

# *On the $L^2$ -Dolbeault Cohomology of Annuli*

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ABSTRACT. For certain annuli in  $\mathbb{C}^n$ ,  $n \geq 2$ , with non-smooth holes, we show that the  $\bar{\partial}$ -operator from  $L^2$  functions to  $L^2(0, 1)$ -forms has closed range. The holes admitted include products of pseudoconvex domains and certain intersections of smoothly bounded pseudoconvex domains. As a consequence, we obtain estimates in the Sobolev space  $W^1$  for the  $\bar{\partial}$ -equation on the non-smooth domains which are the holes of these annuli.

## 1. INTRODUCTION

**1.1. Results.** Let  $n \geq 2$ , let  $\tilde{\Omega}$  be a bounded domain in  $\mathbb{C}^n$ , and  $K \subset \tilde{\Omega}$  be a non-empty compact subset such that  $\Omega = \tilde{\Omega} \setminus K$  is connected. We will refer to  $\Omega$  as the *annulus between  $K$  and  $\tilde{\Omega}$* , and  $K$  as the *hole* of the annulus. Annuli such as  $\Omega$  are of great importance in complex analysis, for example as one of the simplest types of domains to exhibit Hartogs' phenomenon.

The goal of this paper is to study whether the  $\bar{\partial}$ -operator from  $L^2_{0,0}(\Omega)$  to  $L^2_{0,1}(\Omega)$  has closed range, and to characterize the range. As a consequence of such closed-range results, using a duality argument, we can prove estimates for the  $\bar{\partial}$ -problem on the hole in degree  $(0, n - 1)$  in the Sobolev space  $W^1$ . Since our holes can be non-smooth, this leads to Sobolev estimates on certain classes of non-smooth domains, including intersections of smoothly bounded convex domains. Previously, such  $W^1$ -estimates have been known only on domains of class  $C^2$ , and have been unknown even for domains such as the bidisc or the intersection of two balls in  $\mathbb{C}^2$ , on both of which we obtain here  $W^1$  estimates for the  $\bar{\partial}$ -problem.

To state our results we introduce a few definitions. First, for an open subset  $U \subset \mathbb{C}^n$ , we define the  $L^2$ -Dolbeault cohomology group

$$(1.1) \quad H_{L^2}^{p,q}(U) = \frac{\ker(\bar{\partial} : L_{p,q}^2(U) \dashrightarrow L_{p,q+1}^2(U))}{\text{img}(\bar{\partial} : L_{p,q-1}^2(U) \dashrightarrow L_{p,q}^2(U))},$$

where the dashed arrows are a reminder that the  $\bar{\partial}$  operator is defined only on a dense linear subspace of the space  $L_{p,q}^2(U)$ . Then, the quotient topology on  $H_{L^2}^{p,q}(U)$  is Hausdorff if and only if  $\bar{\partial} : L_{p,q-1}^2(U) \dashrightarrow L_{p,q}^2(U)$  has closed range. Similarly, we use the notation  $H_{W^1}^{p,q}(U)$  when we substitute  $L^2$  spaces by  $W^1$  spaces, where  $W^1(U)$  is the Sobolev space of functions in  $L^2(U)$  with all first partial derivatives in  $L^2(U)$ . Note the following strengthening of [21, Proposition 4.7], which characterizes the annuli with Lipschitz holes on which  $\bar{\partial}$  has closed range (see also [27, Theorem 3]).

**Theorem 1.1.** *Let  $V \Subset \bar{\Omega}$  be bounded open subsets of  $\mathbb{C}^n$ ,  $n \geq 2$ . Assume  $V$  has Lipschitz boundary and  $\Omega = \bar{\Omega} \setminus \bar{V}$  is connected; then the following are equivalent:*

- (1)  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff.
- (2)  $H_{W^1}^{0,n-1}(V) = 0$  and  $H_{L^2}^{0,1}(\bar{\Omega})$  is Hausdorff.

Because of Theorem 1.1, the question of closed range in the  $L^2$ -sense on annuli is reduced to an estimate in the  $W^1$  norm for the  $\bar{\partial}$ -problem on the hole, and an  $L^2$ -estimate for the  $\bar{\partial}$ -problem on the domain  $\bar{\Omega}$ . From Kohn’s theory of the weighted  $\bar{\partial}$ -Neumann problem (see [17]), it follows that, for a  $C^\infty$ -smooth bounded pseudoconvex domain  $V$  in  $\mathbb{C}^n$ , we have  $H_{W^1}^{p,q}(V) = 0$ , if  $q \geq 0$ . This therefore gives examples of domains to which Theorem 1.1 applies. In this paper, we give examples of more general non-smooth holes for which the closed range of  $\bar{\partial}$  holds in the annulus. Our first result in this direction is the following.

**Theorem 1.2.** *Let  $\bar{\Omega}$  be a domain in  $\mathbb{C}^n$ ,  $n \geq 2$ , and let  $K \subset \bar{\Omega}$  be a compact set such that  $\Omega = \bar{\Omega} \setminus K$  is connected. Suppose that*

- (1)  $H_{L^2}^{0,1}(\bar{\Omega})$  is Hausdorff.
- (2)  $K = \bigcap_{j=1}^N K_j$ , where for  $1 \leq j \leq N$ ,  $K_j \subset \bar{\Omega}$  is a compact set such that  $\bar{\Omega} \setminus K_j$  is connected, and  $H_{L^2}^{0,1}(\bar{\Omega} \setminus K_j)$  is Hausdorff.
- (3) For each pair of indices  $1 \leq i, j \leq N$ , the set  $\bar{\Omega} \setminus (K_i \cup K_j)$  is connected.

*Then,  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff.*

By Theorem 1.1, if  $K_j$  is the closure of a Lipschitz domain  $V_j$  such that  $H_{W^1}^{0,n-1}(V_j) = 0$ , then hypothesis (2) is satisfied (we can take  $K_j = \bar{V}_j$ , where  $V_j$  is a smoothly bounded pseudoconvex domain). Further, if the sets  $K_j$  are taken to be closures of smoothly bounded convex domains or closures of smoothly bounded pseudoconvex domains which are *star-shaped* with respect to a common point, then the hypothesis (3) will be automatically satisfied.

Our approach to Theorem 1.2 (as well as Theorem 1.4 below) is based on an analog of the Leray theorem in the  $L^2$  setting which allows us to replace questions about the  $L^2$ -Dolbeault cohomology  $H_{L^2}^{0,1}(\Omega)$  with questions about the Čech cohomology  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2})$  (with coefficients in the presheaf  $\mathcal{O}_{L^2}$  of  $L^2$  holomorphic functions), where  $\mathfrak{U}$  is a cover of the domain  $\Omega$  by sets on each of which there is an  $L^2$  estimate for the  $\bar{\partial}$ -operator (see Theorem 3.1 below).

Combined with Theorem 1.1, Theorem 1.2 gives the following estimate for the  $\bar{\partial}$ -problem on a class of non-smooth domains.

**Corollary 1.3.** *Let  $V \Subset \mathbb{C}^n$  be a Lipschitz domain such that  $\mathbb{C}^n \setminus V$  is connected. Suppose that  $\bar{V} = \bigcap_{j=1}^N \bar{V}_j$ , where for  $1 \leq j \leq N$ ,  $V_j \Subset \mathbb{C}^n$  is a Lipschitz domain such that  $H_{W^1}^{0,n-1}(V_j) = 0$ . If  $\mathbb{C}^n \setminus V_j$  is connected for each  $j$ , and  $\mathbb{C}^n \setminus (V_i \cup V_j)$  is connected for each  $1 \leq i, j \leq N$ , then  $H_{W^1}^{0,n-1}(V) = 0$ .*

If we define the Sobolev spaces correctly (see Corollary 4.1 below), one can obtain analogous results for much more general domains.

In the case of intersection of *two* smoothly bounded pseudoconvex domains in  $\mathbb{C}^n$ , it is known that the  $\bar{\partial}$ -Neumann operator is compact in degree  $n - 1$ , provided it is compact on each domain (cf. [2]). It would be interesting to know if this result is related to the above result.

Next, we consider the case when the hole is a *product*.

**Theorem 1.4.** *Let  $N \geq 2$ , and for  $j = 1, \dots, N$ , let  $V_j$  be a bounded Lipschitz domain in  $\mathbb{C}^{n_j}$ ,  $n_j \geq 1$ , such that  $\mathbb{C}^{n_j} \setminus V_j$  is connected. If the dimension  $n_j \geq 2$ , assume further that  $H_{W^1}^{0,n_j-1}(V_j) = 0$ .*

*Set  $n = \sum_{j=1}^N n_j$  and let  $V = V_1 \times \dots \times V_N \Subset \mathbb{C}^n$ . Then,  $H_{W^1}^{0,n-1}(V) = 0$ .*

When the factors are one dimensional, a similar result holds with much less boundary regularity requirement on the factors.

Before Theorem 1.4, it was known by a different method (cf. [6]) that, given  $g \in W_{0,n-1}^N(V)$  such that  $\bar{\partial}g = 0$ , there is a  $u \in W_{0,n-2}^1(V)$  such that  $\bar{\partial}u = g$ . Theorem 1.4 improves this result considerably.

Combining Theorems 1.4 and 1.1, we have the following result.

**Corollary 1.5.** *Let  $V \Subset \mathbb{C}^n$  be a domain which is a product, as in Theorem 1.4. Let  $\tilde{\Omega}$  be a domain such that  $H_{L^2}^{0,1}(\tilde{\Omega})$  is Hausdorff, and  $V \Subset \tilde{\Omega}$ . If  $\Omega = \tilde{\Omega} \setminus \bar{V}$  is connected, then  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff.*

Corollary 1.5 solves the so-called *Chinese Coin Problem*, which is to obtain  $L^2$ -estimates for the  $\bar{\partial}$  operator in an annulus  $\tilde{\Omega} \setminus \bar{V}$  in  $\mathbb{C}^2$ , where  $\tilde{\Omega}$  is a ball, and  $V \Subset \tilde{\Omega}$  is a bidisc (see [22]). Alternatively, the existence of such estimates also follows from Theorem 1.2, since the bidisc can be represented as the intersection of two smoothly bounded convex domains.

The question arises of characterizing the cohomology group  $H_{L^2}^{0,1}(\Omega)$ . When the dimension  $n = 2$ , it is known (see Corollary 2.5 to Theorem 2.4 below) that the  $L^2$ -cohomology in degree  $(0, 1)$  of an annulus in  $\mathbb{C}^2$  is infinite dimensional,

provided that the hole is Lipschitz. Observe, therefore, that we need only to consider the case  $n \geq 3$ . When  $\tilde{\Omega}$  is strongly pseudoconvex and  $K$  is also the closure of a strongly pseudoconvex domain, the annulus  $\Omega = \tilde{\Omega} \setminus K$  satisfies the condition  $Z(q)$  for  $q \neq n - 1$  (see [10, 15]). It follows that for  $n \geq 3$ , we have for such annuli that  $H_{L^2}^{0,1}(\Omega) = 0$ . It was shown in [25] and [26, Theorem 2.2] that even in the situation when  $\tilde{\Omega}$  is a smoothly bounded pseudoconvex domain and  $K$  is the closure of a smoothly bounded pseudoconvex domain, we have  $H_{L^2}^{0,1}(\Omega) = 0$ , when the dimension  $n \geq 3$ . We generalize this to the situation of Theorem 1.2.

**Corollary 1.6.** *Let  $\Omega = \tilde{\Omega} \setminus K$  be an annulus, which for some compact sets  $K_j$ ,  $1 \leq j \leq N$  satisfies the hypotheses (1), (2), and (3) of Theorem 1.2. Suppose further that for each  $j$ , we have  $H_{L^2}^{0,1}(\tilde{\Omega} \setminus K_j) = 0$ . Then,  $H_{L^2}^{0,1}(\Omega) = 0$ .*

For example, if  $\tilde{\Omega}$  is a bounded pseudoconvex domain in  $\mathbb{C}^n$ ,  $n \geq 3$ , and each  $K_j \subset \tilde{\Omega}$  is the closure of a smoothly bounded convex domain, then the assumptions of Corollary 1.6 are satisfied. To state such a vanishing result for the product situation will require further hypotheses. For simplicity, before stating the somewhat technical full result, we first state a special case, when each factor of the product is one dimensional.

**Corollary 1.7.** *Let  $K_1, \dots, K_n$  be compact sets in  $\mathbb{C}$  and  $\tilde{\Omega} \Subset \mathbb{C}^n$  be a bounded domain with  $H_{L^2}^{0,1}(\tilde{\Omega}) = 0$ . Set  $K = K_1 \times \dots \times K_n$  and assume that  $K \subset \tilde{\Omega}$  and that  $\Omega = \tilde{\Omega} \setminus K$  is not empty and connected. Then,  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff, and vanishes if  $n \geq 3$ .*

The general result on the vanishing of  $H_{L^2}^{0,1}(\Omega)$  for product holes is as follows.

**Theorem 1.8.** *Consider an annulus  $\Omega = \tilde{\Omega} \setminus K \subset \mathbb{C}^n$ , where  $K = K_1 \times \dots \times K_N$  is a product of compact sets in  $\mathbb{C}^{n_j}$ ,  $n_j \geq 1$ . Assume the following:*

- (1)  $H_{L^2}^{0,1}(\tilde{\Omega}) = 0$ .
- (2) *There is a neighborhood  $U$  of  $K$  contained in  $\tilde{\Omega}$  of the form*

$$U = U_1 \times \dots \times U_N \Subset \mathbb{C}^n,$$

*where, for each  $j$ , the open set  $U_j$  is a neighborhood of  $K_j$  in  $\mathbb{C}^{n_j}$ , satisfying  $H_{L^2}^{0,1}(U_j) = 0$ .*

- (3) *For each  $j$ ,  $H_{L^2}^{0,1}(U_j \setminus K_j)$  is Hausdorff.*

*Then,  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff, and if  $n \geq 3$ , we have  $H_{L^2}^{0,1}(\Omega) = 0$ .*

Hypotheses (2) and (3) are vacuous if each  $n_j = 1$ , so that Theorem 1.8 reduces to Corollary 1.7 if each of the factors  $K_j$  of  $K$  is one dimensional. Note also that hypothesis (3) is implied (thanks to Theorem 1.1) by the following statement:

- (3)' *If for some  $1 \leq j \leq N$  we have  $n_j \geq 2$ , then  $K_j = \bar{V}_j$ , where  $V_j \subset \mathbb{C}^{n_j}$  is a Lipschitz domain which satisfies  $H_{W^1}^{0,n_j-1}(V_j) = 0$ .*

**1.2. Remarks.** The analogs of Theorem 1.1 and Theorem 1.2 continue to hold for an annulus in a Stein manifold rather than in  $\mathbb{C}^n$ , and the proofs generalize easily. Theorem 1.4 and its proof can also be readily generalized to products  $V_1 \times \cdots \times V_N \subset M$  where, for each  $j$ , there is a Stein manifold  $M_j$  of dimension  $n_j$  such that the factor  $V_j$  is a relatively compact Lipschitz domain in  $M_j$ , which further satisfies  $H_{W^1}^{0, n_j-1}(V_j) = 0$  if  $n_j \geq 2$ , and  $M$  is the product  $M_1 \times \cdots \times M_N$ . The other results can also be generalized to domains in Stein manifolds. We prefer to state the results in the case of domains in  $\mathbb{C}^n$  for clarity of exposition.

One also notes here that the choice of the  $L^2$  topology for estimates on the  $\bar{\partial}$ -problem is a matter of convenience rather than necessity. The methods of this paper are based on duality and gluing of local estimates, and can be generalized to estimates in any norm: for example, the  $L^p$ -norm. The required duality results in the  $L^p$  setting may be found in [20].

On pseudoconvex domains, the closed-range property is a consequence of *a priori* estimates on the  $\bar{\partial}$ -operator (for bounded domains see [15], and see [14] for some recent developments regarding unbounded domains). When  $\Omega$  is the annulus between two smooth strongly pseudoconvex domains in  $\mathbb{C}^n$ ,  $L^2$  theory for  $\bar{\partial}$  is obtained in [10, 15] for all  $(p, q)$ -forms since the boundary satisfies the Andreotti-Grauert condition  $Z(q)$  for all  $q \neq n - 1$ . The closed-range property for  $\bar{\partial}$  when  $q = n - 1$  also follows in this case (see Proposition 3.1.17 in [10]). When the domain  $\Omega$  is the annulus between a bounded pseudoconvex domain and a  $C^2$ -smooth pseudoconvex domain in  $\mathbb{C}^n$ ,  $L^2$  theory for  $\bar{\partial}$  has been established in the works [16, 25, 26]. Duality between the  $L^2$  theory on the annulus and the  $W^1$  estimates for  $\bar{\partial}$  in the hole has been obtained in [21].

The classical approach to regularity in the  $\bar{\partial}$ -problem is via the  $\bar{\partial}$ -Neumann problem. It is difficult to use this method to obtain Sobolev estimates even on simple Lipschitz domains such as the bidisc or the intersection of two balls. The problem arises because a  $(0, 1)$ -form on the bidisc  $\mathbb{D}^2$  which is in the domain of  $\bar{\partial}^*$  and is smooth up to the boundary must vanish along the Šilov boundary  $b\mathbb{D} \times b\mathbb{D}$ , since the complex normal components of the form along two  $\mathbb{C}$ -linearly independent directions must vanish. Thus, *a priori* estimates do not translate into actual estimates. One can show that, on product domains, the  $\bar{\partial}$ -Neumann operator does not preserve the space of  $(0, 1)$ -forms smooth up to the boundary on the bidisc (see [8]). However, for domains represented as intersections of strongly pseudoconvex domains, with boundaries meeting transversely, one may obtain subelliptic estimates with  $\frac{1}{2}$  gain for the canonical solution operator (see [13, 23]).

## 2. GENERAL CONDITION FOR CLOSED RANGE

**2.1. Notation and preliminaries.** We now introduce some notation for the  $L^2$ -version of the  $\bar{\partial}$ -complex.

For an open set  $U \subset \mathbb{C}^n$ , denote

$$A_{L^2}^{p,q}(U) = \{f \in L_{p,q}^2(U) \mid \bar{\partial}f \in L_{p,q+1}^2(U)\},$$

where  $\bar{\partial}$  acts in the sense of distributions. One may recall here that this defines the (weak) maximal realization of the  $\bar{\partial}$ -operator as an unbounded, densely defined, closed operator on  $L_{p,q}^2(U)$ . Let  $Z_{L^2}^{p,q}(U) = \{f \in A_{L^2}^{p,q}(U) \mid \bar{\partial}f = 0\}$  and  $B_{L^2}^{p,q}(U) = \{\bar{\partial}f \mid f \in A_{L^2}^{p,q-1}(U)\}$  denote the subspaces of  $\bar{\partial}$ -closed and  $\bar{\partial}$ -exact forms in  $A_{L^2}^{p,q}(U)$ . Note that  $B_{L^2}^{p,q}(U) \subset Z_{L^2}^{p,q}(U)$  since  $\bar{\partial}^2 = 0$ . As noted in (1.1), we denote  $H_{L^2}^{p,q}(U) = Z_{L^2}^{p,q}(U)/B_{L^2}^{p,q}(U)$ .

Denote by  $\bar{\partial}_c$  the (strong) minimal realization of the  $\bar{\partial}$ -operator as a closed, densely defined, unbounded linear operator from  $L_{p,q}^2(U)$  to  $L_{p,q+1}^2(U)$ . The operator  $\bar{\partial}_c$  is the closure (in graph norm) of the restriction of  $\bar{\partial}$  to the space of smooth compactly supported forms in  $L_{p,q}^2(U)$ . The domain of the operator  $\bar{\partial}_c$  is a dense subspace of  $L_{p,q}^2(U)$ , denoted by  $A_{c,L^2}^{p,q}(U)$ . Then,  $A_{c,L^2}^{p,q}(U)$  is a proper subspace of  $A_{L^2}^{p,q}(U)$  for bounded domains  $U$ , and  $\bar{\partial}_c$  is the restriction to  $A_{c,L^2}^{p,q}(U)$  of the operator  $\bar{\partial} : A_{L^2}^{p,q}(U) \rightarrow B_{L^2}^{p,q+1}(U)$ . We thus denote by  $Z_{c,L^2}^{p,q}(U) = \{f \in A_{c,L^2}^{p,q}(U) \mid \bar{\partial}_c f = 0\}$  the space of  $\bar{\partial}_c$ -closed forms, and by  $B_{c,L^2}^{p,q}(U) = \{\bar{\partial}_c f \mid f \in A_{c,L^2}^{p,q}(U)\}$  the space of  $\bar{\partial}_c$ -exact forms. The quotient  $H_{c,L^2}^{p,q}(U) = Z_{c,L^2}^{p,q}(U)/B_{c,L^2}^{p,q}(U)$  is the  $L^2$ -Dolbeault cohomology with minimal realization, analogous to cohomology with compact support. We also use the following  $L^2$ -analogue of Serre duality. Consider the pairing  $H_{L^2}^{p,q}(U) \times H_{c,L^2}^{n-p,n-q}(U) \rightarrow \mathbb{C}$  given by

$$(\text{class}(f), \text{class}(g)) \mapsto \int_U f \wedge g.$$

Then,  $H_{L^2}^{p,q}(U)$  (respectively,  $H_{c,L^2}^{n-p,n-q}(U)$ ) is Hausdorff if and only if 0 is the only element of  $H_{L^2}^{p,q}(U)$  (respectively,  $H_{c,L^2}^{n-p,n-q}(U)$ ) which is orthogonal to all of  $H_{c,L^2}^{n-p,n-q}(U)$  (respectively,  $H_{L^2}^{p,q}(U)$ ) (cf. [5]).

We say that a nonempty compact subset  $K \subset \mathbb{R}^n$  is *regular* if there is an open set  $V$  with

$$(2.1) \quad K = \bar{V} \quad \text{and} \quad V = \text{interior}(K).$$

For a regular compact subset  $K \subset \mathbb{C}^n$ , we use the notation

$$(2.2) \quad H_{W^1}^{p,q}(K) = 0$$

to denote that the following is true: if  $f \in W_{p,q}^1(\mathbb{C}^n)$  is a form with coefficients in the Sobolev space  $W^1$ , and on the set  $K$  we have  $\bar{\partial}f = 0$ , then there is a form  $u \in W_{p,q-1}^1(\mathbb{C}^n)$  such that on the set  $K$  we have  $\bar{\partial}u = f$ . Note that if  $K = \bar{V}$ , where  $V$  is a Lipschitz domain, then  $H_{W^1}^{p,q}(K) = 0$  if and only if  $H_{W^1}^{p,q}(V) = 0$ . This is so since each function in  $W^1(V)$  can be extended to a function in  $W^1(\mathbb{C}^n)$ .

**2.2. Proof of Theorem 1.1.** We prove below three propositions which together imply Theorem 1.1.

**Proposition 2.1.** *Let  $\bar{\Omega}$  be a bounded open set in  $\mathbb{C}^n$ ,  $n \geq 2$ , and let  $K$  be a regular compact subset of  $\bar{\Omega}$ . If  $\Omega = \bar{\Omega} \setminus K$  is connected such that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff, then  $H_{W^1}^{0,n-1}(K) = 0$ .*

*Proof.* Let  $f \in W_{0,n-1}^1(\mathbb{C}^n)$  be a form with  $W^1$  coefficients, and assume that  $\bar{\partial}f = 0$  on  $K$ . We need to show that there is a  $u \in W_{0,n-2}^1(\mathbb{C}^n)$  such that  $\bar{\partial}u = f$  holds on  $K$ . After multiplying with an appropriate smooth compactly supported function, we assume that  $f$  has compact support in  $\bar{\Omega}$ . Then, the  $(0, n)$ -form  $\bar{\partial}f$  lies in  $A_{c,L^2}^{0,n}(\bar{\Omega})$  and vanishes on  $K$ . We then claim that, for each holomorphic  $(n, 0)$ -form  $\theta \in Z_{L^2}^{n,0}(\Omega)$  on  $\Omega$ , we have

$$(2.3) \quad \int_{\Omega} \theta \wedge \bar{\partial}f = 0.$$

Indeed, since  $\Omega$  is connected, by the Hartogs extension phenomenon, the form  $\theta$  extends through the hole  $K$  to give a holomorphic form  $\bar{\theta} \in Z_{L^2}^{n,0}(\bar{\Omega})$  and

$$\int_{\Omega} \theta \wedge \bar{\partial}f = \int_{\bar{\Omega}} \bar{\theta} \wedge \bar{\partial}f = (-1)^n \int_{\bar{\Omega}} \bar{\partial}\bar{\theta} \wedge f = 0.$$

Observe, now, that by [5, Lemma 3], the hypothesis that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff is equivalent to the fact that  $H_{c,L^2}^{n,n}(\Omega)$  is Hausdorff, which in turn is equivalent to  $H_{c,L^2}^{0,n}(\Omega)$  being Hausdorff. Now, (2.3) shows that under the Serre pairing  $H_{L^2}^{n,0}(\Omega) \times H_{c,L^2}^{0,n}(\Omega) \rightarrow \mathbb{C}$ , the cohomology class of  $\bar{\partial}f$  is orthogonal to all of  $H_{L^2}^{n,0}(\Omega)$ , and therefore there is a  $g \in A_{c,L^2}^{0,n-1}(\Omega)$  such that  $\bar{\partial}f = \bar{\partial}g$ . Let  $\bar{g}$  be the extension of  $g$  by 0 to all of  $\mathbb{C}^n$ . Then,  $\bar{g} \in A_{c,L^2}^{0,n-1}(\mathbb{C}^n)$  and has compact support. Therefore, the form  $f - \bar{g}$  is a compactly supported  $\bar{\partial}$ -closed form in  $Z_{c,L^2}^{0,n-1}(\mathbb{C}^n)$ . Thus, there is a compactly supported  $(0, n - 2)$ -form  $u$  on  $\mathbb{C}^n$  such that  $\bar{\partial}u = f - \bar{g}$ , and by interior regularity of the  $\bar{\partial}$ -problem, we may assume that  $u$  has coefficients in  $W^1$  when restricted to a neighborhood of  $K$ . Noting that by construction  $\bar{g}$  vanishes on  $K$ , we see that  $\bar{\partial}u = f$ , so that  $H_{W^1}^{0,n-1}(K) = 0$ .  $\square$

For a general regular compact set  $K$ , the condition  $H_{W^1}^{0,n-1}(K) = 0$  is only a statement about existence of solutions of the  $\bar{\partial}$  problem, and does not lead to any estimates for these solutions. However, when  $H_{W^1}^{0,n-1}(K) = 0$  can be interpreted as the vanishing of a cohomology defined by a densely defined closed realization of the operator  $\bar{\partial}$  acting between Banach spaces, by the open mapping theorem, we do obtain estimates for the  $\bar{\partial}$ -problem. For example, when  $K = \bar{V}$ , where  $V$  is a Lipschitz domain, the condition  $H_{W^1}^{0,n-1}(K) = 0$  implies that there is a constant

$C > 0$  such that, for each  $f \in W_{0,n-1}^1(V)$  such that  $\bar{\partial}f = 0$  as distributions, there is a  $u \in W_{0,n-2}^1(V)$  such that  $\bar{\partial}u = f$  in the sense of distributions, and  $\|u\|_{W^1} \leq C\|f\|_{W^1}$ .

**Proposition 2.2.** *Let  $\tilde{\Omega}$  be a bounded open set in  $\mathbb{C}^n$ ,  $n \geq 2$ , and let  $K$  be a compact subset of  $\tilde{\Omega}$ . If  $\Omega = \tilde{\Omega} \setminus K$  is connected such that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff, then  $H_{L^2}^{0,1}(\tilde{\Omega})$  is Hausdorff.*

*Proof.* Assuming again that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff, we now show that  $H_{L^2}^{0,1}(\tilde{\Omega})$  is Hausdorff. Let  $f \in Z_{L^2}^{0,1}(\tilde{\Omega})$  be such that for each  $\varphi \in Z_{c,L^2}^{n,n-1}(\tilde{\Omega})$ , we have  $\int_{\tilde{\Omega}} f \wedge \varphi = 0$ . We have to show that  $f$  is  $\bar{\partial}$ -exact, and then the claim will follow by Serre duality.

Now, if  $\psi \in Z_{c,L^2}^{n,n-1}(\Omega)$ , then we clearly have  $\int_{\Omega} f \wedge \psi = 0$ , so that the fact that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff implies that there is a  $g \in A_{L^2}^{0,0}(\Omega)$  satisfying  $\bar{\partial}g = f$ . Let  $\chi \in C_0^\infty(\mathbb{C}^n)$  be a cutoff which is identically equal to 1 on a neighborhood of  $K$  and is supported in a compact subset of  $\tilde{\Omega}$ . We consider the  $(0, 1)$ -form  $\theta$  on  $\mathbb{C}^n$ , defined by extending by zero the compactly supported form  $f - \bar{\partial}((1 - \chi)g)$  on  $\Omega$ . Writing  $f - \bar{\partial}((1 - \chi)g) = \chi f + \bar{\partial}\chi \wedge g$ , we see that  $\theta$  is a compactly supported form in  $L_{0,1}^2(\mathbb{C}^n)$ . Therefore, there is a compactly supported  $L^2$  function  $u$  on  $\mathbb{C}^n$  such that  $\bar{\partial}u = \theta$ . Then,  $(1 - \chi)g + u \in A_{L^2}^{0,0}(\tilde{\Omega})$ , and  $\bar{\partial}((1 - \chi)g + u) = f$ .  $\square$

**Proposition 2.3.** *Let  $V \Subset \tilde{\Omega}$  be bounded open subsets of  $\mathbb{C}^n$ ,  $n \geq 2$ . Assume  $V$  has Lipschitz boundary,  $\Omega = \tilde{\Omega} \setminus \bar{V}$  is connected,  $H_{W^1}^{0,n-1}(V) = 0$ , and  $H_{L^2}^{0,1}(\tilde{\Omega})$  is Hausdorff. Then,  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff.*

*Proof.* First note that, thanks to [5, Lemma 3], the hypothesis that  $H_{L^2}^{0,1}(\tilde{\Omega})$  is Hausdorff is equivalent to the condition that  $H_{c,L^2}^{0,n}(\tilde{\Omega})$  is Hausdorff, and the conclusion that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff is equivalent to  $H_{c,L^2}^{0,n}(\Omega)$  being Hausdorff.

We therefore start with an  $f \in Z_{c,L^2}^{0,n}(\Omega)$  such that, for each  $\varphi \in Z_{L^2}^{n,0}(\Omega)$ , we have  $\int_{\Omega} f \wedge \varphi = 0$ , and want to show that there is a  $u$  in  $A_{c,L^2}^{0,n-1}(\Omega)$  such that  $\bar{\partial}u = f$ .

By Hartogs' phenomenon, each element of  $Z_{L^2}^{n,0}(\Omega)$  extends to an element of  $Z_{L^2}^{n,0}(\tilde{\Omega})$ . It follows that  $\int_{\tilde{\Omega}} \tilde{f} \wedge \varphi = 0$ , where  $\tilde{f}$  is the extension of  $f$  by 0 to all of  $\tilde{\Omega}$ . Since, by hypothesis,  $H_{c,L^2}^{0,n}(\tilde{\Omega})$  is Hausdorff, there is a  $g \in A_{c,L^2}^{0,n-1}(\tilde{\Omega})$  such that  $\bar{\partial}g = \tilde{f}$ , and using interior regularity of  $\bar{\partial}$ , we may assume that  $g$  has  $W^1$  coefficients in a neighborhood of  $\bar{V}$ . Note that, by the definition of  $\tilde{f}$ , we have  $\bar{\partial}g = 0$  on  $V$ . Invoking the hypothesis  $H_{W^1}^{0,n-1}(V) = 0$ , we see that there is an  $h \in W_{0,n-2}^1(V)$  such that  $\bar{\partial}h = g$  on  $V$ . Since  $V$  is Lipschitz, we may extend  $h$  as a form with  $W^1$  coefficients on the whole of  $\mathbb{C}^n$ . Multiplying by a smooth cutoff, we may further assume that  $h$  has compact support in  $\tilde{\Omega}$ . Then,  $u = (g - \bar{\partial}h)|_{\Omega}$  is a form on  $\Omega$  whose extension by zero to  $\tilde{\Omega}$  belongs to the domain of  $\bar{\partial}$  on  $\tilde{\Omega}$ .



Since  $V$  has Lipschitz boundary, it follows by [5, Proposition 2] that  $u$  belongs to  $A_{c,L^2}^{0,n-1}(\Omega)$ , and  $\bar{\partial}u = \bar{\partial}g = f$  on  $\Omega$ . □

**2.3. The two-dimensional case.** For a domain  $D \subset \mathbb{C}^n$ , denote by  $Z^{0,1}(\bar{D})$  the space  $C_{0,1}^\infty(\bar{D}) \cap \ker \bar{\partial}$  of  $\bar{\partial}$ -closed forms which are  $C^\infty$ -smooth up to the boundary on  $D$ , and let  $\tilde{B}^{0,1}(D)$  denote the subspace of  $Z^{0,1}(\bar{D})$  consisting of those forms  $g$  smooth up to the boundary such that there is a distribution  $u \in \mathcal{D}'(D)$  such that  $\bar{\partial}u = g$  on  $D$ . We consider the quotient

$$(2.4) \quad \check{H}^{0,1}(D) = Z^{0,1}(\bar{D}) / \tilde{B}^{0,1}(D).$$

The proof of the following result is known (see Fu [11]).

**Theorem 2.4.** *Let  $D \subset \mathbb{C}^2$  be a domain such that  $\text{interior}(\bar{D}) = D$ . If the vector space  $\check{H}^{0,1}(D)$  of (2.4) is finite dimensional, then  $D$  is pseudoconvex.*

*Proof.* We repeat the proof of [11]. Assume  $D$  is not pseudoconvex; then, there exists a domain  $\tilde{D}$  strictly containing  $D$  such that any holomorphic function on  $D$  extends holomorphically to  $\tilde{D}$ . Since  $\text{interior}(\bar{D}) = D$ , after a translation and a rotation, we may assume that  $0 \in \tilde{D} \setminus \bar{D}$ , and there exists a point  $z_0$  in the intersection of the plane  $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = 0\}$  with  $D$ , which belongs to the same connected component of that plane with  $\tilde{D}$ .

For an integer  $k \geq 0$ , we consider the smooth  $(0, 1)$ -form  $B_k$  on  $\mathbb{C}^2 \setminus \{0\}$ , derived from the Bochner-Martinelli kernel and given by

$$B_k(z_1, z_2) = (k + 1)! \cdot \bar{z}_2^k \cdot \frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{|z|^{2(k+2)}}.$$

These forms are  $\bar{\partial}$ -closed, and if we define

$$u_k(z_1, z_2) = k! \cdot \frac{\bar{z}_2^{(k+1)}}{|z|^{2(k+1)}},$$

we have on  $\mathbb{C}^2 \setminus \{0\}$

$$\bar{\partial}(u_k) = -z_1 B_k.$$

Note that the restriction of  $B_k$  to  $D$  belongs to  $Z^{0,1}(\bar{D})$ . Let  $N$  be an integer such that  $N > \dim \check{H}^{0,1}(D)$ . Then, there exists a non-trivial linear combination

$$B = \sum_{k=1}^N a_k B_k$$

(which belongs to  $Z^{0,1}(\bar{D})$ ), such that there exists a distribution  $v$  on  $D$  satisfying  $\bar{\partial}v = B$ . Set

$$F = z_1 v + \sum_{k=1}^N a_k u_k.$$

Then,  $F$  is a holomorphic function on  $D$ , so it should extend holomorphically to  $\tilde{D}$ , but we have

$$F(0, z_2) = \sum_{k=1}^N a_k \frac{k!}{z_2^{k+1}},$$

which is holomorphic and singular at  $z_2 = 0$ . This then gives the contradiction, since  $0 \in \tilde{D} \setminus \bar{D}$ . □

This allows us to prove the following analog for cohomologies with estimates of a result of Laufer ([19]).

**Corollary 2.5.** *Let  $D \Subset \mathbb{C}^2$  be a bounded domain such that  $\text{interior}(\bar{D}) = D$ . If either  $H_{L^2}^{0,1}(D)$  or  $H_{W^1}^{0,1}(D)$  is finite dimensional, then  $D$  is pseudoconvex.*

*Proof.* Suppose that  $H_{L^2}^{0,1}(D) = Z_{L^2}^{0,1}(D)/B_{L^2}^{0,1}(D)$  is finite dimensional. Then, *a fortiori*, the space  $Z^{0,1}(\bar{D})/(B_{L^2}^{0,1}(\bar{D}) \cap Z^{0,1}(\bar{D}))$  consisting of  $L^2$  cohomology classes representable by forms smooth up to the boundary is also finite dimensional. But,

$$\frac{Z^{0,1}(\bar{D})}{B_{L^2}^{0,1}(\bar{D}) \cap Z^{0,1}(\bar{D})} \supset \frac{Z^{0,1}(\bar{D})}{\tilde{B}^{0,1}(\bar{D})} = \tilde{H}^{0,1}(D),$$

since  $B_{L^2}^{0,1}(\bar{D}) \cap Z^{0,1}(\bar{D}) \subset \tilde{B}^{0,1}(\bar{D})$ . Therefore, it follows that  $\tilde{H}^{0,1}(D)$  is finite dimensional, and we can apply Theorem 2.4 to conclude that  $D$  is pseudoconvex. A similar proof works in the case of the  $W^1$ -cohomology. □

Now, we can give a general characterization of domains in  $\mathbb{C}^2$  on which the  $L^2$   $\bar{\partial}$ -operator has closed range.

**Corollary 2.6.** *Let  $V \Subset \tilde{\Omega}$  be a Lipschitz domain in  $\mathbb{C}^2$ , and suppose that  $\mathbb{C}^2 \setminus \tilde{\Omega}$  and  $\Omega = \tilde{\Omega} \setminus V$  are connected. If  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff, then both  $\tilde{\Omega}$  and  $V$  are pseudoconvex, and the space  $H_{L^2}^{0,1}(\Omega)$  is infinite dimensional.*

*Proof.* By Theorem 1.1, the fact that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff is equivalent to the fact that  $H_{W^1}^{0,1}(V) = 0$  and that  $H_{L^2}^{0,1}(\tilde{\Omega})$  is Hausdorff. By Corollary 2.5 above,  $H_{W^1}^{0,1}(V) = 0$  implies that  $V$  is pseudoconvex. It was shown in [21, Theorem 3.4] that, for a Lipschitz “hole-less” domain such as  $\tilde{\Omega}$ , the space  $H_{L^2}^{0,1}(\tilde{\Omega})$  is Hausdorff if and only if  $\tilde{\Omega}$  is pseudoconvex.

The infinite dimensionality of  $H_{L^2}^{0,1}(\Omega)$  also follows from Corollary 2.5, since we have  $\text{interior}(\bar{\Omega}) = \Omega$ , and if  $H_{L^2}^{0,1}(\Omega)$  were finite dimensional,  $\Omega = \tilde{\Omega} \setminus \bar{V}$  would be pseudoconvex. □

### 3. $L^2$ -DOLBEAULT AND $L^2$ -ČECH COHOMOLOGIES

For an open  $U \subset \mathbb{C}^n$ , let  $A_{L^2}^{p,q}(U)$ ,  $B_{L^2}^{p,q}(U)$ , and  $Z_{L^2}^{p,q}(U)$  be as in Section 2.1 above. Then,  $A_{L^2}^{p,q}$  defines a presheaf of pre-Hilbert spaces on  $\Omega$ , which is not a

sheaf.  $B_{L^2}^{p,q}$  and  $Z_{L^2}^{p,q}$  are sub-presheaves of  $A_{L^2}^{p,q}$ , and  $Z_{L^2}^{p,q}$  is a presheaf of Hilbert spaces. Note that  $Z_{L^2}^{0,0}(U)$  is just the Bergman space on  $U$ , which will also be denoted by  $\mathcal{O}_{L^2}(U)$ . Recall also that the  $L^2$ -Dolbeault cohomology of  $U$  is defined to be the quotient topological vector space  $H_{L^2}^{p,q}(U) = Z_{L^2}^{p,q}(U)/B_{L^2}^{p,q}(U)$ . Given open sets  $V \subset U \subset \Omega$ , and a class  $\gamma \in H_{L^2}^{p,q}(U)$ , we denote by  $\gamma|_V$  the class in  $H_{L^2}^{p,q}(V)$  obtained by restricting  $\gamma$  to  $V$ . More precisely, if  $g \in Z_{L^2}^{p,q}(U)$  is such that  $\gamma = \text{class}(g)$  in  $H_{L^2}^{p,q}(U)$ , then  $\gamma|_V = \text{class}(g|_V)$  in  $H_{L^2}^{0,1}(V)$ .

Now, suppose that we are given a finite collection  $\mathfrak{U} = \{\Omega_j\}_{j=1}^N$  of open sets covering an open set  $\Omega \subset \mathbb{C}^n$  (i.e.,  $\bigcup_{j=1}^N \Omega_j = \Omega$ ). Given a presheaf  $\mathcal{B}$  of normed linear spaces on  $\Omega$  (e.g., the presheaves  $A_{L^2}^{p,q}$ ,  $Z_{L^2}^{p,q}$  and  $B_{L^2}^{p,q}$  of the previous paragraph), we can define for each  $k$  the space  $\check{C}^k(\mathfrak{U}, \mathcal{B})$  of Čech  $k$ -cochains, and the corresponding coboundary map  $\delta_k : \check{C}^k(\mathfrak{U}, \mathcal{B}) \rightarrow \check{C}^{k+1}(\mathfrak{U}, \mathcal{B})$  (cf. [12, p. 187]). As usual, we let  $\check{Z}^k(\mathfrak{U}, \mathcal{B})$  and  $\check{B}^k(\mathfrak{U}, \mathcal{B})$  denote the spaces of Čech cocycles and coboundaries respectively, and then the Čech cohomology of  $\mathcal{B}$  with respect to the cover  $\mathfrak{U}$  is given by  $\check{H}^a(\mathfrak{U}, \mathcal{B}) = \check{Z}^k(\mathfrak{U}, \mathcal{B})/\check{B}^k(\mathfrak{U}, \mathcal{B})$ .

Note that  $\check{C}^k(\mathfrak{U}, \mathcal{B})$  is a topological vector space as the direct sum of the  $\mathcal{B}(\Omega_{i_0, \dots, i_k})$  as  $1 \leq i_0, \dots, i_k \leq N$ , endowed with the direct sum topology, where  $\Omega_{i_0, \dots, i_k} = \bigcap_{\ell=0}^k \Omega_{i_\ell}$ . The topological vector space  $\check{C}^k(\mathfrak{U}, \mathcal{B})$  is a normed linear space in a natural way: for  $k \geq 0$ , a norm on  $\check{C}^k(\mathfrak{U}, \mathcal{B})$  is given by

$$\|g\|_{\check{C}^k(\mathfrak{U}, \mathcal{B})}^2 = \sum_{1 \leq i_0, \dots, i_k \leq N} \|g_{i_0, \dots, i_k}\|_{\mathcal{B}(\Omega_{i_0, \dots, i_k})}^2.$$

Of course, there are many other choices of equivalent norms, but the above choice is appropriate when  $\mathcal{B}$  is a sheaf of pre-Hilbert spaces, which is the only case we consider. Then,  $\check{C}^k(\mathfrak{U}, \mathcal{B})$  is again a pre-Hilbert space, and a Hilbert space if  $\mathcal{B}$  happens to be a sheaf of Hilbert spaces. With this topology, the coboundary map  $\delta$  is continuous, the cocycle space  $\check{Z}^k(\mathfrak{U}, \mathcal{B}) = \ker \delta_k \cap \check{C}^k(\mathfrak{U}, \mathcal{B})$  is a normed linear space, and the coboundary space  $\check{B}^k(\mathfrak{U}, \mathcal{B}) = \text{img } \delta_{k-1} \cap \check{C}^k(\mathfrak{U}, \mathcal{B})$  is a subspace of  $\check{Z}^k(\mathfrak{U}, \mathcal{B})$ . Then, the  $k$ -th cohomology group of this complex  $\check{H}^k(\mathfrak{U}, \mathcal{B})$  is a topological vector space with the quotient topology, and this topology is Hausdorff if and only if  $\check{B}^k(\mathfrak{U}, \mathcal{B})$  is a closed subspace of  $\check{Z}^k(\mathfrak{U}, \mathcal{B})$ .

A relation between these two types of cohomology is given by the following result, whose proof is inspired by that of a well-known classical result of Leray (cf. [12, p. 189]). Related results were obtained for the Fréchet topology in [18].

**Theorem 3.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$ . Let  $\mathfrak{U} = \{\Omega_1, \dots, \Omega_N\}$  be a finite open cover of  $\Omega$  such that, for all  $j = 1, \dots, N$ , the cohomology group  $H_{L^2}^{0,1}(\Omega_j)$  is Hausdorff. Then, we have the following:*

- (1) *If the Čech group  $\check{H}^1(\mathfrak{U}, Z_{L^2}^{0,0})$  is Hausdorff, then the  $L^2$ -Dolbeault cohomology  $H_{L^2}^{0,1}(\Omega)$  is also Hausdorff.*

(2) If  $\check{H}^1(\mathfrak{U}, Z_{L^2}^{0,0}) = 0$ , the map  $H_{L^2}^{0,1}(\Omega) \rightarrow \bigoplus_{k=1}^N H_{L^2}^{0,1}(\Omega_k)$  induced by restriction maps from  $\Omega$  to  $\Omega_k$  maps  $H_{L^2}^{0,1}(\Omega)$  isomorphically onto a subspace of

$$(3.1) \quad \{(\gamma_k)_{k=1}^N, \gamma_k \in H_{L^2}^{0,1}(\Omega_k) \mid \text{for } i \neq j, \gamma_i|_{\Omega_{ij}} = \gamma_j|_{\Omega_{ij}}\}.$$

(3) If  $\check{H}^1(\mathfrak{U}, Z_{L^2}^{0,0}) = 0$  and for each  $j$ ,  $H_{L^2}^{0,1}(\Omega_j) = 0$ , then  $H_{L^2}^{0,1}(\Omega) = 0$ .

Note that this result may be interpreted as saying that, given an  $L^2$ -estimate for the  $\bar{\partial}$ -problem in each open set of the cover  $\mathfrak{U}$ , the obstruction to obtaining a global  $L^2$ -estimate on  $\Omega$  resides in the  $L^2$ -Čech group  $\check{H}^1(\mathfrak{U}, Z_{L^2}^{0,0})$ . When this Čech group is Hausdorff, then the  $L^2$ -Dolbeault group is Hausdorff. In particular, when the Čech group vanishes, the cohomology classes in  $H_{L^2}^{0,1}(\Omega)$  are obtained by “gluing together” the cohomologies in each set in the cover as in (3.1).

We also note that use of the  $L^2$ -topology in Theorem 3.1 is not important, and similar gluing techniques work for estimates in any norm: for example,  $L^p$  estimates or Hölder estimates.

*Proof of Theorem 3.1.* For  $k, q \geq 0$ , define an operator

$$\bar{\partial} = \bar{\partial}_q : \check{C}^k(\mathfrak{U}, A_{L^2}^{p,q}) \rightarrow \check{C}^k(\mathfrak{U}, A_{L^2}^{p,q+1})$$

sectionwise; that is, for  $g \in \check{C}^k(\mathfrak{U}, A_{L^2}^{p,q})$ , set  $(\bar{\partial}g)_{i_0 \dots i_k} = \bar{\partial}(g_{i_0 \dots i_k})$ , and note that, for a given  $k$ ,  $\ker \bar{\partial}_q = \check{C}^k(\mathfrak{U}, Z_{L^2}^{p,q})$  and  $\text{img } \bar{\partial}_q = \check{C}^k(\mathfrak{U}, B_{L^2}^{p,q+1})$ . Then, for each fixed  $k, p \geq 0$ , we have a complex  $(\check{C}^k(\mathfrak{U}, A_{L^2}^{p,q}), \bar{\partial}_q)$ . In fact, we have, for each fixed  $p$ , a double complex of commuting differentials  $(\check{C}^k(\mathfrak{U}, A_{L^2}^{p,q}), \bar{\partial}_q, \delta_k)$ ; that is, we have  $\bar{\partial}\delta = \delta\bar{\partial}$ . This follows, since, for  $g \in \check{C}^k(\mathfrak{U}, A_{p,q})$ ,

$$\bar{\partial}\delta g = \delta\bar{\partial}g = \sum_{j=0}^k (-1)^j \bar{\partial}g_{i_0 \dots \hat{i}_j \dots i_k}$$

where the hat denotes omission. We represent the relevant part of the double complex for  $p = 0$  in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_{L^2}^{0,0}(\Omega) & \xrightarrow{\varepsilon} & \check{C}^0(\mathfrak{U}, Z_{L^2}^{0,0}) & \xrightarrow{\delta} & \check{C}^1(\mathfrak{U}, Z_{L^2}^{0,0}) \longrightarrow \dots \\ & & \downarrow i & & \downarrow i & & \downarrow i \\ 0 & \longrightarrow & A_{L^2}^{0,0}(\Omega) & \xrightarrow{\varepsilon} & \check{C}^0(\mathfrak{U}, A_{L^2}^{0,0}) & \xrightarrow{\delta} & \check{C}^1(\mathfrak{U}, A_{L^2}^{0,0}) \longrightarrow \dots \\ & & \downarrow \bar{\delta} & & \downarrow \bar{\delta} & & \downarrow \bar{\delta} \\ 0 & \longrightarrow & A_{L^2}^{0,1}(\Omega) & \xrightarrow{\varepsilon} & \check{C}^0(\mathfrak{U}, A_{L^2}^{0,1}) & \xrightarrow{\delta} & \check{C}^1(\mathfrak{U}, A_{L^2}^{0,1}) \longrightarrow \dots \\ & & \downarrow \bar{\delta} & & \downarrow \bar{\delta} & & \downarrow \bar{\delta} \end{array}$$

Here, for a presheaf  $\mathcal{B}$ , the map  $\varepsilon : \mathcal{B}(\Omega) \rightarrow \check{C}^0(\mathcal{U}, \mathcal{B})$  is given by  $(\varepsilon f)_i = f|_{\Omega_i}$  for each  $\Omega_i \in \mathcal{U}$ . Note that, then, the sequence  $\mathcal{B}(\Omega) \xrightarrow{\varepsilon} \check{C}^0(\mathcal{U}, \mathcal{B}) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{B})$  is exact at  $\check{C}^0(\mathcal{U}, \mathcal{B})$ , that is,  $\text{img } \varepsilon = \ker \delta$ . The map  $i$  is the inclusion map, so that we have that each vertical column is exact along the second row, that is,  $\text{img } i = \ker \bar{\delta}$ . Our proof will be a ‘‘Topological Diagram Chase’’ with this diagram, where we will need to keep track of the continuity of the maps.

Note that, by hypothesis,  $H_{L^2}^{0,1}(\Omega_j)$  is Hausdorff for each  $j$ , so there is a continuous solution operator  $K_j : B_{L^2}^{0,1}(\Omega_j) \rightarrow A_{L^2}^{0,0}(\Omega_j)$ . For example, we can take  $K_j$  to be the canonical solution operator  $\bar{\partial}^* N_{0,1}$ , where  $\bar{\partial}^*$  is the adjoint of  $\bar{\partial}$ , and  $N_{0,1}$  is the  $\bar{\partial}$ -Neumann operator (see [10]). Now, we define a map  $K = K : \check{C}^0(\mathcal{U}, Z_{L^2}^{0,1}) \rightarrow \check{C}^0(\mathcal{U}, A_{L^2}^{0,0})$  by setting  $(Kg)_j = K_j(g_j)$ . Then,  $K$  is continuous, and we have  $\bar{\partial}Kg = g$ .

Consider the map  $\varepsilon : A_{L^2}^{0,1}(\Omega) \rightarrow \check{C}^0(\mathcal{U}, A_{L^2}^{0,1})$ . Since, by hypothesis,  $B_{L^2}^{0,1}(\Omega_j)$  is a closed subspace of  $A_{L^2}^{0,1}(\Omega_j)$  for each  $j$ , it follows that  $\varepsilon^{-1}(\check{C}^0(\mathcal{U}, B_{L^2}^{0,1}))$  is a closed subspace of  $A_{L^2}^{0,1}(\Omega)$ . Define a continuous linear map

$$\ell : \varepsilon^{-1}(\check{C}^0(\mathcal{U}, B_{L^2}^{0,1})) \rightarrow \check{Z}^1(\mathcal{U}, Z_{L^2}^{0,0})$$

by setting

$$(3.2) \quad \ell = \delta K \varepsilon.$$

From the definitions of  $\varepsilon$ ,  $K$ , and  $\delta$ , it follows that  $\ell$  is a continuous linear map from  $\varepsilon^{-1}(\check{C}^0(\mathcal{U}, B_{L^2}^{0,1}))$  to  $\check{C}^1(\mathcal{U}, A_{L^2}^{0,0})$ . But, for  $g \in \varepsilon^{-1}(\check{C}^0(\mathcal{U}, B_{L^2}^{0,1}))$ , we have

$$\bar{\partial}\ell g = \bar{\partial}\delta K \varepsilon g = \delta \bar{\partial} K \varepsilon g = \delta \varepsilon g = 0,$$

which shows that  $\ell(g) \in \check{C}^1(\mathcal{U}, Z_{L^2}^{0,0})$ . However, since  $\delta\bar{\partial} = 0$ , it follows that  $\ell(g) \in \check{Z}^1(\mathcal{U}, Z_{L^2}^{0,0})$ .

The basic property of  $\ell$  that makes it useful is that

$$(3.3) \quad \ell^{-1}(\check{B}^1(\mathcal{U}, Z_{L^2}^{0,0})) = B_{L^2}^{0,1}(\Omega).$$

To see (3.3), first, let  $g \in B_{L^2}^{0,1}(\Omega)$ . Then, there is a  $u \in A_{L^2}^{0,0}(\Omega)$  such that  $\bar{\partial}u = g$ . Consider  $h = K\varepsilon g - \varepsilon u$ . By construction,  $h \in \check{C}^0(\mathcal{U}, A_{L^2}^{0,0})$ . Note that

$$\bar{\partial}h = \bar{\partial}K\varepsilon g - \bar{\partial}\varepsilon u = \varepsilon g - \bar{\partial}\varepsilon u = 0,$$

since  $\bar{\partial}u = g$ . Therefore, in fact,  $h \in \check{C}^0(\mathcal{U}, Z_{L^2}^{0,0})$ . On the other hand,

$$\delta h = \delta K \varepsilon g - \delta \varepsilon u = \ell(g) - 0 = \ell(g),$$

which shows that  $\ell(g) = \delta(h) \in \check{B}^1(\mathcal{U}, Z_{L^2}^{0,0})$ . It follows that

$$B_{L^2}^{0,1}(\Omega) \subset \ell^{-1}(\check{B}^1(\mathcal{U}, Z_{L^2}^{0,0})).$$

For the opposite inclusion, let  $g \in \ell^{-1}(\check{B}^1(\mathfrak{U}, Z_{L^2}^{0,0}))$ ; then,  $\ell(g) \in \check{B}^1(\mathfrak{U}, Z_{L^2}^{0,0})$ , and there exists  $h \in \check{C}^0(\mathfrak{U}, Z_{L^2}^{0,0})$  such that  $\ell(g) = \delta h$ . We define  $u_0 \in \check{C}^0(\mathfrak{U}, A_{L^2}^{0,0})$  by  $u_0 = K\varepsilon g - h$ . In fact,  $\bar{\partial}u_0 = \bar{\delta}K\varepsilon g - \bar{\partial}h = \varepsilon g$ . Also,

$$\delta u_0 = \delta K\varepsilon g - \delta h = \ell(g) - \delta h = 0.$$

It follows that there is a  $u \in A_{L^2}^{0,0}(\Omega)$  such that  $u_0 = \varepsilon u$ . Since  $\bar{\partial}u_0 = \varepsilon(g)$ , it follows that  $g = \bar{\partial}u$ , so that  $g \in B_{L^2}^{0,1}(\Omega)$ . Equation (3.3) is thus established. It follows from (3.3) that there is a continuous linear injective map

$$(3.4) \quad \bar{\ell} : \frac{\varepsilon^{-1}(\check{C}^0(\mathfrak{U}, B_{L^2}^{0,1}))}{B_{L^2}^{0,1}(\Omega)} \rightarrow \check{H}^1(\mathfrak{U}, Z_{L^2}^{0,0}),$$

induced by the map (3.2).

To prove part (1), suppose that  $\check{H}^1(\mathfrak{U}, Z_{L^2}^{0,0})$  is Hausdorff, that is, the subspace  $\check{B}^1(\mathfrak{U}, Z_{L^2}^{0,0})$  is closed in  $\check{Z}^1(\mathfrak{U}, Z_{L^2}^{0,0})$ . Then, by (3.3), since  $\ell$  is continuous,  $B^{0,1}(\Omega)$  is a closed subspace of  $Z_{L^2}^{0,1}(\Omega)$ , which means that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff. This completes the proof of part (1).

To prove part (2), from the hypothesis that  $\check{H}^1(\mathfrak{U}, Z_{L^2}^{0,0}) = 0$  and the injectivity of the map (3.4), we see that

$$(3.5) \quad B_{L^2}^{0,1}(\Omega) = \varepsilon^{-1}(\check{C}^0(\mathfrak{U}, B_{L^2}^{0,1})).$$

Consider, now, the sequence of Hilbert spaces and continuous linear maps:

$$(3.6) \quad Z_{L^2}^{0,1}(\Omega) \xrightarrow{\varepsilon} \check{C}^0(\mathfrak{U}, Z_{L^2}^{0,1}) \xrightarrow{\delta} \check{C}^1(\mathfrak{U}, Z_{L^2}^{0,1}),$$

which is clearly exact. Therefore, going modulo  $B_{L^2}^{0,1}(\Omega)$ , and using (3.5), we obtain an injective continuous map

$$\bar{\varepsilon} : H_{L^2}^{0,1}(\Omega) \rightarrow \frac{\check{C}^0(\mathfrak{U}, Z_{L^2}^{0,1})}{\check{C}^0(\mathfrak{U}, B_{L^2}^{0,1})} \cong \bigoplus_{j=1}^N H_{L^2}^{0,1}(\Omega_j),$$

whose image, by the exactness of (3.6), is the subspace of  $\check{C}^0(\mathfrak{U}, Z_{L^2}^{0,1})/\check{C}^0(\mathfrak{U}, B_{L^2}^{0,1})$  given by

$$\begin{aligned} \text{img } \bar{\varepsilon} &= \frac{\ker(\delta : \check{C}^0(\mathfrak{U}, Z_{L^2}^{0,1}) \rightarrow \check{C}^1(\mathfrak{U}, Z_{L^2}^{0,1}))}{\check{C}^0(\mathfrak{U}, B_{L^2}^{0,1})} \\ &= \left\{ (g_k)_{k=1}^N \in \bigoplus_{k=1}^N Z_{L^2}^{0,1}(\Omega_k) \mid \text{for } i \neq j, g_j|_{\Omega_{ij}} = g_i|_{\Omega_{ij}} \right\} / \left( \bigoplus_{k=1}^N B_{L^2}^{0,1}(\Omega_k) \right) \\ &\subset \left\{ (\gamma_k)_{k=1}^N \mid \gamma_k \in H_{L^2}^{0,1}(\Omega_k), \gamma_i|_{\Omega_{ij}} = \gamma_j|_{\Omega_{ij}} \text{ for } i \neq j \right\}, \end{aligned}$$

which completes the proof of part (2) of the proposition.

For part (3), note that, under the hypothesis  $H_{L^2}^{0,1}(\Omega_j) = 0$  for each  $j$ , the space  $\bigoplus_{k=1}^N H_{L^2}^{0,1}(\Omega_k)$  vanishes. But by part (2), there is an injective mapping of  $H_{L^2}^{0,1}(\Omega)$  into this space, so the conclusion follows.  $\square$

4. CASE OF A HOLE WHICH IS AN INTERSECTION

For an open set  $D$  in  $\mathbb{C}^n$ , from now on, for convenience of notation we denote the space  $Z_{L^2}^{0,0}(D)$  by  $\mathcal{O}_{L^2}(D)$ . Then,  $\mathcal{O}_{L^2}(D) = \mathcal{O}(D) \cap L^2(D)$  is the Bergman space of  $D$ , the space of all holomorphic functions on  $D$  which are square integrable with respect to the Lebesgue measure.

**4.1. Proof of Theorem 1.2.** Set  $\Omega_j = \tilde{\Omega} \setminus K_j$ . Then,  $\mathfrak{U} = \{\Omega_j \mid 1 \leq j \leq N\}$  is an open cover of  $\Omega$ , each  $\Omega_j$  is connected, and  $H_{L^2}^{0,1}(\Omega_j)$  is Hausdorff for each  $j$  by hypothesis.

We claim that  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2}) = 0$ . Let  $f \in \check{Z}^1(\mathfrak{U}, \mathcal{O}_{L^2})$ , so that  $f = (f_{ij})$ , where  $f_{ij} \in \mathcal{O}_{L^2}(\Omega_{ij})$ ,  $1 \leq i, j \leq N$ . By hypothesis (3), the open set  $\Omega_{ij} = \tilde{\Omega} \setminus (K_i \cup K_j)$  is connected. By Hartogs' phenomenon, each  $f_{ij}$  extends to a  $\tilde{f}_{ij} \in \mathcal{O}_{L^2}(\tilde{\Omega})$ . Now, since  $\delta f = 0$ , we have for  $1 \leq i, j, k \leq N$  the following equation on  $\Omega_{ijk}$ .

$$f_{ij} - f_{jk} + f_{ki} = 0.$$

By analytic continuation, we have on the whole of  $\tilde{\Omega}$ :

$$\tilde{f}_{ij} - \tilde{f}_{jk} + \tilde{f}_{ki} = 0.$$

Define a  $u \in \check{C}^0(\mathfrak{U}, \mathcal{O}_{L^2})$  by setting  $u_1 = 0$  on  $\Omega_1$ , and for  $j \geq 2$ ,  $u_j = \tilde{f}_{1j}|_{\Omega_j}$ . Then, on  $\Omega_{ij}$ , we have

$$(\delta u)_{ij} = u_j - u_i = \tilde{f}_{1j} - \tilde{f}_{1i} = \tilde{f}_{ij}|_{\Omega_{ij}} = f_{ij}.$$

Therefore,  $\delta u = f$ , that is,  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2}) = 0$ . It now follows from Theorem 3.1 that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff. This completes the proof of Theorem 1.2.

**4.2. Proof of Corollary 1.3.** We note that the following stronger form of Corollary 1.3 holds, where the Lipschitz domain  $V$  can be replaced by a compact set with minimal boundary regularity hypothesis.

**Corollary 4.1.** Let  $K \Subset \mathbb{C}^n$  be a regular compact set such that  $\mathbb{C}^n \setminus K$  is connected. Suppose that  $K = \bigcap_{j=1}^N \bar{V}_j$ , where for  $1 \leq j \leq N$ ,  $V_j \Subset \mathbb{C}^n$  is a Lipschitz domain such that  $H_{W^1}^{0,n-1}(V_j) = 0$ . If  $\mathbb{C}^n \setminus V_j$  is connected for each  $j$ , and  $\mathbb{C}^n \setminus (V_i \cup V_j)$  is connected for each  $1 \leq i, j \leq N$ , then  $H_{W^1}^{0,n-1}(K) = 0$ .

*Proof.* Recall that, here, regularity of the compact  $K$  is in the sense of (2.1), and  $H_{W^1}^{0,n-1}(K) = 0$  is in the sense explained after (2.2) above. Apply Theorem 1.2, taking  $\tilde{\Omega}$  to be a ball of sufficiently large radius to contain the closure of all the  $V_j$ . If we set  $K_j = \bar{V}_j$ , it is easy to verify that all the hypotheses of Theorem 1.2 hold, so that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff. The conclusion follows from Proposition 2.1.  $\square$

**4.3. Proof of Corollary 1.6.** By hypothesis, we have  $H_{L^2}^{0,1}(\Omega_j) = 0$  for each  $\Omega_j \in \mathfrak{U}$ , and as we saw in the proof of Theorem 1.2 above,  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2}) = 0$ . Consequently, by Theorem 3.1 (3) we have  $H_{L^2}^{0,1}(\Omega) = 0$ .

### 5. $L^2$ -ČECH COHOMOLOGY OF AN ANNULUS BETWEEN PRODUCT DOMAINS

Let  $N \geq 2$ , and for each  $j \in \{1, \dots, N\}$  let  $U_j \subset \mathbb{C}^{n_j}$  be a bounded domain, and  $K_j$  a compact set in  $\mathbb{C}^{n_j}$  such that  $K_j \subset U_j$ , and such that the annulus  $R_j = U_j \setminus K_j$  is connected. Let  $U = U_1 \times \dots \times U_N$  and  $K = K_1 \times \dots \times K_N$ . In this section we consider the domain  $\mathbb{W} = U \setminus K$ , which is an annulus between a product domain and a product hole. We let

$$\Omega_j = U_1 \times \dots \times R_j \times \dots \times U_N,$$

where the  $j$ -th factor is  $R_j$  and for  $k \neq j$ , the  $k$ -th factor is  $U_k$ . We call the domains  $R_j$  ( $j = 1, \dots, N$ ) the *factor annuli* of  $\mathbb{W}$ . We denote the collection  $\{\Omega_j \mid 1 \leq j \leq N\}$  by  $\mathfrak{U}$ , and note that  $\bigcup_{j=1}^N \Omega_j = \mathbb{W}$ , that is,  $\mathfrak{U}$  is a cover of  $\mathbb{W}$  by open sets. In this section, we prove the following result.

**Proposition 5.1.** *With  $\mathfrak{U}$  as above,  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2})$  is Hausdorff, and vanishes if  $n \geq 3$ .*

Note that pseudoconvexity does not play any direct role in the statement of this result. The proof will be based on a closed-range property of the restriction map on Bergman spaces (see Lemma 5.2). Also, it is true that for  $n = 2$ , the group  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2})$  is infinite dimensional, though we neither prove nor use this fact. A technique similar to that used in the proof of Proposition 5.1 was used to compute the usual Dolbeault cohomology of some non-pseudoconvex domains in [4].

**5.1. Closed range for restriction maps on Bergman spaces.** For open subsets  $U, R \subset \mathbb{C}^n$ , where  $R \subset U$ , we denote by  $\mathcal{O}_{L^2}(U)|_R$  the subspace of  $\mathcal{O}_{L^2}(R)$  consisting of restrictions of functions in  $\mathcal{O}_{L^2}(U)$ .

**Lemma 5.2.** *Let  $U$  be a bounded domain in  $\mathbb{C}^n$ , and let  $K \subset U$  be a compact subset. Set  $R = U \setminus K$ , and when  $n \geq 2$ , assume that  $R$  is connected. Then,  $\mathcal{O}_{L^2}(U)|_R$  is a closed subspace of the Hilbert space  $\mathcal{O}_{L^2}(R)$ .*

*Proof.* If  $n \geq 2$ , then by the Hartogs extension theorem,  $\mathcal{O}_{L^2}(U)|_R = \mathcal{O}_{L^2}(R)$ , so the assertion is correct. For  $n = 1$ , by a classical argument (cf. [1, p. 143] or [7, p. 195, Proposition 1.1]), there is a neighborhood  $W$  of  $K$  such that  $W$  is



contained in  $U$ , the boundary  $bW$  is a union of finitely many closed polygons, and for any holomorphic function in  $U$ , we have a Cauchy representation

$$f(z) = \frac{1}{2\pi i} \int_{bW} \frac{f(\zeta)}{\zeta - z} d\zeta$$

valid for each  $z \in K$ .

Let  $f_\nu \in \mathcal{O}_{L^2}(U)$ , and suppose that  $f_\nu|_R \rightarrow g$  in  $\mathcal{O}_{L^2}(R)$  as  $\nu \rightarrow \infty$ . By Bergman's inequality (cf. [24, p. 155]),  $\{f_\nu\}$  converges uniformly to  $g$  when restricted to the compact subset  $bW$  of  $R$ . Representing the holomorphic function  $f_\nu$  on  $K$  as the Cauchy integral over  $bW$  as in the above formula, we see that  $\{f_\nu\}$  converges uniformly on  $K$  to a function given by the Cauchy integral of  $g$ . It follows that  $g$  extends to a function in  $\mathcal{O}_{L^2}(U)$ , which proves the lemma.  $\square$

Let  $\mathcal{Q} \hat{\otimes} \mathcal{R}$  denote the Hilbert tensor product of Hilbert spaces  $\mathcal{Q}$  and  $\mathcal{R}$ . In our application, we only consider Hilbert tensor products where for some domain  $D \subset \mathbb{C}^m$ , the space  $\mathcal{Q}$  is a closed subspace of  $\mathcal{O}_{L^2}(D)$ , and for some domain  $V \subset \mathbb{C}^n$ , the space  $\mathcal{R}$  is a closed subspace of  $\mathcal{O}_{L^2}(V)$ . Then,  $\mathcal{Q} \hat{\otimes} \mathcal{R}$  is the closure in  $\mathcal{O}_{L^2}(D \times V)$  of the linear span of functions of the form  $(f \otimes g)(z, w) = f(z)g(w)$ , where  $f \in \mathcal{Q}$  and  $g \in \mathcal{R}$ . See [6] for more details on Hilbert tensor products.

**5.2. Proof of Proposition 5.1.** We first consider the case when the number of factors  $N = 2$ . To conclude that  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2})$  is Hausdorff, we need to show that the coboundary map

$$\delta : \check{C}^0(\mathfrak{U}, \mathcal{O}_{L^2}) \rightarrow \check{C}^1(\mathfrak{U}, \mathcal{O}_{L^2})$$

has closed range. As a topological vector space,  $\check{C}^1(\mathfrak{U}, \mathcal{O}_{L^2})$  is simply  $\mathcal{O}_{L^2}(\Omega_{12})$ , since there is only one double intersection. Therefore, the closed range of  $\delta$  will follow, if we show that the map

$$\mathcal{O}_{L^2}(\Omega_1) \oplus \mathcal{O}_{L^2}(\Omega_2) \rightarrow \mathcal{O}_{L^2}(\Omega_{12}),$$

given by  $\delta(h_1, h_2) = h_2|_{\Omega_{12}} - h_1|_{\Omega_{12}}$ , has closed range. Recall that  $\Omega_1 = R_1 \times U_1$ ,  $\Omega_2 = U_1 \times R_2$ , and  $\Omega_{12} = R_1 \times R_2$ .

Since for  $j = 1, 2$  by Lemma 5.2  $\mathcal{O}_{L^2}(U_j)|_{R_j}$  is closed in  $\mathcal{O}_{L^2}(R_j)$ , we obtain a direct sum decomposition

$$\mathcal{O}_{L^2}(R_j) = \mathcal{O}_{L^2}(U_j)|_{R_j} \oplus (\mathcal{O}_{L^2}(U_j)|_{R_j})^\perp,$$

and it follows by the distributivity of the Hilbert tensor product over direct sums that

$$\mathcal{O}_{L^2}(\Omega_{12}) = \mathcal{O}_{L^2}(R_1 \times R_2) = \mathcal{O}_{L^2}(R_1) \hat{\otimes} \mathcal{O}_{L^2}(R_2) = \bigoplus_{j=1}^4 E_j,$$

where

$$\begin{aligned} E_1 &= \mathcal{O}_{L^2}(U_1)|_{R_1} \hat{\otimes} \mathcal{O}_{L^2}(U_2)|_{R_2}, \\ E_2 &= (\mathcal{O}_{L^2}(U_1)|_{R_1})^\perp \hat{\otimes} \mathcal{O}_{L^2}(U_2)|_{R_2}, \\ E_3 &= \mathcal{O}_{L^2}(U_1)|_{R_1} \hat{\otimes} (\mathcal{O}_{L^2}(U_2)|_{R_2})^\perp, \end{aligned}$$

and

$$E_4 = (\mathcal{O}_{L^2}(U_1)|_{R_1})^\perp \hat{\otimes} (\mathcal{O}_{L^2}(U_2)|_{R_2})^\perp,$$

and for  $j = 1, 2$ ,  $(\mathcal{O}_{L^2}(U_j)|_{R_j})^\perp$  denotes the orthogonal complement of the closed subspace  $\mathcal{O}_{L^2}(U_j)|_{R_j}$  in  $\mathcal{O}_{L^2}(R_j)$ . Note that

$$E_1 \oplus E_3 = \mathcal{O}_{L^2}(U_1)|_{R_1} \hat{\otimes} \mathcal{O}_{L^2}(R_2) = \mathcal{O}_{L^2}(U_1 \times R_2)|_{R_1 \times R_2} \subset \text{img } \delta$$

and

$$E_1 \oplus E_2 = \mathcal{O}_{L^2}(R_1) \hat{\otimes} \mathcal{O}_{L^2}(U_2)|_{R_2} = \mathcal{O}_{L^2}(R_1 \times U_2)|_{R_1 \times R_2} \subset \text{img } \delta,$$

and by definition of  $\delta$ ,

$$\begin{aligned} \text{img } \delta &= \mathcal{O}_{L^2}(U_1 \times R_2)|_{R_1 \times R_2} + \mathcal{O}_{L^2}(R_1 \times U_2)|_{R_1 \times R_2} \\ &= (E_1 \oplus E_3) + (E_1 \oplus E_2) = E_1 \oplus E_2 \oplus E_3 = (E_4)^\perp \\ &= ((\mathcal{O}_{L^2}(U_1)|_{R_1})^\perp \hat{\otimes} (\mathcal{O}_{L^2}(U_2)|_{R_2})^\perp)^\perp, \end{aligned}$$

where the outer  $\perp$  in the last line, and the  $\perp$  in the previous-to-last line denote orthogonal complementation in  $\mathcal{O}_{L^2}(\Omega_{12})$ . It follows that  $\text{img } \delta$  is closed, and we obtain an isomorphism of Hilbert spaces

$$\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2}) \cong E_4 = (\mathcal{O}_{L^2}(U_1)|_{R_1})^\perp \hat{\otimes} (\mathcal{O}_{L^2}(U_2)|_{R_2})^\perp,$$

valid when the number of factors  $N = 2$  and the dimension  $n \geq 2$ .

Assume  $n \geq 3$ . Since  $n = n_1 + n_2$ , there is a  $j \in \{1, 2\}$  such that  $n_j \geq 2$ . By Hartogs' phenomenon,  $\mathcal{O}_{L^2}(U_j)|_{R_j} = \mathcal{O}_{L^2}(R_j)$  so we have  $(\mathcal{O}_{L^2}(U_j)|_{R_j})^\perp = \{0\}$ , and consequently,  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2}) = 0$ .

Now, we will show that for  $N \geq 3$  (which forces  $n \geq 3$ ), we again have  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2}) = 0$ . A similar result (with one-dimensional factors) was proved by Frenkel [9, Proposition 31.1] for the structure sheaf  $\mathcal{O}$ .

Let  $N \geq 3$ , and suppose that  $f = (f_{ij})_{i < j} \in \check{Z}^1(\mathfrak{U}, \mathcal{O}_{L^2})$ . We show that there is a  $u \in \check{C}^0(\mathfrak{U}, \mathcal{O}_{L^2})$  such that  $f = \delta u$ .

Now, denote by  $\mathcal{E}$  the subspace of  $\check{Z}^1(\mathfrak{U}, \mathcal{O}_{L^2})$  consisting of  $f$  such that each  $f_{ij} \in \mathcal{O}_{L^2}(\Omega_{ij})$  extends holomorphically to a function in  $\mathcal{O}_{L^2}(U)$ , so that on  $\Omega_{ij}$ ,

$$(5.1) \quad f_{ij} \in \mathcal{O}_{L^2}(U_i)|_{R_i} \hat{\otimes} \mathcal{O}_{L^2}(U_j)|_{R_j} \hat{\otimes} \mathcal{O}_{L^2}(U'_{ij}),$$

where the tensor product has been reordered (we do this in the sequel without further comment) and  $U'_{ij}$  is the product of all the  $U_k$ s except  $U_i$  and  $U_j$ . Define  $u \in \check{C}^0(\mathfrak{U}, \mathcal{O}_{L^2})$  by setting  $u_1 = 0$  on  $\Omega_1$ , and for  $j \geq 2$ ,  $u_j = f_{1j}|_{\Omega_j}$ , where we continue to denote the extension of  $f_{ij}$  to  $U$  by the same symbol. Then, on the set  $\Omega_{ij}$ , we have

$$u_j - u_i = f_{1j} - f_{1i} = f_{ij},$$

since  $\delta f = 0$ . We have, therefore,  $\delta u = f$ .

Hence, we may assume without loss of generality that  $f \in \check{Z}^1(\mathfrak{U}, \mathcal{O}_{L^2})$  actually belongs to the orthogonal complement  $\mathcal{E}^\perp$  of  $\mathcal{E}$  in  $\check{Z}^1(\mathfrak{U}, \mathcal{O}_{L^2})$ . Noting that  $\mathcal{O}_{L^2}(\Omega_{ij}) = \mathcal{O}_{L^2}(R_i) \hat{\otimes} \mathcal{O}_{L^2}(R_j) \hat{\otimes} \mathcal{O}_{L^2}(U'_{ij})$ , and that, by Lemma 5.2, for each  $k$  we have  $\mathcal{O}_{L^2}(R_k) = \mathcal{O}_{L^2}(U_k)|_{R_k} \oplus \mathcal{O}_{L^2}(U_k)|_{R_k}^\perp$ , we have from (5.1) for each  $i < j$

$$(5.2) \quad f_{ij} \in S_1 \oplus S_2 \oplus S_3 \subset \mathcal{O}_{L^2}(\Omega_{ij}),$$

where

$$\begin{aligned} S_1 &= \mathcal{O}_{L^2}(U_i)|_{R_i}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U_j)|_{R_j} \hat{\otimes} \mathcal{O}_{L^2}(U'_{ij}), \\ S_2 &= \mathcal{O}_{L^2}(U_i)|_{R_i} \hat{\otimes} \mathcal{O}_{L^2}(U_j)|_{R_j}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U'_{ij}), \\ S_3 &= \mathcal{O}_{L^2}(U_i)|_{R_i} \hat{\otimes} \mathcal{O}_{L^2}(U_j)|_{R_j} \hat{\otimes} \mathcal{O}_{L^2}(U'_{ij}). \end{aligned}$$

We claim that *the component of  $f_{ij}$  along  $S_3$  vanishes*, that is, we have

$$(5.3) \quad f_{ij} \in S_1 \oplus S_2.$$

To prove the claim, since  $N \geq 3$ , there is a  $k \in \{1, 2, \dots, N\}$  such that  $k$  is distinct from both  $i$  and  $j$ . Let  $U'_{ijk}$  be the product of all the  $U_\ell$  except  $U_i$ ,  $U_j$ , and  $U_k$ . Consider the restriction map  $\rho$  from  $\mathcal{O}_{L^2}(\Omega_{ij})$  to  $\mathcal{O}_{L^2}(\Omega_{ijk})$ . Since  $\Omega_{ij} = R_i \times R_j \times R_k \times U'_{ijk}$  and  $\Omega_{ijk} = R_i \times R_j \times R_k \times R'_{ijk}$ , the restriction map is a tensor product:

$$\rho = \text{id}_{\mathcal{O}_{L^2}(R_i)} \hat{\otimes} \text{id}_{\mathcal{O}_{L^2}(R_j)} \hat{\otimes} \rho_k \hat{\otimes} \text{id}_{\mathcal{O}_{L^2}(U'_{ijk})},$$

where  $\rho_k$  is the restriction map from  $\mathcal{O}_{L^2}(U_k)$  to  $\mathcal{O}_{L^2}(R_k)$ . Consequently, we have a tensor product representation

$$S_3|_{\Omega_{ijk}} = \mathcal{O}_{L^2}(U_i)|_{R_i}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U_j)|_{R_j}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U_k)|_{R_k} \hat{\otimes} \mathcal{O}_{L^2}(U'_{ijk}).$$

Denote by  $p_3$  the orthogonal projection from  $\mathcal{O}_{L^2}(\Omega_{ij})$  to  $S_3$ , and by  $P_3$  the orthogonal projection from  $\mathcal{O}_{L^2}(\Omega_{ijk})$  onto  $S_3|_{\Omega_{ijk}}$ . Then, we have the diagram

$$(5.4) \quad \begin{array}{ccc} \mathcal{O}_{L^2}(\Omega_{ij}) & \xrightarrow{p_3} & S_3 \\ \downarrow \rho & & \downarrow \rho \\ \mathcal{O}_{L^2}(\Omega_{ijk}) & \xrightarrow{P_3} & S_3|_{\Omega_{ijk}} \end{array}$$

which commutes, since both  $\rho \circ p_3$  and  $P_3 \circ \rho$  are equal to  $\pi_i \hat{\otimes} \pi_j \hat{\otimes} \rho_k \hat{\otimes} \text{id}_{\mathcal{O}_{L^2}(U'_{ijk})}$ , where  $\pi_i : \mathcal{O}_{L^2}(U_i) \rightarrow \mathcal{O}_{L^2}(U_i)|_{R_i}^\perp$  and  $\pi_j : \mathcal{O}_{L^2}(U_j) \rightarrow \mathcal{O}_{L^2}(U_j)|_{R_j}^\perp$  are the orthogonal projections.

We show that  $P_3(f_{ij}|_{\Omega_{ijk}}) = 0$ . From  $\delta f = 0$ , we conclude that

$$(5.5) \quad f_{ij}|_{\Omega_{ijk}} = f_{ik}|_{\Omega_{ijk}} - f_{jk}|_{\Omega_{ijk}}.$$

Now,

$$f_{ik}|_{\Omega_{ijk}} \in \mathcal{O}_{L^2}(U_i)|_{R_i}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U_j)|_{R_j} \hat{\otimes} \mathcal{O}_{L^2}(U_k)|_{R_k}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U'_{ijk}),$$

and

$$f_{jk}|_{\Omega_{ijk}} \in \mathcal{O}_{L^2}(U_i)|_{R_i} \hat{\otimes} \mathcal{O}_{L^2}(U_j)|_{R_j}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U_k)|_{R_k}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U'_{ijk}),$$

so that both  $f_{ik}|_{\Omega_{ijk}}$  and  $f_{jk}|_{\Omega_{ijk}}$  lie in subspaces of  $\mathcal{O}_{L^2}(\Omega_{ijk})$  which are orthogonal to  $S_3|_{\Omega_{ijk}}$ . Therefore,  $P_3(f_{ik}|_{\Omega_{ijk}}) = P_3(f_{jk}|_{\Omega_{ijk}}) = 0$ . Therefore, by (5.5), we see that  $P_3(f_{ij}|_{\Omega_{ijk}}) = 0$ .

Now, by the commutativity of the diagram (5.4), we have that

$$\rho(p_3(f_{ij})) = P_3(\rho(f_{ij})) = P_3(f_{ij}|_{\Omega_{ijk}}) = 0,$$

and since, by analytic continuation,  $\rho$  is injective, we have  $p_3(f_{ij}) = 0$ . Therefore, the claim (5.3) follows.

Denote now by  $P_1$  and  $P_2$  the projections from  $\mathcal{O}_{L^2}(\Omega_{ijk})$  onto  $S_1|_{\Omega_{ijk}}$  and  $S_2|_{\Omega_{ijk}}$ , respectively. Note that

$$(5.6) \quad \begin{aligned} S_1|_{\Omega_{ijk}} &= \mathcal{O}_{L^2}(U_i)|_{R_i}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U_j)|_{R_j} \hat{\otimes} \mathcal{O}_{L^2}(U_k)|_{R_k} \hat{\otimes} \mathcal{O}_{L^2}(U'_{ij}), \\ S_2|_{\Omega_{ijk}} &= \mathcal{O}_{L^2}(U_i)|_{R_i} \hat{\otimes} \mathcal{O}_{L^2}(U_j)|_{R_j}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U_k)|_{R_k} \hat{\otimes} \mathcal{O}_{L^2}(U'_{ij}). \end{aligned}$$

By the representation in (5.6) above, we see that  $P_1(f_{ij})$  extends holomorphically to an element  $(-u_i^{ij})$  of  $\mathcal{O}_{L^2}(\Omega_i)$ , and  $P_2(f_{ij})$  extends holomorphically to an element  $u_j^{ij}$  of  $\mathcal{O}_{L^2}(\Omega_j)$ . Therefore, we have

$$(5.7) \quad f_{ij} = u_j^{ij}|_{\Omega_{ij}} - u_i^{ij}|_{\Omega_{ij}}.$$

We note two features of this decomposition. First, it is independent of the choice of the auxiliary index  $k$ , since the decomposition (5.2) in no way depends on  $k$ . Second, by construction,  $u_i^{ij}|_{\Omega_{ijk}}$  and  $u_j^{ij}|_{\Omega_{ijk}}$  belong to the orthogonal subspaces  $S_1|_{\Omega_{ijk}}$  and  $S_2|_{\Omega_{ijk}}$  of  $\mathcal{O}_{L^2}(\Omega_{ijk})$ . (The orthogonality is immediate from (5.6).)

We claim that if  $k$  is an index distinct from  $i$  and  $j$ , we have  $u_i^{ij} = u_i^{ik}$ . To see this, we substitute expressions like (5.7) into (5.5), and obtain that, when restricted to  $\Omega_{ijk}$ , we have

$$(u_k^{jk} - u_j^{jk}) - (u_k^{ik} - u_i^{ik}) + (u_j^{ij} - u_i^{ij}) = 0,$$

which gives rise to the condition that

$$(u_k^{jk} |_{\Omega_{ijk}} - u_k^{ik} |_{\Omega_{ijk}}) + (u_j^{ij} |_{\Omega_{ijk}} - u_j^{jk} |_{\Omega_{ijk}}) + (u_i^{ik} |_{\Omega_{ijk}} - u_i^{ij} |_{\Omega_{ijk}}) = 0.$$

The three terms of the above sum belong to three orthogonal subspaces of  $\mathcal{O}_{L^2}(\Omega_{ijk})$ . The first term is in

$$\mathcal{O}_{L^2}(U_i) |_{R_i} \hat{\otimes} \mathcal{O}_{L^2}(U_j) |_{R_j} \hat{\otimes} \mathcal{O}_{L^2}(U_k) |_{R_k}^\perp \hat{\otimes} \mathcal{O}_{L^2}(U'_{ij}),$$

the second term is in  $S_2|_{\Omega_{ijk}}$ , and the third term is in  $S_1|_{\Omega_{ijk}}$ , where the notation is as in (5.6). Therefore, all three terms vanish. Thus, by analytic continuation, it will follow that there is for each  $i \in \{1, \dots, N\}$  an  $u_i \in \mathcal{O}_{L^2}(\Omega_i)$  such that, if  $u = (u_i)_{i=1}^N \in \check{C}^0(\mathfrak{U}, \mathcal{O}_{L^2})$ , then  $\delta u = f$ . It follows that  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2}) = 0$  if  $N \geq 3$ .

## 6. PROOF OF THEOREM 1.4

**6.1. The  $L^2$ -Dolbeault cohomology of an annulus between product domains.** Let  $\mathbb{W} = U \setminus K \subset \mathbb{C}^n$  be as in Section 5, an annulus between the products  $U = U_1 \times \dots \times U_N$  and  $K = K_1 \times \dots \times K_N$ , and let  $R_j = U_j \setminus K_j \subset \mathbb{C}^{n_j}$  denote the factor annuli for  $j = 1, \dots, N$ . In this section, we compute the  $L^2$ -Dolbeault cohomology of  $\mathbb{W}$ .

**Proposition 6.1.** *Suppose that for each  $j$ ,  $H_{L^2}^{0,1}(R_j)$  is Hausdorff. Then,  $H_{L^2}^{0,1}(\mathbb{W})$  is Hausdorff. Further, if  $H_{L^2}^{0,1}(U_j) = 0$  for each  $j$ , then  $H_{L^2}^{0,1}(\mathbb{W})$  vanishes if  $n \geq 3$ .*

*Proof.* By Proposition 5.1, the Čech group  $\check{H}^1(\mathfrak{U}, Z_{L^2}^{0,0})$  is Hausdorff. Now, in the cover  $\mathfrak{U}$ , we can write  $\Omega_j = U_1 \times \dots \times R_j \times \dots \times U_N$ . If  $n_j = 1$ , then  $H_{L^2}^{0,1}(U_j) = 0$  (and therefore Hausdorff), and if  $n_j \geq 2$ , then by Proposition 2.2, since each  $H_{L^2}^{0,1}(R_j)$  is Hausdorff, and  $R_j = U_j \setminus K_j$  is an annulus, it follows that each  $H_{L^2}^{0,1}(U_j)$  is also Hausdorff. It follows from the results of [3, 6] regarding the  $L^2$ -cohomology of product domains, that since  $\Omega_j$  is a product of domains whose  $L^2$ -Dolbeault cohomology is Hausdorff in degrees  $(0, 0)$  and  $(0, 1)$ , the cohomology  $H_{L^2}^{0,1}(\Omega_j)$  is also Hausdorff. Now, by part (1) of Theorem 3.1, since we have covering  $\mathfrak{U}$  of  $\mathbb{W}$  such that for each  $\Omega_j \in \mathfrak{U}$  we have  $H_{L^2}^{0,1}(\Omega_j)$  Hausdorff, as well as the Čech group  $\check{H}^1(\mathfrak{U}, Z_{L^2}^{0,0})$  Hausdorff, we conclude that  $H_{L^2}^{0,1}(\mathbb{W})$  is Hausdorff.

Now, let  $n \geq 3$ . Apply part (2) of Theorem 3.1, and assume  $H^{0,1}(U_j) = 0$  for each  $j$ . By Proposition 5.1,  $\check{H}^1(\mathfrak{U}, \mathcal{O}_{L^2}) = 0$ , so the cohomology  $H_{L^2}^{0,1}(\Omega)$  is isomorphic to a subspace of  $\bigoplus_{k=1}^N H_{L^2}^{0,1}(\Omega_k)$  contained in

$$\{(\mathcal{Y}_k)_{k=1}^N, \mathcal{Y}_k \in H_{L^2}^{0,1}(\Omega_k) \mid \text{for } i \neq j, \mathcal{Y}_i |_{\Omega_{ij}} = \mathcal{Y}_j |_{\Omega_{ij}}\}.$$

It is therefore sufficient to show that the above space vanishes. Recall that

$$\Omega_1 = R_1 \times U_2 \times U'_{12},$$

and

$$\Omega_2 = U_1 \times R_2 \times U'_{12},$$

Where  $U'_{12} = U_3 \times \dots \times U_N$ . Then,  $\Omega_{12} = R_1 \times R_2 \times U'_{12}$ . We have, by the Künneth formula for  $L^2$ -cohomology (cf. [3, 6]),

$$H_{L^2}^{0,1}(\Omega_1) = H_{L^2}^{0,1}(R_1) \hat{\otimes} H_{L^2}^{0,0}(U_2) \hat{\otimes} H_{L^2}^{0,0}(U'_{12}),$$

and

$$H_{L^2}^{0,1}(\Omega_2) = H_{L^2}^{0,0}(U_1) \hat{\otimes} H_{L^2}^{0,1}(R_2) \hat{\otimes} H_{L^2}^{0,0}(U'_{12}),$$

where the other terms vanish, since  $H^{0,1}(U_j) = 0$  for each  $j$ . Similarly, we obtain

$$(6.1) \quad \begin{aligned} H^{0,1}(\Omega_{12}) &= H_{L^2}^{0,1}(R_1) \hat{\otimes} H_{L^2}^{0,0}(R_2) \hat{\otimes} H^{0,0}(U'_{12}) \\ &\oplus H_{L^2}^{0,0}(R_1) \hat{\otimes} H_{L^2}^{0,1}(R_2) \hat{\otimes} H_{L^2}^{0,0}(U'_{12}). \end{aligned}$$

Note, also, that the two direct summands in (6.1) are orthogonal to each other because of the tensor nature of the inner product on  $H_{L^2}^{0,1}(\Omega_{12})$  (see [6]). Consider now the restriction map

$$(6.2) \quad H_{L^2}^{0,1}(\Omega_1) \rightarrow H_{L^2}^{0,1}(\Omega_{12}),$$

written as  $\gamma \rightarrow \gamma|_{\Omega_{12}}$ , whose image is

$$(6.3) \quad H_{L^2}^{0,1}(\Omega_1)|_{\Omega_{12}} \cong H_{L^2}^{0,1}(R_1) \hat{\otimes} H^{0,0}(U_2)|_{R_2} \hat{\otimes} H_{L^2}^{0,0}(U'_{12}).$$

The map (6.2) may be represented as a tensor product of maps

$$\text{id}_{H_{L^2}^{0,1}(R_1)} \hat{\otimes} \rho \hat{\otimes} \text{id}_{H_{L^2}^{0,0}(U'_{12})},$$

where

$$\rho : H_{L^2}^{0,0}(U_2) \rightarrow H_{L^2}^{0,0}(R_2)$$

is the restriction map  $f \mapsto f|_{R_2}$  from  $H_{L^2}^{0,0}(U_2) = \mathcal{O}_{L^2}(U_2)$  to  $H_{L^2}^{0,0}(R_2) = \mathcal{O}_{L^2}(R_2)$ , which is injective by analytic continuation, since  $R_2$  is connected. Therefore the map (6.2) is also injective, being the tensor product of injective maps. A similar reasoning shows that the restriction map  $H_{L^2}^{0,1}(\Omega_2) \rightarrow H_{L^2}^{0,1}(\Omega_{12})$  is also injective, and has image

$$(6.4) \quad H_{L^2}^{0,1}(\Omega_2)|_{\Omega_{12}} \cong H_{L^2}^{0,0}(U_1)|_{R_1} \hat{\otimes} H_{L^2}^{0,1}(R_2) \hat{\otimes} H_{L^2}^{0,0}(U'_{12}).$$

The subspaces of  $H_{L^2}^{0,1}(\Omega_{12})$  given by (6.3) and (6.4) are orthogonal, being contained in different summands of the orthogonal direct sum (6.1), so that we

have  $H_{L^2}^{0,1}(\Omega_1)|_{\Omega_{12}} \cap H_{L^2}^{0,1}(\Omega_2)|_{\Omega_{12}} = \{0\}$ , and since this reasoning applies if 1 and 2 are replaced by  $i$  and  $j$  with  $i \neq j$ , we see that

$$(6.5) \quad H_{L^2}^{0,1}(\Omega_i)|_{\Omega_{ij}} \cap H_{L^2}^{0,1}(\Omega_j)|_{\Omega_{ij}} = \{0\}, \quad \forall i, j \in \{1, \dots, N\}.$$

Now, in (3.1), we will let the  $N$ -tuple  $(y_k)_{k=1}^N \in \bigoplus_{k=1}^N H_{L^2}^{0,1}(\Omega_k)$  be such that  $y_i|_{\Omega_{ij}} = y_j|_{\Omega_{ij}}$  whenever  $i \neq j$ . Therefore, by (6.5), we have  $y_i|_{\Omega_{ij}} = 0$  for all  $i \neq j$ . Now, the same reasoning that shows that the map (6.2) is injective shows that the restriction map  $H_{L^2}^{0,1}(\Omega_i) \rightarrow H^{0,1}(\Omega_{ij})$  is injective when  $i \neq j$ , and this shows that  $y_i = 0$  for each  $i$ . Part (2) of Theorem 3.1 now shows that  $H_{L^2}^{0,1}(\mathbb{W}) = 0$ .  $\square$

**6.2. Proof of Theorem 1.4.** We begin by noting the following variant of Theorem 1.4 with minimal boundary regularity, when the factors are each one dimensional.

**Proposition 6.2.** *For  $j = 1, \dots, n$ , let  $K_j$  be a compact subset of  $\mathbb{C}$ , and let  $K = K_1 \times \dots \times K_n \subset \mathbb{C}^n$  be their Cartesian product. Then, if  $K$  is regular, we have  $H_{W^1}^{0,n-1}(K) = 0$ .*

*Proof.* For each  $j$ , let  $U_j$  be a large disc containing the compact  $K_j$ . Apply Proposition 2.1, with  $\tilde{\Omega} = U = U_1 \times \dots \times U_n$ , and  $K = K_1 \times \dots \times K_n$ . A simple topological reasoning shows that, for large enough  $U_j$ , the annulus  $U \setminus K$  is connected. Since  $U$  is pseudoconvex, the result follows.  $\square$

The proof of the general case is similar.

*Proof of Theorem 1.4.* For each  $j$ , choose a large ball  $U_j \subset \mathbb{C}^{n_j}$  such that  $V_j \Subset U_j$ . If  $n_j \geq 2$ , since  $H_{W^1}^{0,n_j-1}(V_j) = 0$ , and  $H_{L^2}^{0,1}(U_j) = 0$ , it follows by Theorem 1.1 that  $H_{L^2}^{0,1}(R_j)$  is Hausdorff, where  $R_j = U_j \setminus \bar{V}_j$ . In case  $n_j = 1$ , then  $H_{L^2}^{0,1}(R_j) = 0$ , and so it is Hausdorff. Now, we will let  $\mathbb{W} = U \setminus \bar{V}$ , where  $U = U_1 \times U_2 \times \dots \times U_N$ . Therefore, by Proposition 6.1, we see that  $H_{L^2}^{0,1}(\mathbb{W})$  is Hausdorff. We now invoke Theorem 1.1 again in order to conclude that  $H_{W^1}^{0,n-1}(V) = 0$ .  $\square$

## 7. PROOF OF THEOREM 1.8

**7.1. Extension of solvability.** In this section, to prove Theorem 1.8, we consider the following situation. Let  $D \Subset \tilde{\Omega}$  be a bounded domain in  $\mathbb{C}^n$ , and let  $K$  be a compact set contained in  $D$ . We consider the relation between the cohomologies of the annuli

$$\Omega = \tilde{\Omega} \setminus K \quad \text{and} \quad \mathbb{W} = D \setminus K.$$

**Proposition 7.1.** *Suppose that in some degree  $(p, q)$ ,  $q \geq 1$ , we have that  $H_{L^2}^{p,q}(\tilde{\Omega}) = 0$ , and the  $L^2$ -cohomology  $H_{L^2}^{p,q}(\mathbb{W})$  is Hausdorff. Then, we have the following:*

- (a)  $H_{L^2}^{p,q}(\Omega)$  is Hausdorff.
- (b) The natural restriction map on cohomology  $H_{L^2}^{p,q}(\Omega) \rightarrow H_{L^2}^{p,q}(\mathbb{W})$  is injective. Consequently, if  $H^{p,q}(\mathbb{W}) = 0$  then  $H^{p,q}(\Omega) = 0$ .

The proof is based on the following observation.

**Lemma 7.2.** *With hypothesis and notation as above, there is a constant  $C > 0$  with the following property. Suppose that  $g \in Z_{L^2}^{p,q}(\Omega)$  is such that the restriction  $g|_{\mathbb{W}}$  is in  $B_{L^2}^{p,q}(\mathbb{W})$ . Then, there is a  $u \in A_{L^2}^{p,q-1}(\Omega)$  such that  $\bar{\partial}u = g$  and*

$$\|u\|_{L^2(\Omega)} \leq C \|g\|_{L^2(\Omega)}.$$

*Proof.* We denote by  $C$  any constant that depends solely on the geometry of the domains  $\mathbb{W}$  and  $\tilde{\Omega}$ , and  $C$  may have different values at different occurrences. Since  $H_{L^2}^{p,q}(\mathbb{W})$  is Hausdorff, there is a  $u_0 \in A_{L^2}^{p,q-1}(\mathbb{W})$  such that  $\bar{\partial}u_0 = g|_{\mathbb{W}}$ , and we have an estimate

$$(7.1) \quad \|u_0\|_{L^2(\mathbb{W})} \leq C \|g\|_{L^2(\mathbb{W})}.$$

We take a cutoff  $\chi \in C_0^\infty(D)$  such that  $\chi \equiv 1$  near  $K$ . As usual, we assume that after multiplying by a cutoff, we extend functions and forms by zero outside the support of the cutoff. We note that  $\bar{\partial}(\chi u_0) = \bar{\partial}\chi \wedge u_0 + \chi \cdot g$  on  $\Omega$ , so that we have an estimate

$$(7.2) \quad \|\bar{\partial}(\chi u_0)\|_{L^2(\Omega)} \leq C \|g\|_{L^2(\Omega)}.$$

Let  $h = g - \bar{\partial}(\chi u_0)$ . Since, by hypothesis,  $g \in Z_{L^2}^{p,q}(\Omega)$ , we see that  $h \in Z_{L^2}^{p,q}(\Omega)$ , and near  $K$ , we have  $h = g - \bar{\partial}u_0 = 0$ . If we define

$$h^\# = \begin{cases} h & \text{on } \Omega, \\ 0 & \text{on } K, \end{cases}$$

then  $h^\#$  belongs to  $Z_{L^2}^{p,q}(\tilde{\Omega})$ , and we have

$$(7.3) \quad \|h^\#\|_{L^2(\tilde{\Omega})} = \|h\|_{L^2(\Omega)} \leq C \|g\|_{L^2(\Omega)},$$

where the last estimate follows immediately from the definition of  $h$  and (7.2). Since  $H_{L^2}^{p,q}(\tilde{\Omega}) = 0$ , by the open mapping theorem there is a  $v \in A_{L^2}^{p,q-1}(\tilde{\Omega})$  such that  $\bar{\partial}v = h^\#$  and

$$\|v\|_{L^2(\tilde{\Omega})} \leq C \|h^\#\|_{L^2(\tilde{\Omega})} = C \|h\|_{L^2(\Omega)},$$



where the last equality holds since  $h^\# = 0$  on  $K$ . So, if we set  $u = v|_\Omega + \chi u_0$ , then

$$\bar{\partial}u = \bar{\partial}v|_\Omega + \bar{\partial}(\chi u_0) = h^\#|_\Omega + \bar{\partial}(\chi u_0) = h + \bar{\partial}(\chi u_0) = g,$$

by the definition of  $h$ . Furthermore, we have

$$\begin{aligned} \|u\|_{L^2(\Omega)} &\leq \|v\|_{L^2(\Omega)} + \|\chi u_0\|_{L^2(\Omega)} \\ &\leq C\|h\|_{L^2(\Omega)} + C\|u_0\|_{L^2(\mathbb{W})} \leq C\|g\|_{L^2(\Omega)}, \end{aligned}$$

using (7.1) and (7.3). □

*Proof of Proposition 7.1.*

(a) Let  $\lambda : Z_{L^2}^{p,q}(\Omega) \rightarrow Z_{L^2}^{p,q}(\mathbb{W})$  be the restriction map  $g \mapsto g|_{\mathbb{W}}$ . It follows immediately from Lemma 7.2 that  $B_{L^2}^{p,q}(\Omega) = \lambda^{-1}(B_{L^2}^{p,q}(\mathbb{W}))$ . But, since  $B_{L^2}^{p,q}(\mathbb{W})$  is closed in  $Z_{L^2}^{p,q}(\mathbb{W})$ , it follows that  $B_{L^2}^{p,q}(\Omega)$  is closed in  $Z_{L^2}^{p,q}(\Omega)$ .

(b) Let  $g \in Z_{L^2}^{p,q}(\Omega)$  be such that  $\text{class}(g) \in H_{L^2}^{p,q}(\Omega)$  is in the kernel of the restriction map. Then,  $g|_{\mathbb{W}}$  is in  $B_{L^2}^{p,q}(\mathbb{W})$ , so by Lemma 7.2 above,  $g \in B_{L^2}^{p,q}(\Omega)$ , and  $g$  represents 0 in  $H_{L^2}^{p,q}(\Omega)$ . □

**7.2. Proof of Theorem 1.8.** We let  $\mathbb{W} = U \setminus K$  in Proposition 7.1. Let  $R_j = U_j \setminus K_j$  be the factor annuli. Therefore,  $H_{L^2}^{0,1}(R_j)$  is Hausdorff for each  $j$ , and by Proposition 6.1, we know that  $H_{L^2}^{0,1}(\mathbb{W})$  is Hausdorff and vanishes if  $n \geq 3$ . Since by hypothesis,  $H_{L^2}^{0,1}(\tilde{\Omega}) = 0$ , we conclude by part (a) of Proposition 7.1 that  $H_{L^2}^{0,1}(\Omega)$  is Hausdorff, and by part (b) that  $H_{L^2}^{0,1}(\Omega) = 0$  if  $n \geq 3$ .

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