Hearing Pseudoconvexity in Complex Manifolds

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Outline

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The Cauchy-Riemann equation in $\mathbb{C}$

Let $\mathbb{C}$ be the complex Euclidian space with coordinate $z = x + iy$. We define the *Cauchy-Riemann operator*

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

A function h is *holomorphic (or analytic)* in a domain D of $\mathbb{C}$, if h satisfies the homogeneous Cauchy-Riemann equation

$$\frac{\partial h}{\partial \bar{z}} = 0. \tag{1}$$

Let $h = u + iv$, where u and v are real-valued functions on D. The equation (1) is equivalent to

$$u_x = v_y, \ u_y = -v_x. \tag{2}$$

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The *inhomogeneous Cauchy-Riemann equation*

$$\frac{\partial u}{\partial \bar{z}} = f, \tag{3}$$

where f a given function in D.

Theorem

Let D be a bounded domain in $\mathbb{C}$ and let $f \in C^k(\overline{D})$, for $k \geq 1$. Then the function defined by

$$u(z) := \frac{1}{2\pi i} \iint_D \frac{f(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta},$$

is in $C^k(D)$ and satisfies (3). Moreover, if f is only in $C(\overline{D})$, then $u(z)$ defined as before satisfies (3) in the distribution sense.

Proof: This can be derived from the Generalized Cauchy Integral Formula.

The fundamental solution for $\overline{\partial}$

- Observe that the function

$$u(z) = \frac{1}{\pi} \cdot \frac{1}{z},$$

is a fundamental solution to (1).

- This can be derived by differentiating the fundamental solution for the Laplace operator $\triangle = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$. Since

$$\frac{1}{2\pi} \triangle \log |z| = \frac{2}{\pi} \frac{\partial^2}{\partial \bar{z} \partial z} \log |z| = \delta_0,$$

where δ_0 is the Dirac delta function centered at 0.

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The Cauchy-Riemann equations in $\mathbb{C}^n$

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidian space, $n \geq 2$. We denote coordinates by $z = (z_1, \ldots, z_n)$, where $z_j = x_j + iy_j$, $1 \leq j \leq n$. We can define the *Cauchy-Riemann operator*

$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \quad 1 \leq j \leq n.$$

The *Cauchy-Riemann equations* are

$$\frac{\partial u}{\partial \bar{z}_j} = f_j, \qquad j = 1, \ldots, n, \quad n \geq 2. \tag{4}$$

This system is *overdetermined* (one unknown function with n equations). In order for Equation (4) to be solvable, f must satisfy the following *compatibility conditions*

$$\frac{\partial f_k}{\partial \bar{z}_j} = \frac{\partial f_j}{\partial \bar{z}_k}, \qquad 1 \leq j, k \leq n. \tag{5}$$

(Non)-Riemann Mapping Theorem

Theorem (Riemann Mapping Theorem)

Let D be a simply connected domain in $\mathbb{C}$. Then D is either $\mathbb{C}$ or bihomorphic to the unit disc $\triangle = \{|z| < 1\}$.

When $n \geq 2$, there is no such uniformization theorem in $\mathbb{C}^n$. Let

$$B_n = \{|z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$$

and

$$\triangle^n = \{|z_1| < 1, \ldots, |z_n| < 1\}.$$

Theorem (Poincaré (1907))

The ball B_n and the polydisc $\triangle^n$ in $\mathbb{C}^n$ are not equivalent when $n \geq 2$.

Poincaré proves the theorem by comparing the automorphism groups of the two domains.

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Hartogs' phenonmenon

One of the most striking differences between holomorphic functions in one variable and several variables is the so called *Hartogs phenomenon*.

Theorem (Hartogs' phenomenon (1906))

Let D be an open set in $\mathbb{C}^n$, $n \geq 2$, and let K be a compact subset in D, such that $D \setminus K$ is connected. If $f \in \mathcal{O}(D \setminus K)$, then f can be extended holomorphically to all D.

- This is not true in $\mathbb{C}$.
- The zero set of a non-trivial holomorphic function in $\mathbb{C}$ is isolated. The zero set of a holomorphic function in $\mathbb{C}^n$ is a variety.
- One can solve $\bar{\partial}$ with compact support in $\mathbb{C}^n$, but not in $\mathbb{C}$.

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Domain of holomorphy

Definition

Let Ω be a domain in $\mathbb{C}^n$. D is a domain of holomorphy if for any $p \in bD$, there exists a holomorphic function $f \in \mathcal{O}(D)$ such that f is singular at p.

The following domains are domains of holomorphy:

- (1) Any domain in the complex plain $\mathbb{C}$.
- (2) Product of planar domains in $\mathbb{C}^n$.
- (3) Convex domains in $\mathbb{C}^n$.

The Levi Problem

What kind of domains are domains of holomorphy?

Lemma (Levi)

A C^2 bounded domain Ω in $\mathbb{C}^n$, $n \geq 2$, is a domain of holomorphy only if it is pseudoconvex.

Pseudoconvex Domains

Let Ω be a domain of $\mathbb{C}^n$ with C^2 boundary with a C^2 defining function ρ.

Definition

The domain Ω is *(Levi) pseudoconvex* if

$$\mathcal{L}|_p(\rho; a) = \partial\bar{\partial}\rho_p(a, \bar{a}) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) a_j \bar{a}_k \geq 0, \qquad \forall\, p \in bD,$$

for all vector $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ such that $\sum_{j=1}^{n} a_j \frac{\partial \rho}{\partial z_j} = 0$ $(a \in T_p^{1,0}(b\Omega))$. $\mathcal{L}_p(\rho; a)$ is called the *Levi form* of ρ at p. A domain D is called *strictly pseudoconvex* if the Levi form is positive definite. We call the $\partial\bar{\partial}\rho|_p(a, \bar{a}))$ the complex Hessian.

Definition

A function ρ is called (strictly) plurisubharmonic at p if $\mathcal{L}_p(\rho; a) \geq 0$ (> 0) for all vectors $a \in \mathbb{C}^n$.

Outline

Inhomogeneous Cauchy-Riemann equations

The $\overline{\partial}$-problem

Let Ω be a domain in $\mathbb{C}^n$ (or a complex manifold), $n \geq 2$. For the complexified tangent space $\mathbb{C}T\mathbb{C}^n$, there is a natural decomposition:
$\mathbb{C}T\mathbb{C}^n = T^{1,0}(\mathbb{C}^n) \oplus T^{0,1}(\mathbb{C}^n)$,
$T^{1,0}(\mathbb{C}^n) = \text{span}\left\{\frac{\partial}{\partial z_j}\right\}$, $T^{0,1}(\mathbb{C}^n) = \text{span}\left\{\frac{\partial}{\partial \bar{z}_j}\right\}$.
Let f a C^1 function defined on an open subset D of $\mathbb{C}^n$. We have

$$df = \partial f + \overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}.$$

The $\overline{\partial}$ complex

$$d = \partial + \overline{\partial}$$

$$d^2 = 0, \partial^2 = 0, \overline{\partial}^2 = 0.$$

The $\overline{\partial}$-equation

Given a (p,q)-form g in Ω such that $\overline{\partial} g = 0$, find a $(p,q-1)$-form u such that

$$\overline{\partial} u = g.$$

If g is in $\mathcal{C}^\infty_{p,q}(\Omega)$ (or $g \in \mathcal{C}^\infty_{p,q}(\overline{\Omega})$), one seeks $u \in \mathcal{C}^\infty_{p,q-1}(\Omega)$ (or $u \in \mathcal{C}^\infty_{p,q-1}(\overline{\Omega})$).

Definition

A (p,q)-form g satisfying $\overline{\partial} g = 0$ is called $\overline{\partial}$-closed. A (p,q)-form $g = \overline{\partial} u$ for some $(p,q-1)$-form u is called $\overline{\partial}$-exact.

This is analogous to the problem of solving the real de Rham complex

$$du = g$$

for a q-form g with $dg = 0$ on Ω.

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de Rham Cohomology

Let Ω be a domain in $\mathbb{R}^n$ (or a differential manifold), $n \geq 2$.

de Rham Cohomology

$$H^q_{de}(\Omega) = \frac{\ker\{d : \mathcal{C}^\infty_q(\Omega) \to \mathcal{C}^\infty_{q+1}(\Omega)\}}{\text{range}\{d : \mathcal{C}^\infty_{q-1}(\Omega) \to \mathcal{C}^\infty_q(\Omega)\}}$$

- Obstruction to solving the d-problem on Ω.
- If the boundary of Ω is sufficiently smooth (C^1 or Lipschitz), $H^q_{de}(\Omega)$ is finite dimensional.
- From the de Rham's Theorem: $H^q_{de}(\Omega) \cong H^q_{\text{sing}}(\Omega)$.
- $H^q_{de}(\Omega)$ is a topological invariant.
- $H^0_{de}(\Omega) = \{\mathbb{R}\}$ and $H^1(\Omega) = \beta_1$, the first Betti number.

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- $H^{p,0}(\Omega)$ is the space of holomorphic functions or forms in Ω.
- if $H^{p,q}(\Omega) = 0$, every $\bar\partial$-closed form is $\bar\partial$-exact.
- Natural topology arising as quotients of Fréchet topologies on $\ker(\bar\partial)$ and range$(\bar\partial)$.
- If the range$(\bar\partial)$ is closed, we also say that one can solve $\bar\partial$ (in the functional analysis sense).
- This quotient topology is Hausdorff if and only if range$(\bar\partial)$ is closed in $\mathcal{C}^\infty_{p,q}(\Omega)$

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Solution to the Levi Problem

Theorem

Let Ω be a domain in $\mathbb{C}^n$. The following are equivalent:

- *Ω is pseudoconvex;*
- *Ω is a domain of holomorphy;*
- *For each $0 \le p \le n$, $1 \le q \le n$, if $f \in C^\infty_{p,q}(\Omega)$ with $\bar\partial f = 0$, there is $u \in C^\infty_{p,q-1}(\Omega)$ such that $\bar\partial u = f$.*
- $H^{p,q}(\Omega) = 0, \qquad 0 \le p \le n,\ 1 \le q \le n.$

- Sheaf method: (1950) K. Oka, H. Bremermann and F. Norguet (n=2) and (1958) H. Grauert.
- L^2 method: (1962, 63) Kohn on smooth strictly pseudoconvex domains.
- L^2 method: (1965) Hörmander on pseudoconvex domains.
- Integral Kernel: (1970) Henkin-Grauert on smooth strictly pseudoconvex domains.

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L^2-approach to $\overline{\partial}$

Two ways to close an unbounded operator in L^2

- (1) The (weak) maximal closure of $\overline{\partial}$:
  Realize $\overline{\partial}$ as a closed densely defined (maximal) operator
  $$\overline{\partial} : L^2_{p,q}(\Omega) \to L^2_{p,q+1}(\Omega).$$
  The L^2-Dolbeault Coholomolgy is defined by
  $$H^{p,q}_{L^2}(\Omega) = \frac{\ker\{\overline{\partial} : L^2_{p,q}(\Omega) \to L^2_{p,q+1}(\Omega)\}}{\mathrm{range}\{\overline{\partial} : L^2_{p,q-1}(\Omega) \to L^2_{p,q}(\Omega)\}}$$
- (2) The (strong) minimal closure of $\overline{\partial}$: Let $\overline{\partial}_c$ be the (strong) minimal closed L^2 extension of $\overline{\partial}$.
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The $\overline{\partial}$-Neumann problem

Let $\Box_{p,q}$ ($\overline{\partial}$-Laplacian) be the closed self-adjoint densely defined (unbounded) operator :

$$\Box_{p,q} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : L^2_{p,q}(\Omega) \to L^2_{p,q}(\Omega)$$

and let $\mathcal{H}^{p,q}(\Omega) = \ker \Box_{p,q}$, the space of harmonic (p,q)-forms.

Consequences of the closed range property of $\overline{\partial}$

Suppose that the range of $\overline{\partial}$ is a closed subspace in $L^2_{p,q}(\Omega)$ and $L^2_{p,q+1}(\Omega)$,

- (Hodge Theorem) The space $H^{p,q}_{L^2}(\Omega)$ is isomorphic to the space of harmonic forms $\mathcal{H}^{p,q}(\Omega)$.
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L^2 theory for $\overline{\partial}$ on pseudoconvex domains in $\mathbf{C}^n$

Hörmander 1965

If $\Omega \subset\subset \mathbb{C}^n$ is bounded and pseudoconvex, then

$$H^{p,q}_{L^2}(\Omega) = 0, \qquad q \neq 0.$$

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Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then

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Ideas of the proof

Use the strictly plurisubharmonic weight function $t|z|^2$, $t > 0$, to set up the problem in the weighted L^2 space with respect to weights $L^2(\Omega, e^{-t|z|^2})$.

- In Hörmander's case, we first choose $t > 0$ to obtain the L^2 existence theorem. Set $t = \delta^{-2}$ where δ is the diameter of the bounded pseudoconvex domain Ω to obtain the estimates independent of the weights: The *basic estimates* hold:

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- If $q = n$, this is the Poincaré's inequality: Let Ω be a bounded domain.

$$\|f\|^2 \leq 4\delta^2 \|\nabla f\|^2, \quad f \in C_0^\infty(\Omega).$$

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On a pseudoconvex domain Ω, from Hörmander's Basic Estimates:

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- $\Box$ has closed range and the range is $L^2_{p,q}(\Omega)$.
- $\Box$ has a bounded inverse $\mathcal{N}$, the $\bar{\partial}$-Neumann operator.
- The $\bar{\partial}$-Neumann operator $\mathcal{N}$ is not necessarily compact on a pseudoconvex domain!
- $\mathcal{N}$ is compact (subelliptic) if the domain is *strictly* pseudoconvex (Kohn), or finite type (Kohn, Catlin).
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Outline

Can one hear the shape of a drum?

Eigenvalues of the Laplacian

Mark Kac (1966) asks whether the spectrum (eigenvalues) of the Dirichlet Laplacian determines the shape of a planar domain. This question was answered negatively by Gordon, Webb, and Wolpert (1992).

Not all is lost!

Kac (1966), Kac-Singer (1967)

One still can deduce from the spectrum of the domain its area, perimeter and the number of holes.

- The generalization to domains in higher dimension is to study the spectrum of the Laplace-Beltrami operator

$$\triangle_q = dd^* + d^*d.$$

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Hearing pseudoconvexity with $\Box$

We can characterize pseudoconvex domains via L^2 Dolbeault cohomology groups.

Theorem (Hörmander, Laufer)

Let Ω be a bounded domain in $\mathbb{C}^n$ with connected Lipschitz boundary. The following are equivalent:

1. Ω is pseudoconvex.
2. $H^{0,q}_{L^2}(\Omega) = \{0\}$.
3. The infimum of the spectrum of $\Box_{0,q}$ is positive for all $1 \leq q \leq n-1$.
4. The infimum of the essential spectrum of $\Box_{0,q}$ is positive for all $1 \leq q \leq n-1$.

When $n = 2$, the conditions are equivalent to the following:

- The range of $\Box_{0,1}$ is closed.
- $\overline{\partial} : L^2(\Omega) \to L^2_{0,1}(\Omega)$ has closed range.

Hearing pseudoconvexity with Kohn's Laplacian

Let $\overline{\partial}_b$ be the (induced) tangential Cauchy-Riemann complex on the boundary of $b\Omega$ and let $\Box_b$ be the $\overline{\partial}_b$-Laplacian (or Kohn's Laplacian):

$$\Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b \overline{\partial}_b.$$

Theorem (Fu 2005)

Let Ω be a bounded domain in $\mathbb{C}^n$ with connected Lipschitz boundary. Suppose that the infimum of the essential spectrum of $\Box_b$ is positive for all $1 \leq q \leq n-1$, then Ω is pseudoconvex.

- The converse is also true for domains with smooth boundary.
- L^2 existence for $\overline{\partial}_b$ is proved by Shaw (1985) for $1 \leq q < n-1$ and closed range property by Boas-Shaw and Kohn (1986) for $q = n-1$.
- This is also true if the boundary is Lipschitz with a plurisubharmonic defining function (2003).

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Serre Duality Theorem

A useful tool in complex analysis is the Serre duality. This is the complex version of the Poincaré duality.

Theorem (Serre Duality (1955))

Let Ω be a domain in a complex manifold and let E be a holomorphic vector bundle on $\overline{\Omega}$. Let $\overline{\partial}_E$ has closed range in the Fréchet space $C^\infty_{p,q}(\Omega, E)$ and $C^\infty_{p,q+1}(\Omega, E)$. We have $H^{p,q}(\Omega, E)' \cong H_c^{n-p,n-q}(\Omega, E^*)$.

- The classical Serre duality are duality results of Dolbeault coholomology group $H^{p,q}(\Omega, E)$ for E-valued smooth (p, q)-forms with the Fréchet topology and compactly supported smooth E^*-valued forms with the natural inductive limit topology.
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- It is natural to use the L^2 setting for duality results.

L^2 Serre Duality

Theorem (Chakrabarti-S 2012)

Let Ω be a bounded domain in a complex hermitian manifold of dimension n and let E be a holomorphic vector bundle on $\overline{\Omega}$ with a hermitian metric h. Suppose that $\Box_E$ has closed range on $L^2_{p,q}(\Omega, E)$. Then $\Box_{c,E^*}$ has closed range on $L^2_{n-p,n-q}(\Omega, E^*)$ and $H^{p,q}_{L^2}(\Omega, E) \cong H^{n-p,n-q}_{c,L^2}(\Omega, E^*)$.

- Let $\star_E : C^\infty_{p,q}(\Omega, E) \to C^{n-p,n-q}(\Omega, E^*)$ be the Hodge star operator.

$$\star_E \Box_E = \Box^c_{E^*} \star_E .$$

- This gives the explicit formula:

$$\star_E \mathcal{H}^{p,q}(\Omega, E) = \mathcal{H}^{n-p,n-q}_{c,L^2}(\Omega, E^*).$$

- The theorem follows from the L^2 Hodge theorem.

Non-closed range property for $\bar{\partial}$

Laurent-S (2013)

Let Ω be a bounded Lipschitz domain in $\mathbb{C}^2$ such that $\mathbb{C}^2 \setminus \overline{\Omega}$ is connected. Suppose that Ω is not pseudoconvex. Then $H^{0,1}_{L^2}(\Omega)$ is non-Hausdorff.

Corollary

Either $H^{0,1}_{L^2}(\Omega) = 0$ (and Ω is pseudoconvex) or $H^{0,1}_{L^2}(\Omega)$ is non-Hausdorff.

Similar results also hold for $(0, n-1)$-forms in $\mathbb{C}^n$ when $n \geq 3$ or s Stein manifold.

- Laufer (1975) Let Ω be a domain in $\mathbb{C}^n$ (or a Stein manifold). Then either $H^{0,1}(\Omega) = 0$ or $H^{0,1}(\Omega)$ is infinite dimensional.
- Trapani (1986) obtained similar results in $H^{0,1}(\Omega)$.

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Hearing pseudoconvexity on domains with holes

We can characterize domains with holes with L^2 Dolbeault cohomology groups. Let $\tilde{\Omega}$ be a bounded domain in $\mathbb{C}^n$ such that $\mathbb{C}^n \setminus \tilde{\Omega}$ is connected. Let $D = \cup_1^N D_i$ be the disjoint union of finitely many domains with C^2 boundary and $\overline{D} \subset \tilde{\Omega}$. Let

$$\Omega = \tilde{\Omega} \setminus \overline{D}.$$

Fu-Laurent-Shaw (Math. Zeit. 2017)

The following are equivalent:

- $\tilde{\Omega}$ and D are pseudoconvex.
- $\begin{cases} H^{0,q}_{L^2}(\Omega) = \{0\}, \quad 1 \leq q \leq n-2, \\ H^{0,n-1}_{L^2}(\Omega) \text{ is Hausdorff and infinite dimensional.} \end{cases}$
- 0 is not in the spectrum of $\Box_{0,q}$ when $1 \leq q \leq n-2$ and 0 is not a limit point for $\Box_{0,n-1}$ but 0 is in the essential spectrum of $\Box_{0,n-1}$.

Remark: The number of holes is irrelevant!

Dolbeault cohomology for domains with or without holes

Dolbeault cohomology on domains with holes (Trapani 1986)

The following are equivalent: $\Omega = \tilde{\Omega} \setminus \overline{D}$.

- $\begin{cases} H^{0,q}(\Omega) = \{0\}, \quad 1 \le q \le n-2, \\ H^{0,n-1}(\Omega) \text{ is Hausdorff and infinite dimensional.} \end{cases}$
- $\tilde{\Omega}$ is pseudoconvex and $\overline{D}$ is holomorphically convex (stronger condition).

Domains without holes

Let Ω be a bounded domain in $\mathbb{C}^n$ with Lipschitz boundary. The following are equivalent:

- Ω is pseudoconvex.
- $H^{0,q}(\Omega) = \{0\}, \quad 1 \le q \le n.$
- $H^{0,q}_{L^2}(\Omega) = \{0\}, \quad 1 \le q \le n.$

Outline

The $\bar{\partial}$-equation on an annulus

Annuli between smooth pseudoconvex domains (S. 1985)

- Let $\Omega \Subset \mathbb{C}^n$ be an annulus between two smooth pseudoconvex domains Ω_0 and Ω_1 with
$$\Omega_0 \Subset \Omega_1, \qquad \Omega = \Omega_1 \setminus \Omega_0.$$
- If$1 \leq q < n-1$,
$$H^{p,q}_{W^s}(\Omega) = 0, \qquad s \geq 0.$$
- The proof is based on Kohn-Hörmander's weighted $\bar{\partial}$-Neumann operator with strictly super-harmonic weight function $-t|z|^2$ near the pseudoconcave part of the boundary. The boundary smoothness has been relaxed by using the L^2 Serre duality results (Chakrabarti-S, 2012).
- The hole domain Ω_0 is assumed to have C^2 boundary.
- Hörmander-Kohn (1965) Subelliptic $\frac{1}{2}$ estimates hold for smooth strongly pseudoconvex boundary.

Harmonic spaces for $q = n - 1$ on the annulus

Hörmander 2004

Let $\Omega = B_1 \setminus \overline{B}_0$, where B_1 and B_0 are two concentric balls in $\mathbb{C}^n$. Then $\Box$ has closed range and the harmonic space $H^{p,n-1}_{L^2}(\Omega)$ is isomorphic to the Bergman space $H_{L^2}(B_0)$. The harmonic space
$\mathcal{H}^{n,n-1}(\Omega) = \{\sum_j h(\frac{z}{|z|^2}) \star d\bar{z}_j \mid h \in H_{L^2}(B_0)\}$.

Duality between harmonic and Bergman spaces (2011)

Let $\Omega = \Omega_1 \setminus \overline{\Omega_0} \Subset \mathbb{C}^n$ where Ω_1 is bounded and pseudoconvex and $\Omega_0 \Subset \Omega_1$ is also pseudoconvex but with C^2 smooth boundary, then again closed range holds for $q = n - 1$ and

$$H^{n,n-1}_{L^2}(\Omega) \cong H_{L^2}(\Omega_0).$$

If $b\Omega_0$ is not C^2, it is not known if $H^{0,n-1}(\Omega)$ is Hausdorff.

More on the annulus

Let $T \Subset \mathbb{C}^2$ be the Hartogs triangle

$$T = \{(z, w) \mid |z| < |w| < 1\}.$$

Then T is not Lipschitz at the origin.

- Let Ω be a pseudoconvex domain in $\mathbb{C}^2$ such that $\overline{T} \subset \Omega$. Then $H^{0,1}(\Omega \setminus \overline{T})$ is not Hausdorff (Trapani, Laurent-S).
- If we replace H by the bidisc $\triangle^2$, then $H^{0,1}(\Omega \setminus \overline{\triangle^2})$ is Hausdorff since $\triangle^2$ has a Stein neighborhood basis (Laurent-Leiterer).

Chinese Coin Problem

Let B be a ball of radius two in $\mathbb{C}^2$ and $\triangle^2$ be the bidisc. Determine if the L^2 cohomology $H^{0,1}_{L^2}(B \setminus \overline{\triangle^2})$ is Hausdorff.

Solution to the Chinese Coin Problem

Let $V_1, \ldots, V_n$ be bounded planar domains in $\mathbb{C}$ with Lipschtz boundary and let $V = V_1 \times \cdots \times V_n$.

Theorem (Chakrabarti-Laurent-S)

Let $\tilde{\Omega}$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ such that $V \Subset \tilde{\Omega}$. Let $\Omega = \tilde{\Omega} \setminus \overline{V}$ be the annulus between $\tilde{\Omega}$ and V. Then $H^{0,1}_{L^2}(\Omega)$ is Hausdorff and

- $H^{0,1}_{L^2}(\Omega) = \{0\}$, *if* $n \geq 3$.
- $H^{0,1}_{L^2}(\Omega)$ *is infinite dimensional if* $n = 2$.

Corollary

Let V be the product of bounded planar domains with Lipschitz boundary. Then

$$H^{0,n-1}_{W^1}(V) = \{0\}.$$

Outline

Non-closed range property for some Stein domain

Theorem (Chakrabarti-S, 2015 Math. Ann.)

There exists a pseudoconvex domain Ω in a complex manifold such that

- *Ω is Stein with smooth (real-analytic) Levi-flat boundary.*
- *Any continuous bounded plurisubharmonic function on Ω is a constant.*
- *$\overline{\partial}$ does not have closed range in $L^2_{2,1}(\Omega)$.*
- *$H^{2,1}_{L^2}(\Omega)$ is non-Hausdorff.*

Let

$$X = \mathbb{CP}^1 \times T$$

be a compact complex manifold of dimension 2 endowed with the product metric where T is the torus.
The domain $\Omega \subset X = \mathbb{C}P^1 \times T$ is defined by

$$\Omega = \{(z, [w]) \in \mathbb{C}P^1 \times T : Rezw > 0\}.$$

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Remarks

- Ω is biholomorphic to a punctured plane $\mathbb{C}^*$ and an annulus. Hence Ω is Stein (Ohsawa 1982).
- $$H^{p,q}(\Omega) = 0, \quad q > 0.$$
- We still do not know if $H^{0,1}_{L^2}(\Omega)$ or $H^{1,1}_{L^2}(\Omega)$ is Hausdorff.
- An earlier example (constructed by Grauert) of a pseudoconvex domain in a a two-tori has been shown with non-Hausdorff property by Malgrange (1975). But the domain is not holomorphically convex (not Stein).

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Thank You