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The Hartogs Triangle in Complex Analysis

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Abstract. The Hartogs triangle serves as an important example in several complex variables. The Hartogs triangle is pseudoconvex, but its boundary is not Lipschitz, yet rectifiable. In this paper will analyze the Hartogs triangle in $\mathbb{C}^2$ and in $\mathbb{C}P^2$ using both geometric measure theoretic approach and the $\bar\partial$ approach.

Introduction

The Hartogs triangle is an important example in several complex variables. It provides many interesting phenomena in several complex variables which do not exist in one complex variable. In this paper we will summarize some well-known facts corresponding to the Hartogs triangle. Some of the results are well-known but many are newly obtained results for the Hartogs triangle both in $\mathbb{C}^2$ and in $\mathbb{C}P^2$.

In Chapter 1 we discuss some basic properties for the Hartogs triangle in $\mathbb{C}^2$. The Hartogs triangle is the first example of a pseudoconvex domain which do not admit a Stein neighborhood basis. It also does not admit a bounded plurisubharmonic exhaustion function. The Hartogs triangle as well as its smooth cousins, the so-call Diederich-Fornaess worm domains (see [**DF1**], [**DF2**]), have play important role in the function theory for pseudoconvex domains.

In Chapter 2, L^2 theory for $\bar\partial$ on the Hartogs triangle is summarized. The Hartogs triangle is a bounded pseudoconvex domain. Hence we have the L^2 existence theorem for $\bar\partial$ as well as the $\bar\partial$-Neumann operator from the Hörmander theorem. But the Sobolev regularity for the $\bar\partial$-Neumann operator or weighted $\bar\partial$-Neumann operator are different. Since the domain is not Lipschitz, we need to define the Sobolev spaces carefully. For a general pseudoconvex domain with smooth boundary, the $\bar\partial$-Neumann operator is not smooth on Sobolev spaces W^s for $s > 0$ (see [**Ba2**]). Yet there is some regularity for the $\bar\partial$-Neumann operator on the Hartogs triangle measured in the weighted Sobolev spaces singular near the origin (see [**ChS3**]).

In Chapter 3 we examine the regularity for $\bar\partial$ other than the L^2 approach. It follows from the kernel approach, one does have regularity in certain Hölder spaces. But the global regularity for $\bar\partial$ in the smooth category does not hold. This gives very different Dolbeault cohomology groups when measured in different function spaces. We also discuss the non-Hausdorff property for the annulus between a ball and the Hartogs triangle which was obtained by Laurent-Thiébaut and Shaw (see

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[**LS**]). In Chapter 4, we discuss function theory for the related Hartogs triangle in the complex projective space $\mathbb{C}P^2$. We also raise some open questions concerning function theory for the Hartogs triangle and other domains.

1. Basic properties of the Hartogs triangle in $\mathbb{C}^2$

The Hartogs triangle in $\mathbb{C}^2$ is defined by

$$\mathbb{H} = \{(z, w) \in \mathbb{C}^2 \mid |z| < |w| < 1\}.$$

We first recall the following well-known facts about the Hartogs triangle $\mathbb{H}$.

Proposition 1.1. *Any function holomorphic in a neighborhood of $\overline{\mathbb{H}}$ is holomorphic in the polydisc $D^2 = \{|z| < 1|\} \times \{|w| < 1\}$. In fact, any holomorphic function $f \in C^\infty(\overline{\mathbb{H}})$ extends holomorphically to the unit bidisc D^2.*

Proof. Let $D_* = D \setminus \{0\}$ be the punctured disc. We notice that the Hartogs triangle is biholomorphic to the product $D \times D_*$ via the map

$$(z, w) \in H \to (\frac{z}{w}, w) \in D \times D_*.$$

Thus for any holomorphic function f on $\mathbb{H}$ admits an expansion of the form

$$f = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} a_{ij} z^i w^j = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} a_{ij} \left(\frac{z}{w}\right)^i w^{i+j}. \tag{1.1}$$

If $f \in C^\infty(\overline{\mathbb{H}})$, then $a_{ij} = 0$ for all $j < 0$. Thus f is actually holomorphic on D^2.

Since the Hartogs triangle is biholomorphic to the product of D and D_*, it is a domain of holomorphy. But its boundary is singular at the point (0,0), where it is not Lipschitz (as a graph of a locally Lipschtz function).

Corollary 1.2. *The Hartogs triangle $\mathbb{H}$ is a domain of holomorphy, but it does not admit a Stein neighborhood basis.*

By a Stein neighborhood basis we mean that $\overline{\mathbb{H}} = \cap_k \Omega_k$ where each Ω_k is a pseudoconvex domain. The corollary follows from Proposition 1.1 easily. Thus the Hartogs triangle $\mathbb{H}$ cannot be approximated from outside by domains of holomorphy. For a smooth pseudoconvex domain with no Stein neighborhood, see [**DF1**].

Let $\mathcal{O}(\mathbb{H})$ denote the space of holomorphic functions on $\mathbb{H}$. For each $k \in \mathbb{N}$, the space $C^k(\overline{\mathbb{H}})$ denote the space of functions whose k-th derivatives are continuous in $\overline{\mathbb{H}}$. For $0 < \alpha < 1$, the space $C^{k,\alpha}(\overline{\mathbb{H}})$ denote the space of functions whose k-th derivatives are Hölder continuous with exponent α. A function f is in $C^{k,\alpha}(\overline{\mathbb{H}})$ if and only if f is in $C^k(\overline{\mathbb{H}})$ and for each multiindex m with $|m| \le k$, one has

$$\sup_{x,y \in \mathbb{H}} \frac{|D^m f(x) - D^m f(y)|}{|x - y|^\alpha} < \infty.$$

Proposition 1.3. *For each $k \ge 0$, the space $C^{k+1}(\overline{\mathbb{H}}) \cap \mathcal{O}(\mathbb{H})$ is not dense in $C^k(\overline{\mathbb{H}}) \cap \mathcal{O}(\mathbb{H})$. It is not even dense uniformly on compact subsets of $\mathbb{H}$.*

Proof. Suppose that f is holomorphic and $f \in C^k(\overline{\mathbb{H}})$ for some nonnegative integer k. Using the expansion (1.1), it is easy to see that for all $(i, j) \neq (0, 0)$, we must have

$$a_{ij} = 0, \qquad \text{for all } j \le -i + k,\ i = 0, 1, 2, \ldots.$$

On the other hand, the term

$$\left(\frac{z}{w}\right)^i w^{k+1},$$

where $i = 0, 1, 2, \ldots$ is in $C^k(\overline{\mathbb{H}})$. This shows that $C^{k+1}(\overline{\mathbb{H}}) \cap \mathcal{O}(\mathbb{H})$ is not dense in $C^k(\overline{\mathbb{H}}) \cap \mathcal{O}(\mathbb{H})$. Since the coefficients a_{ij} can be determined by integration on curves inside the domain, it is not even dense uniformly on compact subsets of $\mathbb{H}$.

PROPOSITION 1.4. *There exists no bounded continuous plurisubharmonic exhaustion function on* $\overline{\mathbb{H}}$.

PROOF. To see this, if there exists real-valued plurisubharmonic bounded exhaustion ϕ on $\mathbb{H}$ such that $\phi < 0$ on $\mathbb{H}$ and $\phi = 0$ on bH, we have that the function $h(w) = \phi(0, w)$ is subharmonic with interior maximum, thus it must be constant $h = 0$. This means that $\phi = 0$, a contradiction.

In contrast, on any bounded pseudoconvex domain with C^2 bounary in $\mathbb{C}^n$, there exists a continuous plurisubharmonic exhaustion function ϕ. This is the well-known resuts by Diederich-Fornaess [**DF2**]. The result is also true if the domain has only Lipschitz boundary in $\mathbb{C}^n$ (see [**KeR**], [**De**].) In fact, one can even take ϕ to be Hölder continuous (see [**Ha1**]). We remark that there exists strictly plurisubharmonic exhaustion function on $\mathbb{H}$ since it is pseudoconvex, but it does not have bounded continuous plurisubharmonic exhaustion function.

2. L^2 theory for $\bar\partial$ on the Hartogs triangle

Following Hörmander's L^2 theory for $\bar\partial$, the $\bar\partial$-Neumann operator exists on any bounded pseudoconvex domain Ω. If the boundary of Ω is Lipschitz, then the $\bar\partial$-Neumann operator N is bounded on $W^s(\Omega)$ to itself for some $s > 0$ (see [**BC**], [**CSW**]). For the Hartogs triangle, the $\bar\partial$-Neumann operator N exists on $L^2_{(0,1)}(\mathbb{H})$ and the Hodge decomposition holds. We will study the regularity in the Sobolev spaces of the $\bar\partial$-Neumann operator on the Hartogs triangle.

For each $s \in \mathbb{N}$, let $W^s(\mathbb{H})$ denote the Sobolev space of order s. A function f is in $W^s(\mathbb{H})$ if the weak derivative of f up to order s is in $L^2(\mathbb{H})$.

PROPOSITION 2.1. *The space* $W^1(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$ *is not dense in the Bergman space* $L^2(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$.

PROOF. Let f be any holomorphic function f on $\mathbb{H}$, we expand

$$f = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} a_{ij} z^i w^j = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} a_{ij} \left(\frac{z}{w}\right)^i w^{i+j}.$$

It is easy to see the following few facts:

(1) The function $\frac{1}{w}$ is in $L^2(\mathbb{H})$ but not in $W^1(\mathbb{H})$.
(2) The terms in the Laurent expansion are orthogonal to each other.
(3) If $f \in L^2(\mathbb{H})$, then $a_{ij} = 0$ for all $i + j < -1$.
(4) If $f \in W^1(\mathbb{H})$, then all $a_{ij} = 0$ for $i + j < 0$.

Thus $W^1(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$ is not dense in $L^2(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$.

Recall that the Bergman projection B is defined as the orthogonal projection operator from $L^2(\mathbb{H})$ onto the closed subspace $L^2(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$, where $\mathcal{O}(\mathbb{H})$ is the space of holomorphic functions on $\mathbb{H}$. We also have the following regularity and irregularity results for the Bergman projection on $\mathbb{H}$.

PROPOSITION 2.2. *The Bergman projection B is not bounded from $W^1(\mathbb{H})$ to $W^1(\mathbb{H})$. In fact, $B(C_0^\infty(\mathbb{H}))$ is not contained in $W^1(\mathbb{H})$*

PROOF. We will show that the Bergman projection is not bounded from $C_0^\infty(\mathbb{H})$ to $W^1(\mathbb{H})$. Let $f \in L^2(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$ and let $f_n \in C_0^\infty(\mathbb{H}) \to f$ in $L^2(\mathbb{H})$. Then $Bf_n \to Bf$ in $L^2(\mathbb{H})$.

Suppose that the Bergman projection is bounded in $W^1(\mathbb{H})$, then Bf_n is in $W^1(\mathbb{H})$. This implies that $W^1(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$ is dense in the Bergman space $L^2(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$, a contradiction to Proposition 2.1.

Let δ be the distance function from z to the boundary bH. The distance function is always a Lipschitz function. To define the Sobolev spaces $W^s(\mathbb{H})$ for fractional order s, we recall the following results.

For any domain Ω with Lipschtz boundary, the space $f \in W^{k+\epsilon}(\Omega) \cap \mathcal{O}(\Omega)$, where $k = 0, 1, \dots$ and $0 \le \epsilon < 1$ implies that $\nabla^k f \in L^2(\Omega, \delta^{-2\epsilon})$. In fact, for the space of harmonic functions or holomorphic functions, one can use the equivalent norm (see [**JK**]).

$$\|f\|_{W^{k+\epsilon}} = \sum_{|mI=0}^{k} \|D^m f\| + \|\delta^{-\epsilon} D^m f\|. \tag{2.1}$$

It is known that if Ω is a pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary, the $\bar\partial$-Neumann operator N is bounded on $W^s_{(0,1)}(\Omega)$ for some $s > 0$ (see Berndtsson-Charpentier[**BC**] and [**CSW**]).

DEFINITION. *For each $0 < s < 1$, we define the space $W^s(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$ to be the space of holomorphic functions f satisfying*

$$\|f\|_{W^\epsilon} = \|f\|_{L^2} + \|\delta^{-s} f\|_{L^2} < \infty.$$

THEOREM 2.3. *The space $W^s(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$ is desne in $L^2(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$ for any $0 < s < \frac{1}{2}$. The space $W^s(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$ is not desne in $L^2(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$ for any $s \ge \frac{1}{2}$.*

PROOF. It is easy to show that the function z/w is in $W^1(\mathbb{H})$. The function w^{-1} is square-integrable on $\mathbb{H}$ and in $W^s(\mathbb{H})$ for any $s < \frac{1}{2}$ but not in $s \ge \frac{1}{2}$. The distance function $\delta(z, w) = |w| - |z|$. near 0. We have for $s = \frac{1}{2}$,

$$\begin{aligned}
\int_H \frac{1}{(|w| - |z|)} \frac{1}{|w|^2} dV &= 4\pi^2 \iint_{r_1 < r_2 < 1} \left(\frac{1}{(r_2 - r_1) r_2^2} \right) r_2 dr_2 r_1 dr_1 \\
&= 4\pi^2 \int_0^1 \left(\int_{r_1}^1 \frac{1}{(r_2 - r_1)} - \frac{1}{r_2} dr_2 \right) dr_1 \\
&= \infty.
\end{aligned}$$

On the other hand, if $0 < s < \frac{1}{2}$,

$$\begin{aligned}
\int_H \frac{1}{(|w| - |z|)^{2s}} \frac{1}{|w|^2} dV &= 4\pi^2 \iint_{r_1 < r_2 < 1} \left(\frac{r_1}{(r_2 - r_1)^{2s} r_2} \right) dr_1 dr_2 \\
&= 4\pi^2 \int_0^1 \left(\int_0^{r_2} \frac{r_1}{(r_2 - r_1)^{2s}} dr_1 \right) \frac{1}{r_2} dr_2 \\
&< \infty.
\end{aligned}$$

The theorem is proved.

Corollary 2.4. *The Bergman projection B is not bounded from $B(C_0^\infty(\mathbb{H}))$ to $W^{\frac{1}{2}}(\mathbb{H})$. The weighted Bergman metric is not bounded from $B(C_0^\infty(\mathbb{H}))$ to $W^{\frac{1}{2}}(\mathbb{H})$ for any weight function smooth in a neighborhood of $\overline{\mathbb{H}}$.*

Theorem 2.3 and Corollary 2.4 show that irregularity of the Bergman projection in the Sobolev spaces. But the following theorem obtained in Chakrabarti-Shaw [**ChS3**] shows that the singularity of the Bergman projection does not propagate.

Theorem 2.5. *If $f \in C^\infty(\overline{\mathbb{H}})$, then $Bf \in C^\infty(\overline{H} \setminus \{0\}) \cap \mathcal{O}(\mathbb{H})$. On the other hand, B does not map the space $C_0^\infty(\mathbb{H})$ of smooth functions compactly supported in $\mathbb{H}$ into $W^1(\mathbb{H}) \cap \mathbb{O}(\mathbb{H})$.*

Note that this result shows that the singularity of the Bergman projection for the Hartogs triangle is contained at the singular point (0,0) and it does not propagate.

Proposition 2.6. *On the Hartogs triangle $\mathbb{H}$, The $\bar{\partial}$-Neumann operator N_2 on $L^2_{(0,2)}(\mathbb{H})$ is bounded from $W^s_{(0,2)}(\mathbb{H})$ to itself for $s \le 1$.*

Proof. We notice that the $\bar{\partial}$-Neumann problem for (0,2)-forms is actually the Dirichlet problem. In this case, we have the Poincaré inequality for the bounded domain: There exists some $C > 0$ such that

$$\int_{\mathbb{H}} |u|^2 \le C \int_{\mathbb{H}} |\nabla u|^2, \qquad u \in C_0^\infty(\mathbb{H}).$$

Let $W_0^1(\mathbb{H})$ be the space of completion of $C_0^\infty(\mathbb{H})$ under the $W^1(\mathbb{H})$-norm, we have that the Dirichlet solution is in $W_0^1(\mathbb{H})$. This proves the theorem.

Remark: Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 2$. Suppose that the boundary is Lipschitz, then the Dirichlet operator is bounded from $L^2(\Omega)$ to $W^{\frac{3}{2}}(\Omega)$ (see Jerison-Kenig [**JK**]).

3. Boundary regularity for $\bar{\partial}$ on the Hartogs triangle

Despite the fact that the canonical solution for $\bar{\partial}$ is not regular on the Hartogs triangle, one can always find sufficiently smooth solution in the Hölder spaces. Let Ω be a bounded domain in $\mathbb{C}^n$. We use $C^\infty(\overline{\Omega})$ to denote the set of all functions which are smooth in an open neighborhood of $\overline{\Omega}$. Recall the following definition of Dolbeault cohomology groups with various function spaces.

Definition. *Let Ω be a domain in $\mathbb{C}^n$. We define the Dolbeault cohomology $H^{p,q}(\Omega)$ and $H^{p,q}(\overline{\Omega})$ where $0 \le p, q \le n$ as follows:*

$$H^{p,q}(\Omega) = \frac{\{f \in C^\infty_{(p,q)}(\Omega) \mid \bar{\partial} f = 0\}}{\{\bar{\partial} C^\infty_{(p,q-1)}(\Omega)\}},$$

$$H^{p,q}(\overline{\Omega}) = \frac{\{f \in C^\infty_{(p,q)}(\overline{\Omega}) \mid \bar{\partial} f = 0\}}{\{\bar{\partial} C^\infty_{(p,q-1)}(\overline{\Omega})\}},$$

$$H^{p,q}_{L^2}(\Omega) = \frac{\{f \in L^2_{(p,q)}(\Omega) \mid \bar{\partial} f = 0\}}{\{f \in L^2_{(p,q)}(\Omega) \mid f = \bar{\partial} u \text{ for some } u \in L^2_{(p,q-1)}(\Omega)\}}.$$

It follows from a well-known result of Hörmander [**Hö1**], we have that

THEOREM (Hörmander). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then we have*

$$H^{p,q}(\Omega) = H^{p,q}_{L^2}(\Omega) = \{0\}, \qquad q > 0.$$

If we assume that the boundary is smooth, we have the following theorem of Kohn [**Ko2**].

THEOREM (Kohn). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose that the boundary is C^∞-smooth. For $g \in C^\infty_{p,q+1}(\overline{\Omega})$, $\bar\partial g = 0$, there exists $u \in C^\infty(\overline{\Omega})$ such that $\bar\partial u = g$. In particular, we have*

$$H^{p,q}(\overline{\Omega}) = \{0\}, \qquad q > 0.$$

When the boundary is not smooth, one also has the following result by Dufresnoy [**Du**].

THEOREM (Dufresnoy). *Let Ω be a bounded domain in $\mathbb{C}^n$ such that Ω has a special Stein neighborhood basis, then*

$$H^{p,q}(\overline{\Omega}) = \{0\}, \qquad q > 0.$$

We refer the reader to the papers [**Du**] or [**St**] for the special Stein neighborhood basis. The theorem can be applied to non-smooth domains like polydiscs. We also recall a theorem by Michel-Shaw [**MS**] which shows that one also have the vanishing cohomology groups

$$H^{p,q}(\overline{\Omega}) = \{0\}, \qquad q > 0$$

for any bounded piecewise smooth pseudoconvex domains in $\mathbb{C}^n$. Yet there exists pseudoconvex domain Ω in $\mathbb{C}^2$ such that the cohomology group $H^{0,1}(\overline{\Omega})$ is infinite dimensional. In fact, the Hartogs triangle is such a domain. The following theorem was proved by Chaumat-Chollet [**CC**].

THEOREM 3.1. *On the Hartogs triangle $\mathbb{H}$, for every $f \in C^{k,\alpha}(\mathbb{H})$ with $\bar\partial f = 0$, where $k = 1, 2, \dots$ and $0 < \alpha < 1$, there exists $u \in C^{k,\alpha}(\mathbb{H})$ such that $\bar\partial u = f$. But there exists $f \in C^\infty_{(0,1)}(\overline{\mathbb{H}})$ with $\bar\partial f = 0$ such that there does not exist any $u \in C^\infty(\overline{\mathbb{H}})$ satisfying $\bar\partial u = f$.*

It follows from [**CC**] that for any ζ in the bidisc $P = \Delta \times \Delta$ and $\zeta \in P \setminus \overline{\mathbb{H}}$, there exists a ξ-smooth, $\bar\partial$-closed $(0,1)$-form α_ζ defined in $\mathbb{C}^2 \setminus \{\zeta\}$ such that there does not exist any ξ-smooth function β on $\overline{\mathbb{H}}$ such that $\bar\partial\beta = \alpha_\zeta$. In particular the $\bar\partial$-equation $\bar\partial u = \alpha_\zeta$ cannot be solved in the ξ-smooth category in any neighborhood of $\overline{\mathbb{H}}$.

COROLLARY 3.2. *The cohomology group $H^{0,1}(\overline{\mathbb{H}})$ is infinite dimensional. But $H^{0,1}(\mathbb{H}) = 0$.*

Using an argument due to Laufer [**La**], the Dolbeault cohomology group $H^{0,1}(\overline{\mathbb{H}})$ is either zero or infinite dimensionnal, we can conclude the theorem. Since the Hartogs triangle is pseudoconvex, hence

$$H^{0,1}(\mathbb{H}) = 0.$$

It has been shown by the authors recently that the group is non-Hausdorff (see Theorem 2.3 in the recent paper by the authors [**LS1**]). The following theorem is proved by Laurent-Shaw [**LS**].

THEOREM 3.3. *Let V be a simply-connected open bounded domain in $\mathbb{C}^2$ such that V is not pseudoconvex (i.e., Stein). Then $\bar\partial : C^\infty(V) \to C^\infty_{(0,1)}(V)$ dose not have closed range in $C^\infty_{(0,1)}(V)$. Suppose we further assume that the domain V is Lipschitz, then operator $\bar\partial : L^2(V) \to L^2_{(0,1)}(V)$ does not have closed range in $L^2_{(0,1)}(V)$ either.*

For a proof of Theorem 3.3, see Theorem 3.4 in [**LS**]. The following theorem is also proved in [**LS**] (See Corollary 4.6 in the paper).

THEOREM 3.4. *Let $\Omega_1 \subset\subset \mathbb{C}^2$ such that $\overline{\mathbb{H}} \subset \Omega_1$. Let $\Omega = \Omega_1 \setminus \overline{\mathbb{H}}$. Then $H^{0,1}(\Omega)$ is non-Hausdorff.*

QUESTION. *Let Ω be the same as in Theorem 3.4. Is $H^{0,1}_{L^2}(\Omega)$ non-Hausdorff?*

This is in sharp contrast to the case when the domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, where Ω_2 is pseudoconvex and smooth. In this case, the spaces $H^{0,1}(\Omega)$ and $H^{0,1}_{L^2}(\Omega)$ are both Hausdorff (see [**Hö2**] and [**Sh1**]). However, we need to assume that the boundary of Ω_2 is C^2 smooth. If the boundary of Ω_2 is only Lipschitz, it is not known if the corresponding cohomology groups are Hausdorff. In particular, one has the following specific question.

QUESTION (Chinese Coin Problem). *Let $\Omega = B_2(0) \setminus D^2$ be the domain between the ball $B_2(0)$ of radius 2 centered at 0 and the bidisc D^2. Is $H^{0,1}_{L^2}(\Omega)$ Hausdorff?*

The domain Ω has the shape of an ancient Chinese coin five pence in the Han dynasty. The problem will be answered affirmatively if we can show the following W^1 estimates for $\bar\partial$ on D^2.

QUESTION. *For any $f \in W^1_{(0,1)}(D^2)$ with $\bar\partial f = 0$, can one find a solution $u \in W^1(D^2)$ such that $\bar\partial u = f$?*

We remark that in an earlier paper by Chakrabarti-Shaw [**ChS1**], we obtain estimates for $\bar\partial$ on the bidisc in special Sobolev spaces. For a pseudoconvex domain with C^2 boundary, W^1 estimates for $\bar\partial$ have been obtained in [**Ha2**]. Such simple question on bidisc has not yet been answered.

4. Holomorphic functions on the Hartogs triangle in $\mathbb{C}P^2$

Let Ω be a domain in $\mathbb{C}P^n$ and let δ be the distance function from z to the boundary. For any domain Ω with Lipschtz boundary, the space $f \in W^{k+\epsilon}(\Omega) \cap \mathcal{O}(\Omega)$, where $k = 0, 1, \dots$ and $0 \le \epsilon < 1$ is defined as in (2.1). We first recall some known results for holomorphic extension of functions from domains in $\mathbb{C}P^n$ (see [**CSW**], [**CS1**] or [**ChS2**]).

THEOREM 4.1. *Let Ω be a pseudoconvex domain with Lipschitz boundary in $\mathbb{C}P^n$, $n \ge 2$, with the Fubini-Study metric. Let $\Omega^+ = \mathcal{X} \setminus \overline{\Omega}$ be a pseudoconcave domain with Lipschtiz boundary. For any $f \in W^s_{(p,q)}(\Omega^+)$, where $0 \le p \le n$, $0 \le q < n-1$ and $s > 1$ such that $\bar\partial f = 0$ in Ω^+, there exists $F \in W^{s-1}_{(p,q)}(\mathcal{X})$ with $F|_{\Omega^+} = f$ and $\bar\partial F = 0$ in $\mathcal{X}$ in the distribution sense.*

COROLLARY 4.2. *Let $\Omega \subset\subset \mathcal{X}$ be a pseudoconvex domain with Lipschitz boundary in $\mathbb{C}P^n$ with the Fubini-Study metric. Let $\Omega^+ = \mathcal{X} \setminus \overline{\Omega}$ be a pseudoconcave domain with Lipschtiz boundary. Suppose that $f \in W^{1+\epsilon}_{(p,0)}(\Omega^+)$ and $\bar\partial f = 0$, where $0 \le p \le n$ and $\epsilon > 0$, then f is a constant if $p = 0$ and $f = 0$ if $p > 0$.*

Next we study the boundary values of holomorphic functions on pseudoconvex domains in $\mathbb{C}P^n$. A function f on the Lipschitz boundary is called CR if f is annihilated by any tangential Cauchy-Riemann equations. We start from following jump formula (see also Lemma 9.3.5 in [**CS**]) or [**Sh3**]).

LEMMA 4.3. *Let $M \subset \mathbb{C}P^n$ be a compact Lipschitz hypersurface which divides $\mathbb{C}P^n$ into two connected domains $\mathbb{C}P^n \setminus M = \Omega^+ \cup \Omega^-$, $n \geq 2$. For any $0 \leq \epsilon \leq \frac{1}{2}$ and $f \in W^{\frac{1}{2}+\epsilon}(M)$ which is CR on M, there exist $F^+ \in W^{1+\epsilon}(\Omega^+)$ and $F^- \in W^{1+\epsilon}(\Omega^-)$ such that $\bar\partial F^+ = 0$ in Ω^+, $\bar\partial F^- = 0$ in Ω^- and the following decomposition holds:*

$$F^+ - F^- = f \quad \text{on} \;\; M.$$

PROOF. We extend $f \in W^{\frac{1}{2}+\epsilon}(M)$ to be $\tilde f$ with $\tilde f \in W^{1+\epsilon}(\mathbb{C}P^n)$. We define a $(0,1)$-form g on $\mathbb{C}P^n$ by

$$g = \begin{cases} -\overline\partial \tilde f, & \text{if } z \in \Omega^-, \\ 0, & \text{if } z \in M, \\ \overline\partial \tilde f, & \text{if } z \in \Omega^+. \end{cases}$$

Then $g \in W^{\epsilon}_{(0,2)}(\mathbb{C}P^n)$ and $\overline\partial g = 0$ in the distribution sense in $\mathbb{C}P^n$.

We can solve $\bar\partial G = g$ for some $G \in W^{1+\epsilon}(\mathbb{C}P^n)$ since the space of harmonic $(0,1)$-forms $\mathcal{H}_{(0,1)}(\mathbb{C}P^n)$ is trivial.

Setting

$$F^+ = \frac{1}{2}(\tilde f - G), \quad z \in \Omega^+,$$
$$F^- = -\frac{1}{2}(\tilde f + G), \quad z \in \Omega^-,$$

we see that

$$f = \tilde f = (F^+ - F^-) \qquad \text{on } M.$$

We also have

$$\overline\partial F^+ = \frac{1}{2}(\overline\partial \tilde f - \overline\partial G) = \frac{1}{2}(\overline\partial \tilde f - \overline\partial \tilde f) = 0 \qquad \text{in} \;\; \Omega^+,$$

and

$$\overline\partial F^- = -\frac{1}{2}(\overline\partial \tilde f + \overline\partial G) = -\frac{1}{2}(\overline\partial \tilde f - \overline\partial \tilde f) = 0 \qquad \text{in} \;\; \Omega^-.$$

The lemma is proved.

Notice that in Lemma 4.3, there is no assumption on the pseudoconvexity of M.

THEOREM 4.4. *Let $\Omega \subset \mathbb{C}P^n$ be a domain with Lipschitz boundary $b\Omega$. For any CR function $f \in W^s(b\Omega)$, where $s > \frac{1}{2}$, there exists an $F \in W^s(\Omega)$ such that F is holomorphic in Ω and $F = f$ on $b\Omega$.*

PROOF. Let $f = F^+ - F^-$ where F^+ and F^- are the $\bar\partial$-closed functions on Ω^+ and Ω^- respectively obtained in Lemma 4.1. Then we have $F^+ \in W^{1+s}(\Omega^+)$ and $F^- \in W^{1+s}(\Omega^-)$.

By our assumption, $\Omega^+ = \mathbb{C}P^n \setminus \overline{\Omega}$ is a pseudoconcave domain with Lipschitz boundary. It follows that there exist a $\bar\partial$-closed extension $\tilde F^+ \in (\mathbb{C}P^n)$. But this implies that F^+ is a constant. Thus $f = F^-$ modulo a constant.

The extension results on Lipschitz domain is maximal in the sense that the results might not hold if the Lipschitz condition is dropped. We will analyze the holomorphic extension of functions on a non-Lipschitz domain. Let Ω be the Hartogs' triangle in $\mathbb{C}P^2$ defined by

$$\Omega = \{[z_0, z_1, z_2] \mid |z_1| < |z_2|\}$$

where $[z_0, z_1, z_2]$ are the homogeneous coordinates for $\mathbb{C}P^2$.

THEOREM 4.5. *Let $\Omega \subset \mathbb{C}P^2$ be the Hartogs' triangle. Then we have the following results:*

(1) *The space of holomorphic functions in $L^2(\Omega) \cap Ker(\bar\partial)$ separate points in Ω.*
(2) *Let f be an holomorphic function on Ω and $f \in W^2(\Omega)$. Then f is a constant.*
(3) *There exist holomorphic functions in $W^1(\Omega)$. The space of holomorphic functions in $W^1(\Omega) \cap Ker(\bar\partial)$ does not separate points in Ω and is not dense in $L^2(\Omega) \cap Ker(\bar\partial)$.*

The L^2 theory for $\bar\partial$ on Ω is not fully understood except for $(0,1)$-forms.

QUESTION. *For $p = 1$ or $p = 2$, is $H^{p,1}_{L^2}(\Omega) = \{0\}$?*

We remark that the Hartogs domain Ω is Stein, we have $H^{p,q}(\Omega) = \{0\}$ for all $q > 0$. It also follows that

$$H^{0,1}_{L^2}(\Omega) = 0.$$

For any pseudoconvex domain with C^2 boundary, we also have the vanishing L^2 cohomology for all $q > 0$ (see [**HI**]).

THEOREM 4.6. *Let $\Omega \subset \mathbb{C}P^2$ be the Hartogs' triangle. Let f be a CR function $f \in W^1(b\Omega)$. Then f is a constant. However, there exist nonconstant CR functions in $W^{\frac{1}{2}}(b\Omega)$ on the boundary.*

For a proof of these results see [**ChS2**].

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