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Topology of Dolbeault cohomology groups

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Dedicated to Duong H. Phong

Abstract. In this paper we give a systematic discussion of the Hausdorff property for Dolbeault cohomology groups on a domain in a complex manifold.

1. Introduction

A fundamental problem in function theory on a domain Ω in a complex manifold is the Cauchy-Riemann operator, or the $\overline{\partial}$-equation. The understanding of the existence and regularity of the solutions of the system of inhomogeneous Cauchy-Riemann equations on Ω plays central role in complex analysis. The obstruction to the solvability of the Cauchy-Riemann equations (in a given topology) is measured by the *Dolbeault Cohomology.* We consider cohomology with respect a *cohomological complex* $(E_*, \overline{\partial})$, i.e., we are given for each bidegree (p, q) a topological vector space $E_{p,q}$ of (p, q)-forms, such that the operator $\overline{\partial}_{p,q}$ mapping $E_{p,q}$ to $E_{p,q+1}$ is closed. Then the *Dolbeault Cohomology group* of degree (p, q) with respect to E_* is defined to be:

$$H^{p,q}(\Omega; E_*) = \frac{\ker(\overline{\partial}_{p,q} : E_{p,q} \to E_{p,q+1})}{\mathrm{Range}(\overline{\partial}_{p,q-1} : E_{p,q-1} \to E_{p,q})},$$

which is a topological vector space with the quotient topology.

Such a quotient is Hausdorff if and only if the range $\overline{\partial}_{p,q-1} : E_{p,q-1} \to E_{p,q}$ is closed. The property of $\overline{\partial}$ having closed range has important consequences for function theory on the manifold. First of all, it gives the solution to the $\overline{\partial}$-operator from functional analysis point of veiw. It also has other applications. For example, closed range in appropriate degrees implies certain kinds of duality between cohomology groups, the prototype of this kind of result being the classical Serre duality theorem (see [**Ser55**]). Furthermore, in the L^2 setting, such closed range implies the existence of a *canonical solution*, i.e., a solution with smallest L^2-norm. More generally, closed range of $\overline{\partial}$ and the corresponding Hausdorff property for the groups is equivalent to solving the $\overline{\partial}$-problem with estimates in a given topology.

Among the topologies on the spaces of forms are the C^∞ forms with its natural Fréchet topology. This is the classical Dolbeault cohomology groups. We can also consider the space of forms smooth up to the boundary (with its natural topology),

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the space of forms with L^2 or the space of forms with compact support and so on. A general form of duality may be formulated between these spaces (see Laurent-Shaw [**LTS13**] for a systematic discussion of these spaces and their duals.) The main purpose of this paper is to compare Dolbeault cohomology groups in different topological spaces.

On bounded pseudoconvex domains in $\mathbb{C}^n$, or more generally relatively compact pseudoconvex domains in Stein manifolds, the classical work of Kohn and Hörmander studies the vanishing of Dolbeault cohomology groups with L^2-methods. The following L^2 existence and regularity theorems for $\overline{\partial}$ on pseudoconvex domains in $\mathbb{C}^n$ (or a Stein manifold $\mathcal{X}$) are well known. We use $H^{p,q}(\Omega)$ or $H^{p,q}(\overline{\Omega})$ to denote the cohomology group of (p,q)-forms with $C^\infty(\Omega)$ coefficients or $C^\infty(\overline{\Omega})$ respectively. If $\mathcal{X}$ is hermitian, we use $H^{p,q}_{L^2}(\Omega)$ to denote the cohomology group of (p,q)-forms with L^2 coefficients.

THEOREM 1.1 (**Hörmander** [**Hör65**]). *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain. We have*

$$H^{p,q}_{L^2}(\Omega) = 0. \qquad q \geq 1.$$

Thus the range of $\overline{\partial}$ in $L^2_{(p,q)}(\Omega)$ is equal to the kernel of $\overline{\partial}$. Furthermore, if we assume that the boundary $b\Omega$ is smooth, the following global boundary regularity results also hold for $\overline{\partial}$.

THEOREM 1.2 (**Kohn** [**Koh63**, **Koh73**]). *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary $b\Omega$. We have*

$$H^{p,q}(\overline{\Omega}) = 0, \qquad q \geq 1.$$

Thus for a bounded smooth pseudoconvex domain in $\mathbb{C}^n$ (or in a Stein manifold with a hermitian metric), all the cohomology groups vanish and we have

$$H^{p,q}(\Omega) = H^{p,q}_{L^2}(\Omega) = H^{p,q}(\overline{\Omega}) = 0, \qquad q > 0.$$

In fact, it can be shown that for any bounded Lipschitz domain Ω which is not pseudoconvex in $\mathbb{C}^2$, the L^2 $\overline{\partial}$-operator does not have closed range from L^2 to $L^2_{(0,1)}(\Omega)$. Thus the Dolbeault cohomology $H^{0,1}_{L^2}(\Omega)$ is non-Hausdorff. This and several new examples of this phenomenon were recently obtained in [**LTS13**] (see also Theorem 3.2 in this paper). Moreover there are other properties of the Dolbeault groups which are related to the boundary smoothness and the function theory on a domain. In other words, even for domains in $\mathbb{C}^n$, we do not know the topology of many simple examples in various function spaces. Much less is known for domains in complex manifolds which are not Stein. However, examples have been known for a long time for pseudoconvex domains in which the Dolbeault cohomolgy in Fréchet or L^2 space is not Hausdorff, or equivalently, the $\overline{\partial}$-operator does not have closed range (see Serre [**Ser55**] or Malgrange [**Mal75**]). Recent results by Chakrabarti-Shaw [**CS13**] show that $\overline{\partial}$ does not have closed range in L^2 on a general pseudoconvex domain Ω in a complex manifold, even if the domain Ω is Stein and with smooth boundary. In other words, the L^2 Dolbeault cohomology for a Stein domain could be non-Hausdorff even though the classical Dolbeault cohomology group vanishes in the Fréchet topology. All these examples provide interesting leads for further exploration.

In this paper we survey some recent results on the Hausdorff property for Dolbealt cohomology groups using L^2 methods. We first formulate the maximal closure of $\overline{\partial}$ and the minimal L^2 closure $\overline{\partial}_c$, the closure for forms with compact support and discuss the dual relations between $\overline{\partial}$ and $\overline{\partial}_c$. From this formulation, a simple proof for the L^2 version of the Serre duality theorem will be obtained via the Hodge star operator directly. In Section 3 we discuss some recent results on Dolbeault coholomology groups with respect to various topological complexes, and obtain information regarding the non-closed range property on $\overline{\partial}$ in the L^2 setting in $\mathbb{C}^n$. In Section 4, we give an example of a pseudoconvex domain which is Stein, but whose $\overline{\partial}$ does not have closed range in L^2. In the last section, we analyze holomorphic functions on domains with Levi-flat boundaries in complex manifolds and use it to prove the non-Hausdorff property of some Dolbeault cohomology group.

2. The L^2 $\overline{\partial}$ problem in complex manifolds

Let $\mathcal{X}$ be a complex hermitian manifold of dimension $n \geq 2$ and let Ω be a relatively compact domain in $\mathcal{X}$. Let $L^2_{(p,q)}(\Omega)$ be the space of (p,q)-forms with L^2 coefficients in Ω. We recall the $\overline{\partial}$-Neumann problem and its dual the $\overline{\partial}$-Cauchy problem on Ω. Let $\overline{\partial}$ be the maximal closure of the $\overline{\partial}$ operator

$$\overline{\partial} : L^2_{(p,q-1)}(\Omega) \to L^2_{(p,q)}(\Omega)$$

Let $\overline{\partial}_c$ be the (strong minimal) closure of the $\overline{\partial}$ operator

$$\overline{\partial}_c : \mathcal{D}_{(p,q-1)}(\Omega) \to \mathcal{D}_{(p,q)}(\Omega)$$

where $\mathcal{D}$ is the set of compactly supported functions in Ω. By this we mean that $\overline{\partial}_c$ is the minimal closed extension of the operator such that $\mathrm{Dom}(\overline{\partial}_c)$ contains $\mathcal{D}_{(p,q-1)}$. The $\mathrm{Dom}(\overline{\partial}_c)$ contains elements $f \in L^2_{(p,q-1)}(\Omega)$ such that there exists sequence $f_\nu \in \mathcal{D}_{(p,q-1)}(\Omega)$ such that $f_\nu \to f$ in $L^2_{(p,q-1)}(\Omega)$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in $L^2_{(p,q)}(\Omega)$. The $\overline{\partial}_c$ operator is a closed operator. It is related to solving $\overline{\partial}$ with vanishing boundary data in the weak sense and we refer it as the $\overline{\partial}$-Cauchy problem.

The L^2 adjoint of $\overline{\partial}$ is denoted by $\overline{\partial}^*$, the Hilbert space adjoint. Let ϑ denote the weak maximal closure of the formal adjoint ϑ defined on $\mathrm{Dom}(\vartheta) = \{f \in L^2_{(p,q)}(\Omega) \mid \vartheta f \in L^2_{(p,q-1)}(\Omega)\}$. Similarly, we also define ϑ_c as the strong minimal closure of ϑ by approximation in the graph norm of compactly supported smooth forms only.

Recall that the complex Hodge star operator $\star = \bar{*}$ where $*$ is the Riemannian Hodge star opearator. Then for any $L^2_{(p,q)}$-form f and $L^2_{(n-p,n-q)}$-form g, we have

$$(\star f, g)|_\Omega = (-1)^{p+q} \overline{\int_\Omega g \wedge f} = \overline{\int_\Omega f \wedge g}.$$

The following lemma follows directly from the definition.

LEMMA 2.1. *The operator $\overline{\partial}^*$ is equal to the strong minimal closure ϑ_c. Similarly, the operators $\overline{\partial}_c$ and ϑ are Hilbert space adjoints to each other. i.e., $\overline{\partial}_c = \vartheta^*$. An L^2 form $f \in Dom(\overline{\partial}_c)$ if and only if $\star f \in Dom(\overline{\partial}^*)$ and $\overline{\partial}_c = \star \overline{\partial}^* \star$.*

LEMMA 2.2. *Let Ω be a relatively compact domain in $\mathcal{X}$. The following conditions are equivalent*

(1) $\overline{\partial} : L^2_{(p,q-1)}(\Omega) \to L^2_{(p,q)}(\Omega)$ *has closed range.*
(2) $\overline{\partial}_c : L^2_{(n-p,n-q)}(\Omega) \to L^2_{(n-p,n-q+1)}(\Omega)$ *has closed range.*

PROOF. From Lemma 2.1, we have that $f \in \mathrm{Dom}(\overline{\partial}_c)$ if and only if $\star f \in \mathrm{Dom}(\overline{\partial}^*)$. Suppose (1) holds. Then $\overline{\partial}^* : L^2_{(p,q)}(\Omega) \to L^2_{(p,q-1)}(\Omega)$ has closed range. Thus we have $\star\overline{\partial}^*\star = \overline{\partial}_c$ has closed range. Thus (1) implies (2). The other direction is proved similarly. $\square$

Let $\square = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ be the $\overline{\partial}$-Laplacian. Recall that the inverse of $\square$, denoted by the $\overline{\partial}$-Neumann operator $N_{(p,q)}$, exists if and only if $\overline{\partial} : L^2_{(p,q-1)}(\Omega) \to L^2_{(p,q)}(\Omega)$ and $\overline{\partial} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q+1)}(\Omega)$ have closed range (see [**CS01**]). Let $\mathcal{H}_{(p,q)}(\Omega)$ denote the space of harmonic (p,q)-forms, i.e.,

$$\mathcal{H}_{(p,q)}(\Omega) = \{h \in L^2_{(p,q)}(\Omega) \cap \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) \mid \overline{\partial}h = 0,\ \overline{\partial}^* h = 0\}.$$

We formulate the $\overline{\partial}_c$-Laplacian similarly by setting $\square_c = \overline{\partial}_c\overline{\partial}_c^* + \overline{\partial}_c^*\overline{\partial}_c$ as the closed operator defined on

$$\mathrm{Dom}(\square_c) = \{f \in L^2_{(p,q)}(\Omega) \mid f \in \mathrm{Dom}(\overline{\partial}_c) \cap \mathrm{Dom}(\overline{\partial}_c^*),\\ \overline{\partial}_c f \in \mathrm{Dom}(\overline{\partial}_c^*),\ \overline{\partial}_c^* f \in \mathrm{Dom}(\overline{\partial}_c).\}$$

It is a closed densely defined operator on $L^2_{(p,q)}(\Omega)$. We define the $\mathrm{Ker}(\square_c)$ to be the corresponding harmonic forms, denoted by $\mathcal{H}^c_{(p,q)}(\Omega)$. Thus

$$\mathcal{H}^c_{(p,q)}(\Omega) = \mathrm{Ker}(\square_c) = \{f \in \mathrm{Dom}(\overline{\partial}_c) \cap \mathrm{Dom}(d_c^*),\ \overline{\partial}_c f = 0,\ \overline{\partial}_c^* f = 0\}.$$

Following the same arguments as for the $\overline{\partial}$-Neumann problem, if the range of $\square_c$ is closed, then there exists an inverse operator, denoted by $N^c_{(p,q)}$ which is bounded on $L^2_{(p,q)}(\Omega)$ and vanishes on $\mathcal{H}^c_{(p,q)}(\Omega)$.

DEFINITION. *We define the L^2 cohomology group for (p,q)-forms by*

$$H^{p,q}_{L^2}(\Omega) = \frac{\{f \in L^2_{(p,q)}(\Omega) \mid \overline{\partial} f = 0 \ in\ X\}}{\{f \in L^2_{(p,q)}(\Omega) \mid f = \overline{\partial} u \ for\ some\ u \in L^2_{(p,q-1)}(\Omega)\}}.$$

We also define the L^2 cohomology group with compact support by

$$H^{p,q}_{c,L^2}(\Omega) = \frac{\{f \in L^2_{(p,q)}(\Omega) \mid f \in Dom(\overline{\partial}_c),\ \overline{\partial}_c f = 0 \ in\ \Omega\}}{\{f \in L^2_{(p,q)}(\Omega) \mid f = \overline{\partial}_c u \ in\ \Omega\ for\ some\ u \in L^2_{(p,q-1)}(\Omega) \cap Dom(\overline{\partial}_c)\}}.$$

Using the Hodge star operator, we see that $\square_c$ and $\square$ are naturally related.

THEOREM 2.3. *Let Ω be a relatively compact domain with Lipschitz boundary in a complex hermitian manifold $\mathcal{X}$. Then for each $0 \le p \le n$ and $1 \le q \le n$, $f \in Dom(\square_{(p,q)})$ if and only if $\star f \in Dom(\square_c) \cap L^2_{(n-p,n-q)}$. In particular, we have*

$$\star\square = \square_c \star \quad on\ Dom(\square).$$

The $\bar\partial$-Neumann operator $N_{(n-p,n-q)}$ exists on $L^2_{(p,q)}(\Omega)$ if and only if the $\overline{\partial}$-Cauchy operator $N^c_{(n-p,n-q)}$ exists for $L^2_{(n-p,n-q)}(\Omega)$ with

$$\star N^c_{(n-p,n-q)} = N_{(p,q)} \star .$$

This theorem follows easily from Lemma 2.1 and Lemma 2.2. It immediately gives the following L^2 version of the Serre duality theorem.

COROLLARY 2.4 (L^2 Serre Duality). *Let Ω be a relatively compact domain in a complex hermitian manifold $\mathcal{X}$. Suppose that the $\overline{\partial}$-Neumann operator $N_{(n-p,n-q)}$ exists for some $0 \le p \le n$ and $0 \le q \le n$. We have*

$$\mathcal{H}_{(p,q)}(\Omega) \simeq \mathcal{H}^c_{(n-p,n-q)}(\Omega)$$

and

$$H^{p,q}_{L^2}(\Omega) \simeq H^{n-p,n-q}_{c,L^2}(\Omega).$$

This corollary follows Theorem 2.3 and the Hodge theorem. If $N_{(n-p,n-q)}$ exists, then the Hodge theorem implies $H^{p,q}_{L^2}(\Omega) \simeq \mathcal{H}_{(p,q)}(\Omega)$ and $H^{n-p,n-q}_{c,L^2}(\Omega) \simeq \mathcal{H}^c_{(n-p,n-q)}(\Omega)$. For details of the proof, see [**CS11**].

Let $\overline{\partial}_s$ be the strong L^2 closure of $\overline{\partial}$. A form $f \in \mathrm{Dom}(\overline{\partial}_s) \cap L^2_{p,q}(D)$ if and only if there exists a sequence $f_\nu \in C^\infty_{p,q}(X)$ such that $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in $L^2(D)$ strongly. We denote the L^2 adjoint of the strong closure $\overline{\partial}_s$ by $\overline{\partial}_s^*$. It is well-known that $\mathrm{Dom}(\overline{\partial}_s) \subset \mathrm{Dom}(\overline{\partial})$. If the boundary is Lipschitz as a Lipschtiz graph locally, then the strong closure is equal to the weak closure $\overline{\partial}_s = \overline{\partial}$ (see [**CS01**]). Similarly, we define ϑ_s as the strong L^2 closure of ϑ.

LEMMA 2.5. *Let Ω be a relatively compact domain with Lipschitz boundary in $\mathcal{X}$. Then $f \in Dom(\overline{\partial}_c)$ if and only if $f \in L^2_{(p,q-1)}(\Omega)$ and $\overline{\partial} f^0 \in L^2(\mathcal{X})$ where f^0 is the extension of f to be zero outside Ω and $\overline{\partial} f^0 = (\overline{\partial}_c f)^0$. In particular, the strong minimal closure $\overline{\partial}_c$ is equal to the weak minimal closure ϑ_s^* and we have $\overline{\partial}_c = \vartheta_s^*$ and $\overline{\partial}_c^* = \vartheta_s$.*

From now on we will simply use $\overline{\partial}_c f$ instead of $\overline{\partial}_c f^0$ whenever the derivative is taking in the whole space. We will apply the L^2 Serre duality to study holomorphic extension of CR functions. This extends the results of the work by Kohn-Rossi [**KR65**] where extension of smooth functions are discussed. Let Ω be a bounded domain with Lipschtiz boundary $b\Omega$. Let f be a function defined on the boundary $b\Omega$. We recall the following definition for L^2 CR functions.

DEFINITION. *An function f in $L^2(b\Omega)$ is CR if f is annihilated by the tangential Cauchy-Riemann equations in the weak sense. By this we mean that*

$$\int_{b\Omega} f \wedge \overline{\partial}\phi = 0$$

for any smooth $(n, n-1)$-form ϕ smooth in a neighborhood of $b\Omega$.

THEOREM 2.6. *Let $\mathcal{X}$ be a complex hermitian manifold of dimension $n \ge 2$. Let Ω be a relatively compact domain in $\mathcal{X}$ with Lipschitz boundary. We assume that the $\bar{\partial}$-Neumann operator $N_{(n,n-1)}$ in Ω exists and assume that $\mathcal{H}_{(n,n-1)}(\Omega) = \{0\}$. For every CR function $f \in W^{\frac{1}{2}}(b\Omega)$, one can find a holomorphic function $F \in W^1(\Omega)$ such that $F = f$ on $b\Omega$.*

PROOF. By our assumption, the $\bar{\partial}$-Neumann operator $N_{(n,n-1)}$ in Ω exists and $\mathcal{H}_{(n,n-1)}(\Omega) = \{0\}$. From the L^2 Serre Duality, we have

$$\mathcal{H}^c_{(0,1)}(\Omega) = H^{0,1}_{c,L^2}(\Omega) = \{0\}. \tag{2.1}$$

For any CR function or form f with $W^{\frac{1}{2}}(b\Omega)$ coefficients, we extend f to be $\tilde{f} \in W^1(\Omega)$. This can be done since the boundary is Lipschitz. Let $g = \overline{\partial}\tilde{f}$. Then g is in $L^2_{(0,1)}(\Omega) \cap \mathrm{Dom}(\overline{\partial}_c)$ and $\overline{\partial}_c g = 0$. Thus from (2.1), there exists an u such that $\overline{\partial}_c u = g$. This implies that $F = \tilde{f} - g$ is holomorphic and $F = f$ on the boundary $b\Omega$. This proves the theorem. $\square$

COROLLARY 2.7. *Suppose that $b\Omega$ is smooth. Then there exists a small $\epsilon > 0$ such that for every CR function $f \in W^{\frac{1}{2}-\epsilon}(b\Omega)$, one can extend f holomorphically into Ω.*

If the boundary is smooth, the $\bar{\partial}$-Neumann operator $N_{(n-p,n-q)}$ is regular for some small $\epsilon > 0$. This is true for pseudoconvex domains in $\mathbb{C}^n$ (see [**BC00**] or [**CSW04**]). For domains in complex manifolds, this follows from the arguments in Kohn-Nirenberg [**KN65**]. Then we use the arguments in Theorem 2.6 to see that one can extend any CR functions in $W^{\frac{1}{2}-\epsilon}(b\Omega)$.

3. On the Hausdorff property for Dolbeault cohomology groups for domains in $\mathbb{C}^n$

Let Ω be a domain in a complex manifold $\mathcal{X}$. Suppose X is Stein. It is well-known that $H^{p,q}(\Omega) = 0$ for all $q \geq 1$.

THEOREM 3.1. *Let X be a complex hermitian manifold and let Ω be a relatively compact domain with Lipshitz boundary in X. Suppose that $H^{n,1}_c(X) = 0$ and $X \setminus \Omega$ is connected. Then either $H^{0,n-1}_{L^2}(\Omega) = 0$ or $H^{0,n-1}_{L^2}(\Omega)$ is not Hausdorff.*

PROOF. If $f \in L^2_{(n,1)}(\Omega)$ with $\overline{\partial}_c f = 0$, we may first assume that f is smooth. Using the assumption $H^{n,1}_c(X) = 0$, there exists u with compact support in X such that $\overline{\partial}_c u = 0$. This implies that u is analytic outside its support. Thus from analytic continuation and interior regularity for $\overline{\partial}$, we have $u = 0$ in $X \setminus \overline{\Omega}$ since $X \setminus \Omega$ is connected. We have that u has compact support in $\overline{\Omega}$. By the regularity for $\overline{\partial}$, we have $H^{n,1}_{c,L^2}(\Omega) = 0$ (for details for the proof, see [**LTS13**]). Thus $\overline{\partial}_c : L^2_{(n,0)}(\Omega) \to L^2_{(n,1)}(\Omega)$ has closed range and it is equal to the $\mathrm{Ker}(\overline{\partial}_c)$. From the L^2 duality, this gives that

$$\overline{\partial} : L^2_{(0,n-1)}(\Omega) \to L^2_{(0,n)}(\Omega)$$

has closed range. Suppose $\overline{\partial} : L^2_{(0,n-2)}(\Omega) \to L^2_{(0,n-1)}(\Omega)$ has closed range. The $\overline{\partial}$-Neumann operator $N_{(0,n-1)}(\Omega)$ exists and from the L^2 Serre duality (Corollary 2.4),

$$H^{n,1}_{c,L^2}(\Omega) = H^{0,n-1}_{L^2}(\Omega) = 0.$$

On the other hand, if $\overline{\partial} : L^2_{(0,n-2)}(\Omega) \to L^2_{(0,n-1)}(\Omega)$ does not have closed range, we have that $H^{0,n-1}_{L^2}(\Omega)$ is not Hausdorff. The theorem is proved. $\square$

The following theorem gives a simple criterion for closed range property for $\bar{\partial}$ for domains in $\mathbb{C}^2$.

THEOREM 3.2. *Let Ω be a bounded domain in $\mathbb{C}^2$ with Lipschitz boundary such that $\mathbb{C}^2 \setminus \Omega$ is connected. Then $\overline{\partial} : L^2(\Omega) \to L^2_{(0,1)}(\Omega)$ has closed range if and only if Ω is pseudoconvex. In particular, if Ω is not peudoconvex, then $H^{0,1}_{L^2}(\Omega)$ is non-Hausdorff. Suppose Ω is not pseudoconvex, then $H^{0,1}(\Omega)$ is non-Hausdorff.*

PROOF. It follows from Höremander that if Ω is pseudoconvex, then $H^{0,1}_{L^2}(\Omega) = 0$. This is also necessary for the domain with Lipschitz boundary (see Fu [**Fu05**]). On the other hand, if the range of $\overline{\partial} : L^2(\Omega) \to L^2_{(0,1)}(\Omega)$ is not closed, then $H^{0,1}_{L^2}(\Omega)$ is not zero, thus non-Hausdorff. The result is also true for Dolbeault cohomology in the Fréchet space from similar arguments (see [**LTS13**]).

□

Note that earlier known non-closed range example in $\mathbb{C}^n$ was given in Folland-Kohn's book [**FK72**]. Our results show that from L^2 duality, any bounded non-pseudoconvex domain with Lipschitz boundary will provide such an example. We also remark an earlier papers by Laufer [**Lau67, Lau75**] show that if Ω is not pseudoconvex, then $H^{0,1}(\Omega)$ is infinite dimensional. In fact, it was even proved by Siu [**Siu67**] that $H^{0,1}(\Omega)$ cannot be countably infinite dimensional. Our simple duality arguments actually gives the non-Hausdorff property.

In the case of Hausdorff property for $H^{0,1}(\overline{\Omega})$ for pseudoconvex domain Ω with non-smooth boundary, Kohn's result does not always hold in this case. We first recall a theorem by Dufresnoy [**Duf79**] states that for any bounded pseudoconvex domain in $\mathbb{C}^n$ with a "nice" Stein neighborhood basis, $H^{0,1}(\overline{\Omega}) = 0$. But this is not the case if the domain is non-smooth and does not have a Stein neighborhood basis. The prototype non-smooth domain with no Stein neighborhood basis is the Hartogs triangle. We will discuss some known results for $\overline{\partial}$ on Hartogs triangle sinc it provides some insights to the subtlety of the problem.

Consider the Hartogs triangle

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < |z_2| < 1\}.$$

The Hartogs triangle and it smooth cousins, the Diederich-Fornaess worm domains, provide many counter examples for function theory on pseudoconvex domains in $\mathbb{C}^n$. The Hartogs triangle is not smooth at (0,0), where it is not even Lipschitz (as a graph). The map

$$(z_1, z_2) \to (\frac{z_1}{z_2}, z_2)$$

maps the Hartogs domain bi-holomorphically to a product domain $D \times D^*$, where D^* is the punctured disc. In [**CS12**], we use the weighted space to study the $\overline{\partial}$-Neumann operator on the Hartogs triangle.

However, one does have *almost* smooth solutions to the problem in the following sense. Let $C^{k,\alpha}(\overline{\mathbb{H}})$ denote the Hölder space of restriction of functions in $\mathbb{C}^2$ whose k-th derivatives are C^α in $C^{k,\alpha}(\mathbb{C}^2)$ to $\mathbb{H}$. Let $H^{p,q}_{C^{k,\alpha}}(\mathbb{H})$ denote the Dolbeault cohomology of (p,q)-forms with $C^{k,\alpha}(\overline{\mathbb{H}})$ coefficients. Using the integral kernel method, Chomat-Chollet prove the following results.

THEOREM (see [**CC91**]). *For every $k \in \mathbb{N}$ and $0 < \alpha < 1$, $H^{0,1}_{C^{k,\alpha}}(\mathbb{H}) = 0$, but $H^{0,1}(\overline{\mathbb{H}})$ is infinite dimensional.*

Notice that the intersection $\cap_k C^{k,\alpha}(\overline{\mathbb{H}}) = C^\infty(\overline{\mathbb{H}})$. In other words, for each k, one can have a solution operator bounded in $C^{k,\alpha}(\overline{\mathbb{H}})$. But the solution operator is different for each k. This shows the delicate nature of such problem on non-Lipschitz domains. On the other hand, it is still an open question if Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary, one can conclude that $H^{p,q}(\overline{\Omega}) = 0$ for $q \geq 1$. The following related result is proved in Laurent-Shaw [**LTS13**].

THEOREM 3.3. *Let $\Omega = \Omega_1 \setminus \overline{\mathbb{H}}$ be the annulus between a pseudoconvex domain Ω_1 and $\mathbb{H}$ with $\mathbb{H} \subset\subset \Omega_1 \subset\subset \mathbb{C}^2$. Then the Dolbeault cohomology group $H^{0,1}(\Omega)$ is non-Hausdorff.*

We do not know if the corresponding L^2 cohomology is non-Hausdorff. If the domain is an annulus has C^2-smooth boundary in the interior boundary, the closed range property for $\overline{\partial}$ has been established earlier in [**Sha85**, **Sha10**].

THEOREM 3.4. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}^-$ between two pseudoconvex domains Ω_1 and Ω^- where $\Omega^- \subset\subset \Omega_1$ and Ω^- has C^2 boundary. Then the $\overline{\partial}$-Neumann operator $N_{(p,q)}$ exists on $L^2_{(p,q)}(\Omega)$ for $0 \leq p \leq n$ and $0 \leq q \leq n$. For any $f \in L^2_{(p,q)}(\Omega)$, we have*

$$
\begin{aligned}
f &= \overline{\partial}^*\overline{\partial} N_{(p,0)} f + H_{(p,0)} f, \qquad q = 0.\\
f &= \overline{\partial}\overline{\partial}^* N_{(p,q)} f + \overline{\partial}^*\overline{\partial} N_{(p,q)} f, \qquad 1 \leq q \leq n-2.\\
f &= \overline{\partial}\overline{\partial}^* N_{(p,n-1)} f + \overline{\partial}^*\overline{\partial} N_{(p,n-1)} f + H_{(p,n-1)} f, \qquad q = n-1.\\
f &= \overline{\partial}\overline{\partial}^* N_{(p,n)} f, \qquad q = n.
\end{aligned}
$$

We have used the notation $H_{(p,q)}$ to denote the projection operator from $L^2_{(p,q)}(\Omega)$ onto the harmonic space $\mathcal{H}_{(p,q)}(\Omega) = ker(\Box_{(p,q)})$.

For a proof of Theorem 3.4, see [**Sha85**] and [**Sha10**]. We can further establish the isomorphism between the spaces $H^{p,n-1}_{L^2}(\Omega)$ and the Bergman space $\mathcal{H}^{(n-p,0)}(\Omega^-)$ (see Theorem 3.1 in [**Sha11**]).

THEOREM 3.5. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}^-$ between two pseudoconvex domains Ω_1 and Ω^-, where $\Omega^- \subset\subset \Omega_1$, $n \geq 2$. We assume that the boundary of Ω^- is C^2 smooth. Then we have the isomorphism:*

$$H^{p,n-1}_{L^2}(\Omega) \simeq \mathcal{H}_{(n-p,0)}(\Omega^-). \tag{3.1}$$

Theorem 3.5 is inspired by the paper of Hörmader [**Hör04**], where he proves the isomorphism explicitly between two concentric balls. We do not know if we can relax the condition on the smoothness of Ω^-. Another interesting problem to ask is the following question.

THEOREM (Chinese Coin Problem). *Let $\Omega = B_2(0) \setminus \overline{D}^2$ where $B_2(0)$ is a ball of radius 2 centered at 0 and D^2 is the bidisc contained in B. Determine if $H^{0,1}_{L^2}(\Omega)$ is Hausdorff.*

The domain has the abstract shape of an ancient Chinese coin. We remak that the corresponding Dolbeault cohomology $H^{0,1}(\Omega)$ is Hausdorff (see Laurent-Leiterer [**LTL00**]). The Hausdorff property for this domain is related to the W^1 estimates for the bidisc, which is still unknown. From the work by Chakrabarti-Shaw [**CS11**] on product domains, regularity for $\tilde{W}^s$ is obtained where $\tilde{W}^s$ is the special Sobolev space of order $s \in \mathbb{N}$. We remark that we do have the following isomorphism (see Theorem 2.2 in [**Sha11**]):

$$H^{p,n-1}_{W^1}(\Omega) \simeq \mathcal{H}_{(n-p,0)}(\Omega^-). \tag{3.2}$$

The question is whether we have the following isomorphism:

$$H^{(p,n-1)}_{W^1}(\Omega) \simeq H^{(p,n-1)}(\Omega).$$

We are only beginning to understand these cohomology groups and their relations with each other.

4. Non-closed range property for $\overline{\partial}$ on pseudoconvex domains in complex manifolds

When $\mathcal{X}$ is a Hermitian complex manifold and $\Omega \subset\subset \mathcal{X}$ is a pseudoconvex domain with smooth boundary, the $\overline{\partial}$ problem is very different. We first note that if Ω is strongly pseudoconvex, Grauert proves that the $\overline{\partial}$ has closed range in the Frechét spaces. If the domain Ω is relatively compact strongly pseudoconvex, or more generally of finite type with smooth boundary, the closed range property for the $\overline{\partial}$ equation in the L^2 setting has been established by Kohn [**Koh73**] via the $\overline{\partial}$-Neumann problem. There have been numerous work in the study of the existence and regularity of $\overline{\partial}$, we refer the reader to the books by Chen-Shaw [**CS01**] and Boas-Straube [**BS91**], [**BS99**] and Straube [**Str10**] for references.

For general complex manifolds, function theory on pseudocnvex domains on general complex manifolds can be quite different from that of Stein manifolds. Grauert (see [**Gra58**]) first gives an example of a pseudoconvex domain Ω in a complex torus which is not holomorphically convex. He shows that the only holomorphic function on Ω are constants. The domain in the Grauert's example actually has Levi-flat boundary. The boundary splits the complex two torus into two symmetric parts. Based on the examples of Grauert, Malgrange proves the following theorem.

THEOREM (Malgrange [**Mal75**]). *There exists a pseudoconvex domain Ω with Levi-flat boundary in a complex torus of dimension two whose Dolbeault cohomology group $H^{p,1}(\Omega)$ for every $0 \le p \le 2$ is non-Hausdorff in the Fréchet topology.*

Malgrange shows that for some Grauert's example, the $\overline{\partial}$ equation does not necessarily have closed range in the Frechét space and the corresponding Dolbeault cohomology $H^{p,1}(\Omega)$ is non-Hausdorff. The domain Ω is not holomorphically convex. Recently Chakrabarti-Shaw show that there exists a domain D_∞ in a complex manifold with Levi-flat boundary such that D_∞ is Stein, but the $\overline{\partial}$ equation does not have closed range in $L^2_{2,1}(D_\infty)$. The domain D_∞ is defined as follows.

Let $\alpha > 1$ be a real number and let Γ be the subgroup of $\mathbb{C}^*$ generated by α We will standardize $\alpha = e^{2\pi}$. Let $T = \mathbb{C}^*/\Gamma$ be the torus.

Let $\mathcal{X} = \mathbb{C}P^1 \times T$ be equipped with the product metric ω from the Fubini-Study metric for $\mathbb{C}P^1$ and the flat metric for T. Let D_∞ be the domain in $\mathcal{X}$ defined by

$$D_\infty = \{(z,[w]) \in \mathbb{C}P^1 \times T \mid \Re zw > 0\} \tag{4.1}$$

where z is the inhomogeneous coordinate on $\mathbb{C}P^1$. The domain D_∞ was first introduced by Ohsawa [**Ohs82**] and used in Barrett [**Bar86**].

THEOREM (Ohsawa [**Ohs82**]). *The domain D_∞ is biholomorphic to a product domain of an annulus and puctured disc in $\mathbb{C}^2$. In particular, D_∞ is Stein.*

The domain D_∞ is Stein, we have

$$H^{p,q}(D_\infty) = 0, \qquad q > 0.$$

The following results have been obtained recently by Chakrabarti-Shaw.

THEOREM 4.1 (see [**ChS4**]). *The range of* $\overline{\partial} : L^2_{(2,0)}(D_\infty) \to L^2_{(2,1)}(D_\infty)$ *is not closed. In particular, the space* $H^{2,1}_{L^2}(D_\infty)$ *is non-Hausdorff.*

Theorem 4.1 shows that on a pseudconvex domain in a complex manifold X, there is no connection of the Dolbeault cohomology groups in the classical Fréchet spaces and the L^2 spaces. This is in sharp contrast with the case when the manifold X is Stein. We note that D_∞ is Stein, but the ambient space $\mathcal{X}$ is not Stein. The idea of the proof is to use the L^2 Serre duality and the extension of holomorphic functions. For details of the proof of the theorem, we refer the reader to [**CS13**].

Let $\overline{\partial}_c$ be the minimal closure of the $\overline{\partial}$ operator for compactly supported forms in D_∞. The following lemma was proved in [**ChS4**].

LEMMA 4.2. *Suppose that operator* $\overline{\partial}_c : L^2(D_\infty) \to L^2_{0,1}(D_\infty)$ *has closed range in* $L^2_{0,1}(D_\infty)$. *Then we have*

$$H^{0,1}_{c,L^2}(D_\infty) = \frac{Ker(\overline{\partial}_c) \cap L^2_{0,1}(D_\infty)}{Range(\overline{\partial}_c)} = \{0\}. \tag{4.2}$$

Suppose that $\overline{\partial} : L^2_{(2,0)}(D_\infty) \to L^2_{(2,1)}(D_\infty)$ has closed range. Then $\overline{\partial}_c : L^2(D_\infty) \to L^2_{0,1}(D_\infty)$ has closed range. From the Hodge theorem,

$$H^{2,1}_{L^2}(D_\infty) \simeq \mathcal{H}^{2,1}(D_\infty).$$

From the L^2 Serre duality, we have

$$\mathcal{H}^{0,1}_c(D_\infty) \simeq \mathcal{H}_{2,1}(D_\infty) \simeq H^{2,1}_{L^2}(D_\infty) = \{0\}. \tag{4.3}$$

Thus on D_∞, we have either $H^{2,1}_{L^2}(D_\infty) = \{0\}$ or $H^{2,1}_{L^2}(D_\infty)$ is not Hausdorff. The boundary bD_∞ has two tori, which divide the boundary into two disjoint parts. Thus there exists a CR function which is constant on each disjoint part. By analyzing the holomorphic extension of CR functions on bD_∞ (see Corollary 5.4 and Proposition 5.5), we conclude using Corollary 2.7 that $H^{2,1}_{L^2}(D_\infty)$ cannot be zero, hence non-Hausdorff. In the next section, we will analyze holomorphic functions on the domain D_∞.

Remark: It is still an open question if $\overline{\partial} : L^2_{p,0}(D_\infty) \to L^2_{(p,1)}(D_\infty)$ has closed range or not for $p = 0$ or $p = 1$.

5. Holomorphic functions on domains with Levi-flat boundary

The examples in [**Gra58**] [**Mal75**] or [**ChS4**] are all pseudoconvex domains with Levi-flat boundary. Such domains are both pseudoconvex and pseudoconcave. These domains provide interesting examples to show that function theory on pseudoconvex domains in a non-Stein manifold is very different from the Stein case. In this section, we will analyze holomorphic functions on domains with Levi-flat boundary.

Let Ω be a relatively compact domain in a complex manifold $\mathcal{X}$. We assume that the boundary is smooth and Levi-flat. This implies that locally it is foliated by complex submanifolds of dimension $n-1$. We can also define Levi-flat hypersurfaces for Lipschitz boundary (see [**CS07**]). Let $\mathcal{O}(\Omega)$ be the set of holomorphic functions on Ω.

LEMMA 5.1. *Let Ω be a relatively compact domain in a complex manifold $\mathcal{X}$ with Levi-fflat boundary. Suppose that $h \in \mathcal{O}(\Omega) \cap W^{1+\epsilon}(\Omega)$ for some $\epsilon > 0$. Then h is Hölder continuous on the boundary.*

PROOF. This is essentially a Sobolev embedding theorem for Levi-flat hypersurfaces. From the trace theorem, we have that the restriction of h to the boundary is in $W^{\frac{1}{2}+\epsilon}(b\Omega)$. Locally in a neighborhood U near a point p in the boundary, $b\Omega \cap U$ is foliated by complex submanifold Σ_t, where $0 < t < \delta$, i.e.,

$$b\Omega \cap U = \cup_t \Sigma_t. \tag{5.1}$$

Denote (z', t) as the local coordinates on $b\Omega \cap U$. If we restrict the function h to each leaf, it is in $W^{\epsilon}(\Sigma_t)$. Since the function h is CR on the boundary, it satisfies the Cauchy-Riemann equations on each leaf

$$\bar{\partial}_{z'} h(z', t) = 0, \qquad \text{on } \Sigma_t.$$

Using the regularity for the Cauchy-Riemann equations, we see that h is holomorphic on each leaf. This implies that h is bounded on $b\Omega \cap U$. It remains to see h is Hölder continuous. This is done by using the arguments similar to the proof of Lemma 5.2 and Lemma 5.3 in [**CSW04**]. We refer the reader to the paper for details. □

Let D_∞ be the domain in $\mathcal{X}$ defined as in (4.1). The domain D_∞ is biholomorphic to the product domain Ω in C^2 where

$$\Omega = \mathbb{C}^* \times A = \mathbb{C}^* \times \{w \in \mathbb{C} \mid e^{-\frac{\pi}{2}} < |w| < e^{\frac{\pi}{2}}\} \tag{5.2}$$

via the map $\Phi : \Omega \to D_\infty$ defined by

$$\Phi : (z, w) \to (z, [z^{-1} e^{iw}]). \tag{5.3}$$

THEOREM 5.2. *Let $\mathcal{O}(D_\infty)$ be the set of holomorphic functions on D_∞. Then*
(1) *$\mathcal{O}(D_\infty)$ separates points,.*
(2) *There exist non-constant bounded holomoprhic functions on D_∞.*
(3) *There exist no non-constant holomorphic function in D_∞ which is continuous up to the boundary bD_∞.*

PROOF. Since the domain D_∞ is biholomorphic to the product of two circular domains, using Fourier series expansion, any holomorphic functions on Ω admits a Laurent expansion. Let $(\tilde{z}, \tilde{w})$ be coordinates on Ω. Then any holomorphic function h on Ω can be expressed as

$$h(\tilde{z}, \tilde{w}) = \underset{m,n\in\mathbb{Z}}{\rightarrow} \sum h_{m,n} \tilde{z}^m (\tilde{w})^n.$$

From (5.3), we have that any holomorphic function $h \in \mathcal{O}(D_\infty)$ admits an expansion of the form

$$h(z, [w]) = \underset{m,n\in\mathbb{Z}}{\rightarrow} \sum h_{m,n} z^m (zw)^{in}. \tag{5.4}$$

The term $(zw)^{in}$ is well defined since $\Re zw > 0$ and $(zwe^{2\pi})^{in} = (zw)^{in}$. The term $(zw)^i = e^{i\log(zw)}$ is bounded by $e^{\frac{\pi}{2}}$. Thus all the functions $(zw)^{in}$ are bounded holomorphic functions on D_∞.

The boundary bD_∞ of D_∞ is Levi-flat since it is defined by the real part of a holomorphic function locally. This shows that locally the boundary is foliated by complex curves. The boundary bD_∞ consists of two torus T_0 (when $z = 0$) and T_∞ (when $z = \infty$). We parametrize the boundary $bD_\infty \setminus \{T_0, T_\infty\}$ by

$$(r, \theta, \tau) \in (0, \infty) \times S^1 \times S^1 \times \to (z, [w]) = (re^{i\theta}, ie^{\tau}(r^{-1}e^{-i\theta})]).$$

Let $\Sigma_t \subset bD_\infty$ be the complex curve defined by

$$[w] = [z^{-1}ie^{\tau}] = [\frac{it}{z}], \qquad e^{-\pi} < e^{\tau} = t < e^{\pi},\ z \in \mathbb{C}^*.$$

Then the boundary bD_∞ is the union of complex curves

$$bD_\infty = \cup_{t>0}\Sigma_t \cup T_0 \cup T_\infty.$$

If f is a continuous CR function on bD_∞, then f must be constant on T_0 and T_∞. Since each leave Σ_t is biholomorphic to the punctured disc $\mathbb{C}^*$, the continuous CR function f must be a constant on each leaf Σ_t for each $t > 0$. The closure of each Σ_t intersect T_0 and T_∞, we conclude that f must be constant on bD_∞. We remark that (3) has already been observed in [**Bar86**]. ☐

We next discuss the existence and nonexistence of holomorphic functions in L^2 and Sobolev spaces.

THEOREM 5.3. *On D_∞, we have*

(1) *$\mathcal{O}(D_\infty) \cap L^2(D_\infty)$ is infinite dimensional but does not separates points.*
(2) *$\mathcal{O}(D_\infty) \cap W^1(D_\infty) = \{\mathbb{C}\}$.*
(3) *For any $0 < \epsilon \le 1$, the set $\mathcal{O}(D_\infty) \cap W^{1-\epsilon}$ is infinite dimensional.*

PROOF. From the Fourier expansion (5.4) for any holomorphic function in D_∞, we see that the term $z^{-1} \notin L^2(D_\infty)$. We also can check that the function z is not in L^2 at ∞ for the same reason. Thus the term z^m is in $L^2(D_\infty)$ only when $m = 0$. This shows that the set $\mathcal{O}(D_\infty) \cap L^2(D_\infty)$ consists of functions of the form

$$h(z, [w]) = \underset{n\in\mathbb{Z}}{\rightarrow} \sum c_n (zw)^{in}. \tag{5.5}$$

This set is infinite dimensional and does not separate points.

Further inspection shows that the holomorphic $(zw)^{in}$ is not in $W^1(D_\infty)$ for each $n \neq 0$ since the function z^{-1} is not L^2. Any $W^1(D_\infty)$ holomorphic function is a convergent sequence of the form (5.5). We see that any $f \in \mathcal{O}(D_\infty) \cap W^1(D_\infty)$, each coefficient in (5.5) must be 0 except the constant term.

We will show that the function $(zw)^{in}$ is in $W^{1-\epsilon}(D_\infty)$. It is easy to see that the function is smooth up to the boundary except at the two tori $T_0 \cup T_\infty$. Let δ denote the distance function from a point in D_∞ to the boundary bD_∞. To see this, we will use the fact that $h \in W^{1-\epsilon}(D_\infty) \cap \mathcal{O}(\Omega)$, one can use the equivalent norm (see Jerison-Kenig [**JK95**] or Theorem C.2 in the Appendix of the book by Chen-Shaw [**CS01**]).

$$\|h\|_{W^{1-\epsilon}} = \|\delta^{2\epsilon}\nabla h\| + \|h\|.$$

Since

$$\delta(z, w) \le C|z|,$$

near $z = 0$, it is easy to see that the function $(zw)^{in}$ is in $W^{1-\epsilon}(D_\infty)$. ☐

COROLLARY 5.4. *There exists CR function $f \in W^{\frac{1}{2}-\epsilon}(bD_\infty) \cap L^\infty(bD_\infty)$, where $\epsilon > 0$ such that f does not extend holomorphically to D_∞.*

PROOF. Let $D_\infty^- = \mathcal{X} \setminus \overline{D}_\infty$. From Theorem 5.2, there exist bounded holomorphic functions $(zw)^{in}$ on D_∞^- in $W^{1-\epsilon}(D_\infty^-)$. The restriction of any such function gives a CR function in $W^{\frac{1}{2}-\epsilon}(bD_\infty) \cap L^\infty(bD_\infty)$ which does not extend holomorphically to D_∞. $\square$

In fact, there exist CR functions which do not extend to either side of the boundary bD_∞.

PROPOSITION 5.5. *There exists non-constant bounded CR function* $f \in W^{\frac{1}{2}-\epsilon}(bD_\infty) \cap L^\infty(bD_\infty)$ *for* $\epsilon < \frac{1}{2}$ *which does not extend holomorphically to either* D_∞ *or* D_∞^-.

PROOF. Notice that the two tori on the boundary divides the boundary into two disjoint parts. If we take the value one and zero on each component, one can see that the function is CR and in $W^{\frac{1}{2}-\epsilon}(bD_\infty)$ for any $\epsilon > 0$. The CR function does not extend since such extension must be a constant. $\square$

PROPOSITION 5.6. *There exists no non-constant bounded pluri-subharmonic exhaustion function on* $\overline{D_\infty}$.

PROOF. Suppose that there exists a bounded continuous plurisubharmonic function $\phi : \overline{D}_\infty \to (-L, 0]$, where $L > 0$. Then we parametrize D_∞ by $(z, [w]) \in C^* \times A$ as before. For each fixed w, we have that ϕ is continuous bounded subharmonic function in C^*, this implies that ϕ is constant in z.

On the other hand, $\phi = 0$ on the boundary and each $(z, [w])$ will pass through the boundary point T_0 or T_∞, this implies that $\phi = 0$ on $\overline{D}_\infty$.

$\square$

Remark 1 Recall that a domain is called hyperconvex if there exists a bounded continuous plurisubharmonic exhaustion function. The domain D_∞ is not hyperconvex. This is in sharp contrast with domains in $\mathbb{C}^n$ or $\mathbb{C}P^n$ where pseudoconvex domains with smooth boundary in $\mathbb{C}^n$ are hyperconvex (see Diederich-Fornaess [**DF77**] or Ohsawa-Sibony [**OS98**]).

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