Non-closed Range Property for the Cauchy-Riemann Operator

Christine Laurent-Thiébaut and Mei-Chi Shaw

In memory of M. Salah Baouendi

Abstract In this paper we study the non-closed range of the Cauchy-Riemann operator for relatively compact domains in $\mathbb{C}^n$ or in a complex manifold. We give necessary and sufficient conditions for the L^2 closed range property for $\overline{\partial}$ on bounded Lipschitz domains in $\mathbb{C}^2$ with connected complement. It is proved for the Hartogs triangle that $\overline{\partial}$ does not have closed range for (0, 1)-forms smooth up to the boundary, even though it has closed range in the weak L^2 sense. An example is given to show that $\overline{\partial}$ might not have closed range in L^2 on a Stein domain in complex manifold.

Keywords Cauchy-Riemann operator · Closed range · Duality

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C. Laurent-Thiébaut
University Grenoble Alpes, IF, 38000 Grenoble, France
e-mail: Christine.Laurent@ujf-grenoble.fr

C. Laurent-Thiébaut
CNRS, IF, 38000 Grenoble, France

M. Shaw (✉)
Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA
e-mail: Mei-Chi.Shaw.1@nd.edu

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1 Introduction

A fundamental problem in function theory on a domain Ω in a complex manifold is the study of the Cauchy-Riemann operator, or the $\overline{\partial}$-equation. The understanding of the existence and regularity of the solutions of the system of inhomogeneous Cauchy-Riemann equations on Ω plays central role in complex analysis. On a bounded domain in $\mathbb{C}^n$ (or more generally in a Stein manifold), two theorems (see [13, 14, 16, 17]) for $\overline{\partial}$ on pseudoconvex domains are of paramount importance.

Theorem (Hörmander) *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain. For any $f \in L^2_{p,q}(\Omega)$, where $0 \leq p \leq n$ and $1 \leq q < n$, such that $\overline{\partial} f = 0$ in Ω, there exists $u \in L^2_{p,q-1}(\Omega)$ satisfying $\overline{\partial} u = f$ and $\int_\Omega |u|^2 \leq C \int_\Omega |f|^2$ where C depends only on the diameter of Ω and q.*

Furthermore, if the boundary $b\Omega$ is smooth, we also have the following global boundary regularity results for $\overline{\partial}$.

Theorem (Kohn) *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary $b\Omega$. For any $f \in C^\infty_{p,q}(\overline{\Omega})$, where $0 \leq p \leq n$ and $1 \leq q < n$, such that $\overline{\partial} f = 0$ in Ω, there exists $u \in C^\infty_{p,q-1}(\overline{\Omega})$ satisfying $\overline{\partial} u = f$.*

Let D be a domain in $\mathbb{C}^n$ or a complex manifold. A natural question to ask is when the $\overline{\partial}$ equation has closed range for forms in L^2 or smooth coefficients. The closed range property gives solvability for $\overline{\partial}$ from the point of view of functional analysis. In this paper, we survey the recent progress related to this problem. For closed-range property of the $\overline{\partial}$-equations, we refer the readers to the many books and papers and the references therein (see [2, 6, 10, 13, 14] or [28]). In this paper, we focus on the non-closed range property for the $\overline{\partial}$-equation.

In Sect. 2, we first study the non-closed range property for $\overline{\partial}$ in the L^2 setting for domains in $\mathbb{C}^n$. For any bounded non-pseudoconvex domain D in $\mathbb{C}^2$ such that its complement is connected, the $\overline{\partial}$ equation does not have closed range in L^2 for $(0, 1)$-forms if D. In Sect. 3 we study the Hartogs triangle H for forms smooth up to the boundary. In this case, the $\overline{\partial} : C^\infty(\overline{H}) \to C^\infty_{0,1}(\overline{H})$ does not have closed range. Hence the corresponding cohomology group is non-Hausdorff. The non-hausdorff property is new, since we only knew that the cohomology is infinite dimensional. In Sect. 4, an example of a Stein domain with smooth boundary in a compact complex manifold is given where $\overline{\partial}$ does not have closed range in L^2. In other words, the Hörmander type L^2 results do not hold on a bounded pseudoconvex domain $\Omega \subset\subset \mathcal{X}$ in a complex manifold $\mathcal{X}$ which is not Stein, even though the domain Ω is Stein and with smooth boundary.

2 Non-closed Range in L^2 for $\overline{\partial}$ on Domains in $\mathbb{C}^n$

It is well-known that in $\mathbb{C}^n$, $\overline{\partial} : L^2(\mathbb{C}^n) \to L^2_{0,1}(\mathbb{C}^n)$ does not have closed range. This follows from the fact that the Poincaré inequality does not hold for compactly supported functions in $\mathbb{C}^n$.

For bounded domains in $\mathbb{C}^n$, it is known that there exists a non-pseudoconvex domain on which the L^2 range of $\overline{\partial}$ is not closed (see the example on page 76 in Folland-Kohn [10]). One can show explicitly that $\overline{\partial}$ cannot have closed range by an explicit example of a (0, 1)-form. Using duality, one can show the following result (see [19]).

Theorem 2.1 *Let D be a bounded domain in $\mathbb{C}^2$ such that $\mathbb{C}^2 \setminus D$ is connected. Suppose D is not pseudoconvex and the boundary of D is Lipschitz. Then $\overline{\partial} : L^2(D) \to L^2_{0,1}(D)$ does not have closed range.*

The proof of Theorem 2.1 is based on the Serre duality in the L^2 sense. Let $\overline{\partial}_c$ be the strong minimal closure of the $\overline{\partial}$ operator

$$\overline{\partial}_c : \mathcal{D}_{p,q-1}(\Omega) \to \mathcal{D}_{p,q}(\Omega)$$

where $\mathcal{D}$ is the set of compactly supported functions in Ω. By this we mean that $\overline{\partial}_c$ is the minimal closed extension of the operator such that $\mathrm{Dom}(\overline{\partial}_c)$ contains $\mathcal{D}_{p,q-1}$. The $\mathrm{Dom}(\overline{\partial}_c)$ contains elements $f \in L^2_{p,q-1}(\Omega)$ such that there exists sequence $f_\nu \in \mathcal{D}_{p,q-1}(\Omega)$ such that $f_\nu \to f$ in $L^2_{p,q-1}(\Omega)$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in $L^2_{p,q}(\Omega)$.

Lemma 2.2 *Let Ω be a bounded domain in $\mathbb{C}^n$. The following conditions are equivalent*

(1) $\overline{\partial} : L^2_{p,q-1}(\Omega) \to L^2_{p,q}(\Omega)$ *has closed range.*
(2) $\overline{\partial}_c : L^2_{n-p,n-q}(\Omega) \to L^2_{n-p,n-q+1}(\Omega)$ *has closed range.*

Proof Let $\overline{\partial}^*$ denote the Hilbert space adjoint of $\overline{\partial}$. Following the definition, $f \in \mathrm{Dom}(\overline{\partial}_c)$ if and only if $\star f \in \mathrm{Dom}(\overline{\partial}^*)$. Suppose (1) holds. Then $\overline{\partial}^* : L^2_{p,q}(\Omega) \to L^2_{p,q-1}(\Omega)$ has closed range. Thus we have $\star\overline{\partial}^*\star = \overline{\partial}_c$ has closed range. Thus (1) implies (2). The other direction is proved similarly.

We may also consider $\overline{\partial}_{\tilde{c}}$, the minimal closure of $\overline{\partial}$ in the weak sense, which is related to solving $\overline{\partial}$ with prescribed support in $\overline{\Omega}$ and we refer it as the $\overline{\partial}$-Cauchy problem. When the boundary is Lipschitz, the weak and strong minimal extension are the same (see Lemma 2.4 in [19]).

Definition Let Ω be a domain in a hermitian manifold $\mathcal{X}$. We define the L^2 cohomology group for (p, q)-forms by

$$H^{p,q}_{L^2}(\Omega) = \frac{\{f \in L^2_{p,q}(\Omega) \mid \overline{\partial} f = 0 \text{ in } \Omega\}}{\{f \in L^2_{p,q}(\Omega) \mid f = \overline{\partial} u \text{ for some } u \in L^2_{p,q-1}(\Omega)\}}.$$

We also define the L^2 cohomology group with compact support by

$$H^{p,q}_{c,L^2}(\Omega) = \frac{\{f \in L^2_{p,q}(\Omega) \mid f \in \mathrm{Dom}(\overline{\partial}_c),\ \overline{\partial}_c f = 0 \text{ in } \Omega\}}{\{f \in L^2_{p,q}(\Omega) \mid f = \overline{\partial}_c u \text{ in } \Omega \text{ for some } u \in L^2_{p,q-1}(\Omega) \cap \mathrm{Dom}(\overline{\partial}_c)\}}.$$

Lemma 2.3 *Let D be a bounded Lipschitz domain in $\mathbb{C}^n$, $n \geq 2$, such that $\mathbb{C}^n \setminus D$ is connected, then $H^{0,1}_{c,L^2}(D) = 0$.*

Proof Let $f \in L^2_{0,1}(D) \cap \mathrm{Dom}(\overline{\partial}_c)$ and $\overline{\partial}_c f = 0$. Let f^0 denote the trivial extension of f to $\mathbb{C}^n$ by setting f equal to zero outside D. It follows that $\overline{\partial} f^0 = 0$ in $\mathbb{C}^n$ in the distribution sense. This follows from the assumption that the boundary of D is Lipschtiz (see Lemma 2.4 in [19]). The form f^0 is a compactly supported (0, 1)-form in $\mathbb{C}^n$.

Let B be a large ball in $\mathbb{C}^n$ containing $\overline{D}$. By Hörmander's theorem (see [13]), we can solve $\overline{\partial} u = f^0$ in B and the solution u is in $W^1(D)$ from the interior regularity for $\overline{\partial}$. The function u is holomorphic on $B \setminus \overline{D}$. From our assumption that $\mathbb{C}^n \setminus D$ is connected and $n \geq 2$, the holomorphic function $u \mid_{B\setminus\overline{D}}$ can be extended as a holomorphic function h in B. Let $U = u - h$ in B. Then U is a compactly supported solution in $W^1(D)$ such that $\overline{\partial} U = f^0$ in $\mathbb{C}^n$ in the distribution sense. The solution $U \in \mathrm{Dom}(\overline{\partial}_c)$ and $\overline{\partial}_c U = f$. This proves that $H^{0,1}_{c,L^2}(D) = 0$.

Lemma 2.4 *Let D be a bounded domain in $\mathbb{C}^n$ with Lipschitz boundary. Then the following conditions are equivalent:*

(1) The domain D is pseudoconvex.
(2) $H^{0,q}_{L^2}(D) = 0, \qquad 1 \leq q \leq n-1.$

Proof If $D \subset\subset \mathbb{C}^n$ is bounded pseudoconvex, then $H^{0,q}_{L^2}(D) = 0$ for all $1 \leq q \leq n-1$ by Hörmander L^2-theory. The converse is true provided D has Lipschitz boundary or more generally, D satisfies interior$(\overline{D}) = D$ (see e.g. the remark at the end of the paper in [11]).

Proof of Theorem 2.1. Suppose that

$$\bar{\partial} : L^2(D) \to L^2_{0,1}(D) \tag{2.1}$$

has closed range. Using Lemma 2.2, we have that

$$\bar{\partial}_c : L^2_{2,1}(D) \to L^2_{2,2}(D) \tag{2.2}$$

has closed range.

On the other hand, for top degree (0, 2)-forms, we always have that

$$\bar{\partial} : L^2_{0,1}(D) \to L^2_{0,2}(D) \tag{2.3}$$

has closed range since D is a bounded domain in $\mathbb{C}^2$. From the L^2 Serre duality (see [4]), we have

$$H^{0,1}_{L^2}(D) \simeq H^{2,1}_{c,L^2}(D). \tag{2.4}$$

Since the domain D is in $\mathbb{C}^n$, the exponent p plays no role, the same proof for Lemma 2.3 also holds for $(n, 1)$-forms or any $(p, 1)$-forms. Thus we have $H^{2,1}_{c,L^2}(D) = 0$. From (2.4), this will gives that $H^{0,1}_{L^2}(D) = 0$ and from Lemma 2.4, D is pseudoconvex, a contradiction. The theorem is proved. □

Corollary 2.5 *Let D be a bounded Lipschitz domain in $\mathbb{C}^2$ such that $\mathbb{C}^2 \setminus D$ is connected. Then*

(1) If the domain D is pseudoconvex, then $H^{0,1}_{L^2}(D) = 0$.

(2) If the domain D is non-pseudoconvex, then $H^{0,1}_{L^2}(D)$ is not Hausdorff.

In other words, for bounded Lipschitz domains in $\mathbb{C}^2$ with connected complement we have only two kinds of L^2 cohomology groups $H^{0,1}_{L^2}(D)$: either it is trivial or it is non-Hausdorff. There is nothing in between. We remark that some related results for the Fréchet space cohomology were proved by Trapani ([29], Theorem 2), where a characterization of Stein domains in a Stein manifold of complex dimension 2 is given. In particular he proves that Corollary 2.5 also holds for $H^{0,1}(D)$ the cohomology of C^∞-smooth (0, 1)-forms in D.

Let Ω be a relatively compact pseudoconvex domain with smooth boundary in a Stein manifold with a hermitian metric. We use $H^{p,q}(\Omega)$ or $H^{p,q}(\overline{\Omega})$ to denote the cohomology group of (p, q)-forms with $C^\infty(\Omega)$ coefficients or $C^\infty(\overline{\Omega})$ respectively and since $\mathcal{X}$ is hermitian, we use $H^{p,q}_{L^2}(\Omega)$ to denote the cohomology group of (p, q)-forms with L^2 coefficients. Then

$$H^{p,q}(\Omega) = H^{p,q}_{L^2}(\Omega) = H^{p,q}(\overline{\Omega}) = 0, \qquad q > 0.$$

Next we will compare the L^2-cohomology groups $H^{p,q}_{L^2}(\Omega)$ and $H^{p,q}(\overline{\Omega})$ for a domain Ω when the boundary is not smooth.

3 The Hartogs Triangle

Let us consider the Hartogs triangle H in $\mathbb{C}^2$

$$H = \{(z, w) \in \mathbb{C}^2 \mid |z| < |w| < 1\}.$$

It is a bounded pseudoconvex domain in $\mathbb{C}^2$ with Lipschitz boundary outside the origin. Near the origin, it is not a Lipschitz domain since its boundary is not the graph of a Lipschitz function. The Hartogs triangle and it smooth cousins, the Diederich-Fornaess (see [8]) worm domains, provide many counter examples for function theory on pseudoconvex domains in $\mathbb{C}^n$.

Let $C^{k,\alpha}(H)$ denote the Hölder space of functions in H whose kth derivatives are C^α in H, where $k \in \mathbb{N}$ and $0 < \alpha < 1$. On H, we can consider the space $C^\infty(\overline{H})$ of smooth functions on the closure of H. Several natural definitions could be used.

Definition We define the smooth functions on $\overline{H}$ as follows:

(1) The space of the restrictions to $\overline{H}$ of C^∞-smooth functions on $\mathbb{C}^2$, which can be identified with the quotient of the space of C^∞-smooth functions on $\mathbb{C}^2$ by the ideal of the functions vanishing with all their derivatives on $\overline{H}$.
(2) The intersection for some $0 < \alpha < 1$ of all spaces $C^{k,\alpha}(H)$, $k \in \mathbb{N}$.
(3) The space of C^∞-smooth functions on $\overline{H}$ in the sense of Whitney's jets.

Lemma 3.1 *The three definitions are equivalent.*

Proof The space defined by (1) is clearly contained in the other two. The Whitney extension theorem implies that the spaces defined by (1) and (3) coincide. Using the extension theorem for uniformly continuous functions, it is easy to see that the space defined by (2) is included in the space defined by (3), which implies finally that all the three definitions are equivalent.

For a bounded domain D in $\mathbb{C}^n$ with Lipschitz boundary, it is well-known that the dual of the topological vector space $C^\infty(\overline{D})$ is the space $\mathcal{E}'_{\overline{D}}(\mathbb{C}^n)$ of distributions with compact support in $\overline{D}$ ([19], Lemma 2.3). We will determine the dual of the topological vector space $C^\infty(\overline{H})$ endowed with the Fréchet topology of uniform convergence on $\overline{H}$ of functions and all derivatives.

Theorem 3.2 *The spaces $C^\infty(\overline{H})$ and $\mathcal{E}'_{\overline{H}}(\mathbb{C}^2)$ are dual to each other.*

Proof By the definition of $C^\infty(\overline{H})$, the restriction map $R : \mathcal{E}(\mathbb{C}^2) \to C^\infty(\overline{H})$ is continuous and surjective. Taking the transpose map ${}^tR$ we get an injection from $(C^\infty(\overline{H}))'$ into $\mathcal{E}'(\mathbb{C}^2)$ the space of distributions with compact support in $\mathbb{C}^2$. More precisely the image of $(C^\infty(\overline{H}))'$ by ${}^tR$ is clearly included in $\mathcal{E}'_{\overline{H}}(\mathbb{C}^2)$, the space of distributions on $\mathbb{C}^2$ with support contained in $\overline{H}$.

For any current $T \in \mathcal{E}'_{\overline{H}}(\mathbb{C}^2)$, we set, for $f \in C^\infty(\overline{H})$, $T(f) =< T, \widetilde{f} >$, where $\widetilde{f}$ is a C^∞-smooth extension of f to $\mathbb{C}^2$. We have to prove that $T(f)$ is independent of the choice of the extension $\widetilde{f}$ of f.

Since the difference of two extensions of f is an infinite order flat function on H, we will prove that for any C^∞-smooth function φ flat to infinite order on H and with compact support in $\mathbb{C}^2$, we have $< T, \varphi >= 0$. Since T has compact support, it is a distribution of finite order k_0.

Let r be a positive real number and χ a C^∞-smooth function in $\mathbb{C}^2$ with compact support in the ball $B(0, r)$ of radius r, centered at the origin and equal to 1 on the closed ball $\overline{B(0, r/2)}$. Then $\chi\varphi$ is flat to infinite order on H. Notice that though the Hartogs triangle is not Lipschitz near the origin, it satisfies the exterior cone property at the origin. Thus for any $z \in \mathbb{C}^2 \setminus H \cap B(0, r)$, $d(z, bH) \leq |z|$. This implies that for any $k \in \mathbb{N}$ and any multi-index α with $|\alpha| \leq k$, there exists a positive real constant C_k such that

$$\|D^\alpha(\chi\varphi)\|_\infty \leq C_k r. \tag{3.1}$$

Fix $\varepsilon > 0$ and choose r such that

$$\sup_{k\leq k_0}(C_k r) < \frac{\varepsilon}{2\|T\|}. \tag{3.2}$$

Since $\mathbb{C}^2 \setminus (H \cup B(0,r))$ has Lipschitz boundary, there exists a C^∞-smooth function θ with compact support in $\mathbb{C}^2 \setminus (\overline{H \cup B(0,r)})$ such that

$$\sum_{|\alpha|\leq k_0} \|D^\alpha(1-\chi)\varphi - D^\alpha\theta\|_\infty \leq \frac{\varepsilon}{2\|T\|}. \tag{3.3}$$

Then, since T has support in $\overline{H}$ and θ in $\mathbb{C}^2 \setminus (\overline{H \cup B(0,r)})$, $< T, \theta >= 0$ and we have from (3.2) and (3.3) that

$$\begin{aligned} | < T, \varphi > | &\leq | < T, \chi\varphi > | + | < T, (1-\chi)\varphi > | \\ &\leq | < T, \chi\varphi > | + | < T, (1-\chi)\varphi - \theta > | + | < T, \theta > | \\ &\leq \|T\|\frac{\varepsilon}{2\|T\|} + \|T\|\frac{\varepsilon}{2\|T\|} \leq \varepsilon. \end{aligned} \tag{3.4}$$

This gives that $< T, \varphi >= 0$. Consequently T defines a linear form on $C^\infty(\overline{H})$, which is continuous by the open mapping theorem. This proves that R^t is one-to-one with range equal to $\mathcal{E}'_{\overline{H}}(\mathbb{C}^2)$. □

Since H is a pseudoconvex domain, we have $H^{0,1}(H) = 0$. Moreover it follows from results by Sibony [26, 27] (see also the paper by Chaumat and Chollet [5]) that for any ζ in the bidisc $P = \Delta \times \Delta$ and $\zeta \in P \setminus \overline{H}$, their exists a C^∞-smooth, $\overline{\partial}$-closed $(0,1)$-form α_ζ defined in $\mathbb{C}^2 \setminus \{\zeta\}$ such that there does not exist any C^∞-smooth function β on $\overline{H}$ such that $\overline{\partial}\beta = \alpha_\zeta$, which means that $H^{0,1}(\overline{H}) \neq 0$.

Theorem 3.3 *The cohomology group $H^{0,1}(\overline{H})$ is not Hausdorff.*

Proof By Theorem 3.2, for $0 \leq p \leq 2$, the complexes $(C^\infty_{p,\bullet}(\overline{H}), \overline{\partial})$ and $(\mathcal{E}'^{2-p,\bullet}_{\overline{H}}(\mathbb{C}^2), \overline{\partial})$ are dual from each other. So it follows from Serre duality (see [22] or Corollary 2.6 in [19]) that it is sufficient to prove that we can solve the $\overline{\partial}$ with prescribe support in $\overline{H}$ in the current category.

Lemma 3.4 *For each current $T \in \mathcal{E}'_{2,1}(\mathbb{C}^2)$ with support contained in $\overline{H}$ there exists a $(2,0)$-current S with compact support in $\mathbb{C}^2$, whose support is contained in $\overline{H}$, such that $\overline{\partial}S = T$.*

Proof Let $T \in \mathcal{E}'_{2,1}(\mathbb{C}^2)$ be a current with support contained in $\overline{H}$. Since one can solve the $\overline{\partial}$ equation with compact support for bidegree $(2,1)$ in $\mathbb{C}^2$, there exists a $(2,0)$-current S with compact support in $\mathbb{C}^2$ such that $\overline{\partial}S = T$. The support of T is contained in $\overline{H}$, so the current S is an holomorphic $(2, 0)$-form on $\mathbb{C}^2 \setminus \overline{H}$. Moreover S has compact support in $\mathbb{C}^2$ and hence vanishes on an open subset of $\mathbb{C}^2 \setminus \overline{H}$. By the analytic continuation theorem, the connectedness of $\mathbb{C}^2 \setminus H$ implies that the support of S is contained in $\overline{H}$. □

Remark 1 One does have *almost* smooth solutions to the $\overline{\partial}$ problem on the Hartogs triangle. Let $H^{p,q}_{C^{k,\alpha}}(H)$ denote the Dolbeault cohomology of (p, q)-forms with $C^{k,\alpha}(H)$ coefficients. Using integral kernel method, Chaumat and Chollet [5] prove that $H^{0,1}_{C^{k,\alpha}}(H) = 0$.

Notice that the intersection $\cap_k C^{k,\alpha}(H) = C^\infty(\overline{H})$. This shows the delicate nature of such problem on non-Lipschitz domains.

Remark 2 We also mention that if Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with a good Stein neighborhood basis, then one has that $H^{0,1}(\overline{\Omega}) = 0$ (see [9]). The Hartogs triangle is a prototype of domains without Stein neighborhood basis.

Let $\overline{\partial}_s$ denote the strong maximal extension of $\overline{\partial}$. By this we mean that $\overline{\partial}_s$ is the maximal closed extension of the operator such that $\mathrm{Dom}(\overline{\partial}_s)$ contains $C^\infty_{p,q-1}(\overline{H})$. The $\mathrm{Dom}(\overline{\partial}_s)$ contains elements $f \in L^2_{p,q-1}(H)$ such that there exists sequence $f_\nu \in C^\infty_{p,q-1}(\overline{H})$ such that $f_\nu \to f$ in $L^2_{p,q-1}(H)$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in $L^2_{p,q}(H)$. Since the boundary of H is rectifiable, $\overline{\partial}_{\tilde{c}}$, the weak minimal closure of $\overline{\partial}$, is dual to $\overline{\partial}_s$.

We do know $H^{0,1}_{L^2}(H) = 0$ from Hörmander's result. It is not known if the weak maximal extension $\overline{\partial} : L^2(H) \to L^2_{0,1}(H)$ is the same as the strong maximal extension $\overline{\partial}_s$. So we only get from Lemma 2.3 (the Lipschitz hypothesis is only used to get $\overline{\partial}_c$ at the place of $\overline{\partial}_{\tilde{c}}$) and L^2 Serre duality (see [4]) that

Proposition 3.5 *The cohomology group $H^{0,1}_{\overline{\partial}_s, L^2}(H)$ is either* 0 *or not Hausdorff.*

Again, if the boundary is Lipschitz, then the weak $\overline{\partial}$ and strong $\overline{\partial}_s$ are the same following the Friedrichs' lemma (see [13] or [6]).

Consider the annulus between a pseudoconvex domain and the Hartogs triangle. We have the following result (see Corollary 4.6 in [19]).

Theorem 3.6 *Let Ω be a pseudoconvex domain in $\mathbb{C}^2$ such that $\overline{H} \subset \Omega$. Then $H^{0,1}(\Omega \setminus \overline{H})$ is not Hausdorff.*

Let D be the annulus between two bounded domains $\Omega_1 \subset\subset \Omega \subset\subset \mathbb{C}^2$. Suppose that the Dolbeault cohomology $H^{0,1}(D)$ is Hausdorff. It follows from a result of Trapani (see Theorem 3 in [30]) that both Ω and Ω_1 have to be pseudoconvex. Theorem 3.6 shows that the converse is not true.

If we replace H by the bidisc $\triangle^2$, then $H^{0,1}(\Omega \setminus \overline{\triangle^2})$ is Hausdorff since $\triangle^2$ has a Stein neighborhood basis (see [18] or Corollary 4.3 in [19]).

In fact, Trapani (see Theorem 4 in [30]) proves that, if D has smooth boundary, a sufficient condition for $H^{0,1}(D)$ to be Hausdorff is that Ω is pseudoconvex and Ω_1 is strictly pseudoconvex. It is no longer true if Ω_1 is only pseudoconvex, taking for example Ω_1 to be the Diederich-Fornaes domain [7].

On the other hand, if Ω and Ω_1 are pseudoconvex and we assume that the boundary of Ω_1 is C^2-smooth, then the L^2 cohomology $H^{0,1}_{L^2}(D)$ is Hausdorff (Hörmander [15] or Shaw [23–25]). The following problem remains unsolved.

Question *Let B be a ball of radius two in $\mathbb{C}^2$ and $\triangle^2$ be the bidisc. Determine if the L^2 cohomology $H^{0,1}_{L^2}(B \setminus \overline{\triangle^2})$ is Hausdorff.*

4 Non-closed Range Property for $\overline{\partial}$ on Pseudoconvex Domains in Complex Manifolds

When $\mathcal{X}$ is a Hermitian complex manifold and $\Omega \subset\subset \mathcal{X}$ is a pseudoconvex domain with smooth boundary, the $\overline{\partial}$ problem could be very different if $\mathcal{X}$ is not Stein. We first note that if Ω is strongly pseudoconvex, Grauert proved that the $\overline{\partial}$ has closed range in the Fréchet space of C^∞-smooth forms. If the domain Ω is relatively compact strongly pseudoconvex, (or more generally of finite type) with smooth boundary, the closed range property for the $\overline{\partial}$ equation in the L^2 setting has been established by Kohn [17] via the $\overline{\partial}$-Neumann problem.

However, function theory on general weakly pseudocnvex domains in a complex manifold can be quite different. Grauert (see [12]) first gives an example of a pseudoconvex domain Ω in a complex torus which is not holomorphically convex. He shows that the only holomorphic functions on Ω are constants. The domain in the Grauert's example actually has Levi-flat boundary. The boundary splits the complex two torus into two symmetric parts. Based on the example of Grauert, Malgrange proves the following theorem.

Theorem (Malgrange [20]) *There exists a pseudoconvex domain Ω with Levi-flat boundary in a complex torus of dimension two whose Dolbeault cohomology group $H^{p,1}(\Omega)$ is non-Hausdorff in the Fréchet topology, for every* $0 \leq p \leq 2$.

Malgrange shows that for the Grauert's example, the $\overline{\partial}$ equation does not necessarily have closed range in the Fréchet space of C^∞-smooth forms and the corresponding Dolbeault cohomology $H^{p,1}(\Omega)$ is non-Hausdorff. The domain Ω is not holomorphically convex. Recently the following result is proved in [3].

Theorem 4.1 *There exists a compact complex manifold $\mathcal{X}$ of complex dimension two and a relatively compact, smoothly bounded, Stein domain Ω with smooth boundary in X, such that the range of $\overline{\partial} : L^2_{2,0}(\Omega) \to L^2_{2,1}(\Omega)$ is not closed. Consequently, the L^2-cohomology space $H^{2,1}_{L^2}(\Omega)$ is not Hausdorff.*

The domain Ω is defined as follows. Let $\alpha > 1$ be a real number and let Γ be the subgroup of $\mathbb{C}^*$ generated by α. We will standardize $\alpha = e^{2\pi}$. Let $T = \mathbb{C}^*/\Gamma$ be the torus.

Let $\mathcal{X} = \mathbb{C}P^1 \times T$ be equipped with the product metric ω from the Fubini-Study metric for $\mathbb{C}P^1$ and the flat metric for T. Let Ω be the domain in $\mathcal{X}$ defined by

$$\Omega = \{(z, [w]) \in \mathbb{C}P^1 \times T \mid \Re zw > 0\} \tag{4.1}$$

where z is the inhomogeneous coordinate on $\mathbb{C}P^1$. The domain Ω was first introduced by Ohsawa [21] and used in Barrett [1]. It was proved in [21] that Ω is biholomorphic to the product domain of an annulus and a pictured disc in $\mathbb{C}^2$. In particular, Ω is Stein.

Thus we have

$$H^{p,q}(\Omega) = 0, \qquad q > 0.$$

The range of $\overline{\partial} : L^2_{(2,0)}(\Omega) \to L^2_{(2,1)}(\Omega)$ is not closed (see [3]). Theorem 4.1 shows that on a pseudconvex domain in a complex manifold X, there is no connection between the Dolbeault cohomology groups in the classical Fréchet space of smooth forms and the L^2 space. This is in sharp contrast with the case when the manifold X is Stein. We note that Ω is Stein, but the ambient space $\mathcal{X}$ is not Stein. The idea of the proof is to use the L^2 Serre duality and the extension of holomorphic functions. For details of the proof of the theorem, we refer the reader to [3]. We end the paper by raising the following question.

Question *Let Ω be defined by (4.1). Determine if the range of $\overline{\partial} : L^2_{p,0}(\Omega) \to L^2_{p,1}(\Omega)$ is closed, where $p = 0$ or $p = 1$.*

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