BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 4, Pages 529–533
S 0273-0979(03)00989-3
Article electronically published on July 10, 2003

Partial differential equations in several complex variables, by So-Chin Chen and Mei-Chi Shaw, American Mathematical Society, Providence, RI, 2001, xii + 380 pp., \$49.00, ISBN 0-8218-1062-6

Several complex variables is a moderately old field, dating back at least to the end of the nineteenth century. It got off to a slow start, as there were no tools available that would afford any deep insights. Power series were the primary technical device, and they did little to show the differences between one and several complex variables.

Matters changed dramatically in 1906 with the publication of two seminal papers, one by Poincaré and one by Hartogs. Poincaré proved that the unit ball

$$B = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

and the unit bidisc

$$D^2 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$$

are biholomorphically inequivalent. This discovery deflated any dreams that there would be an analog of the Riemann mapping theorem in several complex variables. We have been attempting to make peace with Poincaré's result for the past 100 years. Hartogs proved an equally shocking, but slightly more subtle fact. An appreciation of his result requires just a little background.

Let $\Omega \subseteq \mathbb{C}$ be a *domain*, that is, a connected open set. Let $\mathcal{W} = \{w_j\}$ be a sequence of points in Ω which has no interior accumulation point in Ω but which accumulates at every boundary point of Ω. By Weierstrass's theorem, there is a holomorphic function h on Ω which vanishes precisely on $\mathcal{W}$ and nowhere else. Now h cannot be analytically continued to any open domain that properly contains Ω; if it could, then there would be a (former) boundary point of Ω which is an interior accumulation point of the zeros, forcing the analytically continued h to be identically zero. That is a contradiction.

Thus we see that *any* domain in $\mathbb{C}$ is a domain of holomorphy—i.e., it supports an analytic function that cannot be analytically continued to any larger domain. Hartogs discovered that if

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 2, |z_2| < 2\} \setminus \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 1\},$$

then any function f that is holomorphic on Ω will perforce analytically continue to the strictly larger domain

$$\widehat{\Omega} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 2, |z_2| < 2\}.$$

So this Ω is *not* a domain of holomorphy.[1]

It became a major program of twentieth century mathematics to give an extrinsic geometric characterization of domains of holomorphy in several complex variables.

2000 *Mathematics Subject Classification.* Primary 35N15; Secondary 35N99, 35J15, 32W05, 32W10.

[1]Hartogs is known for this result and one other: that a separately holomorphic function is in fact jointly holomorphic. Both these theorems appear in the remarkable paper [HAR]. It is a tragic fact that Hartogs (and Hausdorff too) committed suicide in the 1930s rather than live under the Nazi terror.

A condition known as *pseudoconvexity*, which is a biholomorphically invariant version of convexity (see [KRA1], [KRA2]), was formulated by E. E. Levi around 1912 and was soon conjectured to be the sought-after notion. Elementary but tricky arguments were concocted by the Germans in the 1920s and 1930s to show that if a domain is a domain of holomorphy, then it is pseudoconvex. The question of proving the converse result became known as the *Levi problem*.

The Levi problem was cracked by Kiyoshi Oka in the 1940s—first in dimension two and then in all dimensions. To solve the problem he used something called the Cousin problems, which are a natural outgrowth of the combinatorial formulation of cohomology theory and which shoehorns rather naturally into the modern theory of sheaf cohomology. During the course of the 1940s and 1950s, Narasimhan, Bremerman, Norguet, and many others helped to develop and iron out the Levi problem. There are still aspects of the theory, and particular questions, that are open today; study of the Levi problem continues. But the question of pseudoconvex domains and domains of holomorphy remains a cornerstone of the theory.

In point of fact, the Levi problem is so fundamental that new techniques have been developed for studying the matter. One of these is the methodology of differential geometry—invariant metrics and completeness and curvature. Another is the method of partial differential equations. The idea is to study the inhomogeneous Cauchy-Riemann equations. Again, a little background is required.

In one complex variable, define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

It is natural to wonder why the good guys have a minus sign and the bad guys have a plus sign. The answer is that of course we want

$$\begin{array}{ccc} \dfrac{\partial}{\partial z} z = 1 & , & \dfrac{\partial}{\partial z}\overline{z} = 0 \\ \dfrac{\partial}{\partial \overline{z}} z = 0 & , & \dfrac{\partial}{\partial \overline{z}}\overline{z} = 1 . \end{array}$$

Then it is an easy calculation (using the classical Cauchy-Riemann equations) to see that a continuously differentiable function u on a domain $U \subseteq \mathbb{C}$ is holomorphic if and only if $\partial u/\partial \overline{z} = 0$.

In several complex variables there is a similar notation: for $j = 1, \ldots, n$ we set

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Then a continuously differentiable function u on a domain $\Omega \subseteq \mathbb{C}^n$ is holomorphic if $\partial u/\partial \overline{z}_j = 0$ on Ω for $j = 1, \ldots, n$.

It turns out to be useful in several complex variables (somewhat less so in one complex variable) to study the so-called *inhomogeneous* Cauchy-Riemann equations:

$$\frac{\partial u}{\partial \overline{z}_j} = f_j \ , \quad j = 1, \ldots, n . \tag{$*$}$$

Since

$$\frac{\partial^2 g}{\partial \overline{z}_j \partial \overline{z}_k} = \frac{\partial^2 g}{\partial \overline{z}_k \partial \overline{z}_j}$$

for a suitable smooth function g, we see easily that a necessary condition for solving $(*)$ is that

$$\frac{\partial f_j}{\partial \overline{z}_k} = \frac{\partial f_k}{\partial \overline{z}_j} \tag{$\star$}$$

for all $j, k = 1, \ldots, n$. The question is then, subject to the compatibility condition $(\star)$, to find a solution u of $(*)$. In practice, we abbreviate the equations $(*)$ with $\overline{\partial} u = f$ (where $\overline{\partial} u \equiv \frac{\partial u}{\partial \overline{z}_1} d\overline{z}_1 + \cdots \frac{\partial u}{\partial \overline{z}_n} d\overline{z}_n$ and $f = f_1 \overline{z}_1 + \cdots + f_n d\overline{z}_n$) and we phrase the compatibility condition $(\star)$ as "f is a $\overline{\partial}$-closed form".

Why might one wish to solve $(*)$? Let us illustrate with an example. A fundamental result in the subject, essentially due to Hörmander, is this:

Theorem (Hörmander)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be pseudoconvex. If f is a $\overline{\partial}$-closed form on Ω with smooth coefficients, then there is a smooth u on Ω such that $\overline{\partial} u = f$.*

The proof of this theorem is hard work (see, for example, [KRA1]), and we cannot indicate it here. It can be used to establish the following very interesting theorem:

Extension Theorem.[2] *Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain. Assume that the complex hyperplane $\mathbf{p} = \{z : z_n = 0\}$ intersects Ω in such a way that $\omega \equiv \Omega \cap \mathbf{p}$ is a domain of the $(n-1)$ complex variables $z_1, \ldots, z_{n-1}$. Now suppose that g is a holomorphic function on ω. Then there exists a holomorphic function G on all of Ω such that $G\big|_\omega = g$.*

Sketch of Proof of the Extension Theorem. Define $\pi(z) = \pi(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, 0)$. Set $B = \{z \in \Omega : \pi(z) \notin \omega\}$. Then B and ω are disjoint and relatively closed in Ω. Thus, by the C^∞ version of Urysohn's lemma, there is a C^∞ function φ on Ω such that $\varphi \equiv 1$ in a relative neighborhood of ω and $\varphi \equiv 0$ in a relative neighborhood of B.

Now set

$$G(z) = \varphi(z) \cdot g(\pi(z)) + z_n \cdot K(z)\,. \tag{$\dagger$}$$

Notice that the first term on the right-hand side here is a C^∞ extension of g to all of Ω. We include the second term in hopes that (with a propitious choice of K) we can force G to be holomorphic while still preserving this extension property.

Asking that G be holomorphic is the same as asking that $\overline{\partial} G \equiv 0$ on Ω. Doing a little algebra, we find that the condition is thus

$$\overline{\partial} K(z) = -\frac{\overline{\partial}\varphi(z) \cdot g(\pi(z)}{z_n}\,. \tag{$\ddagger$}$$

The apparent difficulty with dividing by zero goes away because $\overline{\partial}\varphi$ vanishes in a neighborhood of ω (just because φ is identically 1 in a neighborhood of ω). Another calculation shows that the right-hand side of this last equation is $\overline{\partial}$-closed. Hörmander's theorem tells us then that $(\ddagger)$ has a solution. Thus we may create G as specified in $(\dagger)$, and that is the holomorphic extension of g that we seek. $\square$

[2]It turns out that this theorem is essentially equivalent to a solution of the Levi problem. For the details of this assertion, see [KRA1]. It is also worth noting that this theorem is so trivial in one complex variable as to be virtually nonsensical.

The theory of the $\overline{\partial}$ problem is extensive and well developed. Pioneers of the subject were Spencer and Morrey. Kohn solved the problem in the early 1960s, and Hörmander developed another point of view a few years later. In the late 1960s and early 1970s, yet another point of view stemming from integral formulas was unfolded. The full story of these various approaches is detailed in [HOR3], [KRA1], [RAN].

The $\overline{\partial}$ equation is but one aspect of the partial differential equations lore of several complex variables. There are also the Monge-Ampère equation, the $\overline{\partial}_b$ equation, and the Laplace equation, and many other aspects and ramifications that cannot be explored here. This material is the *métier* for the book under review.

There are not many books that develop the important symbiotic relationship of complex analysis with partial differential equations. The book [KRA3] has already been mentioned. Of course [FOK] is the primordial reference for Kohn's approach to the $\overline{\partial}$ problem, and [HOR1], [HOR2] are the quintessential sources for Hörmander's approach. Aubin's book [AUB] talks about the Monge-Ampère equation. But this new book of Chen and Shaw is the first definitive and comprehensive treatment of partial differential equations in the subject of several complex variables. As one who works in the subject area, I am immensely grateful that this book was written. The book not only covers the classical material that is treated elsewhere, but it also gives very careful treatments of very modern topics like

- Christ's proof of the failure of Condition R on the worm domain,
- Nirenberg's result on the nonembeddability of certain CR manifolds,
- sharp estimates for the $\overline{\partial}$ equation,
- L^2 existence theorems for the $\overline{\partial}_b$ complex,
- the Lewy unsolvable operator,
- subellipticity of $\Box_b$.

There are many more.

I do not know any polite way to say this: Many of the papers and books in this subject are fraught with errors. They are inconsistent in notation, they are inconsistent with each other, and they are frequently rather opaque. It is really quite a battle for the student to get up to speed in the subject. Chen and Shaw present the *entire picture*, from soup to nuts (with adequate prerequisites and also appendices on ancillary topics such as Sobolev spaces and the Friedrichs extension theorem) in a relatively self-contained manner. The book is lovingly and carefully written. I have yet to find any serious errors. It is a delightful read.

In fact, I am an expert in this subject, and I refer to the Chen/Shaw book every day. It is one of the most valuable books that I own. I fear that I tend to wax euphoric in my praise of this effort. For those who want to learn more about the interaction of partial differential equations and complex analysis, for those who want to learn the partial differential equations approach to solving the Levi problem, or for those hard analysts who want a sympathetic introduction to several complex variables, I can hardly think of a better source.

This book is deep and substantial and difficult. It does not conform to Sammy Eilenberg's dictum that doing mathematics should be "like floating on your back downstream." But its contribution will be a lasting one. The book elucidates and clarifies an important but heretofore obscure subject area. It has enriched the world of several complex variables.

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