Sobolev regularity of the $\bar{\partial}$-equation on the Hartogs triangle

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Abstract The regularity of the $\bar{\partial}$-problem on the domain $\{|z_1| < |z_2| < 1\}$ in $\mathbb{C}^2$ is studied using L^2-methods. Estimates are obtained for the canonical solution in weighted L^2-Sobolev spaces with a weight that is singular at the point $(0, 0)$. In particular, the singularity of the Bergman projection for the Hartogs triangle is contained at the singular point and it does not propagate.

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1 Introduction

The *Hartogs Triangle*, the bounded pseudoconvex domain $\mathbb{H}$ in $\mathbb{C}^2$ given by $\mathbb{H} = \{(w_1, w_2) \in \mathbb{C}^2 \mid |w_1| < |w_2| < 1\}$ is a venerable source of counterexamples to conjectures in complex analysis. The boundary of $\mathbb{H}$ has a serious singularity at $(0, 0)$ near which it cannot be represented as a graph. Though a lot is known about $\mathbb{H}$, not all its mysteries have been uncovered yet. It is an important yet simple model domain which needs to be understood thoroughly in any program of extending classical results

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Dedicated to the memory of Prof. Jianguo Cao.

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of several complex variables from smoothly bounded pseudoconvex domains to more general domains. In this article, we consider the regularity of the $\overline{\partial}$-problem on $\mathbb{H}$ in the L^2-Sobolev topology.

Using integral representations, the regularity of the $\overline{\partial}$-problem has been investigated on $\mathbb{H}$ in [5,16], with estimates in the spaces $\mathscr{C}^{k,\alpha}$ (functions and forms in $\mathscr{C}^k$, whose k-th partial derivatives are Hölder continuous of exponent α). The remarkable outcome of these investigations is that for every $\overline{\partial}$-closed $(0, 1)$-form g on $\mathbb{H}$ of class $\mathscr{C}^{k,\alpha}$, there is a function u on $\mathbb{H}$, also of class $\mathscr{C}^{k,\alpha}$ such that $\overline{\partial} u = g$, and this function u is given by an explicit integral formula. Note that the $\overline{\partial}$-problem is *not* globally regular on $\mathbb{H}$, i.e., there is a $\overline{\partial}$-closed $(0, 1)$-form h on $\mathbb{H}$, such that while $h \in \mathscr{C}^{\infty}(\overline{\mathbb{H}})$, for every u satisfying $\overline{\partial} u = h$, we have $u \notin \mathscr{C}^{\infty}(\overline{\mathbb{H}})$ (see [5]). In contrast, when a domain is pseudoconvex with smooth boundary (see [15]), or its closure has a Stein neighborhood basis (see [7]), one can solve the $\overline{\partial}$-problem to obtain a solution smooth up to the boundary, provided the data is smooth.

However, it is difficult to use the integral representation method to obtain information about regularity in Sobolev spaces. We use a method similar in spirit to that used in [16] to obtain estimates in Sobolev spaces for the canonical solution of the $\overline{\partial}$-equation in $\mathbb{H}$. We use the fact that $\mathbb{H}$ is biholomorphic to a product domain $\mathbb{P}$ to transfer the problem from $\mathbb{H}$ to $\mathbb{P}$ (see Sect. 4 below.) This opens up the possibility of using the technique of [4]. The fact that one of the factors in the product representation of $\mathbb{H}$ is non-Lipschitz causes some technical problems in applying the results of [4] but these are easily overcome. This leads to estimates in Sobolev-type spaces with weights singular at the bad point $(0, 0)$.

The use of weights in the L^2-method is of course classical. In the context of non-smooth domains, it seems that singular weights are a natural device to control the behavior of functions and forms near the singular part of the boundary. Such weights also arise naturally in recent attempts to generalize classical estimates on the $\overline{\partial}$- and $\overline{\partial}$-Neumann problems from smooth to non-smooth strictly pseudoconvex domains (see [8–10]).

While the Hartogs triangle is rather special, right now the method used here seems to be the only technique available to study the question treated in this paper. Of course, we can extend the method to related "Product-type" singularities. It will be very interesting to have a general technique to deal with the regularity in Sobolev spaces of the $\overline{\partial}$-problem on singular domains such as $\mathbb{H}$.

2 Sobolev estimates

Let ℓ be an integer, and let $L^2(\mathbb{H}, \ell\Phi)$ denote the space of locally integrable functions f on $\mathbb{H}$ for which the norm defined by

$$\|f\|^2_{L^2(\mathbb{H},\ell\Phi)} = \int\limits_{\mathbb{H}} |w_2|^{2\ell} |f(w)|^2 \, dV(w) \tag{2.1}$$

is finite, where $w = (w_1, w_2)$ are the standard coordinates on $\mathbb{H}$, and here and in the sequel dV denotes Lebesgue measure on Euclidean space. Φ here denotes the

harmonic function $\Phi(w) = -2\log|w_2|$ whose multiples are used as weights. Other related notation is explained in Sect. 3 below. Then $\ell = 0$ corresponds to the usual unweighted L^2-space on $\mathbb{H}$, positive values of ℓ correspond to allowing functions to blow up in a controlled way at 0, and negative values of ℓ correspond to forcing functions to vanish in a weak sense at the point 0. We let $L^2_{0,1}(\mathbb{H}, \ell\Phi)$ denote the space of $(0, 1)$-forms on $\mathbb{H}$ with coefficients in $L^2(\mathbb{H}, \ell\Phi)$. On a space of forms whose coefficients lie in a Hilbert space (e.g., $L^2_{0,1}(\mathbb{H}, \ell\Phi)$ here, and the spaces $W^k_{0,1}(\mathbb{H}, \ell\Phi)$ and $W^k_{0,1}(\mathbb{P}, \ell\Phi)$ defined below), according to standard convention, we impose a Hilbert space norm whose square is the sum of the squares of the norms of the coefficients. It follows from Hörmander's theory of L^2-estimates for the $\bar{\partial}$-equation (see Sect. 3 below) that given a $\bar{\partial}$-closed f in $L^2_{0,1}(\mathbb{H}, \ell\Phi)$, there is a u in $L^2(\mathbb{H}, \ell\Phi)$ such that $\bar{\partial}u = f$, and we have

$$\|u\|_{L^2(\mathbb{H}, \ell\Phi)} \leq \sqrt{e}\|f\|_{L^2_{0,1}(\mathbb{H}, \ell\Phi)} \tag{2.2}$$

(where e is the base of natural logarithms). By a standard weak compactness argument, among all such solutions u there is a u_ℓ of smallest norm, which is the (*weighted*) *canonical* solution of $\bar{\partial}u = f$, with weight $\ell\Phi$. The aim of this article is to understand the regularity of u_ℓ in terms of that of f.

Let k be a non-negative integer, and let ℓ be an integer. We introduce the weighted Sobolev space $W^k(\mathbb{H}, \ell\Phi)$ of locally integrable functions on $\mathbb{H}$ in the following way. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of non-negative integers, write $|\alpha| = \sum_{j=1}^{4} \alpha_j$, and let

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial w_1^{\alpha_1} \partial \overline{w_1}^{\alpha_2} \partial w_2^{\alpha_3} \partial \overline{w_2}^{\alpha_4}}, \tag{2.3}$$

and define the space $W^k(\mathbb{H}, \ell\Phi)$ by the finiteness of the norm

$$\|f\|^2_{W^k(\mathbb{H}, \ell\Phi)} = \sum_{|\alpha| \leq k} \int_{\mathbb{H}} |w_2|^{2\ell} |D^\alpha f(w)|^2 \, dV(w), \tag{2.4}$$

where the derivatives are in the weak sense. We will refer to $W^k(\mathbb{H}, \ell\Phi)$ as the weighted Sobolev space of order k on $\mathbb{H}$ with weight $\ell\Phi$. We let $W^k_{0,1}(\mathbb{H}, \ell\Phi)$ be the space of $(0, 1)$-forms on $\mathbb{H}$ with coefficients in $W^k(\mathbb{H}, \ell\Phi)$. The main result of this paper is:

Theorem 2.1 *For every non-negative integer k there is a constant $C > 0$, such that for each $\bar{\partial}$-closed g in $W^{2k}_{0,1}(\mathbb{H}, \ell\Phi)$, the canonical solution u_ℓ of $\bar{\partial}u_\ell = g$ is in $W^k(\mathbb{H}, (\ell + 2k)\Phi)$, and satisfies an estimate*

$$\|u_\ell\|_{W^k(\mathbb{H}, (\ell+2k)\Phi)} \leq C\|g\|_{W^{2k}_{0,1}(\mathbb{H}, \ell\Phi)} \tag{2.5}$$

Note that the order of Sobolev differentiability of the solution is half of that of the data, and the weight factor in the norm changes from $|w_2|^{2\ell}$ to $|w_2|^{2(\ell+2k)}$, indicating that the solution u_ℓ may have much more rapid growth near 0 than g has.

No claim can be made for the optimality of the estimate given in (2.5). Indeed, the seeming loss of smoothness from W^{2k} to W^k is illusory, arising from the use of the estimates given in (6.5) below, and could in principle be avoided by introducing special weighted Sobolev spaces adapted to the Hartogs Triangle, but we have chosen to formulate the result in terms of the simpler spaces $W^k(\mathbb{H}, \ell\Phi)$. We are interested in quantifying the possible blowup of the solution of the $\overline{\partial}$-equation on $\mathbb{H}$ with smooth data, and this is deduced in Corollary 2.2 below starting from Theorem 2.1.

The case $\ell = 0$ corresponds to the usual canonical solution. In this case we can deduce the following corollary regarding the blowup of the solution of the $\overline{\partial}$-equation at the point (0,0):

Corollary 2.2 *Let $g \in \mathscr{C}^{\infty}_{0,1}(\overline{\mathbb{H}})$ be a $\overline{\partial}$-closed $(0, 1)$-form smooth up to the boundary on $\mathbb{H}$. Then the canonical solution u_0 of the equation $\overline{\partial}u_0 = g$ is smooth on $\mathbb{H}$ and extends smoothly up to all points of the boundary except possibly at the point $(0, 0)$. If α is a multi-index and D^{α} is as in (2.3), we have*

$$\int_{\mathbb{H}} |w_2|^{4|\alpha|} \left| D^{\alpha} u_0(w) \right|^2 dV(w) < \infty. \tag{2.6}$$

Proof Since $g \in \mathscr{C}^{\infty}_{0,1}(\overline{\mathbb{H}})$, for every nonnegative integer k, we have $g \in W^{2k}_{0,1}(\mathbb{H})$, and consequently by Theorem 2.1, the solution u_0 is in $W^k(\mathbb{H}, 2k\Phi)$. If $\mathbb{B}$ be an open ball in $\mathbb{C}^2$, such that $0 \notin \overline{\mathbb{B}}$, the weight $|w_2|^{4k}$ in the definition of the Sobolev space $W^k(\mathbb{H}, 2k\Phi)$ is smooth and bounded away from zero and therefore the restriction of functions in $W^k(\mathbb{H}, 2k\Phi)$ to $\mathbb{B} \cap \mathbb{H}$ belong to $W^k(\mathbb{B} \cap \mathbb{H})$. Conversely, since $\mathbb{B} \cap \mathbb{H}$ is Lipschitz, by standard extension results, every function in $W^k(\mathbb{B} \cap \mathbb{H})$ may be extended to a function in $W^k(\mathbb{H})$. Since this holds for each k, the restriction of u_0 to $\mathbb{B} \cap \mathbb{H}$ is in $\mathscr{C}^{\infty}(\mathbb{B} \cap \overline{\mathbb{H}})$. So the canonical solution $u_0 \in \mathscr{C}^{\infty}(\overline{\mathbb{H}} \setminus \{(0, 0)\})$. The finiteness of (2.6) now follows from (2.5). □

Recall that the Bergman projection B is defined as the orthogonal projection operator from $L^2(\mathbb{H})$ onto the closed subspace $L^2(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$, where $\mathscr{O}(\mathbb{H})$ is the space of holomorphic functions on $\mathbb{H}$. We also have the following regularity and irregularity results for the Bergman projection on $\mathbb{H}$.

Theorem 2.3 *For $k \geq 0$, the Bergman projection B maps the Sobolev space $W^{2k+1}(\mathbb{H})$ (without weight) continuously into the weighted holomorphic Sobolev space*

$$W^k(\mathbb{H}, 2k\Phi) \cap \mathscr{O}(\mathbb{H}).$$

It follows that if $f \in \mathscr{C}^{\infty}(\overline{\mathbb{H}})$, then $Bf \in \mathscr{C}^{\infty}(\overline{\mathbb{H}} \setminus \{0\}) \cap \mathscr{O}(\mathbb{H})$. On the other hand, B does not map the space $\mathscr{C}^{\infty}_0(\mathbb{H})$ of smooth functions compactly supported in $\mathbb{H}$ into $W^1(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$.

Note that this result shows that the singularity of the Bergman projection for the Hartogs triangle is contained at the singular point and it does not propagate.

3 Hörmander's existence theorem

For a domain Ω in complex Euclidean space, and a real-valued continuous function ψ on Ω, recall that $L^2(\Omega, \psi)$ denotes the space of locally-integrable functions f on Ω for which the *weighted norm*

$$\|f\|^2_{L^2(\Omega,\psi)} = \int_\Omega |f|^2 \, e^{-\psi} dV$$

is finite. We denote by $L^2_{p,q}(\Omega, \psi)$ the space of (p, q)-forms with coefficients in the space $L^2(\Omega, \psi)$. These are Hilbert spaces under the obvious inner products.

In this paper, on a domain in $\mathbb{C}^2$, we will use the harmonic weight function Φ, given by

$$\Phi(w_1, w_2) = -2 \log |w_2|, \tag{3.1}$$

which is continuous provided the domain does not intersect the complex line $\{w_2 = 0\}$. Since $e^{-\ell\Phi(w)} = |w_2|^{2\ell}$, this also explains the notations $L^2(\mathbb{H}, \ell\Phi)$ and $W^k(\mathbb{H}, \ell\Phi)$ adopted in the previous section for the spaces with norms (2.1) and (2.4) respectively.

The cornerstone of the L^2-theory of $\bar{\partial}$-operator is the following famous theorem of Hörmander ([11, Theorem 2.2.1′], see also [1] and the expositions in [6,18]):

Result 3.1 *Let $\Omega \Subset \mathbb{C}^n$ be pseudoconvex, and let $\psi \in \mathscr{C}^2(\Omega)$ be a strictly plurisubharmonic weight function on Ω. For $z \in \Omega$, denote by $\mu(z)$ the smallest eigenvalue of the complex Hessian matrix $(\frac{\partial^2 \psi}{\partial z_j \partial \overline{z_k}}(z))_{1 \le j,k \le n}$. If $\lambda = \inf_{z \in \Omega} \mu(z) > 0$, then for any $\bar{\partial}$-closed $g \in L^2_{p,q}(\Omega, \psi)$, (p, q)-form, $q > 0$, there is a $u \in L^2_{p,q-1}(\Omega, \psi)$ such that $\bar{\partial}u = g$, satisfying the estimate*

$$\|u\|_{L^2_{p,q-1}(\Omega,\psi)} \le \frac{1}{\lambda q} \|g\|_{L^2_{p,q}(\Omega,\psi)}.$$

From this the estimate (2.2) on $\mathbb{H}$ can be deduced as follows. We use the weight ψ on $\mathbb{H}$, where $\psi(w) = \frac{1}{2}|w|^2 + \ell\Phi(w)$. Then the space $L^2(\mathbb{H}, \psi)$ is the same as $L^2(\mathbb{H}, \ell\Phi)$, and the norms are equivalent. In fact it is easy to see that

$$\frac{1}{\sqrt{e}} \|f\|_{L^2(\mathbb{H},\ell\Phi)} \le \|f\|_{L^2(\mathbb{H},\psi)} \le \|f\|_{L^2(\mathbb{H},\ell\Phi)}. \tag{3.2}$$

But ψ is strictly plurisubharmonic, and both eigenvalues of its complex Hessian are identically 1, so in Result 3.1, $\lambda = 1$. Let $g \in L^2_{0,1}(\mathbb{H}, \psi)$, with $\bar{\partial}g = 0$. Therefore, there is a u on $\mathbb{H}$ such that $\bar{\partial}u = g$ and $\|u\|_{L^2(\mathbb{H},\psi)} \le \|g\|_{L^2_{0,1}(\mathbb{H},\psi)}$. Combining this with (3.2), the estimate (2.2) follows.

4 The product model of the Hartogs triangle

The Hartogs Triangle is biholomorphic to the product domain $\mathbb{P} = \mathbb{D} \times \mathbb{D}^*$, where $\mathbb{D}$ is the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathbb{D}^*$ is the punctured unit disc $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$. The explicit map $\mathsf{F} : \mathbb{H} \to \mathbb{P}$ is given by $(w_1, w_2) \mapsto (\frac{w_1}{w_2}, w_2)$, and the inverse $\mathsf{G} = \mathsf{F}^{-1} : \mathbb{P} \to \mathbb{H}$ is given by $(z_1, z_2) \mapsto (z_1 z_2, z_2)$. This product representation allows us to study the regularity of the $\bar{\partial}$-equation on the Hartogs triangle using the technique of [4]. Note that the biholomorphisms F and G are singular at the boundary, so we need to understand how spaces of functions and forms transform under these maps.

Given a locally integrable function or form f on $\mathbb{H}$, we let $\mathsf{G}^* f$ denote the pullback of f to $\mathbb{P}$. Similarly, given a locally integrable function or form g on $\mathbb{P}$, we denote by $\mathsf{F}^* g$ its pullback to a form or function on $\mathbb{H}$. We now consider the mapping properties of the linear mappings F^* and G^* on weighted Sobolev spaces. We denote by F_1^* and F_0^*, the action of the operator F^* on $(0, 1)$-forms and functions respectively, and with a similar meaning for G_1^* and G_0^*.

We define weighted Sobolev spaces $W^k(\mathbb{P}, \ell\Phi)$ on the domain $\mathbb{P}$ by the finiteness of the norm

$$\|g\|_{W^k(\mathbb{H},\ell\Phi)}^2 = \sum_{|\alpha| \leq k} \int_{\mathbb{P}} |z_2|^{2\ell} \left| D^\alpha g(z) \right|^2 dV(z).$$

Lemma 4.1 *For each non-negative integer k, and for $m \in \mathbb{Z}$, the pullback operator F_0^* maps the space $W^k(\mathbb{P}, (m+1)\Phi)$ continuously and injectively to $W^k(\mathbb{H}, (m+k)\Phi)$. Further, for $k = 0$, F_0^* is actually an isometric isomorphism of the Hilbert space $L^2(\mathbb{P}, (m+1)\Phi)$ with the Hilbert space $L^2(\mathbb{H}, m\Phi)$, and consequently the inverse mapping G_0^* is also an isometry from $L^2(\mathbb{H}, m\Phi)$ to $L^2(\mathbb{P}, (m+1)\Phi)$.*

Also, for each non-negative integer k, and for each $\ell \in \mathbb{Z}$, the map G_1^* maps the space of forms $W_{0,1}^k(\mathbb{H}, \ell\Phi)$ continuously and injectively into $W_{0,1}^k(\mathbb{P}, (\ell+1)\Phi)$.

We will allow ourselves, in this proof and the sequel, the standard abuse of notation by which C stands for an arbitrary constant, with possibly different values at different occurrences.

Proof Since G and F are biholomorphisms inverse to each other, it follows that the operators G^* and F^* are also inverses to each other. In particular, they are both injective.

Let f be a locally integrable function on $\mathbb{P}$ and let $g = \mathsf{F}_0^* f$. Then $g(w_1, w_2) = f(\frac{w_1}{w_2}, w_2)$. Using the chain rule repeatedly (i.e., the Faà di Bruno formula, cf. [3]) we see that there is an estimate of the form

$$\left| D_w^\alpha g(w_1, w_2) \right| \leq \frac{C}{|w_2|^{|\alpha|}} \sum_{|\beta| \leq |\alpha|} \left| (D_z^\beta f) \left(\frac{w_1}{w_2}, w_2 \right) \right|.$$

Now, we have

$$\left\| \mathsf{F}_0^* f \right\|_{W^k(\mathbb{H},(m+k)\Phi)}^2 = \sum_{|\alpha| \leq k} \int_{\mathbb{H}} |w_2|^{2(m+k)} \left| D^\alpha g(w) \right|^2 dV(w)$$

$$\leq C \sum_{|\alpha| \leq k} \int_{\mathbb{H}} |w_2|^{2(m+k)} \left(\frac{1}{|w_2|^{2k}} \left| (D_z^\alpha f) \left(\frac{w_1}{w_2}, w_2 \right) \right|^2 \right) dV(w)$$

$$\leq C \sum_{|\alpha| \leq k} \int_{\mathbb{P}} |z_2|^{2m} \left| D^\alpha f \right|^2 |z_2|^2 \, dV(z)$$

$$= C \left\| f \right\|_{W^k(\mathbb{P},(m+1)\Phi)}^2,$$

where, in the last but one line, $|z_2|^2$ represents the Jacobian factor in the change of variables. Considering the case $k = 0$ separately, we have

$$\left\| \mathsf{F}_0^* f \right\|_{L^2(\mathbb{H},m\Phi)} = \int_{\mathbb{H}} |w_2|^{2m} |g(w)|^2 \, dV(w)$$

$$= \int_{\mathbb{P}} |z_2|^{2m} |f(z)|^2 |z_2|^2 \, dV(z)$$

$$= \left\| f \right\|_{L^2(\mathbb{P},(m+1)\Phi)},$$

which proves that F_0^* is an isometry from $L^2(\mathbb{P}, (m+1)\Phi)$ onto $L^2(\mathbb{H}, m\Phi)$.

Now, let $g = g_1 d\overline{w_1} + g_2 d\overline{w_2}$ be a $(0, 1)$-form on $\mathbb{H}$. The pullback $f = \mathsf{G}_1^* g$ is then given by $f = f_1 d\overline{z_1} + f_2 d\overline{z_2}$, where,

$$\begin{cases} f_1(z) = g_1(z_1 z_2, z_2)\overline{z_2} \\ f_2(z) = g_1(z_1 z_2, z_2)\overline{z_1} + g_2(z_1 z_2, z_2). \end{cases}$$

Using the Faà di Bruno formula again, we obtain for some constants depending on α:

$$\begin{cases} |D^\alpha f_1| \leq C \sum_{|\beta| \leq |\alpha|} \left| D^\beta g_1 \right| \\ |D^\alpha f_2| \leq C \sum_{|\beta| \leq |\alpha|} \left(\left| D^\beta g_1 \right| + \left| D^\beta g_2 \right| \right). \end{cases}$$

Therefore,

$$\left\| \mathsf{G}_1^* g \right\|_{W^k(\mathbb{P},(\ell+1)\Phi)}^2 = \sum_{|\alpha| \leq k} \int_{\mathbb{P}} |z_2|^{2(\ell+1)} \left(\left| D^\alpha f_1 \right|^2 + \left| D^\alpha f_2 \right|^2 \right) dV(z)$$

$$\leq C \sum_{|\alpha| \leq k} \int_{\mathbb{H}} |w_2|^{2\ell} \left(\left| D^\alpha g_1 \right|^2 + \left| D^\alpha g_2 \right|^2 \right) dV(w)$$

$$= C \left\| g \right\|_{W^k(\mathbb{H},\ell\Phi)}^2,$$

where again we have used the change of variables formula. $\qquad \square$

5 Canonical solutions on $\mathbb{H}$ and $\mathbb{P}$

5.1 The canonical solution operator

In this paper we are concerned with the situation in which we want to solve on $\mathbb{H}$ the $\overline{\partial}$-problem for a $\overline{\partial}$-closed $(0, 1)$-form g, i.e., find a function u such that $\overline{\partial} u = g$. In view of this, we confine our discussions to the action of the $\overline{\partial}$-operator on functions, noting here that many of these constructions apply to forms of arbitrary degree.

Let Ω be a domain and ψ be a continuous weight function on Ω. As usual, we consider the maximal realization of $\overline{\partial}$, which is a closed densely defined unbounded operator from $L^2(\Omega, \psi)$ to $L^2_{0,1}(\Omega, \psi)$, whose domain $\mathrm{Dom}(\overline{\partial})$ consists of all $f \in L^2(\Omega, \psi)$ such that in the distributional sense $\overline{\partial} f \in L^2_{0,1}(\Omega, \psi)$. If the range $\mathrm{Img}(\overline{\partial})$ of the operator $\overline{\partial}$ is a closed subspace of $L^2_{0,1}(\Omega, \psi)$, we can use general functional analytic methods to define a *bounded* solution operator $K : \mathrm{Img}(\overline{\partial}) \to L^2(\Omega, \psi)$, which maps a $g \in \mathrm{Img}(\overline{\partial}) \subset L^2_{0,1}(\Omega, \psi)$ to the solution of smallest norm of the equation $\overline{\partial} u = g$ (equivalently, we can say that Kg is the unique solution of $\overline{\partial} u = g$ which is orthogonal to the *Bergman Space* $\mathscr{O}(\Omega) \cap L^2(\Omega, \psi)$.) We can extend K to the whole of $L^2(\Omega, \psi)$ by declaring to be zero on $(\mathrm{Dom}(\overline{\partial}))^\perp \subset L^2(\Omega, \psi)$. This K is referred to the *canonical (or Kohn) solution operator* of the $\overline{\partial}$-problem on Ω with weight ψ. In the theory of the $\overline{\partial}$-Neumann problem, we can represent K as $\overline{\partial}^*_\psi \mathsf{N}_{\psi,(0,1)}$, where $\overline{\partial}^*_\psi$ is the Hilbert space adjoint of the $\overline{\partial}$ operator, and $\mathsf{N}_{\psi,(0,1)}$ is the $\overline{\partial}$-Neumann operator on the domain Ω with weight ψ acting on $(0, 1)$-forms. The study of the regularity properties of $\mathsf{N}_{\psi,(0,1)}$ provides a powerful approach to the study of regularity of K itself. Unfortunately this method is not available on the non-smooth domain $\mathbb{H}$ we are considering.

For technical reasons we would sometimes like to think of the canonical solution operator as defined on the orthogonal direct sum $L^2(\Omega, \psi) \oplus L^2_{0,1}(\Omega, \psi)$ and taking values in $L^2(\Omega, \psi)$. This is achieved by declaring the operator to be 0 on the functions in $L^2(\Omega, \psi)$ and extending linearly.

5.2 $K^\ell_{\mathbb{H}}$ and $K^\ell_{\mathbb{P}}$.

From the discussion in Sect. 3 it follows that there exists a canonical solution operator on the domain $\mathbb{H}$ with weight $\ell\Phi$ for each $\ell \in \mathbb{Z}$, where Φ is the harmonic function defined in (3.1). We denote this operator by $K^\ell_{\mathbb{H}}$. Then $K^\ell_{\mathbb{H}}$ is a bounded operator from $L^2_{0,1}(\mathbb{H}, \ell\Phi)$ to $L^2(\mathbb{H}, \ell\Phi)$.

Similarly, there is for each $\ell \in \mathbb{Z}$, a canonical solution operator $K^\ell_{\mathbb{P}}$ for the $\overline{\partial}$-operator on $\mathbb{P}$. Applying Result 3.1 to $\mathbb{P}$, with weight $\psi = \frac{1}{2}|z|^2 + \ell\Phi$ gives us a solution to $\overline{\partial} v = g$ for $g \in L^2_{0,1}(\mathbb{P}, \psi) \cap \ker(\overline{\partial})$, satisfying the estimate $\|v\|_{L^2(\mathbb{P},\psi)} \leq \|g\|_{L^2_{0,1}(\mathbb{P},\psi)}$. But $L^2(\mathbb{P}, \psi)$ and $L^2(\mathbb{P}, \ell\Phi)$ are the same space with equivalent norms, so we can solve $\overline{\partial} v = g$ with $g \in L^2_{0,1}(\mathbb{P}, \ell\Phi) \cap \ker(\overline{\partial})$, where v satisfies the estimate $\|v\|_{L^2(\mathbb{P},\ell\Phi)} \leq C\|g\|_{L^2_{0,1}(\mathbb{P},\ell\Phi)}$. Now the existence of $K^\ell_{\mathbb{P}}$ follows as in Sect. 3.

We note the relation between the canonical operators on $\mathbb{H}$ and $\mathbb{P}$:

Lemma 5.1 *We have*

$$K_{\mathbb{H}}^{\ell} = \mathsf{F}_0^* \circ K_{\mathbb{P}}^{\ell+1} \circ \mathsf{G}_1^*. \tag{5.1}$$

Proof Denote the operator defined by the right hand side of (5.1) by S_ℓ. This maps $(0, 1)$-forms on $\mathbb{H}$ to functions on $\mathbb{H}$ and satisfies

$$K_{\mathbb{P}}^{\ell+1} \circ \mathsf{G}_1^* = \mathsf{G}_0^* \circ S_\ell. \tag{5.2}$$

Since the $\bar{\partial}$ operator commutes with pullbacks by holomorphic mappings, it follows that S_ℓ is a solution operator for $\bar{\partial}$, i.e., $\bar{\partial}(S_\ell g) = g$, if $\bar{\partial}g = 0$ on $\mathbb{H}$. Further, S_ℓ is bounded from $L_{0,1}^2(\mathbb{H}, \ell\Phi)$ to $L^2(\mathbb{H}, \ell\Phi)$, since we know from Lemma 4.1 above that G_1^* is continuous from the space $L_{0,1}^2(\mathbb{H}, \ell\Phi)$ to $L_{0,1}^2(\mathbb{P}, (\ell+1)\Phi)$ and that F_0^* is continuous from $L^2(\mathbb{P}, (\ell+1)\Phi)$ to $L^2(\mathbb{P}, \ell\Phi)$, and by definition $K_{\mathbb{P}}^{\ell+1}$ is continuous from $L_{0,1}^2(\mathbb{P}, (\ell+1)\Phi)$ to $L^2(\mathbb{P}, (\ell+1)\Phi)$.

Suppose now that $S_\ell \neq K_{\mathbb{H},1}^{\ell}$. Then there is a $g \in L_{0,1}^2(\mathbb{H}, \ell\Phi)$ with $\bar{\partial}g = 0$, and a $u \in L^2(\mathbb{H}, \ell\Phi)$ such that $\bar{\partial}u = g$ and $\|u\|_{L^2(\mathbb{H}, \ell\Phi)} < \|S_\ell g\|_{L^2(\mathbb{H}, \ell\Phi)}$. Note that $\mathsf{G}_1^* g$ is $\bar{\partial}$-closed, and consider the $\bar{\partial}$-problem on $\mathbb{P}$ given by $\bar{\partial}v = \mathsf{G}_1^* g$. Both $\mathsf{G}_0^* u$ and $K_{\mathbb{P}}^{\ell+1}(\mathsf{G}_1^* g)$ are solutions of this equation in $L^2(\mathbb{P}, (\ell+1)\Phi)$, and since $K_{\mathbb{P}}^{\ell+1}(\mathsf{G}_1^* g)$ is the canonical solution, we have

$$\left\| K_{\mathbb{P}}^{\ell+1}(\mathsf{G}_1^* g) \right\|_{L^2(\mathbb{P}, (\ell+1)\Phi)} \leq \left\| \mathsf{G}_0^* u \right\|_{L^2(\mathbb{P}, (\ell+1)\Phi)}. \tag{5.3}$$

Since G_0^* is an isometry by Lemma 4.1, it follows that

$$\begin{aligned}
\left\| \mathsf{G}_0^* u \right\|_{L^2(\mathbb{P}, (\ell+1)\Phi)} &= \left\| u \right\|_{L^2(\mathbb{H}, \ell\Phi)} \\
&< \left\| S_\ell g \right\|_{L^2(\mathbb{H}, \ell\Phi)} \\
&= \left\| \mathsf{G}_0^* S_\ell g \right\|_{L^2(\mathbb{P}, (\ell+1)\Phi)} \\
&= \left\| K_{\mathbb{P}}^{\ell+1}(\mathsf{G}_1^* g) \right\|_{L^2(\mathbb{P}, (\ell+1)\Phi)},
\end{aligned}$$

where we have used (5.2) in the last line. But this contradicts (5.3) and we conclude therefore that $S_\ell = K_{\mathbb{H}}^{\ell}$. □

5.3 Representation of the canonical solution on the product domain $\mathbb{P}$

In order to estimate the operator $K_{\mathbb{P}}^{\ell}$ on the product domain $\mathbb{P} = \mathbb{D} \times \mathbb{D}^*$, we want to use [4, Theorem 4.7] to represent it using terms of the canonical solution operators and Bergman projections of the factors $\mathbb{D}$ and $\mathbb{D}^*$. This theorem is as follows ($\widehat{\otimes}$ denotes the Hilbert tensor product of Hilbert spaces, i.e., the completion of the algebraic tensor product under its natural hermitian inner product, see [4]).

Result 5.2 *Let $\Omega_1 \Subset \mathbb{C}^{n_1}$ and $\Omega_2 \Subset \mathbb{C}^{n_2}$ be bounded* Lipschitz *domains, and let ψ_1, ψ_2 be continuous functions on Ω_1, Ω_2 respectively. Suppose that, for $j = 1, 2,$*

the $\bar{\partial}$-operator has closed range as an operator from $L^2(\Omega_j, \psi_j)$ to $L^2_{0,1}(\Omega_j, \psi_j)$. Then the $\bar{\partial}$-operator has closed range from $L^2(\Omega, \psi)$ to $L^2_{0,1}(\Omega, \psi)$, where $\Omega = \Omega_1 \times \Omega_2 \Subset \mathbb{C}^{n_1+n_2}$, and $\psi = \psi_1 + \psi_2$. Further the canonical solution operator $K : L^2_{0,1}(\Omega, \psi) \to L^2(\Omega, \psi)$ restricted to the space of $\bar{\partial}$-closed $(0, 1)$-forms has the representation

$$K = K_1 \widehat{\otimes} I_2 + \sigma_1 P_1 \widehat{\otimes} K_2, \tag{5.4}$$

where K_1, K_2 are the canonical solution operators on Ω_1, Ω_2 respectively, P_1 is the harmonic projection on Ω_1 and σ_1 is a linear operator which restricts to multiplication by $(-1)^d$ on the space of forms of total degree d on Ω_1.

Of course, there is a second representation analogous to (5.4) obtained by switching the roles of Ω_1 and Ω_2.

Unfortunately, one of the factors $\mathbb{D}^*$ of $\mathbb{P}$ is *not* Lipschitz, so Result 5.2 does not apply as stated in the situation we are interested. However, we contend that the conclusion of Result 5.2 still holds for $\Omega_1 = \mathbb{D}$ and $\Omega_2 = \mathbb{D}^*$ with weights $\psi_1 \equiv 0$ and $\psi_2 = \ell\phi$, where ϕ is the harmonic function on $\mathbb{D}^*$ given by

$$\phi(z) = -2\log|z|. \tag{5.5}$$

We first state a general result which we can apply to $\mathbb{P}$. Let H_1 and H_2 be Hilbert spaces, and let $T : \mathsf{H}_1 \to \mathsf{H}_2$ be a densely defined closed linear operator from a subspace $\mathrm{Dom}(T) \subset \mathsf{H}_1$ to a subspace H_2. The *graph norm* $\|u\|_{\Gamma(T)}$ of an element $u \in \mathrm{Dom}(T)$ is defined by $\|u\|^2_{\Gamma(T)} = \|u\|^2_{H_1} + \|Tu\|^2_{H_2}$, and since T is closed, $\mathrm{Dom}(T)$ is a Hilbert space in this norm. Recall that a *core* of a densely defined operator T is a subspace $\mathscr{G} \subset \mathrm{Dom}(T)$ which is dense in $\mathrm{Dom}(T)$ in the graph norm (cf. [13, p. 155].) After these definitions, we can state the slightly more general form of Result 5.2.

Proposition 5.3 *The hypotheses are the same as in Result 5.2, except that Ω_1 and Ω_2 are not assumed to be Lipschitz. Instead we assume that there exists a core $\mathscr{G}_1$ of $\bar{\partial} : L^2(\Omega_1, \psi_1) \to L^2_{0,1}(\Omega_1, \psi_1)$ and a core $\mathscr{G}_2$ of $\bar{\partial} : L^2(\Omega_2, \psi_2) \to L^2_{0,1}(\Omega_2, \psi_2)$, such that the algebraic tensor product $\mathscr{G}_1 \otimes \mathscr{G}_2$ is a core of the operator $\bar{\partial} : L^2(\Omega, \psi) \to L^2_{0,1}(\Omega, \psi)$, then the same conclusion (in particular the representation (5.4) of the canonical solution) holds.*

Proof We only indicate the changes that need to be made in the proof of Result 5.2 as given in [4] in order to verify this more general statement. Note that the only way the Lipschitz condition is used in the proof of Result 5.2 is to provide the cores $\mathscr{C}^\infty(\overline{\Omega_1}), \mathscr{C}^\infty(\overline{\Omega_2}), \mathscr{C}^\infty(\overline{\Omega})$ on Ω_1, Ω_2 and Ω, and to make sure that $\mathscr{C}^\infty(\overline{\Omega_1}) \otimes \mathscr{C}^\infty(\overline{\Omega_2})$ is dense in the graph-norm $\Gamma(\bar{\partial})$ in $\mathscr{C}^\infty(\overline{\Omega})$ and therefore in $\mathrm{Dom}(\bar{\partial})$. It is easy to check all the arguments in [4] continue to hold if we replace $\mathscr{C}^\infty(\overline{\Omega_1})$ by $\mathscr{G}_1$ and $\mathscr{C}^\infty(\overline{\Omega_2})$ by $\mathscr{G}_2$. $\qquad\square$

We now proceed to apply Proposition 5.3 to $\mathbb{P} = \mathbb{D} \times \mathbb{D}^*$. We take Ω_1 to be $\mathbb{D}$ and $\psi_1 \equiv 0$. Then $K_1 = K_{\mathbb{D}}$, the canonical solution operator on the unit disc $\mathbb{D}$ without

any weight, and $P_1 = P_{\mathbb{D}}$, the Bergman projection on $\mathbb{D}$ in degree 0, and the zero operator in other degrees (since harmonic spaces vanish in other degrees.) The closed range property for $\bar{\partial}$ and the existence of the canonical solution operator is immediate from Result 3.1 by using the weight $\psi = \frac{1}{2}|z|^2$.

For Ω_2, we take the punctured disc $\mathbb{D}^*$. Let ϕ be as in (5.5). We take the weight ψ_2 to be $\ell\phi$. Note that then $L^2(\mathbb{D}^*, \ell\phi)$ has the norm

$$\|f\|_{L^2(\mathbb{D}^*, \ell\phi)} = \int_{\mathbb{D}^*} |z|^{2\ell} f(z) dV(z).$$

We need to show that $\bar{\partial} : L^2(\mathbb{D}^*, \ell\phi) \to L^2_{0,1}(\mathbb{D}^*, \ell\phi)$ has closed range. For this we use the same method as used in the proof of (2.2). In Result 3.1, we let the weight ψ to be $\psi = \frac{1}{2}|z|^2 + \ell\phi$. This immediately shows that for any $g d\bar{z} \in L^2_{0,1}(\mathbb{D}^*, \ell\phi)$, there is a $v \in L^2(\mathbb{D}^*, \ell\phi)$ such that $\bar{\partial}v = g d\bar{z}$, i.e. $\frac{\partial v}{\partial \bar{z}} = g$, and v satisfies the estimate

$$\|v\|_{L^2(\mathbb{D}^*, \ell\phi)} \leq \sqrt{e}\|g\|_{L^2(\mathbb{D}^*, \ell\phi)}.$$

The existence of the canonical solution follows as usual. We denote the canonical solution operator by $K^\ell_{\mathbb{D}^*}$. It is a bounded operator from $L^2_{0,1}(\mathbb{D}^*, \ell\phi)$ to $L^2(\mathbb{D}^*, \ell\phi)$.

In order to apply Proposition 5.3 we also need cores $\mathscr{G}_1$ of $\bar{\partial} : L^2(\mathbb{D}) \to L^2_{0,1}(\mathbb{D})$, and $\mathscr{G}_2$ of $\bar{\partial} : L^2(\mathbb{D}^*, \ell\phi) \to L^2_{0,1}(\mathbb{D}^*, \ell\phi)$ such that $\mathscr{G}_1 \otimes \mathscr{G}_2$ is a core of $\bar{\partial} : L^2(\mathbb{P}, \ell\Phi) \to L^2_{0,1}(\mathbb{P}, \ell\Phi)$. We take $\mathscr{G}_1 = \mathscr{C}^\infty(\overline{\mathbb{D}})$. Since $\mathbb{D}$ has smooth boundary, it follows that $\mathscr{G}_1$ is a core for the $\bar{\partial}$ operator on $L^2(\mathbb{D})$. Let $\mathscr{G}_2$ be the space of functions on $\mathbb{D}^*$ of the form $z^{-\ell}f$, where $f \in \mathscr{C}^\infty(\overline{\mathbb{D}})$. We have the following.

Lemma 5.4 (1) $\mathscr{G}_2$ is a core of $\bar{\partial} : L^2(\mathbb{D}^*, \ell\phi) \to L^2_{0,1}(\mathbb{D}^*, \ell\phi)$.
(2) $\mathscr{G}_1 \otimes \mathscr{G}_2$ is a core of $\bar{\partial} : L^2(\mathbb{P}, \ell\Phi) \to L^2_{0,1}(\mathbb{P}, \ell\Phi)$.

Proof Let $g \in \text{Dom}(\bar{\partial}) \subset L^2(\mathbb{D}^*, \ell\phi)$. It follows that $z^\ell g \in L^2(\mathbb{D})$ and $\bar{\partial}(z^\ell g) = z^\ell \bar{\partial}g \in L^2(\mathbb{D})$. Therefore, $z^\ell g$ belongs to the domain of $\bar{\partial}$ as an operator on $L^2(\mathbb{D})$. We take a sequence $\{f_\nu\}$ of forms in $\mathscr{C}^\infty(\overline{\mathbb{D}})$ converging in the graph norm of $\bar{\partial}$ on $L^2(\mathbb{D})$ to $z^\ell g$. It is easy to see that $z^{-\ell}f_\nu$ converges to g in the graph norm of $\bar{\partial}$ on $L^2(\mathbb{D}^*, \ell\phi)$. Part (1) follows.

Let $\mathscr{H}$ denote the forms on $\mathbb{P} = \mathbb{D} \times \mathbb{D}^*$ which are of the type $z_2^{-\ell}f$, where $f \in \mathscr{C}^\infty(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$. An argument analogous to the one in Part (1) above shows that $\mathscr{H}$ is a core of the $\bar{\partial}$ operator acting on $L^2(\mathbb{P}, \ell\Phi)$. Given any $z_2^{-\ell}f \in \mathscr{H}$ we can approximate f in the $\mathscr{C}^1$ norm on $\overline{\mathbb{D}}$ by elements of the algebraic tensor product $\mathscr{C}^\infty(\overline{\mathbb{D}}) \otimes \mathscr{C}^\infty(\overline{\mathbb{D}})$ (cf. [12, page 369].) From this the statement (2) follows immediately. □

Therefore, we obtain the following representation of the canonical solution $K^\ell_{\mathbb{P}}$ in terms of the factor domains $\mathbb{D}$ and $\mathbb{D}^*$. Note that in the second term of (5.4), the only term that is non-zero is the term corresponding to functions on D, since the harmonic projection vanishes in every other degree, and for this remaining term we have $\sigma_1 = 1$.

Proposition 5.5 *On the $\overline{\partial}$-closed $(0, 1)$-forms in $L^2_{0,1}(\mathbb{P}, \ell\Phi)$, we have*

$$K^\ell_{\mathbb{P}} = K_{\mathbb{D}} \widehat{\otimes} I_{\mathbb{D}^*} + P_{\mathbb{D}} \widehat{\otimes} K^\ell_{\mathbb{D}^*},$$

where

- $K_{\mathbb{D}} : L^2_{0,1}(\mathbb{D}) \to L^2(\mathbb{D})$ *is the canonical solution of the $\overline{\partial}$ equation on the unit disc (recall that by convention, we assume that canonical solution operators vanish on functions).*
- $I_{\mathbb{D}^*}$ *is the identity map on functions and forms on $\mathbb{D}^*$.*
- $P_{\mathbb{D}} : L^2(\mathbb{D}) \to L^2(\mathbb{D}) \cap \mathscr{O}(\mathbb{D})$ *is the Bergman projection. It is extended to $L^2_{0,1}(\mathbb{D})$ by setting it equal to 0.*
- $K^\ell_{\mathbb{D}^*} : L^2_{0,1}(\mathbb{D}^*, \ell\phi) \to L^2(\mathbb{D}^*, \ell\phi)$ *is the canonical solution operator on $\mathbb{D}^*$ with weight $\ell\phi$.*

6 Proof of Theorem 2.1

6.1 Expression of $K^\ell_{\mathbb{H}}$ in terms of components

Combining Proposition 5.5 with Lemma 4.1 we obtain the following:

Corollary 6.1 *We have*

$$K^\ell_{\mathbb{H}} = \mathsf{F}^*_0 \circ \left(K_{\mathbb{D}} \widehat{\otimes} I_{\mathbb{D}^*} + P_{\mathbb{D}} \widehat{\otimes} K^{\ell+1}_{\mathbb{D}^*} \right) \circ \mathsf{G}^*_1. \tag{6.1}$$

We need to estimate the various operators appearing in (6.1) in Sobolev spaces in order to prove Theorem 2.1. The regularity of G^*_1 and F^*_0 in partial Sobolev spaces has already been discussed in Lemma 4.1. We consider the Sobolev Space $W^k(\mathbb{D})$ on the unit disc of order $k \geq 0$, which is given by the finiteness of the norm

$$\|f\|^2_{W^k(\mathbb{D})} = \sum_{\alpha+\beta \leq k} \int_{\mathbb{D}} \left| \frac{\partial^{\alpha+\beta} f}{\partial z^\alpha \partial \overline{z}^\beta} \right|^2 dV$$

For the disc $\mathbb{D}$, it is well-known from potential theory that $K_{\mathbb{D}}$ maps $\overline{\partial}$-closed forms in $W^k_{0,1}(\mathbb{D})$ to functions in $W^{k+1}(\mathbb{D})$, and the Harmonic projection, which is non-zero only in degree 0, is identical to the Bergman projection which maps functions in $W^k(\mathbb{D})$ to holomorphic functions in $W^k(\mathbb{D})$ (condition "R").

6.2 Regularity of $K^\ell_{\mathbb{D}^*}$

We use Sobolev spaces with the weight ϕ as in (5.5). The norm in such a space $W^k(\mathbb{D}^*, \ell\phi)$ is given by

$$\|f\|^2_{W^k(\mathbb{D}^*,\ell\phi)} = \sum_{\alpha+\beta\leq k} \int_{\mathbb{D}} |z|^{2\ell} \left|\frac{\partial^{\alpha+\beta}f}{\partial z^\alpha \partial\overline{z}^\beta}\right|^2 dV.$$

With respect to these spaces, we have the following.

Proposition 6.2 *For every nonnegative integer k, the operator $K^\ell_{\mathbb{D}^*}$ is bounded from the Sobolev space $W^k_{0,1}(\mathbb{D}^*, \ell\phi)$ to the Sobolev space $W^k(\mathbb{D}^*, (\ell+k)\phi)$.*

Proof Let $g \in W^k(\mathbb{D}^*, \ell\phi)$, and set $\widetilde{g} = z^{k+\ell}g$. We claim that $\widetilde{g} \in W^k(\mathbb{D})$, the unweighted standard Sobolev space of order k on the disc. Indeed, for $\alpha + \beta \leq k$, we have using the Leibniz rule:

$$\frac{\partial^{\alpha+\beta}\widetilde{g}}{\partial z^\alpha \partial\overline{z}^\beta} = \sum_{j=0}^\alpha \binom{\alpha}{j} z^{k+\ell-j} \frac{\partial^{\alpha+\beta-j}g}{\partial z^{\alpha-j} \partial\overline{z}^\beta}.$$

Note that each term in the sum on the right is in $L^2(\mathbb{D})$, since by hypothesis $g \in W^k(\mathbb{D}^*, \ell\phi)$. Further, using the fact that $\left|z^{k+\ell-j}\right| \leq |z|^\ell$, it easily follows that there is an estimate

$$\|\widetilde{g}\|_{W^k(\mathbb{D})} \leq C \|g\|_{W^k(\mathbb{D}^*,\ell\phi)}. \tag{6.2}$$

Let $\tilde{u}$ denote that canonical solution of the equation $\overline{\partial}\tilde{u} = \tilde{g}d\overline{z}$ in $L^2(\mathbb{D})$ (without any weights). Then we know that $\tilde{u} \in W^{k+1}(\mathbb{D})$, and we have an estimate

$$\|\tilde{u}\|_{W^{k+1}(\mathbb{D})} \leq C \|\tilde{g}\|_{W^k(\mathbb{D})}. \tag{6.3}$$

Set $u = z^{-(k+\ell)}\tilde{u}$. Then, on $\mathbb{D}^*$, we have $\overline{\partial}u = gd\overline{z}$ and

$$\|\widetilde{u}\|^2_{W^{k+1}(\mathbb{D})} = \|z^{k+\ell}u\|^2_{W^{k+1}(\mathbb{D})}$$

$$= \sum_{\alpha+\beta\leq k+1} \int_{\mathbb{D}} \left|\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial\overline{z}^\beta}\left(z^{(k+\ell)}u\right)\right|^2 dV$$

$$\geq C \sum_{\alpha+\beta\leq k+1} \int_{\mathbb{D}} |z|^{2(k+\ell)} \left|\frac{\partial^{\alpha+\beta}u}{\partial z^\alpha \partial\overline{z}^\beta}\right|^2 dV$$

$$\geq C\|u\|_{W^{k+1}(\mathbb{D}^*,(\ell+k)\phi)}.$$

Combining this with (6.2) and (6.3), we see that there is a linear solution operator $g \mapsto u$, for $\frac{\partial}{\partial\overline{z}}$ on the punctured disc $\mathbb{D}^*$ which is continuous from $W^k(\mathbb{D}^*, \ell\phi)$ to $W^{k+1}(\mathbb{D}^*, (\ell+k)\phi)$.

Denote by $v = K^\ell_{\mathbb{D}^*}(gd\overline{z})$ the canonical solution of $\overline{\partial}v = gd\overline{z}$ in the weighted space $L^2(\mathbb{D}^*, \ell\phi)$. Then v is of the form $u + h$, where h is a function in the Bergman

space $\mathcal{O}(\mathbb{D}^*) \cap L^2(\mathbb{D}^*, \ell\phi)$, and u is the solution of $\bar{\partial} u = g d\bar{z}$ found above. But then h must be of the form $h = z^{-\ell} f$, where $f \in \mathcal{O}(\mathbb{D}) \cap L^2(\mathbb{D})$. Denote by $\mathbb{D}^*_{\frac{1}{2}}$ the punctured disc $\{0 < |z| < \frac{1}{2}\}$ of radius $\frac{1}{2}$. A direct computation shows that $h \in W^k(\mathbb{D}^*_{\frac{1}{2}}, (\ell + k)\phi)$. Since by the last paragraph, $u \in W^{k+1}(\mathbb{D}^*, (\ell + k)\phi)$, it now follows that $v \in W^k(\mathbb{D}^*_{\frac{1}{2}}, (\ell + k)\phi)$.

Now let χ be a cutoff on $\mathbb{D}$ which is identically 1 on $\{|z| > \frac{1}{2}\}$, and vanishes in a neighborhood of 0. By standard localization results, $\chi v \in W^{k+1}(\mathbb{D})$. Combining with the fact that $v \in W^k(\mathbb{D}^*_{\frac{1}{2}}, (\ell + k)\phi)$, it follows that $v \in W^k(\mathbb{D}^*, (\ell + k)\phi)$, and the result is proved. □

6.3 Estimates on $K_{\mathbb{H}}^{\ell}$

As in [4], for an integer $k \geq 0$, we introduce the *weighted partial Sobolev space* $\widetilde{W}^k(\mathbb{P}, \ell\Phi)$ by the finiteness of the norm:

$$\|f\|^2_{\widetilde{W}^k(\mathbb{P}, \ell\Phi)} = \sum_{\substack{\alpha_1+\alpha_2 \leq k \\ \alpha_3+\alpha_4 \leq k}} \int_{\mathbb{P}} |z_2|^{2\ell} |D^\alpha f(z)|^2 \, dV(z), \tag{6.4}$$

where D^α is as in (2.3), the derivatives are in the weak sense, and note the special range of summation. It is clear that

$$W^{2k}(\mathbb{P}, \ell\Phi) \subsetneq \widetilde{W}^k(\mathbb{P}, \ell\Phi) \subsetneq W^k(\mathbb{P}, \ell\Phi), \tag{6.5}$$

with continuous inclusions.

We begin with the following lemma which holds for every non-negative k.

Lemma 6.3 $\widetilde{W}^k(\mathbb{P}, \ell\Phi) = W^k(\mathbb{D}) \widehat{\otimes} W^k(\mathbb{D}^*, \ell\phi)$.

Proof Were the domains $\mathbb{D}^*$ and $\mathbb{D}$ both Lipschitz we could use the method of [4, Lemma 5.1] directly. Since $\mathbb{D}^*$ is non-Lipschitz, we proceed as follows. For a multi-index $\nu \in \mathbb{Z}^4$, denote by $\mathbf{z}^\nu$ the Laurent-type monomial $z_1^{\nu_1} \overline{z_1}^{\nu_2} z_2^{\nu_3} \overline{z_2}^{\nu_4}$. For $m \in \mathbb{Z}$, if $S(m)$ denotes the set of monomials $\{\mathbf{z}^\nu \mid \nu_1 \geq 0, \nu_2 \geq 0, \nu_3 + \nu_4 \geq m\}$, it is easy to see that the elements of $S(-\ell)$ form a complete set in $L^2(\mathbb{P}, \ell\Phi)$, i.e. the linear span of $S(-\ell)$ is dense in $L^2(\mathbb{P}, \ell\Phi)$. If α and D^α are as in (2.3) with $\alpha_3 + \alpha_4 \leq k$, then for $\mathbf{z}^\nu \in S(m)$, the derivative $D^\alpha \mathbf{z}^\nu$ is a scalar multiple of an element of $S(m-k)$. Further, every element of $S(m-k)$ arises (up to a multiplicative factor) as a partial derivative of this sort. By definition, a function $f \in \widetilde{W}^k(\mathbb{P}, \ell\Phi)$, if and only if $D^\alpha f \in L^2(\mathbb{P}, \ell\Phi)$, for α with $\alpha_1+\alpha_2 \leq k$ and $\alpha_3+\alpha_4 \leq k$. It now follows easily that $S(-\ell+k)$ is complete in $\widetilde{W}^k(\mathbb{P}, \ell\Phi)$. But an element of $S(-\ell+k)$ may be written as $(z_1^{\nu_1} \overline{z_1}^{\nu_2})(z_2^{\nu_3} \overline{z_2}^{\nu_4})$, where $\nu_1, \nu_2 \geq 0$ and $\nu_3 + \nu_4 \geq -\ell + k$. The first factor is in $W^k(\mathbb{D})$ (indeed for any k) and the second factor is in $W^k(\mathbb{D}^*, \ell\phi)$. It follows that the algebraic tensor product $W^k(\mathbb{D}) \otimes W^k(\mathbb{D}^*, \ell\phi)$ is dense in $\widetilde{W}^k(\mathbb{P}, \ell\Phi)$ and the result follows. □

We can now complete the proof of Theorem 2.1. Since $u_\ell = K_{\mathbb{H}}^\ell(g)$, it is sufficient to show that the operator $K_{\mathbb{H}}^\ell$ is bounded from $W^{2k}(\mathbb{H}, \ell\Phi)$ to $W^k(\mathbb{H}, (l+2k)\Phi)$. We recall that the representation of $K_{\mathbb{H}}^\ell$, given by (6.1), is:

$$K_{\mathbb{H}}^\ell = \mathsf{F}_0^* \circ K_{\mathbb{P}}^{\ell+1} \circ \mathsf{G}_1^* = \mathsf{F}_0^* \circ \left(K_{\mathbb{D}} \widehat{\otimes} I_{\mathbb{D}^*} + P_{\mathbb{D}} \widehat{\otimes} K_{\mathbb{D}^*}^{\ell+1} \right) \circ \mathsf{G}_1^*.$$

Thanks to Lemma 4.1, the operator G_1^* which occurs as the first factor from the right, is known to be continuous from $W_{0,1}^{2k}(\mathbb{H}, \ell\Phi)$ to $W_{0,1}^{2k}(\mathbb{P}, (\ell+1)\Phi)$. Thanks to (6.5), it now follows that the operator G_1^* is continuous from $W_{0,1}^{2k}(\mathbb{H}, \ell\Phi)$ to $\widetilde{W}_{0,1}^k(\mathbb{P}, (\ell+1)\Phi)$.

We claim that the canonical solution operator $K_{\mathbb{P}}^{\ell+1}$, which occurs as the middle factor of the expression for $K_{\mathbb{H}}^\ell$ is bounded from $\widetilde{W}_{0,1}^k(\mathbb{P}, (\ell+1)\Phi)$ to $\widetilde{W}^k(\mathbb{P}, (\ell+k+1)\Phi)$. Using Lemma 6.3 and the fact that forms of different degree are orthogonal by definition, we have

$$\widetilde{W}_{0,1}^k(\mathbb{P}, (\ell+1)\Phi) = W_{0,1}^k(\mathbb{D}) \widehat{\otimes} W^k(\mathbb{D}^*, (\ell+1)\phi) \oplus W^k(\mathbb{D}) \widehat{\otimes} W_{0,1}^k(\mathbb{D}^*, (\ell+1)\phi),$$
(6.6)

where $\oplus$ represents orthogonal direct sum of subspaces. Now we look at the two terms in the expression for $K_{\mathbb{P}}^{\ell+1}$ which is the middle factor of (6.1):

$$K_{\mathbb{D}} \widehat{\otimes} I_{\mathbb{D}^*} + P_{\mathbb{D}} \widehat{\otimes} K_{\mathbb{D}^*}^{\ell+1}.$$

In the first term, $K_{\mathbb{D}}$ is the canonical solution on the disc and maps $W_{0,1}^k(\mathbb{D})$ to $W^{k+1}(\mathbb{D})$. Since the inclusion $W^{k+1}(\mathbb{D}) \subset W^k(\mathbb{D})$ is continuous, it follows that $K_{\mathbb{D}}$ is continuous from $W_{0,1}^k(\mathbb{D})$ to $W^k(\mathbb{D})$. Since the inclusion $W^k(\mathbb{D}^*, (\ell+1)\phi) \subset W^k(\mathbb{D}^*, (\ell+k+1)\phi)$ is continuous, it follows that the identity map also is continuous from $W^k(\mathbb{D}^*, (\ell+1)\phi)$ to $W^k(\mathbb{D}^*, (\ell+k+1)\phi)$. Moreover, we defined the canonical solution operator to be zero on functions. It follows now from Lemma 6.3 that the operator $K_{\mathbb{D}} \widehat{\otimes} I_{\mathbb{D}^*}$ maps the space $\widetilde{W}^k(\mathbb{P}, (\ell+1)\Phi)$ continuously into $\widetilde{W}^k(\mathbb{P}, (\ell+k+1)\Phi)$.

In the second term $P_{\mathbb{D}}$ is the harmonic projection on the disc, which vanishes on $(0,1)$-forms, and consequently, this term acts only on the second summand in the orthogonal decomposition (6.6) of $\widetilde{W}_{0,1}^k(\mathbb{P}, (\ell+1)\Phi)$. For functions, the Bergman projection $P_{\mathbb{D}}$ on the disc preserves the space $W^k(\mathbb{D})$. In Proposition 6.2 we saw that $K_{\mathbb{D}^*}^{\ell+1}$ maps $W_{0,1}^k(\mathbb{D}^*, (\ell+1)\phi)$ continuously into $W^k(\mathbb{D}^*, (\ell+k+1)\phi)$. It follows from Lemma 6.3 that $P_{\mathbb{D}} \widehat{\otimes} K_{\mathbb{D}^*}^{\ell+1}$ also maps the space $\widetilde{W}^k(\mathbb{P}, (\ell+1)\Phi)$ continuously into $\widetilde{W}^k(\mathbb{P}, (\ell+k+1)\Phi)$, and the same is true of the sum $K_{\mathbb{P}}^{\ell+1} = K_{\mathbb{D}} \widehat{\otimes} I_{\mathbb{D}^*} + P_{\mathbb{D}} \widehat{\otimes} K_{\mathbb{D}^*}^{\ell+1}$, which occurs as the middle factor of (6.1).

The continuous inclusions of (6.5) now imply that $K_{\mathbb{P}}^{\ell+1}$ maps $W_{0,1}^{2k}(\mathbb{P}, (\ell+1)\Phi)$ continuously into $W^k(\mathbb{P}, (\ell+k+1)\Phi)$. But the factor F_0^* in (6.1), by Lemma 4.1, maps $W^k(\mathbb{P}, (\ell+k+1)\Phi)$ continuously into $W^k(\mathbb{H}, (\ell+2k)\Phi)$. It follows that $K_{\mathbb{H}}^\ell$ is continuous from $W_{0,1}^{2k}(\mathbb{H}, \ell\Phi)$ to $W^k(\mathbb{H}, (\ell+2k)\Phi)$, and Theorem 2.1 is proved.

7 Proof of Theorem 2.3

By a result of Kohn (see [6, p. 82]), the Bergman projection B on $\mathbb{H}$ can be represented in terms of the $\bar{\partial}$-Neumann operator N as:

$$B = I - \bar{\partial}^* \mathsf{N} \bar{\partial}$$
$$= I - K\bar{\partial},$$

where $K = K_{\mathbb{H}}^0$ is the (unweighted) canonical solution operator on the domain $\mathbb{H}$. Now the $\bar{\partial}$ operator maps $W^{2k+1}(\mathbb{H})$ continuously into $W_{0,1}^{2k}(\mathbb{H})$, and thanks to the regularity result for K established in Theorem 2.1, it follows that K maps $W_{0,1}^{2k}(\mathbb{H})$ continuously into $W^k(\mathbb{H}, 2k\Phi)$. Since the space $W^k(\mathbb{H}, 2k\Phi)$ continuously includes the space $W^{2k+1}(\mathbb{H})$, it follows by Kohn's formula above, B is continuous as well between these spaces. This proves the first statement.

For the second statement, we can either repeat the argument used in the proof of Corollary 2.2, or we can use Kohn's formula and Corollary 2.2 directly: if $f \in \mathscr{C}^\infty(\overline{\mathbb{H}})$, clearly $\bar{\partial} f \in \mathscr{C}_{0,1}^\infty(\overline{\mathbb{H}}) \cap \ker(\bar{\partial})$, so by Corollary 2.2 we have $K\bar{\partial} f \in \mathscr{C}_{0,1}^\infty(\overline{\mathbb{H}} \setminus \{0\})$. Therefore, $Bf = f - K\bar{\partial} f$ is in $\mathscr{C}_{0,1}^\infty(\overline{\mathbb{H}} \setminus \{0\}) \cap \mathscr{O}(\mathbb{H})$.

We claim that to show that B does not map the space $\mathscr{C}_0^\infty(\mathbb{H})$ into $W^1(\mathbb{H})$, it suffices to show that $W^1(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$ is not dense in the Bergman space $L^2(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$ in the L^2-topology. Indeed, if $\{f_n\}$ is a sequence of functions in $\mathscr{C}_0^\infty(\mathbb{H})$ which converge in L^2 to a Bergman function $f \in L^2(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$, then Bf_n converges to f in L^2. If $Bf_n \in W^1(\mathbb{H})$, this would imply that $W^1(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$ is dense in $L^2(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$.

To show that $W^1(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$ is not dense in the Bergman space $L^2(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$, it is sufficient to find a non-zero function $f \in L^2(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$ which lies in the orthogonal complement of $W^1(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$. We can take $f(w) = \dfrac{1}{w_2}$. This f is in the Bergman space $L^2(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$, since using the standard biholomorphism from $\mathbb{H}$ to $\mathbb{P}$ given by $(w_1, w_2) \mapsto (\frac{w_1}{w_2}, w_2)$ we obtain

$$\int_{\mathbb{H}} \left| \frac{1}{w_2} \right|^2 dV(w) = \int_{\mathbb{P}} \frac{1}{|z_2|^2} |z_2|^2 \, dV(z)$$
$$= \pi^2.$$

However, since $\frac{\partial f}{\partial w_2} = \frac{1}{w_2^2}$ is not square integrable on $\mathbb{H}$, it follows that f is not in $W^1(\mathbb{H})$.

Note that any holomorphic function on the domain $\mathbb{P}$ has a Laurent expansion

$$\sum_{k=-\infty}^\infty \sum_{j=0}^\infty a_{j,k} z_1^j z_2^k,$$

which converges uniformly on compact subsets of $\mathbb{P}$. Using again the biholomorphism $(w_1, w_2) \mapsto (\frac{w_1}{w_2}, w_2)$, we see that every function in $\mathscr{O}(\mathbb{H})$ has a Laurent expansion

$$\sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} a_{j,k} \left(\frac{w_1}{w_2}\right)^j w_2^k,$$

converging uniformly on compact subsets. However, if $j \geq 0, k \geq -1$, it is easily seen that each Laurent monomial $(\frac{w_1}{w_2})^j w_2^k$ is in $L^2(\mathbb{H})$ and these monomials are orthogonal. It easily follows from the convergence of the Laurent expansion that these monomials are complete in the Bergman space $L^2(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$, i.e., their span is dense in the Bergman space.

Now let $g \in W^1(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$. Since $\frac{\partial g}{\partial z_2} \in L^2(\mathbb{H})$ it follows that in the Laurent expansion of g, the coefficient of $\frac{1}{w_2}$ must be 0, since otherwise, the expansion of $\frac{\partial g}{\partial z_2}$ will have a term in $\frac{1}{w_2^2}$ which is not in $L^2(\mathbb{H})$. Since the Laurent monomials are orthogonal in $L^2(\mathbb{H})$, it follows that g is orthogonal to f (which is a Laurent monomial $\frac{1}{w_2}$), and our result is proved.

Remark For a bounded pseudoconvex domain Ω in $\mathbb{C}^n$ with smooth boundary, the space $\mathscr{C}^\infty(\overline{\Omega}) \cap \mathscr{O}(\Omega)$ is dense in $L^2(\Omega) \cap \mathscr{O}(\Omega)$. This follows from results due to Kohn (see [15]) on the regularity of the weighted $\bar{\partial}$-Neumann operator N_t, where the weight function $t|z|^2$ with large $t > 0$ is used (see [17, Theorem 8.1] for a detailed discussion.) Using the proof in [17] and the fact $W^1(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$ is not dense in the Bergman space $L^2(\mathbb{H}) \cap \mathscr{O}(\mathbb{H})$, we see that the weighted Bergman projection B_t on the Hartogs triangle is also not bounded from $\mathscr{C}_0^\infty(\mathbb{H})$ to $W^1(\mathbb{H})$. The weights $t|z|^2$ can be substituted by any functions smooth up to the boundary.

We also mention that using a result of Barrett (see [2]), the Bergman projection on each smooth Diederich–Fornaess worm domain Ω is not regular from W^s to W^s for some $s > 0$. But it is still an open question whether on each worm $B(C_0^\infty(\Omega))$ is not contained in $W^s(\Omega)$. Our example $\mathbb{H}$ is not smooth. On the other hand, such examples exist for pseudoconvex domains with smooth boundary in complex manifolds (see [14]).

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