using polar coordinates for t variables. To prove (4), we integrate t_1,t_2 first and then use polar coordinates for $t''=(t_3,\cdots,t_{2n-2})$ and apply (1) to obtain

$$\begin{aligned}
&\int_0^\infty\int_{|t|\leq A}\frac{\mu^{-\epsilon}}{(\sigma+\mu+|t_2|)^2(|t_1|+|t|^2)^{n-1-q}|t|^{2q-1}}dtd\mu\\
&\leq C\int_0^\infty\int_{|t''|\leq A}\frac{|\log|t''||\mu^{-\epsilon}}{(\sigma+\mu)(|t''|^2)^{n-2-q}|t''|^{2q-1}}dt''d\mu\\
&\leq C\int_0^\infty\left(\int_{0<v\leq A}\frac{|\log v|v^{2n-5}}{v^{2n-5}}dv\right)\frac{\mu^{-\epsilon}}{(\sigma+\mu)}d\mu\\
&\leq c_\epsilon^3\sigma^{-\epsilon}.
\end{aligned}$$

Estimate (3) can be similarly proved.

Thus (11.5.17) is proved when there are no characteristic points. Similarly one can prove (11.5.18) by reversing the roles of Φ and Ψ, ζ and z.

Near any characteristic point $z \in \Sigma$, we cannot choose $r(\zeta_n)$, $\mathrm{Im}\Phi\ (\zeta, z)$ and $\mathrm{Im}\tilde\eta(\zeta, z)$ as coordinates since they are linearly dependent at $\zeta = z$. The kernel K_1 is less singular than K_2 or K_3 and can be estimated as before by choosing $r(\zeta_n)$ and $\mathrm{Im}\ \Phi = t_1$ as coordinates. To estimate (11.5.17) and (11.5.18) when $i = 2, 3$, one observes that at characteristic points, the numerator of each $K_i(\zeta, z)$, $i = 2, 3$ also vanishes. We shall prove (11.5.17) for K_3 and the case for K_2 is similar.

We write

$$g_1(\zeta, z) = \mathrm{Im}\Phi(\zeta, z),$$
$$g_2(\zeta, z) = \mathrm{Im}\tilde\eta(\zeta, z) = -\frac{1}{2}\mathrm{Re}(\zeta_n - z_n).$$

It is easy to see that $d_\zeta \mathrm{Im}\tilde\eta(\zeta, z) \wedge d\mu = \frac{-1}{i}\partial r \wedge \overline{\partial} r$. Since $\partial\rho = (\partial\rho - \bar\partial\rho)/2 = i d_\zeta \mathrm{Im}\Phi(\zeta, z) + O(|\zeta - z|)$, we have, setting $\mu(\zeta_n) = \delta - r(\zeta_n)$,

$$\begin{aligned} dg_1(\zeta, z) \wedge dg_2(\zeta, z) \wedge d\mu(\zeta)\big|_{\zeta=z} &= -d_\zeta \mathrm{Im}\Phi(\zeta, z) \wedge d_\zeta \mathrm{Im}\tilde\eta(\zeta, z) \wedge dr(\zeta)\big|_{\zeta=z} \\ &= \partial_\zeta \rho \wedge \partial r(\zeta) \wedge \overline{\partial} r(\zeta)\big|_{\zeta=z}. \end{aligned}$$

Thus (11.5.17) will be proved for $i = 3$ if we can prove that

$$\tilde J_3(z) \le C\sigma^{-\epsilon}, \tag{11.5.22}$$

where $\tilde J_3(z)$ is the integral

$$\int_{\zeta \in U \cap \omega_\delta} \frac{\mu^{-\epsilon}|dg_1(\zeta, z) \wedge dg_2(\zeta, z) \wedge d\mu \wedge V_{2n-4}(\zeta)|}{(|g_2(\zeta, z)| + \mu + \sigma)^2 (|g_1(\zeta, z)| + |\zeta - z|^2)^{n-1-q} |\zeta - z|^{2q-1}}.$$

The other terms are less singular and can be estimated as before.

Let $x = (x_1, \cdots, x_{2n-2}, \mu) = (x', \mu)$ be real coordinates on $U \cap \omega_\delta$ such that $z = (0, \cdots, 0, \mu(z))$ where $x_i = \mathrm{Re}(\zeta_j - z_j)$ or $x_i = \mathrm{Im}(\zeta_j - z_j)$ for some $j = 1, \cdots, n$. In this coordinate system, we have, for some $A > 0$, $\tilde J_3$ is bounded by the integral

$$\tilde{\mathcal{J}}_3(z) = \int_{|x|<A} \frac{\mu^{-\epsilon}|dg_1(x, z) \wedge dg_2(x, z) \wedge d\mu \wedge V_{2n-4}(x)|}{(|g_2(x, z)| + \mu + \sigma)^2 (|g_1(x, z)| + |x|^2)^{n-1-q} |x|^{2q-1}},$$

where V_{2n-4} is a monomial of degree $2n-4$ in $dx_1, \cdots, dx_{2n-2}$. Without loss of generality, we can assume that $V_{2n-4} = dx_3 \wedge \cdots \wedge dx_{2n-2}$. Let $B_A = \{x \in \mathbb{R}^{2n-1} \mid |x| < A\}$. The integral $\tilde{\mathcal{J}}_3(z)$ is the pull-back of the integral $\mathcal{I}$,

$$\mathcal{I} = \int_{t=(t', \mu) \in G(B_A)} \frac{\mu^{-\epsilon} dt_1 \wedge dt_2 \cdots \wedge dt_{2n-2} d\mu}{(\sigma + \mu + |t_2|)^2 (|t_1| + |t|^2)^{n-1-q} |t|^{2q-1}},$$

by the map

$$G : x \in B_A \to G(x) = (g_1, g_2, x_3, \cdots, x_{2n-2}, \mu).$$

$\mathcal{I}$ can be estimated by

$$\begin{aligned}\mathcal{I} &= \int_{t=(t',\mu)\in G(B_A)} \frac{\mu^{-\epsilon}dt_1\wedge dt_2\cdots\wedge dt_{2n-2}d\mu}{(\sigma+\mu+|t_2|)^2(|t_1|+|t|^2)^{n-1-q}|t|^{2q-1}}\\ &\le \int_0^\infty\int_{|t'|\le A'} \frac{\mu^{-\epsilon}dt_1\wedge dt_2\cdots\wedge dt_{2n-2}d\mu}{(\sigma+\mu+|t_2|)^2(|t_1|+|t|^2)^{n-1-q}|t|^{2q-1}} \le c_\epsilon^3\sigma^{-\epsilon},\end{aligned}$$

using (4) in Lemma 11.5.6. However, the set $\{(g_1, g_2, x_3, \cdots, x_{2n-2},\ \mu)||x| < A\}$ may cover the image $G(B_A)$ infinitely many times. We have to modify the function g_1 to guarantee a finite covering. We approximate $g_1(x,z)$ by the polynomial $\tilde{g}_1(x,z)$ where $\tilde{g}_1(x,z)$ is the second-order Taylor polynomial in x of $g_1(x,z)$ at the point z. From (11.3.15) we have for $|x|$ sufficiently small,

$$|\Phi(x,z)| \ge c(|\tilde{g}_1(x,z)|+|x|^2) \tag{11.5.23}$$

where c is independent of z.

For each fixed $a_1\in\mathbb{R}$ and fixed $x_2, x_3, \cdots, x_{2n-2}, \mu > 0$, the equation $g_2(x,z) = a_1$ has at most two solutions by the strict convexity of the defining function ρ for ω_δ. For each fixed $a\in G(B_A)$, $(\tilde{g}_1, g_2, x_3, \cdots, x_{2n-2}, \mu) = a$ has at most four solutions from Bezout's theorem (see [GrHa 1]) since $\tilde{g}_1$ is a second-order polynomial in x. With (11.5.23), one can estimate (11.5.22) by substituting $g_1(x,z)$ by $\tilde{g}_1(x,z)$ plus remaining terms which are less singular. We have

$$\int_{|x|<A} \frac{\mu^{-\epsilon}|d\tilde{g}_1(x,z)\wedge dg_2(x,z)\wedge d\mu\wedge V_{2n-4}(x)|}{(|g_2(x,z)|+\mu+\sigma)^2(|\tilde{g}_1(x,z)|+|x|^2)^{n-1-q}|x|^{2q-1}} \le C\mathcal{I} \le C_\epsilon\sigma^{-\epsilon}$$

for any $0<\epsilon<1$. This proves (11.5.22). Reversing the roles of Φ and Ψ, and ζ and z, one can show (11.5.18) for $i=3$ similarly. This proves (11.5.16) for $i=3$ and

$$\| I_2(\alpha) \|_{L^p(\omega_\delta)} \le C \| \alpha \|_{L^p(\omega_\delta)}\ . \tag{11.5.24}$$

Combining (11.5.6) and (11.5.24), (11.5.4) follows with $u = S_q\alpha$. We have proved Proposition 11.5.4.

In order to prove Theorem 11.5.1, we need the following density lemma:

Lemma 11.5.7. *Under the same assumption as in Theorem 11.5.1, the set of $\overline{\partial}_b$-closed forms in $C_{(0,q)}(\overline{\omega}_\delta)$ is dense in the set of $\overline{\partial}_b$-closed $L^p_{(0,q)}(\omega_\delta)$ forms in the $L^p_{(0,q)}(\omega_\delta)$ norm where $1<p<\infty$, $0\le q<n-2$.*

Proof. Let $\alpha\in L^p_{(0,q)}(\omega_\delta)$ and $\overline{\partial}_b\alpha = 0$ on ω_δ. We approximate α by smooth $(0,q)$-forms $\alpha_k\in C^\infty_{(0,q)}(\overline{\omega}_\delta)$ such that $\alpha_k\to\alpha$ in $L^p_{(0,q)}(\omega_\delta)$ and $\overline{\partial}_b\alpha_k\to 0$ in $L^p_{(0,q+1)}(\omega_\delta)$. This is possible by Friedrichs' Lemma (see Appendix D). Since $\overline{\partial}_b\alpha_k$ is a smooth $\overline{\partial}_b$-closed form on a slightly larger set $\omega_{\delta_k}\supset\omega_\delta$ where $\delta_k\searrow\delta$ and δ_k is sufficiently close to δ, we can apply Proposition 11.5.4 to $\overline{\partial}_b\alpha_k$ on ω_{δ_k} (since $1\le q+1<n-1$) to find $(0,q)$-forms v_k such that

$$\begin{cases}\overline{\partial}_b v_k = \overline{\partial}_b\alpha_k & \text{on } \omega_{\delta_k},\\ \|v_k\|_{L^p_{(0,q)}(\omega_{\delta_k})} \le c_p\|\overline{\partial}_b\alpha_k\|_{L^p_{(0,q+1)}(\omega_{\delta_k})},\end{cases} \tag{11.5.25}$$

where c_p is a constant independent of k and δ. This is true since the constant proved in Proposition 11.5.4 is independent of small perturbation of δ. We set

$$\alpha'_k = \alpha_k - v_k, \tag{11.5.26}$$

then $\alpha'_k \in C_{(0,q)}(\overline{\omega}_\delta)$. It follows from (11.5.25) that α'_k is $\overline{\partial}_b$-closed and $\alpha'_k \to \alpha$ in $L^p_{(0,q)}(\omega_\delta)$. This proves the lemma.

For the case when $q = n-2$, we have the following density lemma:

Lemma 11.5.8. *For every $\overline{\partial}_b$-closed form $\alpha \in L^p_{(0,n-2)}(\omega_\delta)$, there exists a sequence of $\overline{\partial}_b$-closed forms $\{\alpha_k\}$ such that $\alpha_k \in C^\infty_{(0,n-2)}(\overline{\omega}_\delta)$ and α_k converges to α in $L^p_{(0,n-2)}(\omega_\delta, loc)$, $1 < p < \infty$.*

Proof. Let B denote the space of all $\overline{\partial}_b$-closed $L^p_{(0,n-2)}(\omega_\delta)$ forms. We note that the dual of $L^p(\omega_\delta, \mathrm{loc})$ in the Fréchet norm is the space of compactly supported functions in $L^{p'}(\omega_\delta)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Consider the linear functional L on $L^p_{(0,n-2)}(\omega_\delta, \mathrm{loc})$ defined by

$$L(\beta) = \int_{\omega_\delta} \beta \wedge g \quad \text{for} \quad \beta \in B \cap L^p_{(0,n-2)}(\omega_\delta, \mathrm{loc}), \tag{11.5.27}$$

where $g \in L^{p'}(\omega_\delta)$ such that g has compact support in ω_δ. We assume that $L(\beta) = \int_{\omega_\delta} \beta \wedge g = 0$ for every $\beta \in B \cap C^\infty_{(0,n-2)}(\overline{\omega}_\delta)$. From the Hahn-Banach theorem, the lemma will be proved if one can show that

$$L(\beta) = 0 \quad \text{ for every } \beta \in B.$$

Let D be a strictly convex set in $\mathbb{C}^n$ such that the boundary of D, denoted by M, contains ω_δ. Let $K = \operatorname{supp} g \subset\subset \omega_\delta$. Since (11.5.27) holds for all $\beta = \overline{\partial}_b v$ for any $v \in C^\infty_{(0,n-3)}(\overline{\omega}_\delta)$, $\overline{\partial}_b g = 0$ on ω_δ in the distribution sense. We extend g to be zero on $M \setminus \omega_\delta$, then $\overline{\partial}_b g = 0$ on M in the distribution sense. Applying Theorem 11.3.9 for $(n,1)$ forms with $L^{p'}(M)$ coefficients on M, we can find $u = R_q f \in L^{p'}_{(n,0)}(M)$ such that $\overline{\partial}_b u = g$ on M. It follows from (11.3.27) that $u \in C^\infty_{(n,0)}(M \setminus K)$. Let $0 < \delta_0 < \delta_1 < \delta$ be chosen such that $K \subset \omega_{\delta_0} \subset \omega_{\delta_1} \subset \overline{\omega}_{\delta_1} \subset \omega_\delta$ and let χ be a cut-off function such that $\chi \in C^\infty_0(\omega_{\delta_1})$ and $\chi \equiv 1$ on K. We set $u_1 = \chi u$ and $u_2 = (1-\chi)u$, then u_1, $\overline{\partial}_b u_1$ and $\overline{\partial}_b u_2$ have compact support. Thus, we can write

$$\int_{\omega_\delta} \alpha \wedge g = \int_{\omega_\delta} \alpha \wedge \overline{\partial}_b u = \int_{\omega_\delta} \alpha \wedge \overline{\partial}_b u_1 + \int_{\omega_\delta} \alpha \wedge \overline{\partial}_b u_2.$$

We shall prove that for every $\alpha \in B$,

$$\int_{\omega_\delta} \alpha \wedge \overline{\partial}_b u_1 = 0 \tag{11.5.28}$$

$$\int_{\omega_\delta} \alpha \wedge \overline{\partial}_b u_2 = 0. \tag{11.5.29}$$

Since u_1 has compact support, we regularize u_1 and (11.5.28) follows easily from Friedrichs' lemma and integration by parts.

To prove (11.5.29), we note that the coefficients of u_2 are CR functions on $M\setminus\omega_{\delta_1}$ since $\overline{\partial}_b u_2 = g = 0$ on $M\setminus\omega_{\delta_1}$. It follows from Theorem 3.3.2 (Lewy's extension) that one can extend u_2 holomorphically into the set $D_{\delta_1} = D\cap\{z\in\mathbb{C}^n \mid r(z_n) > \delta_1\}$. The set D_{δ_1} is convex. We can approximate $u_2 \in C^\infty_{(n,0)}(\overline{D}_{\delta_1})$ by $(n,0)$-forms P_n with polynomial coefficients and the convergence is in $C^\infty(M\setminus\omega_{\delta_1})$. Let χ_1 be a cut-off function such that $\chi_1\in C_0^\infty(\omega_\delta)$ and $\chi_1\equiv 1$ on $\overline{\omega}_{\delta_1}$. Since $\overline{\partial}_b u_2$ is supported on $\omega_{\delta_1}\setminus K$, we have

$$
\begin{aligned}
\int_{\omega_\delta}\alpha\wedge\overline{\partial}_b u_2 &= \int_{\omega_\delta}\alpha\wedge\chi_1\overline{\partial}_b(u_2-P_n) \\
&= \int_{\omega_\delta}\alpha\wedge\overline{\partial}_b(\chi_1(u_2-P_n)) - \int_{\omega_\delta}\alpha\wedge(\overline{\partial}_b\chi_1)\wedge(u_2-P_n).
\end{aligned}
\tag{11.5.30}
$$

The first term on the right-hand side of (11.5.30) vanishes from the same arguments of (11.5.28). Thus

$$
\int_{\omega_\delta}\alpha\wedge\overline{\partial}_b u_2 = -\int_{\omega_\delta}\alpha\wedge(\overline{\partial}_b\chi_1)\wedge(u_2-P_n)\to 0
$$

as $n\to\infty$, since u_2-P_n converges to 0 uniformly on the $\mathrm{supp}(\overline{\partial}_b\chi_1)$. This proves (11.5.29) and Lemma 11.5.8.

Proof of Theorem 11.5.1. Theorem 11.5.1 can be proved for any $\overline{\partial}_b$-closed α with $L^p(\omega_\delta)$ coefficients using an approximation argument. We first assume $1\le q<n-2$.

Using Lemma 11.5.7, there exists a sequence of $\bar{\partial}_b$-closed forms $\alpha'_m\in C_{(0,q)}(\overline{\omega}_\delta)$ such that $\alpha'_m\to\alpha$ in $L^p_{(0,q)}(\omega_\delta)$. We can apply Proposition 11.5.4 to α'_m to find $(0,q-1)$-form u_m such that

$$
\overline{\partial}_b u_m = \alpha'_m \quad \text{on } \omega_\delta, \tag{11.5.31}
$$

and

$$
\|u_m\|_{L^p_{(0,q-1)}(\omega_\delta)} \le c_p\|\alpha'_m\|_{L^p_{(0,q)}(\omega_\delta)}. \tag{11.5.32}
$$

From (11.5.31) and (11.5.32), u_m must converge to some $(0,q-1)$-form u such that u satisfies $\bar{\partial}_b u=\alpha$ on ω_δ and

$$
\|u\|_{L^p_{(0,q-1)}(\omega_\delta)} \le c_p\|\alpha\|_{L^p_{(0,q)}(\omega_\delta)}. \tag{11.5.33}
$$

Theorem 11.5.1 is proved for $1\le q<n-2$.

When $q=n-2$, from Lemma 11.5.8, there exists $\alpha_k\in C^\infty_{(0,n-2)}(\overline{\omega}_\delta)$ such that $\overline{\partial}_b\alpha_k=0$ on ω_δ and $\alpha_k\to\alpha$ in $L^p_{(0,n-2)}(\omega_{\delta'})$ for any $0<\delta'<\delta$. Let δ^m be an increasing sequence such that $0<\delta^m\nearrow\delta$. Applying Proposition 11.5.4 to α_k on ω_{δ^m}, there exists a solution $u_k^m\in C_{(0,n-3)}(\omega_{\delta^m})$ such that

$$
\overline{\partial}_b u_k^m = \alpha_k \quad \text{on } \omega_{\delta^m} \tag{11.5.34}
$$

and

$$\|u_k^m\|_{L^p_{(0,n-3)}(\omega_{\delta^m})} \le C\|\alpha_k\|_{L^p_{(0,n-2)}(\omega_{\delta^m})}, \tag{11.5.35}$$

where C is independent of m and k. It follows that u_k^m converges strongly to an element $u^m \in L^p_{(0,n-3)}(\omega_{\delta^m})$ for every m and

$$\overline{\partial}_b u^m = \alpha \quad \text{on } \omega_{\delta^m}. \tag{11.5.36}$$

Furthermore, we have

$$\|u^m\|_{L^p_{(0,n-3)}(\omega_{\delta^m})} \le C\|\alpha\|_{L^p_{(0,n-2)}(\omega_\delta)}, \tag{11.5.37}$$

where C is independent of m. There exists a subsequence of u^m which converges weakly in $L^p_{(0,n-3)}(\omega_\delta)$ to a limit $u \in L^p_{(0,n-3)}(\omega_\delta)$. It follows from (11.5.36) that $\overline{\partial}_b u = \alpha$ on ω_δ in the distribution sense. From Fatou's lemma and (11.5.37) we have

$$\|u\|_{L^p_{(0,n-3)}(\omega_\delta)} \le C\|\alpha\|_{L^p_{(0,n-2)}(\omega_\delta)}.$$

Theorem 11.5.1 is proved for $q = n-2$. This proves Theorem 11.5.1.

Corollary 11.5.2 follows easily from Theorem 11.5.1. To prove Corollary 11.5.3, we define

$$\tilde{S}_q\alpha = I_1(\alpha) + \tilde{I}_2(\alpha)$$

where

$$I_1(\alpha) = \int_{\omega_\delta} \Omega^{-+}_{q-1}(\zeta, z) \wedge \alpha(\zeta),$$

and

$$\tilde{I}_2(\alpha) = \int_{\zeta\in\omega_\delta} \overline{\partial}_\zeta \tilde{\Omega}^{\flat-+}_{q-1} \wedge \alpha(\zeta).$$

Corollary 11.5.3 follows from the proof of Proposition 11.5.4 and Lemma 11.5.7.

11.6 The $\bar{\partial}_b$-Neumann Problem

Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary bD. Let ω_δ be an open connected subset in bD with smooth boundary $b\omega_\delta$ defined in (11.5.2). The L^2 existence theory for $\bar{\partial}_b$ can be applied to obtain the Hodge theorem for $\bar{\partial}_b$ on ω_δ. We shall set up the $\overline{\partial}_b$-Neumann problem along the lines of the $\overline{\partial}$-Neumann problem for pseudoconvex complex manifolds. Let $\overline{\partial}_b$ be the linear, closed, densely defined operator

$$\overline{\partial}_b : L^2_{(0,q-1)}(\omega_\delta) \to L^2_{(0,q)}(\omega_\delta).$$

The formal adjoint of $\overline{\partial}_b$ is denoted by ϑ_b and defined on smooth $(0,q)$-forms by the requirement that $(\vartheta_b\phi, \psi) = (\phi, \overline{\partial}_b\psi)$ for all smooth ψ with compact support in ω_δ. The Hilbert space adjoint of $\overline{\partial}_b$, denoted by $\overline{\partial}_b^*$, is a linear, closed, densely defined operator defined on $\mathrm{Dom}(\overline{\partial}_b^*) \subset L^2_{(0,q)}(\omega_\delta)$. An element ϕ belongs to $\mathrm{Dom}(\overline{\partial}_b^*)$ if there exists a $g \in L^2_{(0,q-1)}(\omega_\delta)$ such that for every $\psi \in \mathrm{Dom}(\overline{\partial}_b) \cap L^2_{(0,q-1)}(\omega_\delta)$, we have $(\phi, \overline{\partial}_b \psi) = (g, \psi)$. We then define $\overline{\partial}_b^* \phi = g$. We have the following description of the smooth forms in $\mathrm{Dom}(\overline{\partial}_b^*)$:

For all $\phi \in \mathcal{E}^{(0,q)}(\overline{\omega}_\delta)$, $\psi \in \mathcal{E}^{(0,q-1)}(\overline{\omega}_\delta)$, integration by parts gives

$$(\vartheta_b \phi, \psi) = (\phi, \overline{\partial}_b \psi) + \int_{b\omega_\delta} \langle \sigma(\vartheta_b, dr)\phi, \psi \rangle ds, \tag{11.6.1}$$

where ds is the surface measure of $b\omega_\delta$ and $\sigma(\vartheta_b, dr)$ denote the symbol of ϑ_b in the dr direction. More explicitly, for every $x \in b\omega_\delta$, $\sigma(\vartheta_b, dr)\phi\big|_x = \vartheta_b(r\phi)\big|_x$. The following characterization of $\mathrm{Dom}(\overline{\partial}_b^*) \cap \mathcal{E}^{(0,q)}(\overline{\omega}_\delta)$ uses arguments similar to those in Lemma 4.2.1:

Proposition 11.6.1. *$\phi \in Dom(\overline{\partial}_b^*) \cap \mathcal{E}^{(0,q)}(\overline{\omega}_\delta)$ if and only if $\sigma(\vartheta_b, dr)\ \phi = 0$ on $b\omega_\delta$. If $\phi \in Dom(\overline{\partial}_b^*) \cap \mathcal{E}^{(0,q)}(\overline{\omega}_\delta)$, $\overline{\partial}_b^* \phi = \vartheta_b \phi$.*

We next define the $\bar{\partial}_b$-Laplacian $\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$ from $L^2_{(0,q)}(\omega_\delta)$ to $L^2_{(0,q)}(\omega_\delta)$ such that $\mathrm{Dom}(\Box_b) = \{f \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*);\ \bar{\partial}_b f \in \mathrm{Dom}(\bar{\partial}_b^*)$ and $\bar{\partial}_b^* f \in \mathrm{Dom}(\bar{\partial}_b)\}$. Repeating the proof of Proposition 4.2.3, we have the following proposition:

Proposition 11.6.2. *$\Box_b$ is a linear, closed, densely defined self-adjoint operator.*

We note that the smooth forms in $\mathrm{Dom}(\Box_b)$ must satisfy two boundary conditions, namely, the $\overline{\partial}_b$-Neumann boundary conditions. To be precise, we have the following:

Proposition 11.6.3. *$\varphi \in \mathcal{E}^{(0,q)}(\overline{\omega}_\delta)$ is in $Dom(\Box_b)$ if and only if $\sigma(\vartheta_b, dr)\varphi = 0$ on $b\omega_\delta$ and $\sigma(\vartheta_b, dr)\overline{\partial}_b \varphi = 0$ on $b\omega_\delta$.*

The $\bar{\partial}_b$-Neumann problem is formulated in exactly the same way as the $\bar{\partial}$-Neumann problem. However, due to the existence of characteristic points, it is much harder to study this boundary value problem using *a priori* estimates by imitating the $\bar{\partial}$-Neumann problem. By applying the L^2 existence result proved in Theorem 11.5.1, we have the following L^2 existence theorem for the $\bar{\partial}_b$-Neumann operator on ω_δ:

Theorem 11.6.4. *Let M be a strongly pseudoconvex hypersurface in $\mathbb{C}^n$, $n \geq 4$, and $z_0 \in M$. Let $\{\omega_\delta\}$, $\omega_\delta \subset M$, be the neighborhood base of z_0 obtained in Theorem 11.5.1. Then for each fixed δ, $1 \leq q \leq n-3$, there exists a linear operator $\mathcal{N}_\delta$: $L^2_{(0,q)}(\omega_\delta) \to L^2_{(0,q)}(\omega_\delta)$ such that*

(1) *$\mathcal{N}_\delta$ is bounded and $Range(\mathcal{N}_\delta) \subset Dom(\Box_b)$.*
(2) *For any $\alpha \in L^2_{(0,q)}(\omega_\delta)$, $\alpha = \overline{\partial}_b \overline{\partial}_b^* \mathcal{N}_\delta \alpha + \overline{\partial}_b^* \overline{\partial}_b \mathcal{N}_\delta \alpha$.*
(3) *$\mathcal{N}_\delta \Box_b = \Box_b \mathcal{N}_\delta = I$ on $Dom(\Box_b)$;*
$\overline{\partial}_b \mathcal{N}_\delta = \mathcal{N}_\delta \overline{\partial}_b$ on $Dom(\overline{\partial}_b)$, $1 \leq q \leq n-4$;
$\overline{\partial}_b^* \mathcal{N}_\delta = \mathcal{N}_\delta \overline{\partial}_b^*$ on $Dom(\overline{\partial}_b^*)$, $2 \leq q \leq n-3$.
(4) *If $\alpha \in L^2_{(0,q)}(\omega_\delta)$ and $\overline{\partial}_b \alpha = 0$, then $\alpha = \overline{\partial}_b \overline{\partial}_b^* \mathcal{N}_\delta \alpha$. The form $u = \overline{\partial}_b^* \mathcal{N}_\delta \alpha$ gives the canonical solution (i.e., the unique solution which is orthogonal to $Ker(\overline{\partial}_b)$) to the equation $\bar{\partial}_b u = \alpha$.*

Using Theorem 11.5.1 and Corollary 11.5.2, the $\bar{\partial}_b$ operator has closed range in $L^2_{(0,q)}(\omega_\delta)$ when $1 \le q \le n-2$. Theorem 11.6.4 can be proved by repeating the arguments of the proof of Theorem 4.4.1, Thus the L^2 $\overline{\partial}_b$-Neumann problem is solved for $1 \le q \le n-3$ and $\mathcal{N}_\delta$ is called the $\overline{\partial}_b$-Neumann operator.

Thus the Hodge decomposition theorem for compact strongly pseudoconvex CR manifolds proved in Theorem 8.4.10 has been extended to strongly pseudoconvex CR manifolds with boundaries.

We next study the interior regularity of $\mathcal{N}_\delta$ with applications to the regularity of the solutions of $\overline{\partial}_b$ and the related Szegö projection. Let $W^s(\omega_\delta)$ denote the Sobolev s space and $W^s(\omega_\delta, \mathrm{loc})$ denote the Fréchet space of functions which are in W^s on every compact subset of ω_δ.

Theorem 11.6.5. *Under the hypothesis of Theorem 11.6.4, given $\alpha \in W^s_{(0,q)}(\omega_\delta)$, $s \ge 0$, then $\phi = \mathcal{N}_\delta \alpha$ satisfies the following estimates: for any $\zeta, \zeta_1 \in C_0^\infty(\omega_\delta)$ such that $\zeta_1 = 1$ on the support of ζ, there exists a $C_s > 0$ such that*

$$\|\zeta\phi\|^2_{s+1} \le C_s(\|\zeta_1\alpha\|^2_s + \|\alpha\|^2).$$

Proof. Since $\Box_b\phi = (\bar{\partial}_b\vartheta_b + \vartheta_b\bar{\partial}_b)\phi = \alpha$ in the distribution sense on ω_δ and ω_δ is strongly pseudoconvex, the theorem follows from the interior regularity results for $\Box_b$ on strongly pseudoconvex manifolds proved in Theorem 8.4.3.

Corollary 11.6.6. *Let $\alpha \in L^2_{(0,q)}(\omega_\delta) \cap W^s_{(0,q)}(\omega_\delta, loc)$ and $\overline{\partial}_b\alpha = 0$, $1 \le q \le n-3$, then there exists $u \in L^2_{(0,q-1)}(\omega_\delta) \cap W^{s+1/2}_{(0,q-1)}(\omega_\delta, loc)$ satisfying $\overline{\partial}_b u = \alpha$. In particular, if $\alpha \in L^2_{(0,q)}(\omega_\delta) \cap C^\infty_{(0,q)}(\omega_\delta)$, then there exists $u \in L^2_{(0,q-1)}(\omega_\delta) \cap C^\infty_{(0,q-1)}(\omega_\delta)$ satisfying $\overline{\partial}_b u = \alpha$.*

Proof. Let $u = \overline{\partial}_b^* \mathcal{N}_\delta\alpha$ and let $\zeta_2 \in C_0^\infty(\omega_\delta)$ such that $\zeta_2 = 1$ on supp ζ_1, then from Theorem 11.6.5, we have

$$\begin{aligned}
\|\zeta u\|^2_s &= \|\zeta\overline{\partial}_b^*\phi\|^2_s \\
&\le 2\|\overline{\partial}_b^*(\zeta\phi)\|^2_s + 2\|[\zeta, \overline{\partial}_b^*]\phi\|^2_s \\
&\le c(\|\zeta\phi\|^2_{s+1} + \|\zeta_1\phi\|^2_s) \\
&\le c(\|\zeta_2\alpha\|^2_s + \|\alpha\|^2).
\end{aligned}$$

Thus, $u \in W^s_{(0,q-1)}(\omega_\delta, \mathrm{loc})$. To show that $u \in L^2_{(0,q-1)}(\omega_\delta)$, we note that

$$\begin{aligned}
\|u\|^2 &= (\overline{\partial}_b^*\mathcal{N}_\delta\alpha, \overline{\partial}_b^*\mathcal{N}_\delta\alpha) \\
&= (\overline{\partial}_b\overline{\partial}_b^*\mathcal{N}_\delta\alpha, \mathcal{N}_\delta\alpha) = (\alpha, \mathcal{N}_\delta\alpha) \\
&\le \|\alpha\|\,\|\mathcal{N}_\delta\alpha\| \le c\|\alpha\|^2.
\end{aligned}$$

To show that $u \in W^{s+1/2}_{(0,q-1)}(\omega_\delta, \mathrm{loc})$, we assume first that $s = 0$. Let Λ^k be the pseudodifferential operator of order k. Then, from Theorem 11.6.5 and the discussion above

$$\begin{aligned}\|\zeta u\|_{1/2}^2 &\le c(\Lambda^{1/2}\zeta\overline{\partial}_b^*\mathcal{N}_\delta\alpha, \Lambda^{1/2}\zeta\overline{\partial}_b^*\mathcal{N}_\delta\alpha)\\ &\le c((\zeta\overline{\partial}_b^*\mathcal{N}_\delta\alpha, \Lambda^1\zeta\overline{\partial}_b^*\mathcal{N}_\delta\alpha) + \|\alpha\|^2)\\ &\le c((\zeta\overline{\partial}_b\overline{\partial}_b^*\mathcal{N}_\delta\alpha, \Lambda^1\zeta\mathcal{N}_\delta\alpha) + \|\alpha\|^2)\\ &\le c(\|\zeta\alpha\|\,\|\zeta\mathcal{N}_\delta\alpha\|_1 + \|\alpha\|^2)\\ &\le c\|\alpha\|^2.\end{aligned}$$

For general $s \in \mathbb{N}$, one can prove that $u \in W^{s+1/2}_{(0,q-1)}(\omega_\delta, \text{loc})$ similarly by induction and we omit the details. If $\alpha \in L^2_{(0,q)}(\omega_\delta) \cap C^\infty_{(0,q)}(\omega_\delta)$, then $\alpha \in W^s_{(0,q)}(\omega_\delta, \text{loc})$ for every $s \in \mathbb{N}$. Thus the solution $u = \overline{\partial}_b^*\mathcal{N}_\delta\alpha \in W^{s+1/2}_{(0,q-1)}(\omega_\delta, \text{loc})$ for every $s \in \mathbb{N}$. It follows from the Sobolev embedding theorem that $u \in C^\infty_{(0,q-1)}(\omega_\delta)$ and the corollary is proved.

Definition 11.6.7. *Let $\mathcal{H}_b(\omega_\delta) = \{f \in L^2(\omega_\delta) \mid \overline{\partial}_b f = 0\}$ and let S_b denote the orthogonal projection from $L^2(\omega_\delta)$ onto $\mathcal{H}_b(\omega_\delta)$. We shall call S_b the Szegö projection on ω_δ.*

S_b is the natural analogue of the global Szegö projection. We have the following expression for S_b which is an analogue of the formula for the Bergman projection using the $\overline{\partial}$-Neumann operator:

Theorem 11.6.8. *Let $f \in L^2(\omega_\delta)$. Then $S_b f = (I - \overline{\partial}_b^*\mathcal{N}_\delta\overline{\partial}_b)f$. In particular, if $f \in C^\infty(\overline{\omega}_\delta)$, then $S_b f \in C^\infty(\omega_\delta)$.*

Proof. Since

$$\overline{\partial}_b\overline{\partial}_b^*\mathcal{N}_\delta\overline{\partial}_b f = (\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b)\mathcal{N}_\delta\overline{\partial}_b f = \overline{\partial}_b f$$

by (3) in Theorem 11.6.4, we have

$$\overline{\partial}_b(f - \overline{\partial}_b^*\mathcal{N}_\delta\overline{\partial}_b f) = \overline{\partial}_b f - \overline{\partial}_b f = 0.$$

This implies that $(I - \overline{\partial}_b^*\mathcal{N}_\delta\overline{\partial}_b)f \in \mathcal{H}_b(\omega_\delta)$. On the other hand, for any $h \in \mathcal{H}_b(\omega_\delta)$,

$$(\overline{\partial}_b^*\mathcal{N}_\delta\overline{\partial}_b f, h) = (\mathcal{N}_\delta\overline{\partial}_b f, \overline{\partial}_b h) = 0.$$

It follows that $(I - \overline{\partial}_b^*\mathcal{N}_\delta\overline{\partial}_b)f = S_b f$. The interior regularity proved in Theorem 11.6.5 implies $S_b f \in C^\infty(\omega_\delta)$ if $f \in C^\infty(\overline{\omega}_\delta)$. In fact one can show that if $f \in W^s(\omega_\delta)$, then $S_b f \in W^s(\omega_\delta, \text{loc})$ following the same argument as for the Bergman projection and we omit the details.

NOTES

The use of explicit kernels to solve the Cauchy-Riemann equations in several variables is a different approach parallel to the L^2 method. It is an attempt to generalize the Cauchy integral formula in one variable to several variables. Starting from the Bochner-Martinelli formula, the integral formula stated in Corollary 11.2.3 for holomorphic functions was discovered by J. Leray in [Ler 1,2]. G. M. Henkin [Hen 1] and E. Ramirez [Ram 1] introduced Cauchy-type integral formulas for strictly pseudoconvex domains. Subsequently, H. Grauert and I. Lieb [GrLi 1] and G. M. Henkin [Hen 2] constructed the integral solution formulas for $\bar{\partial}$ on strictly pseudoconvex domains with uniform estimates. Our exposition of the first three sections in this chapter follows that of R. Harvey and J. Polking [HaPo 1,2] (see also the book of A. Boggess [Bog 2]) without referring to currents. It is their notation that we adopt here.

The so-called Bochner-Martinelli-Koppelman formula was proved by S. Bochner [Boc 1], E. Martinelli [Mar 1] independently for functions (when q=0) and W. Koppelman [Kop 1] for forms. Our proof is due to N. N. Tarkhanov [Tark 1]. The jump formula of the Bochner-Martinelli-Koppelman formula was proved in R. Harvey and B. Lawson [HaLa 1] for continuous functions. For more discussion on the Bochner-Martinelli-Koppelman formula, see the book by A. M. Kytmanov [Kyt 1].

The Hölder estimates for $\bar{\partial}$ in strongly pseudoconvex domains were proved in N. Kerzman [Ker 1] using the integral solution operators for $\bar{\partial}$ constructed by Grauert and Lieb [GrLi 1] and Henkin [Hen 2]. L^p estimates were obtained by N. Kerzman for $q = 1$ and by N. Øvrelid [Øvr 1]. Exact Hölder 1/2-estimates for $\bar{\partial}$ were proved by G. M. Henkin and A. V. Romanov [HeRo 1] for $(0,1)$-forms and by R. M. Range and Y.-T. Siu [RaSi 1] for the general case. Sup-norm and Hölder estimates for derivatives of solution for $\bar{\partial}$ are obtained in Siu [Siu 1] and Lieb-Range [LiRa 1]. Hölder estimates for $\bar{\partial}$ on piecewise strongly pseudoconvex domains are discussed in Michel-Perotti [MiPe 1], Polyakov [Poly 1], and Range-Siu [RaSi 1]. Optimal Hölder and L^p estimates for $\bar{\partial}$ was proved by S. G. Krantz [Kra 1], where a theorem similar to Theorem 11.3.9 was proved for $\bar{\partial}$. There are also many results on integral kernels for $\bar{\partial}$ on weakly pseudoconvex domains (see Chaumat-Chollet [ChCh 1], Michel [Mic 1], and Range [Ran 1,3,7,8]). We refer the reader to the books by G. M. Henkin and J. Leiterer [HeLe 1], S. G. Krantz [Kra 2] and R. M. Range [Ran 6] for more discussion and references on integral representations for $\bar{\partial}$.

The homotopy formula for $\bar{\partial}_b$ on compact strictly pseudoconvex boundaries was constructed by G. M. Henkin [Hen 3], A. V. Romanov [Rom 1] and H. Skoda [Sko 1] where Hölder and L^p estimates for $\bar{\partial}_b$ are obtained. Our proof of Theorem 11.3.9 was based on [Hen 3]. These estimates have also been obtained by a different method by L. P. Rothschild and E. M. Stein [RoSt 1]. Using the estimates for $\bar{\partial}_b$, G. M. Henkin [Hen 3] and H. Skoda [Sko 1] have constructed holomorphic functions in the Nevanlinna class with prescribed zeros in strongly pseudoconvex domains. There is another proof of the Henkin-Skoda theorem using estimates for $\bar{\partial}$ directly by R. Harvey and J. Polking [HaPo 1]. When the domain is a ball, this is treated explicitly in the book of W. Rudin [Rud 2]. The Henkin-Skoda theorem has been extended to finite type domains in $\mathbb{C}^2$ by D.-C. Chang, A. Nagel and E. M. Stein [CNS 1] and for convex domains of finite type recently by J. Bruna, P. Charpentier and Y. Dupain [BCD 1].

There are many results on the Hölder and L^p estimates for $\bar{\partial}$ and $\bar{\partial}_b$ on convex boundaries using kernel methods. In particular, Hölder estimates for $\bar{\partial}$ on convex domains in C^2 and for complex ellipsoids in $\mathbb{C}^n$ are proved by R. M. Range [Ran 3,8]. Range's results have been generalized by J. Bruna and J. del Castillo [BrCa 1]. Hilbert integrals were used by J. Polking [Pol 1] to prove L^p estimates for $\bar{\partial}$ on convex domains in $\mathbb{C}^2$. Sharp Hölder estimates for $\bar{\partial}$ on real ellipsoids are obtained in Diederich-Fornaess-Wiegerinck [DFW 1]. Hölder estimates for $\bar{\partial}$ on convex finite type domains are proved in A. Cumenge [Cum 1](see also [DiFo 4] and [DFF 1]). Hölder and L^p estimates for $\bar{\partial}_b$ on the boundaries of real ellipsoids are proved in M.-C. Shaw [Sha 4] (for related results for $\bar{\partial}$, see [CKM 1]). Hölder estimates for $\bar{\partial}_b$ on convex boundaries in $\mathbb{C}^2$ are proved in D. Wu [Wu 1].

J. E. Fornaess [For 4] first obtained the sup-norm estimates for $\bar{\partial}$ on certain finite type domains in $\mathbb{C}^2$ which are not convexible, including the Kohn-Nirenberg domains [KoNi 3]. Using pseudodifferential operators, Hölder estimates for $\bar{\partial}$ and $\bar{\partial}_b$ were obtained by C. Fefferman and J. J. Kohn in [FeKo 1] for finite type domains in $\mathbb{C}^2$ (see also [CNS 1]) and for domains in $\mathbb{C}^n$ with diagonalizable Levi forms (see Fefferman-Kohn-Machedon [FKM 1]). L^p estimates for $\bar{\partial}_b$ for finite type domains in $\mathbb{C}^2$ were obtained by M. Christ [Chr 1].

We also note that N. Sibony [Sib 1] has given an example to show that the sup-norm estimates for $\bar{\partial}$ in general fail for smooth pseudoconvex domains of infinite type. The example in [Sib 1] is not convex and is strongly pseudoconvex except at one boundary point. It is still unknown if sup-norm estimates hold for $\bar{\partial}$ on convex domains in $\mathbb{C}^2$. J. E. Fornaess and N. Sibony [FoSi 1] also showed that L^p estimates, $1 < p \leq \infty$, also do not hold in general for pseudoconvex domains in $\mathbb{C}^2$ with smooth boundaries except for $p = 2$.

The local homotopy formula discussed in Section 11.4 was derived in G. M. Henkin [Hen 3]. This homotopy formula is useful in proving the embeddability of abstract CR structures (see the notes in Chapter 12). When $q = n-2$, A. Nagel and J. P. Rosay [NaRo 1]) showed that there does not exist any homotopy formula for $\bar{\partial}_b$ locally on a strictly convex hypersurface. The additional compatibility condition (11.4.2 a) derived for $q = n-2$ was observed in M.-C. Shaw [Sha 7]. The example given at the end of Section 11.4 was due to J. P. Rosay [Rosa 2].

The L^p estimates for the local solution discussed in Section 11.5 was based on the paper of M.-C. Shaw [Sha 3]. It is proved there that there does not exist any solution operator which maps $\bar{\partial}_b$-closed forms with L^p, $p < 2$, coefficients to solutions with L^2 coefficients. It is also proved in [Sha 3] that the closed range property in L^2 for $\bar{\partial}_b$ is equivalent to the local embeddability of abstract strongly pseudoconvex CR structures. Lemma 11.5.5 was based on the work of D. H. Phong and E. M. Stein [PhSt 1] on Hilbert integral operators. Theorem 11.5.1 is also true for $p = \infty$ (see the paper by L. Ma and J. Michel [MaMi 1]).

The $\bar{\partial}_b$-Neumann problem on strongly pseudoconvex CR manifolds with boundaries follows the paper by M.-C. Shaw [Sha 5]. The $\bar{\partial}_b$-Neumann problem with weights was discussed earlier in M. Kuranishi [Kur 1] in order to prove the embedding theorem for abstract CR structures. The weight functions used in [Kur 1] are singular in the interior. Boundary regularity for the Dirichlet problem for $\Box_b$ is discussed by D. Jerison [Jer 1].

Solvability of $\bar{\partial}_b$ on a weakly pseudoconvex CR manifold near a point of finite type is discussed in [Sha 6]. It is proved there that near a point of finite type, there exists a neighborhood base ω_δ such that $\bar{\partial}_b$ is solvable on ω_δ with interior

Sobolev estimates. C^∞ solvability for $\bar\partial_b$ on weakly pseudoconvex manifolds with flat boundaries were proved by J. Michel and M.-C. Shaw [MiSh 2] based on the barrier functions constructed in [MiSh 3]. When the boundary is piecewise flat, solvability for $\bar\partial_b$ is discussed by J. Michel and M.-C. Shaw [MiSh 3,4]. Integral kernels on a domain in a convex hypersurface with piecewise smooth boundary are constructed by S. Vassiliadou in [Vas 1].

For integral formulas for $\bar\partial$ on domains which are not pseudoconvex, we refer the reader to the paper by W. Fischer and I. Lieb [FiLi 1] and the book by G. M. Henkin and J. Leiterer [HeLe 2]. There are also results on the local solvability for $\bar\partial_b$ when the Levi form is not definite. For more discussion on the integral representation for local solutions for $\bar\partial_b$ under condition $\mathrm{Y}(q)$, we refer the reader to the papers by R. A. Airapetyan and G. M. Henkin [AiHe 1], A. Boggess [Bog 1], A. Boggess and M.-C. Shaw [BoSh 1] and M.-C. Shaw [Sha 8,9]. The reader should consult the book by A. Boggess [Bog 2] for more discussions on integral representations for $\bar\partial_b$ and CR manifolds.

CHAPTER 12

EMBEDDABILITY OF ABSTRACT CR STRUCTURES

The purpose of this chapter is to discuss the embeddability of a given abstract CR structure. This includes local realization of any real analytic CR structure. In Section 12.2, using the subelliptic estimate for $\Box_b$ obtained in Chapter 8, global CR embeddability into complex Euclidean space of any compact strongly pseudoconvex CR manifold of real dimension $2n-1$ with $n \geq 3$ is proved. In Sections 12.4 and 12.5, we present three dimensional counterexamples to the CR embedding either locally or globally.

12.1 Introduction

Let $(M, T^{1,0}(M))$ be a smooth CR manifold of real dimension $2n-1$, $n \geq 2$, as defined in Section 7.1. If M is diffeomorphic to another manifold M_1 of equal dimension via a map φ, then clearly φ induces a CR structure $\varphi_*(T^{1,0}(M))$ on M_1. Since the most natural CR structures are those induced from complex Euclidean spaces on a smooth hypersurface, it is of fundamental importance to see whether a given abstract CR structure $T^{1,0}(M)$ on M can be CR embedded into some $\mathbb{C}^N$ or not. Namely, can one find a smooth embedding φ of M into $\mathbb{C}^N$ so that the induced CR structure $\varphi_*(T^{1,0}(M))$ on $\varphi(M)$ coincides with the CR structure $T^{1,0}(\mathbb{C}^N) \cap \mathbb{C}T(\varphi(M))$ from the ambient space $\mathbb{C}^N$. More precisely, we make the following definition:

Definition 12.1.1. *Let $(M, T^{1,0}(M))$ be a CR manifold. A smooth mapping φ from M into $\mathbb{C}^N$ is called a CR embedding if*

(1) *φ is an embedding, namely, φ is a one-to-one mapping and the Jacobian of φ is of full rank everywhere,*
(2) *$\varphi_*(T^{1,0}(M)) = T^{1,0}(\mathbb{C}^N) \cap \mathbb{C}T(\varphi(M))$.*

The CR embedding problem could be formulated either locally or globally. The following lemma shows that condition (2) in Definition 12.1.1 is equivalent to the fact that each component φ_j of φ is a CR function.

Lemma 12.1.2. *Let $(M, T^{1,0}(M))$ be a CR manifold and let $\varphi = (\varphi_1, \cdots, \varphi_N)$ be a smooth embedding of M into $\mathbb{C}^N$. Then φ is a CR embedding if and only if φ_j is a CR function for $1 \leq j \leq N$.*

Proof. If φ is a CR embedding, then for any type $(0,1)$ vector field $\overline{L}$ on M, we have $\overline{L}(\varphi_j) = \varphi_* \overline{L}(z_j) = 0$ for $1 \le j \le N$. Thus φ_j is a CR function. By reversing the arguments we obtain the proof for the other direction.

To conclude this section, we prove that any real analytic CR structure is locally realizable. Let $(M, T^{1,0}(M))$ be a CR manifold of real dimension $2n-1$, $n \ge 2$, and let $p \in M$. Locally near p, a basis for $T^{1,0}(M)$ can be described by

$$(12.1.1) \qquad L_j = \sum_{k=1}^{2n-1} a_{jk}(x) \frac{\partial}{\partial x_k} \quad \text{for} \quad j = 1, \cdots, n-1,$$

and the integrability condition is then equivalent to

$$(12.1.2) \qquad [L_j, L_k] = \sum_{p=1}^{n-1} b_{jkp}(x) L_p,$$

for all $1 \le j, k \le n-1$. The real analyticity of the CR structure means that the coefficient functions $a_{jk}(x)$ defined in (12.1.1) are real analytic. The real analyticity of the b_{jkp}'s then follows.

Theorem 12.1.3. *Any real analytic CR manifold $(M, T^{1,0}(M))$ of dimension $2n-1$ with $n \ge 2$ can locally be CR embedded as a hypersurface in $\mathbb{C}^n$.*

Proof. We may assume that

$$\frac{\partial}{\partial x_{2n-1}} \notin T^{1,0}(M) \oplus T^{0,1}(M),$$

and that p is the origin. Choose a small neighborhood U_0 of the origin in $\mathbb{R}^{2n-1}$, and a $\epsilon > 0$ small enough so that, when the variable x_{2n-1} is complexified, i.e., replacing x_{2n-1} by $x_{2n-1} + it$, the power series of the real analytic functions that are involved in the expressions of (12.1.1) and (12.1.2) converge on $U_0 \times (-\epsilon, \epsilon)$.

Define

$$X_j = \sum_{k=1}^{2n-1} a_{jk}(x_1, \cdots, x_{2n-2}, x_{2n-1} + it) \frac{\partial}{\partial x_k} \quad \text{for} \quad 1 \le j \le n-1,$$

and

$$X_n = \frac{\partial}{\partial x_{2n-1}} + i \frac{\partial}{\partial t}.$$

Then we have

$$[X_j, X_k] = \sum_{p=1}^{n-1} b_{jkp}(x_1, \cdots, x_{2n-2}, x_{2n-1} + it) X_p,$$

and

$$[X_j, X_n] = 0 \quad \text{for} \quad 1 \le j \le n-1.$$

Hence, by the Newlander-Nirenberg theorem proved in Section 5.4, there is a complex structure defined on $U_0 \times (-\epsilon, \epsilon)$, and M is embedded as the hypersurface $\{t = 0\}$ in this complex structure. This completes the proof of the theorem.

One should note that a compact real analytic CR manifold of real dimension $2n-1$, in general, can not be globally CR embedded into $\mathbb{C}^N$ for any N. A counterexample will be provided in Section 12.4.

12.2 Boutet de Monvel's Global Embeddability Theorem

Let $(M, T^{1,0}(M))$ be a compact strongly pseudoconvex CR manifold of real dimension $2n-1$ with $n \geq 2$. Choose a purely imaginary vector field T defined on M so that T_p is complementary to $T_p^{1,0}(M) \oplus T_p^{0,1}(M)$ at each point $p \in M$. Fix a Hermitian metric on $\mathbb{C}T(M)$ so that $T^{1,0}(M)$, $T^{0,1}(M)$ and T are mutually orthogonal. Let S be the orthogonal projection, called the Szegö projection, from $L^2(M)$ onto the closed subspace $\mathcal{H}(M)$, where $\mathcal{H}(M)=\{f \in L^2(M)|\ \overline{\partial}_b f = 0$ in the sense of distribution$\}$. Denote by $\mathcal{E}^{p,q}(M)$ the space of smooth (p,q)-forms on M. Then we have the following global embeddability theorem of the CR structures:

Theorem 12.2.1 (Boutet de Monvel). *Let $(M, T^{1,0}(M))$ be a compact strongly pseudoconvex CR manifold of real dimension $2n-1$ with $n \geq 3$. Then $(M, T^{1,0}(M))$ can be globally CR embedded into $\mathbb{C}^k$ for some $k \in \mathbb{N}$.*

Theorem 12.2.1 will follow from the next theorem.

Theorem 12.2.2. *Let $(M, T^{1,0}(M))$ be a compact strongly pseudoconvex CR manifold of real dimension $2n-1$ with $n \geq 2$. Suppose that*

(1) *$\overline{\partial}_b : \mathcal{E}^{0,0}(M) \to \mathcal{E}^{0,1}(M)$ has closed range in the C^∞ topology, and that*

(2) *S maps $C^\infty(M)$ into $C^\infty(M)$ continuously in the C^∞ topology.*

Then $(M, T^{1,0}(M))$ can be globally CR embedded into complex Euclidean space. Also, CR functions separate points on M.

Proof. The first step is to show that CR functions separate points on M. By assumption (2) we have the following orthogonal, topological direct sum decomposition:

$$C^\infty(M) = (\mathrm{Ker}(S) \cap C^\infty(M)) \oplus (\mathrm{Range}(S) \cap C^\infty(M)).$$

Let the range of $\overline{\partial}_b$ on $\mathcal{E}^{0,0}$ be denoted by $\mathcal{R}$ which is a closed subspace of $\mathcal{E}^{0,1}$ in the C^∞ topology. Since both $\mathrm{Ker}(S) \cap C^\infty(M)$ and $\mathcal{R}$ are Fréchet spaces, the open mapping theorem implies that the isomorphism

$$\overline{\partial}_b : \mathrm{Ker}(S) \cap C^\infty(M) \xrightarrow{\sim} \mathcal{R} \tag{12.2.1}$$

and its inverse are continuous.

For each $p \in M$ we claim that there exists a $\phi_p \in C^\infty(M)$ satisfying

(a) $\phi_p(p) = 0$ and $\overline{\partial}_b \phi$ vanishes to infinite order at p,

(b) for some coordinate neighborhood system centered at p, we have

$$\mathrm{Re}\phi_p(x) \geq c|x|^2,$$

in some neighborhood of p, where c is a positive constant,

(c) $\mathrm{Re}\phi_p(x) \geq 1$ outside a small neighborhood of p on M.

Proof of the claim. If M is the boundary of a smooth bounded strongly pseudoconvex domain D in $\mathbb{C}^n$, and let r be a strictly plurisubharmonic defining function for D, then we may take $\phi(z)$ to be the Levi polynomial $g_p(z)$ at p in some small neighborhood of p and extend it suitably to M to satisfy (c). Namely, define

$$\begin{aligned}\phi_p(z) &= g_p(z)\\ &= \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)(p_j - z_j) - \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(p)(p_j - z_j)(p_k - z_k),\end{aligned}$$

in a small open neighborhood of p. Using Taylor's expansion, it is easily verified that for $z \in M$ near p we have

$$\mathrm{Re}\phi_p(z) \geq c|z-p|^2.$$

Thus ϕ_p satisfies (b), and (c) is done by an appropriate extension to M. This proves the claim for the embedded case.

If M is an abstract CR manifold, we can first find functions $\varphi_1, \cdots, \varphi_n \in C^\infty(M)$ such that $\varphi_j(p) = 0$, $d\varphi_1(p), \cdots, d\varphi_n(p)$ are linearly independent at p and $\overline{\partial}_b \varphi_j$ vanishes to infinite order at p for $j = 1, \cdots, n$. Then

$$\varphi = (\varphi_1, \cdots, \varphi_n) : M \to \mathbb{C}^n$$

is a smooth embedding of a small neighborhood of p on M into $\mathbb{C}^n$ with $\varphi(p) = 0$ and $\varphi(M)$ is strongly pseudoconvex at the origin. Let $g_0(z)$ be the Levi polynomial for $\varphi(M)$ defined at the origin, then the pullback $\phi_p(x) = g_0 \circ \varphi(x)$ is defined in some small neighborhood of p and satisfies conditions (a) and (b) on a small coordinate neighborhood. Condition (c) is satisfied by a suitable extension of ϕ_p to M. This completes the proof of the claim.

Now if $p, q \in M$ with $p \neq q$, let $\phi_p(x)$ be the function satisfying (a), (b) and (c) so that $\mathrm{Re}\phi_p(q) \geq 1$. Consider the function

$$u_t = e^{-t\phi_p} \quad \text{for} \quad t > 0.$$

Then $u_t \in C^\infty(M)$, $u_t(p) = 1$ and $u_t(q)$ is close to 0 for large $t > 0$. Write

$$u_t = S(u_t) + (I - S)(u_t).$$

Applying $\overline{\partial}_b$ to u_t we obtain

$$\overline{\partial}_b u_t = -te^{-t\phi_p}(\overline{\partial}_b \phi_p).$$

We claim that $\overline{\partial}_b u_t$ converges to zero in the C^∞ topology as $t \to +\infty$. First we note that any kth derivative of $\overline{\partial}_b u_t$ can be written in the following form:

$$I_k = \pm t^j e^{-t\phi_p} D^\beta(\overline{\partial}_b \phi_p)\chi(x), \tag{12.2.2}$$

where $\chi(x)$ is a smooth function on M and $1 \le j \le k+1$, $|\beta| \le k$. Hence, by (b), (12.2.2) is bounded in some open neighborhood V_1 of p by

$$|I_k| \le C_k (ct|x|^2)^j e^{-ct|x|^2} \cdot |x|^{-2j} |D^\beta(\overline{\partial}_b \phi_p)(x)|,$$

for some positive constant $C_k > 0$. Given any $\epsilon > 0$, since $\overline{\partial}_b \phi_p$ vanishes to infinite order at p and $(ct|x|^2)^j e^{-ct|x|^2}$ is uniformly bounded for all x and $t > 0$, one may choose a sufficiently small neighborhood $V_2 \Subset V_1$ so that $|I_k| < \epsilon$ on V_2. For $x \notin V_2$, we have $|x| \ge \delta > 0$ for some constant δ. Letting t be sufficiently large, we see also that $|I_k| < \epsilon$ for $x \notin V_2$. This proves the claim.

It follows that, by (12.2.1), $(I-S)(u_t)$ also converges to zero in the C^∞ topology and that the CR function $S(u_t)$ for sufficiently large $t > 0$ will separate p and q.

By the same reasoning as above, we see that the functions

$$h^j = S(\varphi_j e^{-t\phi_p}) \quad \text{for} \quad j = 1, \cdots, n,$$

for sufficiently large $t > 0$, satisfy

(1) $\overline{\partial}_b h^j = 0$ for $j = 1, \cdots, n$, and
(2) $dh^1(p), \cdots, dh^n(p)$ are linearly independent, and
(3) $h^j(p) = 0$ for $j = 1, \cdots, n$, if necessary, by a translation in $\mathbb{C}^n$.

Hence, for each $p \in M$, there exists an open neighborhood U_p of p on M and smooth CR functions $h_p^1, \cdots, h_p^n$ such that $dh_p^1(x), \cdots, dh_p^n(x)$ are linearly independent for all $x \in U_p$.

Now cover M by a finite number of such U_{p_i}, $i = 1, \cdots, k$, and let $g_1, \cdots, g_s$ be the CR functions that separate points a, b with distance $d(a,b) \ge \delta > 0$ for some constant δ. Then set

$$F = (h_{p_1}^1, \cdots, h_{p_1}^n, h_{p_2}^1, \cdots, h_{p_2}^n, \cdots, h_{p_k}^1, \cdots, h_{p_k}^n, g_1, \cdots, g_s).$$

It is easily verified that F is a global CR embedding of M into $\mathbb{C}^{nk+s}$. The proof of Theorem 12.2.2 is now complete.

We now return to the proof of Theorem 12.2.1.

Proof of Theorem 12.2.1. By the hypothesis of the theorem Condition $Y(1)$ (see Definition 8.3.3) holds on $(M, T^{1,0}(M))$ if the real dimension of M is at least five. Hence, Corollary 8.4.11 shows that the range of $\overline{\partial}_b$ on $W^0_{(0,0)}(M)$ is closed in $W^0_{(0,1)}(M)$ in the L^2 sense. The formula for the Szegö projection S,

$$S = I - \overline{\partial}_b^* N_b \overline{\partial}_b$$

together with Theorem 8.4.14 shows that S maps $C^\infty(M)$ continuously into itself in the C^∞ topology. Theorem 8.4.14 also shows that the range of $\overline{\partial}_b$ on $\mathcal{E}^{0,0}(M)$ is closed in $\mathcal{E}^{0,1}(M)$ in the C^∞ topology. It follows that conditions (1) and (2) in Theorem 12.2.2 are established for any compact strongly pseudoconvex CR manifold $(M, T^{1,0}(M))$ of real dimension $2n-1$ with $n \ge 3$. This proves Theorem 12.2.1.

12.3 Spherical Harmonics

In this section we will review the spherical harmonics in $\mathbb{R}^n$. For any $k \in \mathbb{N} \cup \{0\}$, denote by $\mathcal{P}_k$ the vector space of all homogeneous polynomials of degree k over the complex number field. A basis for $\mathcal{P}_k$ is given by all monomials $\{x^\alpha\}_{|\alpha|=k}$ of degree k, and it is easily seen that the dimension d_k of $\mathcal{P}_k$ over $\mathbb{C}$ is equal to

$$d_k = \binom{n+k-1}{n-1} = \frac{(n+k-1)!}{(n-1)!k!}.$$

We define an inner product on $\mathcal{P}_k$ as follows. For any $P(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha, Q(x) = \sum_{|\alpha|=k} b_\alpha x^\alpha$, the inner product between $P(x)$ and $Q(x)$ is defined by

$$\langle P, Q\rangle = \sum_{|\alpha|=k} a_\alpha \bar{b}_\alpha \alpha!. \tag{12.3.1}$$

If $P(x) = \sum\limits_{\alpha} a_\alpha x^\alpha$ is any polynomial, set

$$P(D) = \sum_{\alpha} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

Then the inner product (12.3.1) can be realized as a differentiation

$$\langle P, Q\rangle = P(D)(\overline{Q}(x)). \tag{12.3.2}$$

Lemma 12.3.1. *For any $P(x) \in \mathcal{P}_k$, we can write*

$$P(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2l} P_l(x), \tag{12.3.3}$$

where each polynomial $P_j(x)$ is homogeneous and harmonic of degree $k-2j$ for $0 \le j \le l$ with l being the largest integer less than or equal to $k/2$.

Proof. We may assume that $k \ge 2$. Define a map Λ_k

$$\begin{aligned} \Lambda_k : \mathcal{P}_k &\to \quad \mathcal{P}_{k-2}, \\ P(x) &\mapsto \quad \Delta P(x), \end{aligned}$$

where Δ is the classical Laplacian. The adjoint operator Λ_k^* of Λ_k is then defined by

$$\langle Q, \Lambda_k P\rangle = \langle \Lambda_k^* Q, P\rangle,$$

where $P(x) \in \mathcal{P}_k$ and $Q(x) \in \mathcal{P}_{k-2}$. A direct computation shows that

$$\begin{aligned} \langle Q, \Lambda_k P\rangle &= Q(D)(\Delta \overline{P}) \\ &= \Delta Q(D)(\overline{P}) \\ &= \Lambda_k^* Q(D)(\overline{P}). \end{aligned}$$

This implies that

$$\Lambda_k^* Q(x) = |x|^2 Q(x).$$

It follows that Λ_k^* is one-to-one, and the following decomposition holds:

$$\mathcal{P}_k \simeq \mathrm{Ker}\Lambda_k \oplus \mathrm{Range}\Lambda_k^*.$$

Hence for any $P(x) \in \mathcal{P}_k$, we can write $P(x)$ as

$$P(x) = P_0(x) + |x|^2 Q(x),$$

where $P_0(x)$ is a homogeneous harmonic polynomial of degree k and $Q(x) \in \mathcal{P}_{k-2}$. The proof of the lemma is then completed by an induction argument.

Lemma 12.3.1 shows that the restriction of any polynomial $P(x)$ to the unit sphere S^{n-1} in $\mathbb{R}^n$ is given by a sum of restrictions of homogeneous harmonic polynomials to S^{n-1}.

Definition 12.3.2. *Denote by $\mathcal{SH}_k$ the space of the restrictions to the unit sphere S^{n-1} of all homogeneous harmonic polynomials of degree k, i.e., $\mathcal{SH}_k = \mathcal{HP}_k|_{S^{n-1}}$, where $\mathcal{HP}_k = Ker\Lambda_k$.*

The restriction is clearly an isomorphism from $\mathcal{HP}_k$ onto $\mathcal{SH}_k$, and

$$\begin{aligned}\dim\mathcal{SH}_k &= \dim\mathcal{HP}_k \\ &= d_k - d_{k-2} \\ &= \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1},\end{aligned}$$

for $k \geq 2$. In particular, $\dim\mathcal{SH}_0 = 1$ and $\dim\mathcal{SH}_1 = n$.

The elements in $\mathcal{HP}_k$ are called solid spherical harmonics and the elements in $\mathcal{SH}_k$ are called surface spherical harmonics, or simply spherical harmonics. As an easy consequence of the Stone-Weierstrass theorem, we obtain the following proposition:

Proposition 12.3.3. *The finite linear combination of elements in $\cup_{k=0}^{\infty}\mathcal{SH}_k$ is uniformly dense in $C(S^{n-1})$, and L^2 dense in $L^2(S^{n-1}, d\sigma)$.*

Proposition 12.3.4. *If $Y^{(j)} \in \mathcal{SH}_j$ and $Y^{(k)} \in \mathcal{SH}_k$ with $j \neq k$, then*

$$\int_{S^{n-1}} Y^{(j)}(x')Y^{(k)}(x')\, d\sigma(x') = 0.$$

Proof. The proof will rely on the following two facts:

(i) (Green's identity) Let D be a bounded domain with C^2 boundary. If $f, g \in C^2(\overline{D})$, we have

$$\int_{\partial D}\left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) d\sigma = \int_D (f\Delta g - g\Delta f)\, dV,$$

where $\partial/\partial n$ is the unit outward normal derivative on the boundary ∂D.

(ii) If $f \in C^1(\overline{B}_n)$ and is harmonic on B_n, then

$$\int_{\partial B_n} \frac{\partial f}{\partial n} \, d\sigma = 0.$$

Here B_n denotes the unit ball in $\mathbb{R}^n$.

For $x \in \mathbb{R}^n$, write $x = rx'$ with $r = |x|$ and $|x'| = 1$. If $Y^{(j)} \in \mathcal{SH}_j$ and $Y^{(k)} \in \mathcal{SH}_k$, define

$$u_j(x) = |x|^j Y^{(j)}(x') = r^j Y^{(j)}(x'),$$

and

$$u_k(x) = |x|^k Y^{(k)}(x') = r^k Y^{(k)}(x').$$

Case (I). If one of j or k is zero, say, $j = 0$, then $u_j(x) = c$, a constant, and

$$\frac{\partial}{\partial n} u_k(x') = \frac{\partial}{\partial r}(r^k Y^{(k)}(x')) = kY^{(k)}(x').$$

Thus, by fact (ii) we have

$$\int_{S^{n-1}} Y^{(j)}(x') Y^{(k)}(x') \, d\sigma = \frac{c}{k} \int_{S^{n-1}} \frac{\partial u_k}{\partial n}(x') \, d\sigma = 0.$$

Case (II). If both j and k are nonzero with $j \neq k$, then

$$\begin{aligned} (k-j) \int_{S^{n-1}} Y^{(j)}(x') Y^{(k)}(x') \, d\sigma &= \int_{S^{n-1}} \left(u_j \frac{\partial u_k}{\partial n} - u_k \frac{\partial u_j}{\partial n} \right) d\sigma \\ &= \int_{B_n} (u_j \Delta u_k - u_k \Delta u_j) \, dV \\ &= 0. \end{aligned}$$

This completes the proof of the proposition.

Let $L^2(S^{n-1}, d\sigma)$ be equipped with the usual inner product. For each $k \in \mathbb{N} \cup \{0\}$, let $\{Y_1^{(k)}, \cdots, Y_{m_k}^{(k)}\}$ be an orthonormal basis for $\mathcal{SH}_k$, where $m_k = d_k - d_{k-2}$. It follows from Proposition 12.3.3 that

$$\bigcup_{k=0}^{\infty} \{Y_1^{(k)}, \cdots, Y_{m_k}^{(k)}\}$$

forms a complete orthonormal basis for $L^2(S^{n-1}, d\sigma)$. Hence, for $f \in L^2(S^{n-1}, d\sigma)$, we have a unique representation

$$f = \sum_{k=0}^{\infty} Y^{(k)}$$

such that the series converges to f in the L^2 norm, and $Y^{(k)} \in \mathcal{SH}_k$ can be expressed in terms of the Fourier coefficients

$$Y^{(k)} = \sum_{p=1}^{m_k} \langle Y^{(k)}, Y_p^{(k)} \rangle Y_p^{(k)}.$$

When $n = 2$, we have $d_k - d_{k-2} = 2$ for all $k \geq 2$. It is easily seen that $\mathcal{HP}_k = \{z^k, \bar{z}^k\}$. This implies, by normalization,

$$\left\{ \frac{1}{\sqrt{2\pi}} \right\} \cup \left(\bigcup_{k=0}^{\infty} \left\{ \frac{1}{\sqrt{\pi}} \cos k\theta, \frac{1}{\sqrt{\pi}} \sin k\theta \right\} \right)$$

is a complete orthonormal basis for $L^2(S^1)$.

12.4 Rossi's Global Nonembeddability Example

We shall present in this section a compact real analytic three dimensional CR manifold which can not be globally CR embedded into $\mathbb{C}^n$ for any dimension n. In view of Theorem 12.1.3 one sees that the nature of global embedding of a CR structure is quite different from that of local embedding. Global properties of the CR structure should be taken into account in the set up of the global embedding problem.

Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | \ |z_1|^2 + |z_2|^2 = 1\}$ be the boundary of the unit ball in $\mathbb{C}^2$, and let the induced CR structure $T^{1,0}(S^3)$ be generated by $L = \bar{z}_2(\partial/\partial z_1) - \bar{z}_1(\partial/\partial z_2)$. Thus, $(S^3, T^{1,0}(S^3))$ forms a compact strongly pseudoconvex CR manifold of real dimension three. For each $t \in \mathbb{R}$, $|t| < 1$, define a new CR structure $T_t^{1,0}(S^3)$ on S^3 by letting $T_t^{1,0}(S^3)$ be generated by the vector field $L_t = L + t\overline{L}$. If $t = 0$, $T_0^{1,0}(S^3)$ coincides with the induced standard CR structure $T^{1,0}(S^3)$. It is easily verified that for $|t| < 1$, $(S^3, T_t^{1,0}(S^3))$ is a compact real analytic strongly pseudoconvex CR manifold of real dimension three.

The next theorem shows that any L^2 integrable CR function f on S^3 with respect to the CR structure $(S^3, T_t^{1,0}(S^3))$ for $0 < |t| < 1$ must be even. Obviously, this implies that, for $0 < |t| < 1$, $(S^3, T_t^{1,0}(S^3))$ can not be globally CR embedded into any $\mathbb{C}^n$.

Theorem 12.4.1. *Any L^2 integrable CR function $f(z)$ on S^3 with respect to the CR structure $T_t^{1,0}(S^3)$, $0 < |t| < 1$, is even, i.e., $f(z) = f(-z)$.*

Proof. Notice first that we can decompose the space of homogeneous harmonic polynomials of degree k as follows.

$$\mathcal{HP}_k = \bigoplus_{p+q=k} \mathcal{HP}_k^{p,q},$$

where $\mathcal{HP}_k^{p,q}$ denotes the space of all homogeneous harmonic polynomials of degree k that can be expressed as a linear combination of terms $z^\alpha \bar{z}^\beta$ with $|\alpha| = p$, $|\beta| = q$ and $p + q = k$. We set $\mathcal{SH}_k^{p,q} = \mathcal{HP}_k^{p,q}|_{S^3}$, then we have

$$\mathcal{SH}_k = \bigoplus_{p+q=k} \mathcal{SH}_k^{p,q}.$$

If h is a harmonic function on $\mathbb{C}^2$, then a simple computation shows that both Lh and $\overline{L}h$ are also harmonic on $\mathbb{C}^2$. It follows that

$$L(\mathcal{SH}_k^{p,q}) \subset \mathcal{SH}_k^{p-1,q+1},$$

and

$$\overline{L}(\mathcal{SH}_k^{p,q}) \subset \mathcal{SH}_k^{p+1,q-1}.$$

Thus, if $f(z)$ is a square integrable CR function on S^3 with respect to the CR structure $T_t^{1,0}(S^3)$ for $0 < |t| < 1$, according to Proposition 12.3.3, there is a unique representation

$$f(z) = \sum_{m \geq 0} f_m(z),$$

where $f_m(z) \in \mathcal{SH}_m$ and the series converges to f in the L^2 norm. Since $\overline{L}_t f(z) = 0$ in the distribution sense on S^3, we obtain $\overline{L}_t f_m(z) = 0$ on S^3 for all $m \geq 0$. For $m = 2k+1$, we can write

$$f_{2k+1} = f_{2k+1,0} + f_{2k,1} + \cdots + f_{k+1,k} + f_{k,k+1} + \cdots + f_{1,2k} + f_{0,2k+1},$$

where $f_{p,q} \in \mathcal{SH}_{p+q}^{p,q}$. Here we have identified $f_{p,q}$ with its preimage in $\mathcal{HP}_{p+q}^{p,q}$.

Since $\overline{L}_t f_{2k+1}(z) = 0$ on S^3, we obtain

$$\overline{L} f_{2k,1}(z) = 0 \quad \text{on} \quad S^3.$$

Hence, $f_{2k,1}(z)$ is a real analytic CR function on S^3. By Theorem 3.2.2, $f_{2k,1}|_{S^3}$ extends smoothly to a holomorphic function $F_{2k,1}(z)$ defined on B_2. Then, by harmonicity of $f_{2k,1}(z)$ and the maximum modulus principle, we obtain $f_{2k,1}(z) = F_{2k,1}(z)$ on B_2. It follows that $f_{2k,1}(z)$ is holomorphic on $\mathbb{C}^2$ and that no $\overline{z}$ terms appear in $f_{2k,1}(z)$. This implies that $f_{2k,1}(z) \equiv 0$ on $\mathbb{C}^2$.

Similarly, we obtain $f_{1,2k}(z) \equiv 0$ on $\mathbb{C}^2$. Inductively, one can show

$$f_{2k-2,3}(z) = f_{2k-4,5}(z) = \cdots = f_{2,2k-1}(z) = f_{0,2k+1}(z) \equiv 0,$$

and

$$f_{3,2k-2}(z) = f_{5,2k-4}(z) = \cdots = f_{2k-1,2}(z) = f_{2k+1,0}(z) \equiv 0.$$

Therefore, $f_m(z) \equiv 0$ for all odd indices m, and $f(z)$ must be even. This completes the proof of the theorem.

Theorem 12.4.1 indicates that a three dimensional compact strongly pseudoconvex CR manifold in general can not be globally CR embedded into a complex Euclidean space. However, we shall show now for any $0 < |t| < 1$, $(S^3, T_t^{1,0}(S^3))$ can always be CR immersed into $\mathbb{C}^3$.

We have seen that the only possible solutions to the $\overline{L}_t$ equation on $(S^3, T_t^{1,0}(S^3))$ are the even functions. By reasoning similarly, one can show that for k even, if $u \in \mathcal{SH}_k$ such that $\overline{L}_t u = 0$ and $u_{k,0} = 0$, then $u = 0$. It follows that the space of solutions of $\overline{L}_t(u) = 0$ in $\mathcal{SH}_2$ is of dimension three which is spanned by

$$\begin{aligned}
X &= \frac{\sqrt{2}}{2i}(z_1^2 + z_2^2 + t(\overline{z}_1{}^2 + \overline{z}_2{}^2)), \\
Y &= \frac{\sqrt{2}}{2}(-z_1^2 + z_2^2 + t(\overline{z}_1{}^2 - \overline{z}_2{}^2)), \\
Z &= \sqrt{2}(z_1 z_2 - t\overline{z}_1 \overline{z}_2).
\end{aligned}$$

A direct computation shows that

$$X^2 + Y^2 + Z^2 = -2t, \tag{12.4.1}$$

and

$$|X|^2 + |Y|^2 + |Z|^2 = 1 + t^2. \tag{12.4.2}$$

For each fixed t, $0 < |t| < 1$, equation (12.4.1) defines a two dimensional complex submanifold M_t in $\mathbb{C}^3$. We claim that the map

$$\begin{aligned} \pi : (S^3, T_t^{1,0}(S^3)) &\to M_t \subset \mathbb{C}^3 \\ z = (z_1, z_2) &\mapsto (X(z), Y(z), Z(z)) \end{aligned} \tag{12.4.3}$$

is a two-to-one CR immersion.

Proof of the claim. First we show that π is two-to-one. If $z = (z_1, z_2)$, $w = (w_1, w_2)$ are two points on S^3 such that $\pi(z) = \pi(w)$, then we have

$$z_1^2 + z_2^2 + t(\overline{z}_1^2 + \overline{z}_2^2) = w_1^2 + w_2^2 + t(\overline{w}_1^2 + \overline{w}_2^2), \tag{12.4.4}$$

$$-z_1^2 + z_2^2 + t(\overline{z}_1^2 - \overline{z}_2^2) = -w_1^2 + w_2^2 + t(\overline{w}_1^2 - \overline{w}_2^2), \tag{12.4.5}$$

and

$$z_1 z_2 - t\overline{z}_1\overline{z}_2 = w_1 w_2 - t\overline{w}_1\overline{w}_2. \tag{12.4.6}$$

From (12.4.4) and (12.4.5) we obtain

$$t^2(w_1^2 - z_1^2) = w_1^2 - z_1^2.$$

Hence, $w_1 = \pm z_1$. If $w_1 = z_1$ and $w_2 = z_2$, then $w = z$. Otherwise, we have $w_1 = z_1$ and $w_2 = -z_2$. From (12.4.6) this implies

$$z_1 z_2 - t\overline{z}_1\overline{z}_2 = 0.$$

Hence, $z_1 z_2 = 0$. If $z_1 = 0$ and $z_2 \neq 0$, then $w = -z$. If $z_1 \neq 0$ and $z_2 = 0$, then $w = z$. Similarly, if $w_1 = -z_1$, we have either $w = -z$ or $w = z$. Thus, π is a two-to-one mapping.

Next we show that the Jacobian of π is of full rank at each point $z \in S^3$. Since $\mathbb{C}T(S^3)$ is spanned by L_t, $\overline{L}_t$ and $L_2 - \overline{L}_2$, where $L_2 = z_1(\partial/\partial z_1) + z_2(\partial/\partial z_2)$, it suffices to show that the images $\pi_*(L_t)$, $\pi_*(\overline{L}_t)$ and $\pi_*(L_2 - \overline{L}_2)$ are linearly independent for each point $z \in S^3$.

Let $w = (w_1, w_2, w_3)$ be the coordinates for $\mathbb{C}^3$. Suppose that we have

$$a\pi_*(L_t) + b\pi_*(\overline{L}_t) + c\pi_*(L_2 - \overline{L}_2) = 0. \tag{12.4.7}$$

Case (i). For $|z_1| \neq |z_2|$, we apply dw_2 and dw_3 respectively to (12.4.7) to get

$$-a(1-t^2)(z_1\overline{z}_2 + \overline{z}_1 z_2) + c(-z_1^2 + z_2^2 + t(-\overline{z}_1^2 + \overline{z}_2^2)) = 0, \tag{12.4.8}$$

and

$$a(1-t^2)(|z_2|^2 - |z_1|^2) + c(2z_1 z_2 + t(2\overline{z}_1\overline{z}_2)) = 0. \tag{12.4.9}$$

A direct calculation shows the determinant of the coefficient matrix given by (12.4.8) and (12.4.9) is

$$\begin{aligned} -(1-t^2)(|z_1|^2+|z_2|^2)&((z_1^2+z_2^2)+t(\overline{z}_1^2+\overline{z}_2^2)) \\ &= -(1-t^2)((z_1^2+z_2^2)+t(\overline{z}_1^2+\overline{z}_2^2)) \\ &\neq 0. \end{aligned}$$

It follows that $a = c = 0$, and hence $b = 0$.

Case (ii). For $z_2 = e^{i\theta} z_1 \neq 0$ with $\theta \neq \pi/2, 3\pi/2$, we obtain similarly from Case (i) that $a = c = 0$. Then by applying $d\overline{w}_2$ to (12.4.7) we get

$$\begin{aligned} 0 &= b(1-t^2)(z_1\overline{z}_2 + \overline{z}_1 z_2) \\ &= b(1-t^2)|z_1|^2(e^{i\theta} + e^{-i\theta}). \end{aligned}$$

Hence, we have $b = 0$.

Case (iii). For $z_2 = \pm i z_1 \neq 0$, we have

$$\begin{aligned} dw_1(\pi_*(L_t)) &= a(\pm 2\sqrt{2})(1-t^2)|z_1|^2 = 0, \\ d\overline{w}_1(\pi_*(\overline{L}_t)) &= b(\pm 2\sqrt{2})(1-t^2)|z_1|^2 = 0, \\ dw_3(\pi_*(L_2 - \overline{L}_2)) &= c(\pm 2\sqrt{2}i)(z_1^2 - t\overline{z}_1^2) = 0. \end{aligned}$$

Thus, $a = b = c = 0$. It shows that π is a two-to-one CR immersion of $(S^3, T_t^{1,0}(S^3))$ into M_t in $\mathbb{C}^3$.

12.5 Nirenberg's Local Nonembeddability Example

In this section we shall construct strongly pseudoconvex CR structures which are not locally embeddable.

As in Section 7.3, the following notation will be used: The Siegel upper half space Ω_2 in $\mathbb{C}^2$ is defined by

$$\Omega_2 = \{(z,w) \in \mathbb{C}^2 |\ s > |z|^2\},$$

where $z = x + iy$ and $w = t + is$. The boundary of Ω_2 will be denoted by M, and will be identified with the Heisenberg group $\mathbb{H}_2$ via the mapping

$$\pi : (z, t + i|z|^2) \mapsto (z, t). \tag{12.5.1}$$

Hence, the tangential Cauchy-Riemann operator on M is generated by

$$\overline{L} = \frac{\partial}{\partial \overline{z}} - 2iz\frac{\partial}{\partial \overline{w}}, \tag{12.5.2}$$

and the corresponding Lewy operator on $\mathbb{H}_2$ is

$$\overline{Z} = \frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial t}. \tag{12.5.3}$$

We start working on $\mathbb{H}_2 = \mathbb{C}\times\mathbb{R}$ with coordinates given by (x, y, t). Construct a sequence of disjoint closed discs $\overline{D}_k$ on the xt-plane with centers $(1/k, 0, 1/k)$. The radii of these discs are chosen to be so small that $\overline{D}_i \cap \overline{D}_j = \emptyset$ if $i \neq j$, and that $\overline{D}_j$ has no intersection with the t-axis for all j. Denote by C_k the boundary of $\overline{D}_k$ and by D_k the interior of $\overline{D}_k$. Then denote by T_k the open solid torus obtained by sweeping D_k around the t-axis. The topological boundary of T_k is denoted by S_k which is given by sweeping C_k around the t-axis.

Now lift these objects from $\mathbb{H}_2$ via the mapping π to M, namely, set

$$\tilde{C}_k = \pi^{-1}(C_k),\ \tilde{S}_k = \pi^{-1}(S_k) \text{ and } \tilde{T}_k = \pi^{-1}(T_k).$$

Next, let P be the projection from $\mathbb{C}^2$ onto the second component, i.e., $P(z, w) = (0, w)$, and set

$$C'_k = P(\tilde{C}_k),\ S'_k = P(\tilde{S}_k) \text{ and } T'_k = P(\tilde{T}_k).$$

It is then easily seen that $\{C'_k = S'_k\}$ is a sequence of disjoint simple closed curves in the first quadrant of the w-plane converging to the origin, and T'_k is exactly the open region bounded by C'_k. Obviously, we have the following lemma:

Lemma 12.5.1. *$P(M \setminus \cup_{k=1}^{\infty}\tilde{T}_k)$ is a connected subset of $\{w \in \mathbb{C} \mid s \geq 0\}$ which contains the t-axis.*

For any function $f : \mathbb{H}_2 \to \mathbb{C}$, let $\tilde{f}$ be the lifting of f to M, namely, $f = \tilde{f}\circ\pi^{-1}$. Hence $\tilde{f}$ is a CR function on M, i.e., $\overline{L}\tilde{f} = 0$ on M, if and only if $\overline{Z}f = 0$ on $\mathbb{H}_2$. Then we have

Lemma 12.5.2. *Let $\tilde{f} : M \to \mathbb{C}$ be a C^1 function.*

(1) *If $\overline{L}\tilde{f} = 0$ on an open subset V of M, then the function*

$$F(w) = \int_{\Gamma(w)} \tilde{f}\ dz$$

is holomorphic on $\{w \in \mathbb{C} \mid \Gamma(w) \subset V\}$, where $\Gamma(w) = M \cap P^{-1}(w)$.

(2) *If $\overline{L}\tilde{f} = 0$ on $M \setminus \cup_{k=1}^{\infty}\tilde{T}_k$, and $\Gamma(w) \subset M \setminus \cup_{k=1}^{\infty}\tilde{T}_k$, then*

$$\int_{\Gamma(w)} \tilde{f}\ dz = 0.$$

(3) *If $\overline{L}\tilde{f} = 0$ on $M \setminus \cup_{k=1}^{\infty}\tilde{T}_k$, then for each $k \geq 1$, we have*

$$\iint_{\tilde{S}_k} \tilde{f}\ dz \wedge dw = 0.$$

Proof. For (1), notice that $\tilde{f}$ is a CR function of class C^1. Hence, $\tilde{f}$ can be extended to a C^0 function, denoted also by $\tilde{f}$, in an ambient neighborhood so that $D\tilde{f}$ exists and is continuous on V and $\overline{\partial}\tilde{f}$ vanishes on V. Also, the circle

$$\Gamma(w = t + is) = \{(z, t + is)|\ s = |z|^2\}$$

can be parameterized by $z = \sqrt{s}e^{i\theta}, 0 \le \theta \le 2\pi$. Then, we have

$$\begin{aligned}
\frac{\partial}{\partial \overline{w}} F(w) &= \frac{\partial}{\partial \overline{w}} \int_{\Gamma(w)} \tilde{f}\ dz \\
&= \frac{\partial}{\partial \overline{w}} \int_0^{2\pi} \tilde{f}(z, w) \frac{\partial z}{\partial \theta}\ d\theta \\
&= \int_0^{2\pi} \left(\frac{\partial \tilde{f}}{\partial z}(z, w) \frac{\partial z}{\partial \overline{w}} \frac{\partial z}{\partial \theta} + \tilde{f}(z, w) \frac{\partial^2 z}{\partial \overline{w} \partial \theta} \right) d\theta \\
&= \int_0^{2\pi} \frac{d}{d\theta} \left(\tilde{f}(z, w) \frac{\partial z}{\partial \overline{w}} \right) d\theta \\
&= 0.
\end{aligned}$$

The assertion in (2) follows now from (1). First, $F(w)$ is holomorphic on the interior of the set $D = \{w \in \mathbb{C}|\ \Gamma(w) \subset M \setminus \cup_{k=1}^{\infty} \tilde{T}_k\}$ and continuous up to the boundary of D. Observe that $\Gamma(w)$ degenerates to just a point on the t-axis, this implies $F(w) = 0$ on the t-axis, and hence $F(w) = 0$ on D.

For (3), we parameterize C'_k by $w(\phi)$ for $0 \le \phi \le 2\pi$. Then $\tilde{S}_k$ is parameterized by $(z(\phi, \theta), w(\phi)) = (\sqrt{\mathrm{Im} w(\phi)} e^{i\theta}, w(\phi))$, and using the fact that $\Gamma(w(\phi)) \subset M \setminus \cup_{k=1}^{\infty} \tilde{T}_k$, we obtain

$$\begin{aligned}
\iint\limits_{\tilde{S}_k} \tilde{f}\ dz \wedge dw &= \int_0^{2\pi} \int_0^{2\pi} \tilde{f}(z, w) \frac{dw}{d\phi} \frac{\partial z}{\partial \theta}\ d\theta d\phi \\
&= \int_0^{2\pi} \left(\int_0^{2\pi} \tilde{f}(z, w) \frac{\partial z}{\partial \theta}\ d\theta \right) \frac{dw}{d\phi}\ d\phi \\
&= \int_0^{2\pi} \left(\int\limits_{\Gamma(w(\phi))} \tilde{f}\ dz \right) \frac{dw}{d\phi}\ d\phi \\
&= 0.
\end{aligned}$$

This completes the proof of Lemma 12.5.2.

Lemma 12.5.3. *Let D be a domain with C^1 boundary on M. If $\tilde{f} : \overline{D} \to \mathbb{C}$ is a C^1 function, then*

$$\iint\limits_{bD} \tilde{f}\ dz \wedge dw = 2i \iiint\limits_{D} (\overline{L}\tilde{f})\ dt dx dy,$$

where $\overline{L}$ is defined in (12.5.2).

Proof. Notice that on M, $w = t + i|z|^2$. Hence,

$$dw = dt + i\overline{z}dz + izd\overline{z}.$$

Then, by Stokes' theorem we have

$$\begin{aligned}\iint\limits_{bD} \tilde{f}\ dz\wedge dw &= \iiint\limits_{D} d\tilde{f}\wedge dz\wedge dw\\ &= \iiint\limits_{D}\left(\frac{\partial\tilde{f}}{\partial\overline{z}}d\overline{z}+\frac{\partial\tilde{f}}{\partial\overline{w}}d\overline{w}\right)\wedge dz\wedge dw\\ &= \iiint\limits_{D}\left(-\frac{\partial\tilde{f}}{\partial\overline{z}}+2iz\frac{\partial\tilde{f}}{\partial\overline{w}}\right)\ dt\wedge dz\wedge d\overline{z}\\ &= 2i\iiint\limits_{D}(\overline{L}\tilde{f})\ dtdxdy.\end{aligned}$$

The proof of Lemma 12.5.3 is thus completed.

Now let g be a smooth function on $\mathbb{H}_2$ with support contained in $\cup_{k=1}^{\infty}\overline{T}_k$ such that g is positive on $\cup_{k=1}^{\infty}T_k$ and vanishes to infinite order at the origin. Define the operator $\overline{Z}_g$ on $\mathbb{H}_2$ by

$$\overline{Z}_g=\overline{Z}+g\frac{\partial}{\partial t}. \tag{12.5.4}$$

There exists a neighborhood U of the origin such that Z_g and $\overline{Z}_g$ are linearly independent and (U,Z_g) defines a strongly pseudoconvex CR structure on U. The next theorem shows that (U,Z_g) can not be realized as a three dimensional CR submanifold of $\mathbb{C}^n$ for any $n\geq 2$.

Theorem 12.5.4 (Nirenberg). *Let $\overline{Z}_g$ be defined as in (12.5.4). Suppose that f_1 and f_2 are two C^1 functions on $\mathbb{H}_2$ such that $\overline{Z}_gf_1=\overline{Z}_gf_2=0$ on U. Then $df_1\wedge df_2=0$ at the origin. In particular, the CR structure (U,Z_g) is not embeddable.*

Proof. The corresponding vector field of $\overline{Z}_g$ on M is given by

$$\overline{L}_g=\overline{L}+\tilde{g}\frac{\partial}{\partial t}.$$

It follows that

$$\overline{L}_g\tilde{f}_1=\overline{L}_g\tilde{f}_2=0$$

on $\pi^{-1}(U)$. Hence, by the construction of g, $\overline{L}\tilde{f}_1=-\tilde{g}(\partial\tilde{f}_1/\partial t)$ vanishes on $M\setminus\cup_{k=1}^{\infty}\tilde{T}_k$. Lemma 12.5.2 then implies that for all $k\geq 1$,

$$\begin{aligned}0=\iint\limits_{\tilde{S}_k}\tilde{f}_1\ dz\wedge dw &= 2i\iiint\limits_{\tilde{T}_k}(\overline{L}\tilde{f}_1)\ dtdxdy\\ &= -2i\iiint\limits_{\tilde{T}_k}\tilde{g}\frac{\partial\tilde{f}_1}{\partial t}\ dtdxdy.\end{aligned}$$

Since $\tilde{g}$ is positive on $\tilde{T}_k$, each of the functions $\mathrm{Re}(\partial \tilde{f}_1/\partial t)$ and $\mathrm{Im}(\partial \tilde{f}_1\ /\partial t)$ must vanish at some point in $\tilde{T}_k$ for all k. Equivalently, both $\mathrm{Re}(\partial f_1/\partial t)$ and $\mathrm{Im}(\partial f_1/\partial t)$ vanish at some point in T_k for all k. Hence, $(\partial f_1/\partial t)(0) = 0$. The fact that f_1 is a CR function with respect to the CR structure (U, Z_g) implies $(\partial f_1/\partial \overline{z})(0) = 0$. Thus, we obtain

$$df_1(0) = \frac{\partial f_1}{\partial z}(0) dz|_0.$$

A similar argument also holds for f_2. Therefore, $df_1(0)$ and $df_2(0)$ are always linearly dependent for any two CR functions f_1 and f_2 of class C^1 on $\mathbb{H}_2$. This proves the theorem.

We now extend the local nonembeddability example to higher dimensions. Let M be a smooth nondegenerate CR manifold in $\mathbb{C}^{n+1}$, $n \geq 2$, with signature $n-2$ near a point p, namely, the Levi form at $p \in M$ has either $n-1$ negative eigenvalues and one positive eigenvalue or $n-1$ positive eigenvalues and one negative eigenvalue. We may assume p is the origin. Let $r(z)$ be a local defining function for M. As in the proof of Theorem 3.3.2 we may write

$$r(z) = \mathrm{Im} z_{n+1} + \sum_{j,k=1}^{n} c_{jk} z_j \overline{z}_k + O(|z'||t| + |t|^2 + |(z',t)|^3)$$

in local coordinates $z = (z', z_{n+1})$, where $z_{n+1} = t + is$. Another linear change of coordinates will turn the defining function $r(z)$ locally to the form

$$r(z) = s - |z_1|^2 + \sum_{j=2}^{n} |z_j|^2 - \Psi(z', \overline{z}', t), \tag{12.5.5}$$

where $\Psi(z', \overline{z}', t) = O(|z'||t| + |t|^2 + |(z',t)|^3)$. Then we show that a small perturbation of the induced CR structure will in general yield a nonembeddable new CR structure on M.

Theorem 12.5.5 (Jacobowitz-Treves). *Let M be the nondegenerate CR manifold with signature $n-2$ defined locally near the origin in $\mathbb{C}^{n+1}$ by (12.5.5). Then there exists a new nonembeddable CR structure on M which agrees with the induced CR structure $\mathbb{C}T(M) \cap T^{1,0}(\mathbb{C}^{n+1})$ to infinite order at the origin.*

Proof. We shall identify M locally with an open subset U containing the origin in $\mathbb{C}^n \times \mathbb{R}$ via the map

$$\pi : (z', t + i\phi(z', \overline{z}', t)) \to (z', t),$$

where

$$\phi(z', \overline{z}', t) = |z_1|^2 - \sum_{j=2}^{n} |z_j|^2 + \Psi(z', \overline{z}', t). \tag{12.5.6}$$

It is easily verified that type $(0,1)$ vector fields on M are spanned by

$$\overline{L}_j = \frac{\partial}{\partial \overline{z}_j} - i\lambda_j \frac{\partial}{\partial \overline{z}_{n+1}}, \quad j = 1, \cdots, n, \tag{12.5.7}$$

where

$$\lambda_1 = 2\left(\frac{z_1 + \frac{\partial\Psi}{\partial\bar{z}_1}}{1 + i\frac{\partial\Psi}{\partial t}}\right) \quad \text{and} \quad \lambda_j = 2\left(\frac{-z_j + \frac{\partial\Psi}{\partial\bar{z}_j}}{1 + i\frac{\partial\Psi}{\partial t}}\right) \quad \text{for } j = 2, \cdots, n.$$

It follows that the corresponding embeddable CR structure on U is spanned by

$$\overline{Z}_j = \frac{\partial}{\partial\bar{z}_j} - i\lambda_j\frac{\partial}{\partial t}, \quad j = 1, \cdots, n, \tag{12.5.8}$$

where

$$\lambda_1 = \frac{z_1 + \frac{\partial\Psi}{\partial\bar{z}_1}}{1 + i\frac{\partial\Psi}{\partial t}} \quad \text{and} \quad \lambda_j = \frac{-z_j + \frac{\partial\Psi}{\partial\bar{z}_j}}{1 + i\frac{\partial\Psi}{\partial t}} \quad \text{for } j = 2, \cdots, n.$$

To get a nonembeddable CR structure we shall perturb the induced CR structure on U. Let h be any smooth function in z_{n+1} with support contained in $\{t + is \in \mathbb{C} |\ |t| \le s\}$. Note that h must vanish to infinite order at the origin. We set $w = t + i\phi(z', \bar{z}', t)$, the restriction of z_{n+1} to M, where $\phi(z', \bar{z}', t)$ is defined in (12.5.6). Then, the composition function $h \circ w$ is supported in $\{(z', t) \in U |\ |t| \le \phi(z', \bar{z}', t)\}$. Define

$$g = \frac{h\circ w}{z_1\frac{\partial w}{\partial t} - h\circ w} \quad \text{and} \quad \tilde{\lambda}_j = \lambda_j(1+g) \quad \text{for } j = 1, \cdots, n, \tag{12.5.9}$$

and set

$$\overline{Z}_{gj} = \frac{\partial}{\partial\bar{z}_j} - i\tilde{\lambda}_j\frac{\partial}{\partial t} = \overline{Z}_j - i\lambda_j g\frac{\partial}{\partial t}. \tag{12.5.10}$$

We claim that $\overline{Z}_{g1}, \cdots, \overline{Z}_{gn}$ defines a new CR structure on an open neighborhood, denoted still by U, containing the origin in $\mathbb{C}^n \times \mathbb{R}$ which agrees with $\overline{Z}_1, \cdots, \overline{Z}_n$ to infinite order at the origin. We shall show that

(1) $[\overline{Z}_{gj}, \overline{Z}_{gk}] = 0$ for $1 \le j, k \le n$, and
(2) the coefficients of $\overline{Z}_{gj}$ are smooth and agree with those of $\overline{Z}_j$ to infinite order at the origin.

Proof of the claim. Since the problem is purely local, we may assume that by shrinking the domain, if necessary, the open set U is sufficiently small,

$$U = \{(z', t) \in \mathbb{C}^n \times \mathbb{R} |\ |z_1| + |z''| + |t| < \epsilon\},$$

for some sufficiently small $\epsilon > 0$, where $z'' = (z_2, \cdots, z_n)$.

First, we show that the function g is well defined and smooth. The constant c that appears below may be different at each occurrence. We estimate the function $w = t + is$ with $s = \phi(z', \bar{z}', t)$ as follows:

$$\begin{aligned} s &\le |z_1|^2 - |z''|^2 + c(|z_1||t| + |z''||t| + |t|^2) + \epsilon(|z_1|^2 + |z''|^2 + |t|^2) \\ &\le (1+\epsilon)|z_1|^2 - (1-\epsilon)|z''|^2 + c(|z_1|^2 + |t|^2 + |z''||t|) \\ &\le (1+\epsilon+c)|z_1|^2 + 2c\epsilon|t|. \end{aligned}$$

Since $|t| \leq s$, we obtain if ϵ is sufficiently small,

$$s \leq c|z_1|^2,$$

and

$$|w| \leq c|z_1|^2,$$

on the support of $h \circ w$.

Since $h = O(|w|^k)$ for any $k \in \mathbb{N}$, we have $z_1 \neq 0$ if $h \circ w \neq 0$. Noting that $(\partial w/\partial t) = O(1)$, we see that the denominator of g is never zero if $h \circ w \neq 0$. Hence, we get $g = O(|z_1|^j |w|^k)$ for any $j, k \in \mathbb{N}$ which in turn implies that g is smooth and vanishes to infinite order at the origin. This proves (2).

To prove (1) we note that

$$[\overline{Z}_j, \overline{Z}_k] = 0, \quad \text{for } 1 \leq j, k \leq n.$$

Hence, a direct calculation shows that

$$[\overline{Z}_{gj}, \overline{Z}_{gk}] = \left\{ \lambda_k \left(-i\overline{Z}_j(g) + \frac{\partial \lambda_j}{\partial t}(g + g^2) \right) - \lambda_j \left(-i\overline{Z}_k(g) + \frac{\partial \lambda_k}{\partial t}(g + g^2) \right) \right\} \frac{\partial}{\partial t}.$$

Thus, for the integrability of the new CR structure it suffices to show that

$$\overline{Z}_j(g) = \lambda_j A - i\frac{\partial \lambda_j}{\partial t}(g + g^2), \quad j = 1, \cdots, n,$$

for some function A independent of j. Since $\overline{Z}_j w = 0$ for $j = 1, \cdots, n$, we get

$$\overline{Z}_j \left(\frac{\partial w}{\partial t} \right) = i \frac{\partial \lambda_j}{\partial t} \frac{\partial w}{\partial t}.$$

Note that

$$\overline{Z}_j \overline{w} = \overline{Z}_j (w + \overline{w}) = -2i\lambda_j,$$

hence,

$$\overline{Z}_j (h \circ w) = \left(\frac{\partial h}{\partial \overline{w}} \circ w \right) \overline{Z}_j(\overline{w}) = -2i\lambda_j \left(\frac{\partial h}{\partial \overline{w}} \circ w \right).$$

It follows that

$$\begin{aligned}
\overline{Z}_j(g) &= \overline{Z}_j \left(\frac{h \circ w}{z_1 \frac{\partial w}{\partial t} - h \circ w} \right) \\
&= \frac{-2i\lambda_j (\frac{\partial h}{\partial \overline{w}} \circ w)}{z_1 \frac{\partial w}{\partial t} - h \circ w} - \frac{(h \circ w)\overline{Z}_j (z_1 \frac{\partial w}{\partial t} - h \circ w)}{(z_1 \frac{\partial w}{\partial t} - h \circ w)^2} \\
&= \lambda_j A - i\frac{\partial \lambda_j}{\partial t}(g + g^2),
\end{aligned}$$

where

$$A = -2i(1+g) \frac{\frac{\partial h}{\partial \overline{w}} \circ w}{z_1 \frac{\partial w}{\partial t} - h \circ w}$$

is independent of j. This proves (1), and hence the claim.

Thus, we have shown that, for each smooth function h in z_{n+1} with support contained in $\{t+is \in \mathbb{C}|\ |t| \le s\}$, Equations (12.5.9) and (12.5.10) define a new CR structure on U. With an appropriate choice of h, we shall show that this new CR structure is not realizable locally near the origin. Let f be a CR function of class C^1 with respect to the new CR structure, namely, $\overline{Z}_{gj} f = 0$ for $j = 1, \cdots, n$. In particular, we have

$$\overline{Z}_{g1} f = 0. \tag{12.5.11}$$

We may set $z'' = (z_2, \cdots, z_n) = 0$ in (12.5.11), and reduces the problem to the case when $n = 1$. Obviously, we have $\overline{Z}_1 w = 0$ for $w = t + i(|z_1|^2 + \Psi(z_1, \overline{z}_1, t))$ with $\Psi = O(|z_1||t| + |t|^2 + |(z_1, t)|^3)$. Then, as in the three dimensional local nonembeddability example we study the intersection of $M_1 = M|_{z''=0}$ with the complex line $z_{n+1} = \mu$. Writing $\mu = \alpha + i\beta$, this intersection is given by

$$\Gamma(\mu) = P^{-1}(\mu) \cap M_1 = \{(z_1, \mu)|\ \beta = |z_1|^2 + \Psi(z_1, \overline{z}_1, \alpha)\},$$

where $P(z_1, z_{n+1}) = (0, z_{n+1})$ is the projection from $\mathbb{C}^2$ onto the second component. Then we have

Lemma 12.5.6. *In the μ-plane there is a smooth curve γ given by $\beta = \beta(\alpha)$ such that*

(1) *for $\beta < \beta(\alpha)$, $\Gamma(\alpha + i\beta) = \emptyset$,*
(2) *for $\beta = \beta(\alpha)$, $\Gamma(\alpha + i\beta)$ is a point which varies smoothly in α,*
(3) *for $\beta > \beta(\alpha)$, $\Gamma(\alpha + i\beta)$ is a simple closed curve which varies smoothly in μ.*

Proof. Let $z_1 = x + iy$, we write

$$\begin{aligned} F(x, y, \alpha) &= |z_1|^2 + \Psi(z_1, \overline{z}_1, \alpha) \\ &= x^2 + y^2 + \Psi(x, y, \alpha). \end{aligned}$$

Since Ψ vanishes at the origin to the order at least two, it is easily seen that for each fixed α the minimum of F occurs at a point $(x(\alpha), y(\alpha))$ which varies smoothly with α. Set $\beta(\alpha) = F(x(\alpha), y(\alpha), \alpha)$. This proves (1) and (2). For (3), we write

$$F(x, y, \alpha) = \beta(\alpha) + Q(x, y, \alpha) + \cdots,$$

where Q is a positive definite quadratic in $x_1 = x - x(\alpha)$ and $y_1 = y - y(\alpha)$. It follows that if $\beta > \beta(\alpha)$, then the level sets $\beta = F$ are smooth simple closed curves which vary smoothly with α and β. This completes the proof of Lemma 12.5.6.

Note that $\{t + is \in \mathbb{C}|\ |t| < s \text{ and } s > \beta(t)\}$ is an open subset in the z_{n+1}-plane with piecewise smooth boundary passing through the origin. Therefore, as in the three dimensional local nonembeddability example one may construct a sequence of disjoint open discs T'_k in this open set which converges to the origin, and let T_k be the corresponding solid open topological torus $\pi(\Gamma(T'_k))$ in U. Now let h be a smooth nonnegative function in the z_{n+1}-plane with support contained in $\cup_{k=1}^{\infty}\overline{T}'_k$ such that h is positive on $\cup_{k=1}^{\infty}T'_k$. Define g by (12.5.9), and let the new CR structure be defined by (12.5.10). Obviously, when restricted to $\{z''=0\}$, g is supported in $\cup_{k=1}^{\infty}\overline{T}_k$. Thus, it follows from the same arguments that Lemma 12.5.2 and 12.5.3 hold in this setting. Hence, for any solution f of class C^1 to $\overline{Z}_{gj}f=0$, $1\le j\le n$, we have

$$\int_{\pi(\Gamma(\mu))} f\ dz_1 = 0,$$

provided that $\mu\notin\cup_{k=1}^{\infty}T'_k$, and

$$\iiint\limits_{T_k}(\overline{Z}_1 f)\ dxdydt = 0. \tag{12.5.12}$$

Since $\overline{Z}_1 f = i\lambda_1 g(\partial f/\partial t)$ on T_k, we get

$$\iiint\limits_{T_k}\lambda_1 g\frac{\partial f}{\partial t}\ dxdydt = 0. \tag{12.5.13}$$

On T_k, the previous estimate shows that both λ_1 and $z_1(\partial w/\partial t)-h\circ w$ are given by $z_1+O(|z_1|^2)$. Thus, (12.5.13) becomes

$$\begin{aligned} 0 &= \iiint\limits_{T_k}(z_1+O(|z_1|^2))\frac{h\circ w}{z_1+O(|z_1|^2)}\frac{\partial f}{\partial t}\ dxdydt \\ &= \iiint\limits_{T_k}(1+O(|z_1|))(h\circ w)\frac{\partial f}{\partial t}\ dxdydt. \end{aligned}\tag{12.5.14}$$

Equation (12.5.14) holds for all k. Hence, we must have $(\partial f/\partial t)(0)=0$. Since f is a CR function of class C^1 with respect to this new CR structure, we conclude that $(\partial f/\partial\overline{z}_1)(0)=\cdots=(\partial f/\partial\overline{z}_n)(0)=0$. This implies $df(0)=(\partial f/\partial z_1)(0)dz_1+\cdots+(\partial f/\partial z_n)(0)dz_n$. Obviously, this new CR structure locally can not be CR embedded into $\mathbb{C}^N$ for any $N\ge n+1$. This proves Theorem 12.5.5.

NOTES

Boutet de Monvel's global embeddability theorem 12.2.1 for compact strongly pseudoconvex CR manifolds with dimension at least five is proved in [BdM 1]. Our presentation here follows that of J. J. Kohn [Koh 7]. Based on the ideas of Boutet de Monvel the formulation of Theorem 12.2.2 for $n=2$ can be found in [Bur 1]. For more details concerning various properties of the spherical harmonics the reader is referred to [StWe 1].

The nonembeddable compact strongly pseudoconvex CR manifold $(S^3, T_t^{1,0}(S^3))$ of dimension three for $0<|t|<1$ is due to H. Rossi [Ros 1]. We proved in Section 12.4 that Rossi's nonembeddable example can be CR immersed into a two dimensional complex submanifold M_t sitting in $\mathbb{C}^3$. The image of $(S^3, T_t^{1,0}(S^3))$,

$0 < |t| < 1$, under the map π defined by (12.4.3) is precisely described by (12.4.1) and (12.4.2). In particular, the image bounds a relatively compact domain Ω_t in M_t. Thus, by combining a theorem proved by L. Boutet de Monvel and J. Sjöstrand in [BdSj 1], one can show that the Szegö projection S on $(S^3, T_t^{1,0}(S^3))$ must map $C^\infty(S^3)$ into $C^\infty(S^3)$ continuously in the C^∞ topology (see also [Bur 1]). It follows from Kohn's work [Koh 10] that the nonclosedness of the range of $\overline{\partial}_b$ on $L^2(S^3)$ in the L^2 sense is the only obstruction to the global CR embeddability of $(S^3, T_t^{1,0}(S^3))$.

The three dimensional local nonembeddable strongly pseudoconvex CR structure was discovered by L. Nirenberg [Nir 4]. Theorem 12.5.5 which generalizes Nirenberg's local nonembeddability example to higher dimension is due to H. Jacobowitz and F. Treves [JaTr 1].

The local CR embedding problem for a strongly pseudoconvex CR manifold of dimension $2n-1$ with $n \geq 3$ is more complicated. M. Kuranishi showed in [Kur 1,2,3] that if $n \geq 5$, the answer is affirmative. Later, it was proved by T. Akahori [Aka 1] that the theorem remains true for $n = 4$. By employing Henkin's homotopy formula proved in Theorem 11.4.1 and using interior estimates of the solution operator, S. Webster presents in [Web 2,3] a simplified proof of the theorem for the cases $n \geq 4$ (see also [MaMi 2]). The remaining case $n = 3$ is still open. When the Levi form has mixed signature, CR embedding problems are discussed in [Cat 5]. Local homotopy formulas for $\overline{\partial}_b$ on CR manifolds with mixed Levi signatures have been obtained in [Sha 8,9] and [Tre 5].

APPENDIX

A. Sobolev Spaces

We include a short summary of the basic properties of the Sobolev spaces for the convenience of the reader. Our goal is to give precise definitions and statements of all theorems or lemmas about the Sobolev spaces which have been used in this book. Since most of the results are well-known and due to the vast amount of literature on this subject, we will provide very few proofs.

Let $f \in L^1(\mathbb{R}^N)$, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\ dx, \tag{1.1}$$

where $x \cdot \xi = \sum_{j=1}^{n} x_j \xi_j$. The estimate

$$\| \hat{f} \|_\infty \le \| f \|_{L^1}$$

is clear from the definition. We now list some basic properties of the Fourier transform whose proofs are left to the reader or can be found in any standard text. For instance, see Stein-Weiss [StWe 1].

Theorem A.1 (Riemann-Lebesgue). *Suppose that $f \in L^1(\mathbb{R}^N)$, then $\hat{f}(\xi) \in C_0$, where C_0 denotes the space of continuous functions on $\mathbb{R}^N$ that vanish at infinity.*

Theorem A.2 (Fourier Inversion). *Suppose that $f \in L^1(\mathbb{R}^N)$ and that $\hat{f}(\xi) \in L^1(\mathbb{R}^N)$. Then*

$$f(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} \hat{f}(\xi)\ d\xi, \quad a.e.$$

In other words, $f(x)$ can be redefined on a Lebesgue measure zero set so that $f(x) \in C_0$.

Theorem A.3 (Uniqueness). *If $f \in L^1(\mathbb{R}^N)$ and $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}^N$, then $f(x) = 0$ almost everywhere.*

Denote by $\mathcal{S}$ the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^N$, i.e., $\mathcal{S}$ consists of all smooth functions f on $\mathbb{R}^N$ with

$$\sup_{\mathbb{R}^N} |x^\beta D^\alpha f(x)| < \infty,$$

for all multiindices α, β, where $\alpha = (\alpha_1, \cdots, \alpha_N)$, $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ and $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_N}^{\alpha_N}$, each α_i is a nonnegative integer. Obviously, any smooth function with compact support belongs to $\mathcal{S}$ and we have the following formulas:

$$\begin{aligned} &(\widehat{D^\alpha f})\ (\xi) = (i\xi)^\alpha \hat{f}(\xi). \\ &D^\alpha \hat{f}(\xi) = ((\widehat{-ix)^\alpha} f)(\xi). \end{aligned} \tag{1.2}$$

Theorem A.4. *The Fourier transform is an isomorphism from $\mathcal{S}$ onto itself.*

Since $L^2(\mathbb{R}^N) \nsubseteq L^1(\mathbb{R}^N)$, the Fourier transform defined by (1.1) in general cannot be applied to L^2 functions directly. Using the following fundamental theorem of the Fourier transform, one can extend the definition to L^2 functions easily:

Theorem A.5 (Plancherel's Theorem). *The Fourier transform can be extended to be an automorphism of $L^2(\mathbb{R}^N)$ with*

$$\| \hat{f} \|^2 = (2\pi)^N \| f \|^2 \quad \textit{for all } f \in L^2(\mathbb{R}^N). \tag{1.3}$$

Equation (1.3) is called the Parseval's identity.

We collect a few results about the Sobolev spaces. For a detailed treatment of the Sobolev spaces $W^s(\Omega)$ for any real s, we refer the reader to Chapter 1 in Lions-Magenes [LiMa 1] for smooth domains or to Grisvard [Gri 1] for nonsmooth domains.

We first define the Sobolev spaces in $\mathbb{R}^N$. Let

$$p(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

be a differential operator of order m with constant coefficients. Then, by (1.2), it is easy to see that for any $f \in \mathcal{S}$,

$$(\widehat{p(D)f})(\xi) = p(i\xi)\hat{f}(\xi). \tag{1.4}$$

Here, the polynomial $p(i\xi)$ is obtained by replacing the operator D in $p(D)$ by $i\xi$.

For any $s \in \mathbb{R}$, we define $\Lambda^s : \mathcal{S} \to \mathcal{S}$ by

$$\Lambda^s u(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi)\, d\xi. \tag{1.5}$$

Set $\sigma(\Lambda^s) = (1 + |\xi|^2)^{s/2}$. $\sigma(\Lambda^s)$ is called the symbol of Λ^s. Define the scalar product $(u, v)_s$ on $\mathcal{S} \times \mathcal{S}$ by

$$(u, v)_s = (\Lambda^s u, \Lambda^s v)$$

and the norm

$$\| u \|_s = \sqrt{(u, u)_s} \quad \text{for } u \in \mathcal{S}.$$

The Sobolev space $H^s(\mathbb{R}^N)$ is the completion of $\mathcal{S}$ under the norm defined above. In particular, $L^2(\mathbb{R}^N) = H^0(\mathbb{R}^N)$. The Sobolev norms $\| \|_{H^s(\mathbb{R}^N)}$ for any $u \in C_0^\infty(\mathbb{R}^N)$ is given by

$$\| u \|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi. \tag{1.6}$$

Next, we define the Sobolev spaces for domains in $\mathbb{R}^N$. Let $\Omega \subset\subset \mathbb{R}^N$ be a domain with C^k boundary, $k = 1, 2, \cdots$. By this we mean that there exists a real-valued C^k function ρ defined in $\mathbb{R}^N$ such that $\Omega = \{z \in \mathbb{R}^N | \rho(z) < 0\}$ and $|\nabla\rho| \neq 0$ on $b\Omega$.

The implicit function theorem shows that locally, $b\Omega$ can always be expressed as a graph of a C^k function. If the boundary can be expressed locally as the graph of a Lipschitz function, then it is called a Lipschitz domain or a domain with Lipschitz boundary.

For any domain Ω in $\mathbb{R}^N$, let $H^s(\Omega)$, $s \geq 0$, be defined as the space of the restriction of all functions $u \in H^s(\mathbb{R}^N)$ to Ω. We define the norm of $H^s(\Omega)$ by

$$\| u \|_{H^s(\Omega)} = \inf_{\substack{\mathcal{U} \in H^s(\mathbb{R}^N) \\ \mathcal{U}|_\Omega = u}} \| \mathcal{U} \|_{s(\mathbb{R}^N)} . \tag{1.7}$$

When s is a positive integer, there is another way to define the Sobolev spaces by weak derivatives. For any domain $\Omega \subset \mathbb{R}^N$, we define $W^s(\Omega)$ to be the space of all the distributions u in $L^2(\Omega)$ such that

$$D^\alpha u \in L^2(\Omega), \quad |\alpha| \leq s,$$

where α is a multiindex and $|\alpha| = \alpha_1 + \cdots + \alpha_N$. We define the norm $\| \ \|_{W^s(\Omega)}$ by

$$\| u \|^2_{W^s(\Omega)} = \sum_{|\alpha| \leq s} \| D^\alpha u \|^2_{(\Omega)} < \infty. \tag{1.8}$$

The space $C^\infty(\overline{\Omega})$ denotes the space of functions which are restrictions of functions in $C^\infty(\mathbb{R}^N)$ to $\overline{\Omega}$. If Ω is a bounded Lipschitz domain, then $C^\infty(\overline{\Omega})$ is dense in $W^s(\Omega)$ in the $W^s(\Omega)$ norm (see Theorem 1.4.2.1 in Grisvard [Gri 1]). Thus $W^s(\Omega)$ can also be defined as the completion of the functions of $C^\infty(\overline{\Omega})$ under the norm (1.8) when Ω has Lipschitz boundary.

When $\Omega = \mathbb{R}^N$, we have $H^s(\mathbb{R}^N) = W^s(\mathbb{R}^N)$ for any positive integer s. This follows from Plancherel's theorem and the inequality

$$\frac{1}{C} \sum_{|\alpha| \leq s} |\xi^\alpha|^2 \leq (1 + |\xi|^2)^s \leq C \sum_{|\alpha| \leq s} |\xi^s|^2,$$

where $C > 0$.

Obviously for any bounded domain Ω, we have $H^s(\Omega) \subseteq W^s(\Omega)$ for any Ω. If $b\Omega$ is Lipschitz, the following theorem shows that the two spaces are equal:

Theorem A.6 (Extension Theorem). *Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary. For any positive integer s, there exists a continuous linear operator P_s from $W^s(\Omega)$ into $W^s(\mathbb{R}^N)$ such that*

$$P_s u|_\Omega = u.$$

The extension operator P_s can be chosen to be independent of s. In particular, we have

$$W^s(\Omega) = H^s(\Omega).$$

For a proof of Theorem A.6, see Chapter 6 in Stein [Ste 2] or Grisvard [Gri 1]. Thus when s is a positive integer and Ω is bounded Lipschitz, the Sobolev spaces will be denoted by $W^s(\Omega)$ with norm $\| \ \|_{s(\Omega)}$, or simply $\| \ \|_s$

Theorem A.7 (Sobolev Embedding). *If Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, then there is an embedding*

$$W^k(\Omega) \hookrightarrow C^m(\overline{\Omega}) \quad \textit{for any interger } m, \;\; 0 \le m < k - N/2.$$

Theorem A.8 (Rellich Lemma). *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. If $s > t \ge 0$, the inclusion $W^s(\Omega) \hookrightarrow W^t(\Omega)$ is compact.*

Theorem A.9 (Trace Theorem). *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. For $s > 1/2$, the restriction map $f \to f\big|_{b\Omega}$ for any $f \in C^\infty(\overline{\Omega})$ can be extended as a bounded operator from $W^s(\Omega)$ to $W^{s-1/2}(b\Omega)$. For any $f \in W^s(\Omega)$, $f\big|_{b\Omega} \in W^{s-1/2}(b\Omega)$ and there exists a constant C_s independent of f such that*

$$\| f \|_{s-\frac{1}{2}(b\Omega)} \le C_s \| f \|_{s(\Omega)} .$$

We remark that in general, the trace theorem does not hold for $s = 1/2$. However, if $f \in W^{1/2}(\Omega)$ and f is harmonic or f satisfies some elliptic equations, then the restriction of f to $b\Omega$ is in L^2 (c.f. Lemma 5.2.3).

Let Ω be a bounded domain in $\mathbb{R}^N$. We introduce other Sobolev spaces. Let $W_0^s(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under $W^s(\Omega)$ norm. When $s = 0$, since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, it follows that $W_0^0(\Omega) = W^0(\Omega) = L^2(\Omega)$. If $s \le 1/2$, we also have $C_0^\infty(\Omega)$ is dense in $W^s(\Omega)$. Thus

$$W^s(\Omega) = W_0^s(\Omega), \quad s \le \frac{1}{2}.$$

This implies that the trace theorem does not hold for $s \le 1/2$. When $s > 1/2$, $W_0^s(\Omega) \subsetneq W^s(\Omega)$.

We define $W^{-s}(\Omega)$ to be the dual of $W_0^s(\Omega)$ when $s > 0$ and the norm of $W^{-s}(\Omega)$ is defined by

$$\| f \|_{-s(\Omega)} = \sup \frac{|(f, g)|}{\| g \|_{s(\Omega)}},$$

where the supremum is taken over all functions $g \in C_0^\infty(\Omega)$. We note that the generalized Schwarz inequality for $f \in W^s(\Omega),\ g \in W^{-s}(\Omega)$,

$$|(f, g)_\Omega| \le \| f \|_{s(\Omega)} \| g \|_{-s(\Omega)}$$

holds only when $s \le 1/2$ for a bounded domain Ω. The proof of these results can be found in Lions-Magenes [LiMa 1] or Grisvard [Gri 1].

The Sobolev spaces can also be defined for functions or forms on manifolds. Let M be a compact Riemannian manifold of real dimension N. Choose a finite number of coordinate neighborhood systems $\{(U_i, \varphi_i)\}_{i=1}^m$, where

$$\varphi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^N$$

is a homeomorphism from U_i onto an open subset V_i contained in $\mathbb{R}^N$. For each i, $1 \le i \le m$, let $\{\eta_j^i\}_{j=1}^N$ be an orthonormal basis for $\mathbb{C}T^*(M)$ on U_i, and let $\{\zeta_i\}_{i=1}^m$

be a partition of unity subordinate to $\{U_i\}_{i=1}^m$. Thus, locally on each coordinate chart U_i, one may express a smooth r-form ϕ as

$$\phi = \sum_{|I|=r}{}' \phi_I^i \eta_I^i, \tag{1.9}$$

where $I = (i_1, \cdots, i_r)$ and $\eta_I^i = \eta_{i_1}^i \wedge \cdots \wedge \eta_{i_r}^i$. Then, we define the Sobolev s norm of $\phi \in \mathcal{E}^r(M)$, for $s \in \mathbb{R}$, by

$$\| \phi \|_s^2 = \sum_{i=1}^m \sum_{|I|=r}{}' \| (\zeta_i \phi_I^i) \circ \varphi_i^{-1} \|_s^2 \ . \tag{1.10}$$

Denote by $W_r^s(M)$ the completion of $\mathcal{E}^r(M)$ under the norm $\| \cdot \|_s$. The definition of $W_r^s(M)$ is highly nonintrinsic. Obviously, it depends on the choice of the coordinate neighborhood systems, the partition of unity and the local orthonormal basis $\{\eta_j^i\}$. However, it is easily seen that different choices of these candidates will come up with an equivalent norm. Therefore, $W_r^s(M)$ is a well-defined topological vector space. If M is a complex manifold of dimension n and Ω is a relatively compact subset in M, the space $W_{(p,q)}^s(\Omega)$, $0 \le p, q \le n$ and $s \in \mathbb{R}$, are defined similarly. The Sobolev embedding theorem and the Rellich lemma also hold for manifolds.

B. Interpolation Theorems and some Inequalities

There is yet another way to define the Sobolev spaces $W^s(\Omega)$ when s is not an integer and $s > 0$. Let k_1 and k_2 be two nonnegative integers and $k_1 > k_2$. On any domain Ω in $\mathbb{R}^N$, we have $W^{k_1}(\Omega) \subset W^{k_2}(\Omega)$. The space $W^s(\Omega)$ for $k_2 < s < k_1$ can be defined by interpolation theory. We shall describe the procedure in detail for the interpolation between W^1 and L^2 (i.e., when $k_1 = 1$ and $k_2 = 0$).

For each $v \in W^1(\Omega)$ and $u \in W^1(\Omega)$,

$$(u, v)_1 \ = \ (u, v) + \sum_{i=1}^N (D_i u, D_i v),$$

where $D_i = \partial/\partial x_i$. Let $D(\mathcal{L})$ denote the set of all functions u such that the linear map

$$v \longrightarrow (u, v)_1, \quad v \in W^1(\Omega)$$

is continuous in $L^2(\Omega)$. From the Hahn-Banach theorem and the Riesz representation theorem, there exists $\mathcal{L}u \in L^2(\Omega)$ such that

$$(u, v)_1 = (\mathcal{L}u, v), \quad v \in W^1(\Omega). \tag{2.1}$$

If $u \in C_0^\infty(\Omega)$, then $u \in D(\mathcal{L})$ and $\mathcal{L}u = (-\triangle + 1)u$. It is easy to see that $\mathcal{L}$ is a densely defined, unbounded self-adjoint operator and $\mathcal{L}$ is strictly positive since

$$(\mathcal{L}u, u) = \|u\|_1^2 \ge \|u\|^2.$$

Using the spectral theory of positive self-adjoint operators (see e.g. Riesz-Nagy [RiNa 1]), we can define $\mathcal{L}^\theta$ of $\mathcal{L}$ for $\theta \in \mathbb{R}$. Let

$$\Lambda = \mathcal{L}^{1/2}.$$

Then Λ is self-adjoint and positive in $L^2(\Omega)$ with domain W^1. From (2.1), we have

$$(u, v)_1 = (\Lambda u, \Lambda v), \quad \text{for every } u, \ v \in W^1(\Omega).$$

Definition B.1. *Let $W^\theta(\Omega)$ be the interpolation space between the spaces $W^1(\Omega)$ and $L^2(\Omega)$ defined by*

$$W^\theta(\Omega) \equiv [W^1(\Omega), L^2(\Omega)]_\theta = Dom(\Lambda^{1-\theta}), \quad 0 \le \theta \le 1,$$

with norm

$$\|u\| + \|\Lambda^{1-\theta}u\| = \textit{the norm of the graph of } \Lambda^{1-\theta},$$

where $Dom(\Lambda^{1-\theta})$ denotes the domain of $\Lambda^{1-\theta}$.

From the definition, we have the following interpolation inequality:

$$\|\Lambda^{1-\theta}u\| \le \|\Lambda u\|^{1-\theta}\|u\|^\theta \tag{2.2}$$

Thus

$$\|u\|_\theta \le C\|u\|_1^{1-\theta}\|u\|^\theta. \tag{2.3}$$

The general case for arbitrary integers k_1 and k_2 can be done similarly. Thus, this gives another definition for the Sobolev spaces $W^s(\Omega)$ when s is not an integer. If $b\Omega$ is bounded Lipschitz, this space is the same Sobolev space as the one introduced in Appendix A (see [LiMa 1] for details for the equivalence of these spaces). For a bounded Lipschitz domain, we can use any of the definitions for $W^s(\Omega)$, $s \ge 0$.

The following interpolation inequality holds for general Sobolev spaces:

Theorem B.2 (Interpolation Inequality). *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. For any $\epsilon > 0$, $f \in W^{s_1}(\Omega)$, $s_1 > s > s_2 \ge 0$, we have the following inequality:*

$$\| f \|_s^2 \le \epsilon \| f \|_{s_1}^2 + C_\epsilon \| f \|_{s_2}^2, \tag{2.4}$$

where C_ϵ is independent of f.

Theorem B.3 (Interpolation Theorem). *Let T be a bounded linear operator from $W^{s_i}(\Omega)$ into $W^{t_i}(\Omega)$, $i = 1,\ 2$, and*

$$s_1 > s_2 \ge -\frac{1}{2},\ t_1 > t_2 \ge -\frac{1}{2},$$

then T is bounded from $[W^{s_1}(\Omega),\ W^{s_2}(\Omega)]_\theta$ into $[W^{t_1}(\Omega), W^{t_2}(\Omega)]_\theta$, $0 \le \theta \le 1$.

We warn our reader of the danger of interpolation of spaces if the assumption $s_i \ge -1/2$ and $t_i \ge -1/2$ is dropped! (See [LiMa 1].) Next we discuss the interpolation between L^p spaces and some applications.

Definition B.4. *Let (X, μ) and (Y, ν) be two measure spaces and let T be a linear operator from a linear subspace of measurable functions on (X, μ) into measurable functions defined on (Y, ν). T is called an operator of type (p, q) if there exists a constant $M > 0$ such that*

$$\| Tf \|_{L^q} \le M \| f \|_{L^p} \tag{2.5}$$

for all $f \in L^p(X)$.

The least M for which inequality (2.5) holds is called the (p, q)-norm of T. If f is a measurable function on (X, μ), we define its *distribution function* $\lambda_f : (0, \infty) \to [0, \infty]$ by

$$\lambda_f(\alpha) = \mu(\{x \mid |f(x)| > \alpha\}).$$

Definition B.5. *Let (X, μ) and (Y, ν) be two measure spaces and let T be a linear operator from a linear subspace of measurable functions on (X, μ) into measurable functions defined on (Y, ν). T is a linear operator of weak type (p, q), $1 \le p \le \infty$ and $1 \le q < \infty$, if there exists a constant k such that*

$$\lambda(s) \le \left(\frac{k \parallel f \parallel_{L^p}}{s} \right)^q \quad \textit{for every } f \in L^p(X),$$

where λ is the distribution function of Tf.

We have the following interpolation theorems:

Theorem B.6 (Riesz-Thorin). *Let (X, μ) and (Y, ν) be two measure spaces and $p_0,\ p_1,\ q_0,\ q_1$ be numbers in $[1, \infty]$. If T is of type (p_i, q_i) with (p_i, q_i)-norm M_i, $i = 0, 1$, then T is of type (p_t, q_t) and*

$$\parallel Tf \parallel_{L^{q_t}} \le M_0^{1-t} M_1^t \parallel f \parallel_{L^{p_t}}, \tag{2.6}$$

provided

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \textit{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

with $0 < t < 1$.

For proof of this fact, see Theorem 1.3 in Chapter 5 in Stein-Weiss [StWe 1].

Theorem B.7 (Marcinkiewicz). *Let (X, μ) and (Y, ν) be two measure spaces and $p_0,\ p_1,\ q_0,\ q_1$ be numbers such that $1 \le p_i \le q_i \le \infty$ for $i = 0, 1$ and $q_0 \ne q_1$. If T is of weak type (p_i, q_i), $i = 0, 1$, then T is of type (p_t, q_t) provided*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \textit{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

with $0 < t < 1$.

For a proof of this theorem, see Appendix B in Stein [Ste 3].

Theorem B.8 (Hardy's Inequality). *If $f \in L^p(0, \infty)$, $1 < p \le \infty$ and*

$$Tf(x) = \frac{1}{x} \int_0^x f(t)dt, \quad x > 0,$$

then

$$\|Tf\|_{L^p} \le \frac{p}{p-1} \|f\|_{L^p}.$$

Proof. We use a change of variables and Minkowski's inequality for integrals,

$$\begin{aligned} \|Tf(x)\|_{L^p} &= \left\| \int_0^1 f(tx)dt \right\|_p \le \int_0^1 \|f(tx)\|_p dt \\ &= \int_0^1 \|f\|_p \frac{1}{t^{\frac{1}{p}}} dt = \frac{p}{p-1} \|f\|_{L^p}. \end{aligned}$$

Theorem B.9. *Let*

$$Tf(x) = \int_0^\infty K(x,y)f(y)dy, \quad x > 0,$$

where $K(x,y)$ is homogeneous of degree -1, that is, $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$, for $\lambda > 0$. If for each $1 \leq p \leq \infty$,

$$\int \mid K(1,y) \mid y^{-1/p}dy = A_p < \infty,$$

then

$$\|Tf\|_{L^p} \leq A_p\|f\|_{L^p}, \quad \text{for every } f \in L^p(0,\infty).$$

In particular, the Hilbert integral defined by

$$Tf(x) = \int_0^\infty \frac{f(y)}{x+y}dy, \quad x > 0,$$

is a bounded operator of type (p,p) for each $1 < p < \infty$.

Proof. Since

$$Tf(x) = \int_0^\infty K(1,y)f(xy)dy,$$

using Minkowski's inequality for integrals, we get

$$\|Tf\|_{L^p} \leq \left(\int \mid K(1,y) \mid y^{-1/p}dy\right)\|f\|_{L^p} = A_p\|f\|_{L^p}.$$

The Hilbert integral is of type (p,p) since, for $1 < p < \infty$, using contour integration, we have

$$\int \frac{y^{-1/p}}{1+y}dy = \frac{\pi}{\sin(\pi/p)}.$$

Theorem B.10. *Let (X,μ) and (Y,ν) be two measure spaces and let $K(x,y)$ be a measurable function on $X \times Y$ such that*

$$\int_X |K(x,y)| \; d\mu \leq C, \quad \text{for a.e.} \quad y,$$

and

$$\int_Y |K(x,y)| \; d\nu \leq C, \quad \text{for a.e.} \quad x,$$

where $C > 0$ is a constant. Then, for $1 \leq p \leq \infty$, the operator T defined by

$$Tf(x) = \int_Y K(x,y)f(y) \; d\nu$$

is a bounded linear operator from $L^p(Y,d\nu)$ into $L^p(X,d\mu)$ with

$$\| Tf \|_{L^p(X)} \leq C \| f \|_{L^p(Y)} .$$

For a proof of Theorem B.10, we refer the reader to Theorem 6.18 in Folland [Fol 3].

Theorem B.11. *Let (X, μ) and (Y, ν) be two measure spaces and $1 < q < \infty$. Let $K(x, y)$ be a measurable function on $X \times Y$ such that*

$$\nu\{\ y \in Y \mid K(x,y) > s\} \le \left(\frac{C}{s}\right)^q, \quad \text{for a.e. } x \in X,$$

and

$$\mu\{\ x \in X \mid K(x,y) > s\} \le \left(\frac{C}{s}\right)^q, \quad \text{for a.e. } y \in Y,$$

where $C > 0$ is a constant. Then the operator T defined by

$$Tf(x) = \int_Y K(x,y) f(y)\ d\nu$$

is a bounded linear operator from $L^p(Y)$ into $L^r(X)$ provided

$$1 < p < r < \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1.$$

T is bounded from $L^1(Y)$ to $L^{q-\epsilon}(X)$ for any $\epsilon > 0$.

The proof of this theorem is based on the Marcinkiewicz Interpolation Theorem B.7. We refer the reader to Theorem 15.3 in Folland-Stein [FoSt 1] or Theorem 6.35 in Folland [Fol 3].

C. Hardy-Littlewood Lemma and its Variations

We first prove the Hardy-Littlewood lemma for bounded Lipschitz domains.

Theorem C.1 (Hardy-Littlewood Lemma). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $\delta(x)$ denote the distance function from x to the boundary of Ω. If u is a C^1 function in Ω and there exists an $0 < \alpha < 1$ and $C > 0$ such that*

$$| \nabla u(x) | \le C\ \delta(x)^{-1+\alpha} \text{ for every } x \in \Omega, \tag{3.1}$$

then $u \in C^\alpha(\Omega)$, i.e., there exists some constant C_1 such that

$$| u(x) - u(y) | \le C_1 \mid x - y \mid^\alpha \ \text{ for } x, y \in \Omega.$$

Proof. Since u is C^1 in the interior of Ω, we only need to prove the assertion when x and y are near the boundary. Using a partition of unity, we can assume that u is supported in $U \cap \overline{\Omega}$, where U is a neighborhood of a boundary point $x_0 \in b\Omega$. After a linear change of coordinates, we may assume $x_0 = 0$ and for some $\varepsilon > 0$,

$$U \cap \Omega = \{x = (x', x_N) \mid x_N > \phi(x'), \mid x' \mid < \varepsilon, \mid x_N \mid < \varepsilon\},$$

where $\phi(0) = 0$ and ϕ is some Lipschitz function with Lipschitz constant M. The distance function $\delta(x)$ is comparable to $x_N - \phi(x')$, i.e., there exists a constant $C > 0$ such that

$$\frac{1}{C}\delta(x) \le x_N - \phi(x') \le C\delta(x) \quad \text{for } x \in \Omega. \tag{3.2}$$

We set $\tilde{x}' = \theta x' + (1-\theta)y'$ and $\tilde{x}_N = \theta x_N + (1-\theta)y_N$. Let $d = |x-y|$. If $x = (x', x_N),\ y = (y', y_N) \in \Omega$, then the line segment L defined by $\theta(x', x_N + Md) + (1-\theta)(y', y_N + Md) = (\tilde{x}', \tilde{x}_N + Md)$, $0 \le \theta \le 1$, lies in Ω since

$$\begin{aligned}
&\theta(x_N + Md) + (1-\theta)(y_N + Md) \\
&\qquad \ge Md + \theta\phi(x') + (1-\theta)\phi(y') \\
&\qquad \ge Md + \theta(\phi(x') - \phi(\tilde{x}')) + (1-\theta)(\phi(y') - \phi(\tilde{x}')) + \phi(\tilde{x}') \\
&\qquad \ge \phi(\tilde{x}').
\end{aligned}$$

Since u is C^1 in Ω, using the mean value theorem, there exists some $(\tilde{x}', \tilde{x}_N + Md) \in L$ such that

$$| u(x', x_N + Md) - u(y', y_N + Md) | \le | \nabla u(\tilde{x}', \tilde{x}_N + Md) | \, d.$$

From (3.1) and (3.2), it follows that

$$\begin{aligned}
| u(x', x_N + Md) - u(y', y_N + Md) | &\le C\delta(\tilde{x}', \tilde{x}_N + Md)^{-1+\alpha} \cdot d \\
&\le \tilde{C}((M+1)d)^{-1+\alpha} \cdot d \le C_M d^\alpha.
\end{aligned}$$

Also we have

$$\begin{aligned}
&| u(x) - u(x', x_N + Md) | \\
&\quad = \left| \int_0^{Md} \frac{\partial u(x', x_N + t)}{\partial t} dt \right| \\
&\quad \le C \int_0^{Md} \delta(x', x_N + t)^{-1+\alpha} dt \le C \int_0^{Md} (x_N + t - \phi(x'))^{-1+\alpha} dt \\
&\quad \le C \int_0^{Md} t^{-1+\alpha} dt \le C(Md)^\alpha.
\end{aligned}$$

Thus for any $x,\ y \in \Omega$,

$$\begin{aligned}
| u(x) - u(y) | &\le | (u(x) - u(x', x_N + Md) | + | u(y', u_N + Md) - u(y) | \\
&\quad + | u(x', x_N + Md) - u(y', y_N + Md) | \\
&\le C_M d^\alpha.
\end{aligned}$$

This proves the theorem.

The following is a variation of the Hardy-Littlewood lemma for Sobolev spaces.

Theorem C.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $\delta(x)$ be the distance function from x in Ω to the boundary $b\Omega$. If $u \in L^2(\Omega) \cap W^1_{\mathrm{loc}}(\Omega)$ and there exists an $0 < \alpha < 1$ such that*

$$\int_\Omega \delta(x)^{2-2\alpha} \mid \nabla u \mid^2 dV < \infty, \tag{3.3}$$

then $u \in W^\alpha(\Omega)$. *Furthermore, there exists a constant* C, *depending only on* Ω, *such that*

$$\| u \|^2_{\alpha(\Omega)} \le C \left(\int_\Omega \delta(x)^{2-2\alpha} \mid \nabla u \mid^2 dV + \int_\Omega |u|^2 dV \right).$$

Proof. For $0 < \alpha < 1$, $W^\alpha(\Omega) = [W^1(\Omega), L^2(\Omega)]_{1-\alpha}$. The interpolation norm of a function u in $W^\alpha(\Omega)$ (see Lions-Magenes [LiMa 1]) is comparable to the infimum over all functions

$$f \ : [0,\infty) \to L^2(\Omega) + W^1(\Omega) \quad \text{with } f(0) = u$$

of the norm I_f where I_f is defined to be

$$I_f = \left(\int_0^\infty \|t^{1-\alpha} f(t)\|^2_{W^1(\Omega)} t^{-1} dt \right)^{\frac{1}{2}} + \left(\int_0^\infty \|t^{1-\alpha} f'(t)\|^2_{L^2(\Omega)} t^{-1} dt \right)^{\frac{1}{2}}. \tag{3.4}$$

From (3.3), we have $u \in W^1(\Omega')$ for any $\Omega' \subset\subset \Omega$. Thus we only need to estimate u in a small neighborhood of the boundary. Using a partition of unity and a change of coordinates as in Theorem C.1, we can assume $U \cap \Omega = \{x_N > \phi(x')\}$. Let $\eta \in C_0^\infty(-\varepsilon, \varepsilon)$ such that $0 \le \eta \le 1, \eta \equiv 1$ when $\mid t \mid < \varepsilon/2$. We define

$$f(t) = u(x', x_N + t)\eta(t).$$

Then $f(0) = u(x)$ and $f(t) \in W^1(\Omega)$ for $t > 0$. To compute the norm defined by (3.4), we have

$$\begin{aligned} |I_f|^2 \le \ C \Bigg(&\int_0^\varepsilon \int_{\Omega\cap U} \mid u(x', x_N + t) \mid^2 dx \ t^{1-2\alpha} dt \\ &+ \int_0^\varepsilon \int_{\Omega \cap U} t^{1-2\alpha} \mid \nabla u(x', x_N + t) \mid^2 dx dt \Bigg). \end{aligned} \tag{3.5}$$

Since $1 - 2\alpha > -1$, the first integral on the right-hand side of (3.4) is bounded by $\|u\|^2_{L^2(\Omega)}$. To estimate the second integral on the right-hand side of (3.5), we first note that for $x \in \Omega \cap U$, using (3.2), there exists $C_1 > 0$,

$$\begin{aligned} \delta(x', x_N + t) &\ge C_1(x_N + t - \phi(x')) \\ &\ge C_1 t. \end{aligned}$$

Thus, after changing variables and the order of integration, we have

$$\begin{aligned} &\int_0^\varepsilon \int_{\Omega\cap U} t^{1-2\alpha} |\nabla u(x', x_N + t)|^2 dx dt \\ &\quad \le \int_{\Omega \cap U} \int_0^{C\delta(x)} t^{1-2\alpha} |\nabla u(x)|^2 dt dx \\ &\quad \le C \int_{\Omega\cap U} \delta(x)^{2-2\alpha} |\nabla u(x)|^2 dx \\ &\quad < \infty. \end{aligned}$$

This implies that $I_f < \infty$ and $u \in W^\alpha(\Omega)$. Theorem C.2 is proved.

Theorem C.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^∞ boundary and let s be a positive integer. If $u \in W_0^s(\Omega)$, then we have*

$$\delta^{-s+|\alpha|} D^\alpha u \in L^2(\Omega), \qquad \text{for every } \alpha \text{ with } |\alpha| \le s,$$

where δ is the distance function to the boundary, α is a multiindex and D^α is defined as in Appendix A.

Proof. If $f \in C_0^\infty(0,\infty)$, using Taylor's theorem, we have

$$f(x) = \frac{1}{(s-1)!} \int_0^x f^{(s)}(t)(x-t)^{s-1} dt.$$

Applying Hardy's inequality (Theorem B.8 in the Appendix), we see that

$$\begin{aligned} \left\| \frac{f(x)}{x^s} \right\|_{L^2} &\le \left\| \frac{1}{(s-1)!\, x} \int_0^x |f^{(s)}(t)| dt \right\|_{L^2} \\ &\le \frac{2}{(s-1)!} \|f^{(s)}(t)\|_{L^2}. \end{aligned}$$

Using localization and a partition of unity, we can assume that u is supported in a compact set in the upper half space $\{x = (x', x_N) \mid x_N \ge 0\}$. Applying the argument to the Taylor expansion in the x_N variable, we have for any $u \in C_0^\infty(\Omega)$,

$$\|\delta^{-s+|\alpha|} D^\alpha u\|_{L^2(\Omega)} \le C \|u\|_{W_0^s(\Omega)}.$$

The theorem follows by approximating $u \in W_0^s(\Omega)$ by functions in $C_0^\infty(\Omega)$.

Theorem C.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^∞ boundary. Let s be any positive number such that $s \ne n - 1/2$ for any $n \in \mathbb{N}$. If $u \in L^2(\Omega, loc)$ and*

$$\int_\Omega \delta^{2s} |u|^2 dV < \infty, \tag{3.6}$$

where δ is the distance function to the boundary, then $u \in W^{-s}(\Omega)$.

When $s = n - 1/2$ for some positive integer n, if we assume in addition that u is harmonic, the same statement also holds.

Proof. We first assume that s is a positive integer. For any $\upsilon \in W_0^s(\Omega)$, we have from Theorem C.3,

$$\begin{aligned} |(u, \upsilon)| &\le \|\delta^s u\| \|\delta^{-s} \upsilon\| \\ &\le C_s \|\delta^s u\| \|\upsilon\|_{W_0^s}. \end{aligned}$$

Thus, $u \in W^{-s}(\Omega)$ from definition.

For other s when $s \ne n - 1/2$, we use interpolation between $W^{-s}(\Omega)$. For $s_2 > s_1 \ge 0$, s_1, s_2 integers, if $(1-\theta)s_1 + \theta s_2 \ne n - 1/2$, then

$$[W^{-s_1}(\Omega), W^{-s_2}(\Omega)]_\theta = W^{-(1-\theta)s_1 - \theta s_2}(\Omega). \tag{3.7}$$

When $(1-\theta)s_1+\theta s_2 = n-1/2$, (3.7) no longer holds (see Lions-Magenes [LiMa 1]) and we restrict ourselves to harmonic functions.

We first prove for $s=1/2$. Using a partition of unity, we may assume that Ω is star-shaped and $0\in\Omega$. Define

$$v(x)=\int_0^1 \frac{1}{s}u(sx)ds.$$

Then v is harmonic and

$$\langle x,\nabla v(x)\rangle=\sum_{i=1}^N\int_0^1 x_i\frac{\partial u}{\partial x_i}(sx)ds=\int_0^1\frac{\partial}{\partial s}u(sx)ds=u(x)-u(0).$$

Without loss of generality, we may assume that $u(0)=0$. We have expressed u as a linear combination of the derivatives of some harmonic function v and, from our assumption,

$$\int_\Omega \delta(x)|\langle x,\nabla v\rangle|^2dV=\int_\Omega\delta(x)|u|^2dV<\infty, \tag{3.8}$$

where C is some positive constant. We claim that

$$\int_\Omega\delta(x)|\nabla v|^2dV\le C\left(\int_\Omega\delta(x)|\langle x,\nabla v\rangle|^2dV+\int_\Omega\delta(z)|v(x)|^2dV\right). \tag{3.9}$$

To prove (3.9), we apply the Rellich identity to the harmonic function v on the boundary $b\Omega_\eta$, where $\Omega_\eta=\{x\in\Omega\mid\delta(x)>\eta\}$ for small $\eta>0$. We have

$$\int_{b\Omega_\eta}\left(|\nabla v|^2\langle x,n\rangle-2\langle x,\nabla v\rangle\frac{\partial u}{\partial n}-(N-2)v\frac{\partial v}{\partial n}\right)dS=0, \tag{3.10}$$

where n is the outward normal on $b\Omega_\eta$ and dS is the surface element on $b\Omega_\eta$. Identity (3.10) follows from the equality

$$\begin{aligned}&\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|\nabla v|^2x_i-2\frac{\partial v}{\partial x_i}\langle x,\nabla v\rangle-(N-2)v\frac{\partial v}{\partial x_i}\right)\\&\quad=-2\triangle v\langle x,\nabla v\rangle-(N-2)v\triangle v=0\end{aligned}$$

and Stokes' theorem. If η is sufficiently small, we have $\langle x,n\rangle>C_0>0$ for some $C_0>0$ uniformly on $b\Omega_\eta$, it follows from (3.10) that

$$\begin{aligned}C_0\int_{b\Omega_\eta}|\nabla v|^2dS&\le\int_{b\Omega_\eta}\left|2\langle x,\nabla v\rangle\frac{\partial v}{\partial n}+(N-2)v\frac{\partial v}{\partial n}\right|dS\\&\le\epsilon\int_{b\Omega_\eta}\left|\frac{\partial v}{\partial n}\right|^2dS+C_\epsilon\left(\int_{b\Omega_\eta}|v|^2dS+\ |\langle x,\nabla v\rangle|^2dS\right),\end{aligned} \tag{3.11}$$

where $\epsilon > 0$. If ϵ is sufficiently small, the first term on the right-hand side of (3.11) can be absorbed by the left-hand side and we obtain

$$\int_{b\Omega_\eta} |\nabla v|^2 \leq C \left(\int_{b\Omega_\eta} |v|^2 dS + |\langle x, \nabla v\rangle|^2 dS \right). \tag{3.12}$$

Multiplying (3.12) by η and integrating over η, (3.9) is proved. Using (3.8) and (3.9), we get

$$\int_\Omega \delta(x)|\nabla v|^2 dV \leq C \int_\Omega \delta(x)|u|^2 dV < \infty. \tag{3.13}$$

It follows from Theorem C.2 that $v \in W^{\frac{1}{2}}(\Omega)$. Since for any first order derivative D with constant coefficients, we have

$$D : HW^{\frac{1}{2}}(\Omega) \to HW^{-\frac{1}{2}}(\Omega), \tag{3.14}$$

where $HW^s(\Omega) = W^s(\Omega) \cap \{u \in C^\infty(\Omega) \mid \triangle u = 0\}$. This implies that $u \in W^{-1/2}(\Omega)$. The cases for other integers can be proved similarly and this completes the proof of Theorem C.4.

We remark that (3.14) does not hold without restricting to the subspace of harmonic functions (see [LiMa 1]). The technique used in the proof of Theorem C.2 involves real interpolation, while the proof of (3.14) uses complex interpolation. We refer the reader to Jerison-Kenig [JeKe 1] and Kenig [Ken 3] for more discussion on these matters.

D. Friedrichs' Lemma

Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be a function with support in the unit ball such that $\chi \geq 0$ and

$$\int \chi dV = 1. \tag{4.1}$$

We define $\chi_\varepsilon(x) = \varepsilon^{-N}\chi(x/\varepsilon)$ for $\varepsilon > 0$. Extending f to be 0 outside D, we define for $\varepsilon > 0$ and $x \in \mathbb{R}^N$,

$$\begin{aligned} f_\varepsilon(x) = f * \chi_\varepsilon(x) &= \int f(x')\chi_\varepsilon(x - x')dV(x') \\ &= \int f(x - \varepsilon x')\chi(x')dV(x'). \end{aligned}$$

In the first integral defining f_ε we can differentiate under the integral sign to show that f_ε is $C^\infty(\mathbb{R}^N)$. From Young's inequality for convolution, we have

$$\| f_\varepsilon \| \leq \| f \| . \tag{4.2}$$

Since χ_ϵ is an approximation of the identity, we have $f_\varepsilon \to f$ uniformly if $f \in C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is a dense subset of $L^2(\mathbb{R}^N)$, this implies that

$$f_\varepsilon \to f \quad \text{in } L^2(\mathbb{R}^N) \quad \text{for every } f \in L^2(\mathbb{R}^N).$$

A very useful fact concerning approximating solutions of a first order differential operator by regularization using convolution is given by the following lemma (see Friedrichs [Fri 1] or Hörmander [Hör 2]):

Lemma D.1 (Friedrichs' Lemma). *If $v \in L^2(\mathbb{R}^N)$ with compact support and a is a C^1 function in a neighborhood of the support of v, it follows that*

$$aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon \to 0 \quad \text{in } L^2(\mathbb{R}^N) \quad \text{as } \epsilon \to 0, \tag{4.3}$$

where $D_i = \partial/\partial x_i$ and $aD_i v$ is defined in the sense of distribution.

Corollary D.2. *Let*

$$L = \sum_{i=1}^{N} a_i D_i + a_0$$

be a first order differential operator with variable coefficients where $a_i \in C^1(\mathbb{R}^N)$ and $a_0 \in C(\mathbb{R}^N)$. If $v \in L^2(\mathbb{R}^N)$ with compact support and $Lv = f \in L^2(\mathbb{R}^N)$ where Lv is defined in the distribution sense, the convolution $v_\epsilon = v * \chi_\epsilon$ is in $C_0^\infty(\mathbb{R}^N)$ and

$$v_\epsilon \to v, \quad Lv_\epsilon \to f \quad \text{in } L^2(\mathbb{R}^N) \quad \text{as } \epsilon \to 0. \tag{4.4}$$

Proof of Friedrichs' lemma. First note that if $v \in C_0^\infty(\mathbb{R}^N)$, we have from the discussion above,

$$D_i(v * \chi_\epsilon) = (D_i v) * \chi_\epsilon \to D_i v, \quad (aD_i v) * \chi_\epsilon \to aD_i v,$$

with uniform convergence. We claim that

$$\| aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon \| \leq C \| v \|, \quad v \in L^2(\mathbb{R}^N), \tag{4.5}$$

where C is some positive constant independent of ϵ and v. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, (4.3) will be proved if one can prove (4.5). To see this, we approximate v by a sequence $v_j \in C_0^\infty(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ and observe that if (4.5) holds, we have

$$\begin{aligned} &\| aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon \| \\ &\quad \leq C(\| v - v_j \| + \| aD_i(v_j * \chi_\epsilon) - (aD_i v_j) * \chi_\epsilon \|). \end{aligned}$$

Thus, it remains to prove (4.5). Without loss of generality, we may assume that $a \in C_0^1(\mathbb{R}^N)$ since v has compact support. We have for $v \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} &aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon \\ &\quad = a(x) D_i \int v(x-y)\chi_\epsilon(y)dy - \int a(x-y)\frac{\partial v}{\partial x_i}(x-y)\chi_\epsilon(y)dy \\ &\quad = \int (a(x) - a(x-y))\frac{\partial v}{\partial x_i}(x-y)\chi_\epsilon(y)dy \\ &\quad = -\int (a(x) - a(x-y))\frac{\partial v}{\partial y_i}(x-y)\chi_\epsilon(y)dy \\ &\quad = \int (a(x) - a(x-y))\, v(x-y)\frac{\partial}{\partial y_i}\chi_\epsilon(y)dy \\ &\qquad - \int \left(\frac{\partial}{\partial y_i} a(x-y)\right) v(x-y)\chi_\epsilon(y)dy. \end{aligned}$$

Let M be the Lipschitz constant for a such that $|a(x) - a(x-y)| \leq M|y|$ for all $x,\ y$. We obtain

$$|aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon| \leq M \int |v(x-y)|\,(\chi_\epsilon(y) + |yD_i\chi_\epsilon(y)|)\,dy.$$

Using Young's inequality for convolution, we have

$$\begin{aligned}\| \, aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon \, \| &\leq M \parallel v \parallel \int (\chi_\epsilon(y) + |yD_i\chi_\epsilon(y)|)\,dy \\ &= M(1 + m_i) \parallel v \parallel,\end{aligned}$$

where

$$m_i = \int |yD_i\chi_\epsilon(y)|dy = \int |y(D_i\chi)(y)|dy.$$

This proves (4.5) when $v \in C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, we have proved (4.5) and the lemma.

Proof of the Corollary. Since $a_0 v \in L^2(\mathbb{R}^N)$, we have

$$\lim_{\epsilon \to 0} a_0(v * \chi_\epsilon) = \lim_{\epsilon \to 0} (a_0 v * \chi_\epsilon) = a_0 v \quad \text{in } L^2(\mathbb{R}^N).$$

Using Friedrichs' lemma, we have

$$Lv_\epsilon - Lv * \chi_\epsilon = Lv_\epsilon - f * \chi_\epsilon \to 0 \quad \text{in } L^2(\mathbb{R}^N) \quad \text{as } \epsilon \to 0.$$

The corollary follows easily since $f * \chi_\epsilon \to f$ in $L^2(\mathbb{R}^N)$.

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