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The Diederich–Fornæss Exponent and Non-existence of Stein Domains with Levi-Flat Boundaries

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Abstract We study the Diederich–Fornæss exponent and relate it to non-existence of Stein domains with Levi-flat boundaries in complex manifolds. In particular, we prove that if the Diederich–Fornæss exponent of a smooth bounded Stein domain in an n-dimensional complex manifold is greater than k/n, then it has a boundary point at which the Levi-form has rank greater than or equal to k.

Keywords Diederich–Fornaess exponent · Levi-flat hypersurface · Oka property · Stein manifold

Mathematics Subject Classification 32T35 · 32V40

1 Introduction

Diederich and Fornæss showed in 1977 that for any bounded pseudoconvex domain Ω with C^2 boundary in a Stein manifold, there exist a positive constant η and a defining function r such that $\hat{r} = -(-r)^\eta$ is plurisubharmonic on Ω ([12]; see also [24]). In particular, any bounded pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$ is necessarily hyperconvex (i.e., there exists a bounded plurisubharmonc exhaustion function on the domain). This result of Diederich and Fornæss was generalized to bounded pseudoconvex domains with C^1 boundary by Kerzman and Rosay [18] and with Lipschitz

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boundary by Demailly [11] and Harrington [16]. The constant η is called a Diederich–Fornæss exponent. The supremum of all Diederich–Fornæss exponents is called the Diederich–Fornæss index of Ω. The Diederich–Fornæss index has implications in regularity theory in the $\bar{\partial}$-Neumann problem. Kohn established a quantitative relationship between global regularity in the $\bar{\partial}$-Neumann problem and the Diederich–Fornæss exponents ([19]; see also [17,23]). In particular, he provided an effective approach to an earlier result of Boas and Straube [3] on global regularity of the $\bar{\partial}$-Neumann operator on a smooth bounded pseudoconvex domain with a defining function that is plurisubharmonic on the boundary. Berndtsson and Charpentier further showed that for a bounded pseudoconvex domain Ω with Lipschitz boundary in $\mathbb{C}^n$, the Bergman projection and the canonical solution operator for the $\bar{\partial}$-operator is bounded on L^2-Sobolev spaces $W^s(\Omega)$ for any s less than one half of the Diederich–Fornæss index ([4]; see also [6]). The Diederich–Fornæss index also plays a role in estimates of the pluri-complex Green function [5] and comparison of the Bergman and Szegö kernels [10].

For a given bounded pseudoconvex domain in $\mathbb{C}^n$, it is difficult to compute the Diederich–Fornæss index in general. Diederich and Fornæss showed that the Diederich–Fornæss index of the worm domain Ω_γ goes to 0 as $\gamma \to \infty$, where γ is the total winding of Ω_γ [13]. Indeed, it follows from the work of Barrett [1] and the aforementioned work of Berndtsson and Charpentier that the Diederich–Fornæss index of Ω_γ is less than or equal to $2\pi/\gamma$. Sibony proved that for a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ that satisfies property (P) in the sense of Catlin, the Diederich–Fornæss index is one (see [9,26]). More recently, Fornæss and Herbig [14] showed that a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function that is plurisubharmonic on the boundary also has Diederich–Fornæss index one.

In this paper, we first establish an effective lower bound for the Diederich–Fornæss index on a C^2-smoothly bounded domain that satisfies the strong Oka property (see Sect. 2 below for detail). It was shown by Ohsawa and Sibony that such a domain has positive Diederich–Fornæss index [22]. We then relate the Diederich–Fornæss index to non-existence of Stein domains with Levi-flat boundaries in complex manifolds. Our main result can be stated as follows:

Theorem 1.1 *Let Ω be a bounded Stein domain with C^2 boundary in a complex manifold M of dimension n. If the Diederich–Fornæss index of Ω is greater than k/n, $1 \le k \le n-1$, then Ω has a boundary point at which the Levi form has rank greater than or equal to k.*

In particular, we have the following corollary:

Corollary 1.2 *If the Diederich–Fornæss index is greater than $1/n$, then its boundary cannot be Levi flat; and if the Diederich–Fornæss index is greater than $1-1/n$, then its boundary must have at least one strongly pseudoconvex boundary point.*

Lins Neto [20] first proved the nonexistence of real-analytic Levi-flat hypersurfaces in $\mathbb{CP}^n$ with $n \ge 3$. The nonexistence of smooth Levi-flat hypersurfaces in $\mathbb{CP}^n$ with $n \ge 3$ was established by Siu [25]. Subsequently, it was proved by Cao et al. [6] that there exist no C^2 Levi-flat hypersurfaces in $\mathbb{CP}^n$, $n \ge 3$. The nonexistence of

Lipschitz Levi-flat hypersurfaces in $\mathbb{CP}^n$ with $n \geq 3$ was proved by Cao and Shaw in [8]. In [6], it was stated that there exist no C^2 Levi-flat hypersurfaces in $\mathbb{CP}^n$ for all $n \geq 2$, but the proof only works for $n \geq 3$. The nonexistence of Levi-flat hypersurfaces in $\mathbb{CP}^2$ remains open.

Our result above was inspired by the work of Nemirovskii who showed that any smooth bounded Stein domain with a defining function that is plurisubharmonic on the domain cannot have Levi-flat boundary ([21, Corollary]). We thank Professor Takeo Ohsawa for informing us that similar results were obtained by Adachi and Brinkschulte independently using different methods. [1]

2 The Diederich–Fornæss Index

Let M be an n-dimensional complex manifold with hermitian metric ω. Let Ω be a bounded domain in M. A continuous real-valued function r on M is called a defining function of Ω if $r < 0$ on Ω, $r > 0$ on $M \setminus \overline{\Omega}$, and $C_1\delta(z) \leq |r(z)| \leq C_2\delta(z)$ near $b\Omega$, where $\delta(z)$ is the geodesic distance from z to the boundary $b\Omega$. We will also assume that the defining function r is in the same smoothness class as that of the boundary $b\Omega$. A defining function r is said to be normalized if $\lim_{z\to b\Omega} |r(z)|/\delta(z) = 1$. Note that the signed distance function $\rho(z) = -\delta(z)$ on Ω and $\rho(z) = \delta(z)$ on $M \setminus \Omega$ is a normalized defining function for Ω.

A constant $0 < \eta \leq 1$ is called a *Diederich–Fornæss exponent* of a defining function r of Ω if there exists a neighborhood U of $b\Omega$ such that

$$i\partial\overline{\partial}(-(-r)^\eta) \geq 0 \tag{2.1}$$

on $U \cap \Omega$ in the sense of distribution. The supremum of all such η's is called *the Diederich–Fornæss index* of r and is denoted by $\mathrm{I}_{\mathrm{DF}}(r)$. The supremum of $\mathrm{I}_{\mathrm{DF}}(r)$ over all defining functions of Ω is called *the Diederich–Fornæss index* of Ω and is denoted by $\mathrm{I}_{\mathrm{DF}}(\Omega)$. Notice that in the above definition of the Diederich–Fornæss index, we only assume $-(-r)^\eta$ to be plurisubharmonic on Ω near the boundary. When the underlying complex manifold M is Stein, the above definition is equivalent to the one that requires $-(-r)^\eta$ to be *strongly* plurisubharmonic on Ω. This equivalence can be seen easily by first replacing r with $\widetilde{r} = re^{-\varepsilon\psi(z)}$, where $\psi(z)$ is a smooth strongly plurisubharmonic exhaustion function for M and ε a sufficiently small positive constant, and then extending $\widetilde{r}$ to the whole domain Ω (see, e.g., [12, p. 133]).

A defining function r is said to satisfy the *strong Oka property* if there exist a constant K and a neighborhood U of $b\Omega$ such that

$$i\partial\overline{\partial}(-\log(-r)) \geq K\omega \tag{2.2}$$

on $U \cap \Omega$ in the sense of distribution. Denote by $K(r)$ the supremum of all constants K such that (2.2) holds. By Takeuchi's theorem, the signed distance function of a (proper)

[1] See: Adachi and Brinkschulte, *A global estimate for the Diederich–Fornæss index of weakly pseudoconvex domains*, Preprint, 2014.

pseudoconvex domain in $\mathbb{CP}^n$ with the Fubini-Study metric satisfies the strong Oka property ([29]; see also [7,15]). Hereafter, the Fubini–Study metric is normalized so that its holomorphic sectional curvature is 2 and hence its holomorphic bisectional curvature is greater than or equal to 1. In this case, one can take, for example, $K = 1/6$ (see [7, Theorem 2.5]).

Let $\Omega \subset\subset M$ be a bounded domain with C^2-boundary. Let r be a defining function of Ω. Let $\omega_\nu = \partial r/|\partial r|_\omega$. Let L_ν be the dual vector of ω_ν. For any $(1,0)$-vector X near $b\Omega$, let $X_\nu = \langle X, L_\nu\rangle_\omega L_\nu$ be the complex normal component of X and $X_\tau = X - X_\nu$ the complex tangential component. Write $T^{1,0}(r) = \{(z, X) \in T^{1,0}(M) \mid Xr = 0\}$. For $z \in b\Omega$, we further decompose $X_\tau = X_s + X_l$, where X_l is in the null space $\mathcal{N}_z$ of the Levi-form $\partial\bar\partial r$ at z and $X_s \perp X_l$. Let $S^{1,0}(M) = \{(z, X) \in T^{1,0}(M), |X|_\omega = 1\}$. Let W be the set of all weakly pseudoconvex points on $b\Omega$. Let

$$S(r) = \max\{|\partial\bar\partial r(X_l, \overline{L}_\nu)(z)|;\ |X_l|_\omega = 1,\ X_l \in \mathcal{N}_z,\ z \in W\}.$$

If $b\Omega$ is strongly pseudoconvex, we set $S(r) = 0$. Define

$$\mathrm{I}_0(r) = \max\left\{\min\left\{\frac{K(r)}{8(S(r))^2}, \frac{1}{2}\right\},\ 1 - \frac{2(S(r))^2}{K(r)}\right\} > 0. \tag{2.3}$$

With the above notations, our main result in this section can be stated as follows:[2]

Theorem 2.1 *Let Ω be a bounded domain with C^2-boundary in a complex hermitian manifold with a normalized defining function r that satisfies the strong Oka property. Then $\mathrm{I}_{\mathrm{DF}}(\Omega) \ge \mathrm{I}_{\mathrm{DF}}(r) \ge \mathrm{I}_0(r)$.*

Proof A simple computation yields that

$$\partial\bar\partial(-\log(-r)) = \frac{\partial\bar\partial r}{-r} + \frac{\partial r \wedge \bar\partial r}{r^2} \tag{2.4}$$

and

$$\begin{aligned}\partial\bar\partial(-(-r)^\eta) &= \eta(-r)^\eta\left(\frac{\partial\bar\partial r}{-r} + (1-\eta)\frac{\partial r \wedge \bar\partial r}{r^2}\right)\\ &= \eta(-r)^\eta\left(\partial\bar\partial(-\log(-r)) - \eta\frac{\partial r \wedge \bar\partial r}{r^2}\right).\end{aligned} \tag{2.5}$$

It follows from (2.5) that (2.1) is equivalent to

$$i\partial\bar\partial(-\log(-r)) \ge \eta\frac{i\partial r \wedge \bar\partial r}{r^2}. \tag{2.6}$$

Let c_0 be a constant such that $0 < c_0 < K(r)$. Then

$$i\partial\bar\partial(-\log(-r)) \ge c_0\omega \tag{2.7}$$

[2] We refer the reader to related work of Biard [2] which we became aware of after this work was completed.

for $z \in \Omega$ near the boundary. It follows from (2.4) that

$$\frac{\partial\overline{\partial}r(X_\tau, \overline{X}_\tau)}{-r} \geq c_0|X_\tau|^2_\omega. \tag{2.8}$$

Let C_1 be any constant such that $C_1 > S(r)$. Then there exists a neighborhood U of $\mathcal{N}^{1,0}(W) = \{(z, X) \mid z \in W, X \in \mathcal{N}_z, |X|_\omega = 1\}$ in $S^{1,0}(M)$ such that

$$|\partial\overline{\partial}r(X, \overline{L}_\nu)| \leq C_1, \qquad (z, X) \in U. \tag{2.9}$$

For $(z, X_\tau) \in S^{1,0}(\overline{\Omega}) \setminus U$ with z near $b\Omega$,

$$\partial\overline{\partial}r(X_\tau, \overline{X}_\tau) \geq C_2|X_\tau|^2_\omega \tag{2.10}$$

for some constant $C_2 > 0$. We write $X = X_\tau + X_\nu$ with $X_l \in \mathcal{N}_z$ as before. Then

$$\begin{aligned}\partial\overline{\partial}(-\log(-r))(X, \overline{X}) &= \frac{\partial\overline{\partial}r(X_\tau, \overline{X}_\tau)}{-r} + \frac{\partial\overline{\partial}r(X_\nu, \overline{X}_\nu)}{-r} \\ &\quad + \frac{2\,\mathrm{Re}\ \partial\overline{\partial}r(X_\tau, \overline{X}_\nu)}{-r} + \frac{|Xr|^2}{r^2}.\end{aligned} \tag{2.11}$$

Note that $|Xr| = |X_\nu|_\omega \cdot |\partial r|_\omega$. Let $K_0 = \sup\{|\partial\overline{\partial}r|_\omega;\ z \in \overline{\Omega}\}$. Then

$$|\partial\overline{\partial}r(X_\nu, X_\nu)| \leq K_0|Xr|^2/|\partial r|^2_\omega \tag{2.12}$$

Similarly,

$$|\,\mathrm{Re}\ \partial\overline{\partial}r(X_\tau, \overline{X}_\nu)| \leq K_0|X_\tau|_\omega \cdot |Xr|/|\partial r|_\omega. \tag{2.13}$$

We first deal with the strongly pseudoconvex directions. For $(z, X) \in T^{1,0}(\Omega)$ with $(z, X_\tau/|X_\tau|_\omega) \in S^{1,0}(\Omega) \setminus U$ with z near $b\Omega$, it follows from (2.13) and (2.10) that for any positive constant M,

$$\begin{aligned}|2\,\mathrm{Re}\ \partial\overline{\partial}r(X_\tau, \overline{X}_\nu)| &\leq K_0\left(\frac{1}{M}|X_\tau|^2_\omega + \frac{M}{|\partial r|^2_\omega}|Xr|^2\right) \\ &\leq \frac{K_0}{MC_2}\partial\overline{\partial}r(X_\tau, \overline{X}_\tau) + \frac{K_0 M}{|\partial r|^2_\omega}|Xr|^2.\end{aligned} \tag{2.14}$$

Therefore,

$$\begin{aligned}\partial\overline{\partial}(-\log(-r))(X, \overline{X}) &\geq \left(1 - \frac{K_0}{MC_2}\right)\frac{\partial\overline{\partial}r(X_\tau, \overline{X}_\tau)}{-r} \\ &\quad + \left(1 - \frac{K_0(M+1)|r|}{|\partial r|^2_\omega}\right)\frac{|Xr|^2}{r^2}.\end{aligned} \tag{2.15}$$

By choosing M sufficiently large and then letting z be sufficiently close to $b\Omega$, we know that (2.6) holds for any $\eta < 1$.

We now deal with weakly pseudoconvex directions. For $(z, X) \in T^{1,0}(\Omega)$ with $(z, X_\tau/|X_\tau|_\omega) \in U$, we have

$$2|\partial\overline{\partial}r(X_\tau, \overline{X}_\nu)| \le 2C_1|X_\tau|_\omega|Xr|/|\partial r|_\omega \le C_1\left(\frac{|r|}{\varepsilon}|X_\tau|_\omega^2 + \frac{\varepsilon}{|r|}\frac{|Xr|^2}{|\partial r|_\omega^2}\right), \tag{2.16}$$

where ε is a positive constant to be chosen. Since r is a normalized defining function, $|\partial r|_\omega = 1/\sqrt{2}$ on $b\Omega$. Combining (2.16) with (2.8), we have for any $\widetilde{C}_1 > C_1$,

$$\begin{aligned}\partial\overline{\partial}(-\log(-r))(X, \overline{X}) &\ge (c_0 - C_1/\varepsilon)|X_\tau|_\omega^2 + \frac{1-(C_1\varepsilon + K_0|r|)|\partial r|_\omega^{-2}}{r^2}|Xr|^2 \\ &\ge (c_0 - \widetilde{C}_1/\varepsilon)|X_\tau|_\omega^2 + \frac{1-2\widetilde{C}_1\varepsilon - K'|r|}{r^2}|Xr|^2 \end{aligned} \tag{2.17}$$

for some positive constant K', after possible shrinking of U.

We consider two cases: $4\widetilde{C}_1^2 \le c_0$ and $4\widetilde{C}_1^2 > c_0$. When $4\widetilde{C}_1^2 \le c_0$, we take $\varepsilon = \widetilde{C}_1/c_0$. Then

$$\partial\overline{\partial}(-\log(-r))(X, \overline{X}) \ge (1 - 2\widetilde{C}_1^2/c_0 - K'|r|)|Xr|^2/r^2. \tag{2.18}$$

When $4\widetilde{C}_1^2 > c_0$, we take $\varepsilon = 1/4\widetilde{C}_1 < \widetilde{C}_1/c_0$. Then combining (2.17) with (2.7), we have

$$\partial\overline{\partial}(-\log(-r))(X, \overline{X}) \ge -\left(\frac{\widetilde{C}_1}{c_0\varepsilon} - 1\right)\partial\overline{\partial}(-\log(-r))(X, \overline{X}) + \frac{1-2\widetilde{C}_1\varepsilon - K'|r|}{r^2}|Xr|^2.$$

Therefore,

$$\partial\overline{\partial}(-\log(-r))(X, \overline{X}) \ge \left(\frac{c_0\varepsilon(1-2\widetilde{C}_1\varepsilon)}{\widetilde{C}_1} - \frac{K'c_0\varepsilon|r|}{\widetilde{C}_1}\right)\frac{|Xr|^2}{r^2}.$$

Hence

$$\partial\overline{\partial}(-\log(-r))(X, \overline{X}) \ge \left(\frac{c_0}{8\widetilde{C}_1^2} - \frac{K'c_0\varepsilon|r|}{2\widetilde{C}_1^2}\right)\frac{|Xr|^2}{r^2}. \tag{2.19}$$

Note that when $4\widetilde{C}_1^2 \le c_0$, we have

$$1 - \frac{2\widetilde{C}_1^2}{c_0} \ge \frac{1}{2} \quad\text{and}\quad \frac{c_0}{4\widetilde{C}_1^2} \ge \frac{1}{2}. \tag{2.20}$$

Furthermore, when $4\widetilde{C}_1^2 > c_0$,

$$\frac{1}{2} > \frac{c_0}{8\widetilde{C}_1^2} > 1 - \frac{2\widetilde{C}_1^2}{c_0}. \tag{2.21}$$

Combining (2.18)–(2.21), we know that (2.6) holds for any $\eta < \mathrm{I}_0(r)$. We thus conclude the proof of Theorem 2.1 □

Combining Theorem 2.1 with Takeuchi's theorem (taking $K = 1/6$), we then have:

Corollary 2.2 *Let Ω be a proper pseudoconvex domain in $\mathbb{CP}^n$ with C^2 boundary. Then its Diederich–Fornaess index*

$$\mathrm{I}_{\mathrm{DF}}(\Omega) \geq \mathrm{I}_0(\rho) = \max\left\{\min\left\{\frac{1}{48(S(\rho))^2}, \frac{1}{2}\right\},\ 1 - 12(S(\rho))^2\right\} > 0,$$

where ρ is the signed distance function to $b\Omega$ with respect to the Fubini–Study metric.

Let z be a point in Ω near the boundary and $\pi(z)$ be its closest point on $b\Omega$. Let $\gamma(t)$ be the geodesic through z parameterized by arc-length such that $\gamma(0) = \pi(z)$. For any $(1, 0)$ tangent vector X at z near $b\Omega$, let $X(t)$ be the vector at $\gamma(t)$ obtained by parallel translation (of the real and imaginary parts) of X along the geodesic from z to $\gamma(t)$ and let $X^0 = X(0)$.

Proposition 2.3 *Let $\Omega \subset\subset M$ be a bounded domain with C^2 boundary and let r be a normalized defining function. Let W be the set of weakly pseudoconvex boundary points. Suppose* (2.2) *holds and there exists a positive constant $K_1 > 1$ such that*

$$\liminf_{\Omega \ni w \to z} \frac{\partial\overline{\partial} r(X_\tau, \overline{X_\tau})(w)}{|r(w)|} \leq K K_1 |X_\tau|^2_\omega, \tag{2.22}$$

for any $z \in W$ and $(1, 0)$-vector field X near z such that $X_\tau \in \mathcal{N}_z$. Then

$$\mathrm{I}_{\mathrm{DF}}(\Omega) \geq \max\left\{\min\left\{\frac{1}{8(K_1 - 1)}, \frac{1}{2}\right\},\ 3 - 2K_1\right\}. \tag{2.23}$$

Proof From (2.2), we know that

$$\Theta = i\partial\overline{\partial}(-\log(-r)) - K\omega$$

is positive semi-definite. Applying the Cauchy–Schwarz inequality to $\Theta(X_\tau, \overline{L}_\nu)$ at w, we then have

$$|\Theta(X_\tau, \overline{L}_\nu)| \leq |\Theta(X_\tau, X_\tau)|^{1/2} |\Theta(L_\nu, \overline{L}_\nu)|^{1/2}.$$

(See [27, Proof of Theorem 1] for a related argument.) Therefore,

$$\left|\frac{\partial\overline{\partial} r(X_\tau, \overline{L}_\nu)}{r(w)}\right|^2 \leq \left(\frac{\partial\overline{\partial} r(X_\tau, X_\tau)}{-r(w)} - K|X_\tau|^2_\omega\right)\left(\frac{\partial\overline{\partial} r(L_\nu, \overline{L}_\nu)}{-r(w)} + \frac{|L_\nu r|^2}{(r(w))^2} - K|L_\nu|^2_\omega\right).$$

Multiplying both sides by $(r(w))^2$ and taking the limit, we then have at z:

$$\left|\partial\overline{\partial} r(X_\tau, \overline{L}_\nu)\right| \leq ((K_1 - 1)K)^{1/2} |X_\tau|_\omega.$$

The Inequality (2.23) then follows by applying Theorem 2.1 with $S(r) = ((K_1 - 1)K)^{1/2}$. ☐

Let $f \in C^2(M)$. Recall that the real Hessian H_f is defined by

$$H_f(\xi, \zeta)(z) = \langle \nabla_\xi(\nabla f), \zeta \rangle$$

for $\xi, \zeta \in T_{\mathbb{R}}(M^{2n})$, where ∇_ξ denotes the covariant derivative. For any $X \in T^{1,0}_{\mathbb{C}}(M)$, we write $X = \frac{1}{\sqrt{2}}(\xi_X - \sqrt{-1}J\xi_X)$ where J is the complex structure.

Proposition 2.4 *Let Ω be a proper pseudoconvex domain with C^2 boundary in $\mathbb{CP}^n$. Let ρ be the signed distance function to $b\Omega$ with respect to the Fubini–Study metric. Let*

$$M(X) = |\nabla_{\xi_X}(\nabla\rho)|^2_\omega + |\nabla_{J\xi_X}(\nabla\rho)|^2_\omega + R(\nabla\rho, J\nabla\rho, \xi_X, J\xi_X)$$

where R is the Riemannian curvature tensor and let

$$K_2 = \max\{M(X);\ z \in W, X \in \mathcal{N}_z, |X|_\omega = 1\}.$$

Then

$$\mathrm{I_{DF}}(\Omega) \geq \max\left\{\min\left\{\frac{1}{8(K_2-1)}, \frac{1}{2}\right\},\ 3 - 2K_2\right\}.$$

Proof It follows from the Riccati equation that

$$\lim_{t\to 0^+} \frac{1}{t}\left(\partial\overline{\partial}\rho(X_\tau(t), \overline{X_\tau(t)}) - \partial\overline{\partial}\rho(X^0, \overline{X^0})\right) = M(X^0).$$

(The above identity was proved in [28] for Ω in $\mathbb{C}^n$. For Ω in $\mathbb{CP}^n$, see [7, pp. 66–69] for related arguments.) We then conclude the proof by applying Proposition 2.3 with $K = 1$ and any $K_1 > K_2$. □

From Proposition 2.1, we also obtain the following slight variation of a result of Ohsawa and Sibony ([22]; see also [6,8]):

Corollary 2.5 *Let Ω be a bounded domain in M with C^2 boundary. Suppose r is a normalized defining function that satisfies (2.2). Then for any $c \in (0,\ K)$ and $\eta \in (0,\ I_0(r))$, there exists a neighborhood V of $b\Omega$ such that*

$$i\partial\overline{\partial}(-\log(-r)) \geq c\omega + \left(1 - \frac{c}{K}\right)\eta\frac{i\partial r \wedge \overline{\partial} r}{r^2}$$

and

$$i\partial\overline{\partial}(-(-r)^\eta) \geq \eta(-r)^\eta\left(c\omega + (1 - \frac{c}{K})\eta\frac{i\partial r \wedge \overline{\partial} r}{r^2}\right).$$

3 Non-Existence of Stein Domains with Levi-Flat Boundaries

We prove Theorem 1.1 in this section. We first recall the following well-known simple lemma. Let Ω be a bounded domain with C^2 boundary in a complex hermitian manifold M of dimension n. Let ρ be a defining function for Ω. For $t > 0$, let $\Omega_{-t} = \{z \in \Omega;\ \rho < -t\}$. Let $i_t : b\Omega_{-t} \to M$ be the inclusion map. Let $1 \leq k \leq n$ be an integer.

Lemma 3.1 *If the rank of the Levi form of $b\Omega$ is less than or equal to $k-1$ at all $z \in b\Omega$, then*

$$i_t^*(d^c\rho \wedge (dd^c\rho)^{n-1}) = O(t^{n-k})dS_t \tag{3.1}$$

where dS_t is the surface element of $b\Omega_{-t}$.

We sketch the proof for the reader's convenience. Note that $dS_t = i_t^*(*d\rho)/|d\rho|_\omega$ and

$$i_t^*(d^c\rho \wedge (dd^c\rho)^{n-1}) = \nu \lrcorner ((d\rho/|d\rho|) \wedge d^c\rho \wedge (dd^c\rho)^{n-1})$$

where ν is the dual vector of $d\rho/|d\rho|_\omega$. By choosing local holomorphic coordinates that diagonalize the Levi form, we then obtain (3.1).

We now prove Theorem 1.1. Let ρ be a defining function of Ω such that $\hat\rho = -(-\rho)^\eta$ is plurisubharmonic on Ω for some constant $\eta > k/n$. Let $\Omega_{-t} = \{\rho < -t\}$, $t > 0$. Since Ω is Stein, Ω_{-t} has at least a strongly pseudoconvex boundary point for sufficiently small t. Let

$$f(t) = \int_{\Omega_{-t}} (dd^c\hat\rho)^n.$$

Then $f(t) \geq 0$ and $f(t)$ is decreasing. By Stokes's theorem,

$$f(t) = \int_{b\Omega_{-t}} i_t^*(d^c\hat\rho \wedge (dd^c\hat\rho)^{n-1}).$$

Since

$$d^c\hat\rho = i\eta(-\rho)^{\eta-1}(\bar\partial\rho - \partial\rho) \quad \text{and} \quad dd^c\hat\rho = 2i\eta\rho^\eta\left(\frac{\partial\bar\partial\rho}{-\rho} + (1-\eta)\frac{\partial\rho\wedge\bar\partial\rho}{\rho^2}\right),$$

we have

$$d^c\hat\rho \wedge \left(dd^c\hat\rho\right)^{n-1} = \eta^n(-\rho)^{n(\eta-1)} d^c\rho \wedge \left(dd^c\rho\right)^{n-1}.$$

Suppose the Levi rank of $b\Omega$ is less than or equal to $k-1$ at all boundary points, then by Lemma 3.1,

$$i_t^*(d^c\rho \wedge \left(dd^c\rho\right)^{n-1}) = O(t^{n-k})dS_t.$$

Thus

$$f(t) = O(t^{n\eta-k}).$$

Therefore, $\lim_{t\to 0^+} f(t) = 0$ and hence $f(t) = 0$ for small $t > 0$. This implies that $b\Omega_{-t}$ has Levi rank less than or equal to $n-2$ at each point, which leads to a contradiction. This concludes the proof of Theorem 1.1.

Corollary 1.2 follows easily. The following theorem is a variation of Theorem 1.1.

Theorem 3.2 *Let M be a complex manifold of dimension n with a hermitian metric ω. Let Ω be a bounded Stein domain in M with C^2 boundary. Suppose there exist a defining function ρ, a constant $\eta > 0$, and a neighborhood U of $b\Omega$ such that*

$$i\partial\bar\partial(-(-\rho)^\eta) \geq c(-\rho)^\eta\left(\omega + \frac{i\partial\rho\wedge\bar\partial\rho}{\rho^2}\right) \tag{3.2}$$

on $U \cap \Omega$ for some constant $c > 0$. If $\eta \geq 1/n$, then Ω cannot have Levi-flat boundary.

Proof In light of Theorem 1.1, it remains to prove the case when $\eta = 1/n$. We follow the notations as in the above proof of Theorem 1.1. Let ε_0 be sufficiently small such that $\Omega \setminus \Omega_{-\varepsilon_0} \subset U \cap \Omega$. We set

$$f(t) = \int_{\Omega_{-t} \setminus \Omega_{-\varepsilon_0}} (dd^c \widehat{\rho})^n$$

for $0 < t < \varepsilon_0$. Suppose $b\Omega$ is Levi-flat, then as in the proof of Theorem 1.1,

$$d^c \hat{\rho} \wedge \left(dd^c \hat{\rho}\right)^{n-1}\Big|_{b\Omega_{-t}} = \eta^n (-\rho)^{n(\eta-1)} d^c \rho \wedge \left(dd^c \rho\right)^{n-1}\Big|_{b\Omega_{-t}} = O(t^{n\eta-1})\, dS_t \leq C\, dS_t.$$

By Stokes's theorem,

$$f(t) = \int_{b\Omega_{-t}} d^c \hat{\rho} \wedge (dd^c \hat{\rho})^{n-1} - \int_{b\Omega_{-\varepsilon_0}} d^c \hat{\rho} \wedge (dd^c \hat{\rho})^{n-1} \leq C. \tag{3.3}$$

On the other hand, it follows from (3.2) that

$$(dd^c \widehat{\rho})^n \geq C(-\rho)^{n\eta} \Big(\omega + \frac{i\partial\rho \wedge \overline{\partial}\rho}{\rho^2}\Big)^n \geq C(-\rho)^{n\eta-2} dV,$$

where dV is the volume element. Thus

$$\begin{aligned} f(t) = \int_{\Omega_{-t} \setminus \Omega_{-\varepsilon_0}} (dd^c \widehat{\rho})^n &\geq C \int_{\Omega_{-t} \setminus \Omega_{-\varepsilon_0}} (-\rho)^{n\eta-2} dV \\ &\geq C \int_{-\varepsilon_0}^{-t} (-\rho)^{-1} d\rho \geq C(-\log t + \log \varepsilon_0). \end{aligned}$$

Therefore, $\lim_{t \to 0^+} f(t) = \infty$, which leads to a contradiction with (3.3). This concludes the proof of Proposition 3.2. □

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