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THE $\overline{\partial}$-EQUATION ON AN ANNULUS WITH MIXED BOUNDARY CONDITIONS

XIAOSHAN LI[1,a] AND MEI-CHI SHAW[2,b]

[1]Department of Mathematics, Wuhan University, Hubei, China.

[a]E-mail: xiaoshanli@whu.edu.cn

[2]Department of Mathematics, University of Notre Dame, Notre Dame, IN 46656, USA.

[b]E-mail: mei-chi.shaw.1@nd.edu

Abstract

In this paper we study the $\overline{\partial}$-equation with mixed boundary conditions on an annulus $\Omega = \Omega_1 \setminus \overline{\Omega}_2 \subset\subset \mathbb{C}^n$ between two pseudoconvex domains satisfying $\Omega_2 \subset\subset \Omega_1$. We prove L^2-existence theorems for $\overline{\partial}_{\text{mix}}$ for any $\overline{\partial}_{\text{mix}}$-closed (p,q)-form with $2 \le q \le n$. For the critical case when $q = 1$ on the annulus Ω, we show that the space of harmonic forms is infinite dimensional and $H^{(p,1)}_{\overline{\partial}_{\text{mix}},L^2}(\Omega)$ is isomorphic to the quotient of $H^{(p,0)}_{W^1}(\Omega_2)$ with coefficients in $W^1(\Omega_2)$ over the Bergman space $H^{(p,0)}_{L^2}(\Omega_1)$ on the pseudoconvex domain Ω_1. Boundary regularity for the corresponding operators is also obtained.

1. Introduction

Let Ω_1 and Ω_2 be two bounded pseudoconvex domains in $\mathbb{C}^n (n \ge 3)$ satisfying $\Omega_2 \subset\subset \Omega_1$. In this paper, we study the $\overline{\partial}$-equation on the annulus $\Omega = \Omega_1 \setminus \overline{\Omega}_2$ with mixed boundary conditions. This paper is inspired by a recent paper of Chakrabarti and Shaw [5], where an abstract version of the L^2-serre duality theorem which generalizes the duality of $\overline{\partial}$ and $\overline{\partial}_c$ has been studied.

The $\overline{\partial}$-equations are the basic tools to study the many geometric or analytic problems in several complex variables and complex geometry. In

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the Hilbert space setting, there is a natural maximal closure of the Cauchy-Riemann equation, denoted by $\overline{\partial}$ and the minimal closure, denoted by $\overline{\partial}_c$. There is a vast amount of work in the literature. Here, we only refer the reader to the books by Folland-Kohn [7], Hörmander [12], Demailly [6] and Chen-Shaw [4], as well as the many references therein.

The $\overline{\partial}$-Neumann problem on an annulus between two pseudoconvex domains in $\mathbb{C}^n$ has been studied by the second author in [16]. In the case $1 \le q \le n-2$, the space of harmonic forms is trivial. In [17], the critical case $q = n-1$ has been completely analyzed and it was proved that the harmonic space is infinite dimensional on the annulus. In particular, the author prove that the space of harmonic forms in the critical case is isomorphic to the Bergman space on Ω_2. When Ω is the domain between two concentric balls, this result was first obtained in Hörmander in [13]. When considering the holomorphic extension of $\overline{\partial}_b$-closed forms from the boundary to complex manifold, the $\overline{\partial}$-Cauchy problem has also been studied by Chakrabarti and Shaw in [5].

The $\overline{\partial}$-equation with mixed boundary conditions has also been studied in the literature before. In order to study various extension of CR structure, Catlin and Cho in [2], [3] studied the $\overline{\partial}$-equation over non-smooth domain with mixed boundary conditions. In a recent paper of Huang-Luk-Yau [10], solving $\overline{\partial}$-equation with mixed boundary conditions also plays an important role for the study of deformation problems for compact strongly pseudoconvex CR manifolds of dimension at least five. Other work related to the $\overline{\partial}$-equation with mixed boundary conditions can be found in the recent paper by [9]. These problems deal with the $\overline{\partial}$-equations with mixed boundary conditions for domains with singularities.

In this paper, we study the $\overline{\partial}$-equation on an annulus between two pseudoconvex domains with mixed boundary conditions. We consider an operator $\overline{\partial}_{\rm mix}$, which is a L^2 closed extension satisfying $\overline{\partial}_c \subseteq \overline{\partial}_{\rm mix} \subseteq \overline{\partial}$. An element $u \in L^2_{(p,q)}(\Omega)$ is in the domain of $\overline{\partial}_{\rm mix}$ if and only if there exists $v \in L^2_{(p,q+1)}(\Omega)$ and a sequence $\{u_n\}$ with u_n vanishing near $\partial\Omega_2$ such that $u_n \to u$ in $L^2_{(p,q)}(\Omega)$ and $\overline{\partial}u_n \to v$ in $L^2_{(p,q+1)}(\Omega)$. Our goal is to obtain L^2-existence theorems for the $\overline{\partial}_{\rm mix}$ when $2 \le q \le n$. In the critical case $q = 1$, we show that the harmonic space associated with the operator $\overline{\partial}_{\rm mix}$ is infinite dimensional. In particular, we prove that the harmonic space in the critical case is isomorphic to the quotient of the Bergman space $H^{(p,0)}_{W^1}(\Omega_2)$ on domain Ω_2 with W^1 coefficients over the Bergman space $H^{(p,0)}_{L^2}(\Omega_1)$.

The plan of this paper is as follows. In Section 2, we obtain the L^2-existence theorems for the $\overline{\partial}_{\text{mix}}$-equation on the annulus $\Omega=\Omega_1\setminus\overline{\Omega}_2$ between two pseudo-convex domains. Here we only require that the boundary of Ω_2 to be C^2-smooth, while Ω_1 is only assumed to be psuedo-convex and bounded. It is still unknown if the boundary smoothness of Ω_2 can be relaxed. In Section 3, we will prove the basic estimate of $\overline{\partial}_{\text{mix}}$ and obtain the regularity results of the weighted $\overline{\partial}_{\text{mix}}$-Neumann operator on the annulus when the boundary is smooth.

2. The $\overline{\partial}_{\text{mix}}$-equation on an Annulus, L^2 Theory

Let Ω_1, Ω_2 be two bounded pseudo-convex domains in $\mathbb{C}^n$ such that $\overline{\Omega}_2 \subset\subset \Omega_1$. We consider the annulus Ω between Ω_1 and Ω_2, i.e., $\Omega=\Omega_1\setminus\overline{\Omega}_2$. We consider an operator $\overline{\partial}_{\text{mix}}$ which is a closed realization of $\overline{\partial}$ and satisfies that $\overline{\partial}_c \subseteq \overline{\partial}_{\text{mix}} \subseteq \overline{\partial}$, where $\overline{\partial}_c$ and $\overline{\partial}$ are the minimal and maximal realization of the differential operator $\overline{\partial}$.

Definition 2.1. For $0\le p\le n$, $0\le q\le n$ and $u\in L^2_{(p,q)}(\Omega)$, $u\in \text{Dom}(\overline{\partial}_{\text{mix}})$ if and only if there exists $v\in L^2_{(p,q+1)}(\Omega)$ and a sequence $\{u_n\}\subset L^2_{(p,q)}(\Omega)$ which vanish near $\partial\Omega_2$ such that $u_n\to u$ in $L^2_{(p,q)}(\Omega)$ and $\overline{\partial}u_n\to v$ in $L^2_{(p,q+1)}(\Omega)$, then we say $u\in\text{Dom}(\overline{\partial}_{\text{mix}})$ and $\overline{\partial}_{\text{mix}}u=v$.

Let $\overline{\partial}^*_{\text{mix}}$ be the Hilbert-space adjoint of $\overline{\partial}_{\text{mix}}$.

Theorem 2.2. *Let $\Omega\subset\subset\mathbb{C}^n$ be the annulus domain $\Omega=\Omega_1\setminus\overline{\Omega}_2$ between two pseudo-convex domains $\Omega_2\subset\subset\Omega_1$, where Ω_2 has C^2 boundary. Then for any $0\le p\le n, 2\le q\le n$ or $q=0$, the space of L^2-cohomology $H^{(p,q)}_{\overline{\partial}_{\text{mix}},L^2}(\Omega)$ is zero where*

$$H^{(p,q)}_{\overline{\partial}_{\text{mix}},L^2}(\Omega)=\frac{\text{Ker}(\overline{\partial}_{\text{mix}})\cap L^2_{(p,q)}(\Omega)}{\text{Img}(\overline{\partial}_{\text{mix}})\cap L^2_{(p,q)}(\Omega)}. \tag{2.1}$$

Proof. We first assume that $1<q\le n$. Let $f\in\text{Ker}(\overline{\partial}_{\text{mix}})\cap L^2_{(p,q)}(\Omega)$. Extending f to be zero in Ω_2, denoted by f^0, we have that $f^0\in L^2_{(p,q)}(\Omega_1)$ and $\overline{\partial}f^0=0$ in Ω_1. This follows from the assumption that Ω_2 has C^2 boundary and the strong $\overline{\partial}_c$ and weak $\overline{\partial}_c$ are equal. Here we only need the

boundary Ω_2 to be Lipschitz. For a proof of such weak equal strong results, see e.g. Lemma 2.4 in [15].

Thus we have from the Hörmander's L^2 theory for bounded pseudoconvex domains, there exists a solution $v \in L^2_{(p,q-1)}(\Omega_1)$ such that $\overline{\partial} v = f^0$ in Ω_1. From the elliptic regularity in the interior for $\overline{\partial}$, we can assume that the form v is in $W^1_{(p,q-1)}(\overline{\Omega}_2)$.

The form v satisfies $\overline{\partial} v = 0$ on Ω_2. Since $q > 1$ and the boundary of Ω_2 is C^2-smooth, there exists a solution $w \in W^1_{(p,q-1)}(\Omega_2)$ such that

$$\overline{\partial} w = v$$

in Ω_2. This follows from a result of Kohn [14] for sufficiently smooth boundary and from Harrington [8] when the boundary is only C^2. Let $\tilde{w}$ be a W^1 extension of w to Ω_1. We set

$$u = v - \overline{\partial}\tilde{w}$$

in Ω_1. Then u is in $L^2_{(p,q-1)}(\Omega_1)$ with $\overline{\partial} u = f^0$ in Ω_1. But $u = 0$ on Ω_2. This implies that $u \in \mathrm{Dom}(\overline{\partial}_{\mathrm{mix}})$ and $\overline{\partial}_{\mathrm{mix}} u = f$. $\square$

We remark that there is no boundary regularity assumption on Ω_1.

Lemma 2.3. *Let Ω be the same as before. Then $H^{(p,1)}_{\overline{\partial}_{\mathrm{mix}},L^2}(\Omega) \neq 0$.*

Proof. First, we prove that $H^{(p,1)}_{\overline{\partial}_{\mathrm{mix}},L^2}(\Omega) \neq \{0\}$. There exists a holomorphic function $g \in \mathrm{Hol}(\Omega_1)$ such that $Z(g) \cap \overline{\Omega}_2 = \emptyset$ and $Z(g) \cap \Omega_1 \neq \emptyset$, where $Z(g)$ is the zero set of the holomorphic function g in Ω_1. Let

$$w = \frac{1}{g(z)} dz_1 \wedge \cdots \wedge dz_p, \qquad z \in \overline{\Omega}_2. \tag{2.2}$$

We smoothly extend $\frac{1}{g}$ to h in Ω_1. Then $\widetilde{w} = h dz_1 \wedge \cdots \wedge dz_p$ is a smooth $(p,0)$-form on Ω_1. Then $\overline{\partial}\tilde{w}$ satisfies $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$. If $H^{(p,1)}_{\overline{\partial}_{\mathrm{mix}},L^2}(\Omega) = 0$, then there exists $(p,0)$-form u on Ω which satisfies $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$ such that

$$\overline{\partial}_{\mathrm{mix}} u = \overline{\partial}\tilde{w}. \tag{2.3}$$

Let

$$u^0(z) = \begin{cases} u, & z \in \Omega \\ 0, & z \in \overline{\Omega}_2 \end{cases} \tag{2.4}$$

Since u satisfies $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$, then

$$\overline{\partial} u^0 = \overline{\partial} \tilde{w}. \tag{2.5}$$

Thus $\overline{\partial}(\tilde{w} - u^0) = 0$. Thus the coefficients of $\tilde{w} - u^0$ is a holomorphic function in Ω_1 which implies that $\frac{1}{g}$ can be extended to a holomorphic function in Ω_1. This is a contradiction. Thus, there exists $[h] \in H^{(p,1)}_{\overline{\partial}_{\mathrm{mix}}, L^2}(\Omega)$ which is not exact. Thus $H^{(p,1)}_{\overline{\partial}_{\mathrm{mix}}, L^2}(\Omega) \neq 0$. □

In fact, we will prove the following stronger results. Let $H^{(p,0)}_{W^1}(\Omega_2)$ denote the space of holomorphic $(p,0)$-forms in $W^1(\Omega_2)$ and let $H^{(p,0)}_{L^2}(\Omega_1)$ be the space of L^2 holomorphic forms on Ω_1.

Theorem 2.4. *Let Ω be the same as in Theorem 2.2. The space $H^{(p,1)}_{\overline{\partial}_{\mathrm{mix}}, L^2}(\Omega)$ is infinite dimensional. Furthermore, we have the following isomorphism:*

$$H^{(p,1)}_{\overline{\partial}_{\mathrm{mix}}, L^2}(\Omega) \cong H^{(p,0)}_{W^1}(\Omega_2) / H^{(p,0)}_{L^2}(\Omega_1). \tag{2.6}$$

Proof. For any $f \in H^{(p,0)}_{W^1}(\Omega_2)$, we extend f from Ω_2 to $\tilde{f}$ in Ω_1. We have that

$$\|\tilde{f}\|_{W^1(\Omega_1)} \leq \|f\|_{W^1(\Omega_2)}.$$

Then $\overline{\partial}\tilde{f} \in L^2_{(p,1)}(\Omega)$ and satisfies $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$. Then $\overline{\partial}_{\mathrm{mix}}\overline{\partial}\tilde{f} = 0$ on Ω. Now we define a map $l : H^{(p,0)}_{W^1}(\Omega_2) \to H^{(p,1)}_{\overline{\partial}_{\mathrm{mix}}, L^2}(\Omega)$ by

$$l(f) = [\overline{\partial}\tilde{f}], \qquad f \in H^{(p,0)}_{W^1}(\Omega_2). \tag{2.7}$$

First, we show that l is well defined. If $\tilde{f}_1$ is another extension of f, then

$$\overline{\partial}\tilde{f} - \overline{\partial}\tilde{f}_1 = \overline{\partial}(\tilde{f} - \tilde{f}_1).$$

Since $\tilde{f} = \tilde{f}_1 = f$ in $W^1(\Omega_2)$, $\tilde{f} - \tilde{f}_1$ satisfies $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$.

We have that

$$\overline{\partial}\tilde{f} - \overline{\partial}\tilde{f}_1 = \overline{\partial}_{\mathrm{mix}}(\tilde{f} - \tilde{f}_1)$$

on Ω. That is

$$[\overline{\partial}\tilde{f}] = [\overline{\partial}\tilde{f}_1]$$

in $H^{(p,1)}_{\overline{\partial}_{\mathrm{mix}},L^2}(\Omega)$. Thus the map l is well-defined.

We will show that the kernel of the map l is $H^{(p,0)}_{L^2}(\Omega_1)$. Let $f \in H^{(p,0)}_{W^1}(\Omega_2)$ such that $l(f) = [0]$. First we extend f from Ω_2 to $\tilde{f}$ in Ω_1. We assume $\tilde{f}$ vanishes near $\partial\Omega_1$. Since $\tilde{f} \in W^1(\Omega_1)$, we have that $\overline{\partial}\tilde{f}$ is a $\overline{\partial}_{\mathrm{mix}}$-closed form in Ω. Now, $\overline{\partial}\tilde{f}$ is $\overline{\partial}_{\mathrm{mix}}$-exact. Thus there exists a solution $g \in L^2_{(p,0)}(\Omega)$ such that

$$\overline{\partial}_{\mathrm{mix}} g = \overline{\partial}\tilde{f}. \tag{2.8}$$

Let

$$g^0 = \begin{cases} g, & \text{on } \Omega, \\ 0, & \text{on } \overline{\Omega}_2. \end{cases}$$

Since g satisfies $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$, $\overline{\partial}g^0 = \overline{\partial}\tilde{f}$. Let $F = \tilde{f} - g^0$. Then F is holomorphic in Ω_1 and $F = f$ on Ω_2. Thus $[\overline{\partial}\tilde{f}] = 0$ in Ω implies that f can be extended as a holomorphic $(p,0)$-form in Ω_1.

Next, we prove that l is surjective. Let $f \in L^2_{(p,1)}(\Omega) \cap \mathrm{Ker}(\overline{\partial}_{\mathrm{mix}})$. Then $\overline{\partial}_{\mathrm{mix}} f = 0$. Let

$$f^0 = \begin{cases} f, & \text{on } \Omega \\ 0, & \text{on } \Omega_2. \end{cases} \tag{2.9}$$

Since f satisfies $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$, we have that

$$\overline{\partial}f^0 = 0$$

in Ω_1. Since Ω_1 is a pseudoconvex domain, there exists a $(p,0)$-form $u \in L^2_{(p,0)}(\Omega_1)$ such that

$$\overline{\partial}u = f^0. \tag{2.10}$$

Then u is a holomorphic $(p,0)$-form in Ω_2. Choose a cut-off function $\eta \in C_0^\infty(\Omega_1)$ such that $\eta \equiv 1$ in a neighborhood of $\overline{\Omega}_2$. $\overline{\partial}(\eta u) \in L^2_{(p,1)}(\Omega_1)$ for

$\overline{\partial}u \in L^2_{(p,1)}(\Omega_1)$. Since ηf has compact support in Ω_1, Integrating by parts we have that $\partial(\eta u) \in L^2_{(p,1)}(\Omega_1)$. Thus

$$\eta u \in W^1_{(p,0)}(\Omega_1). \tag{2.11}$$

Since

$$\overline{\partial}u - \overline{\partial}(\eta u) = \overline{\partial}(u - \eta u) = \overline{\partial}_{\text{mix}}(u - \eta u),$$

we have that

$$[\overline{\partial}u] = [\overline{\partial}(\eta u)]$$

in $H^{(p,1)}_{\overline{\partial}_{\text{mix}},L^2}(\Omega)$. Hence, we have that

$$[f] = [\overline{\partial}(\eta u)]$$

in $H^{(p,1)}_{\overline{\partial}_{\text{mix}},L^2}(\Omega)$ with $\eta u \in H^{(p,0)}_{W^1}(\Omega_2)$. Thus we get the conclusion of Theorem 2.4. $\square$

3. Boundary Regularity for the Weighted $\overline{\partial}_{\text{mix}}$-Operator on an Annulus

Throughout this section, Ω will denote the annulus in $\mathbb{C}^n$ between Ω_1 and Ω_2 with C^3 boundary. Let $\rho \in C^3(\overline{\Omega})$ be the defining function of Ω. For every $p \in \partial\Omega$, there exists a small neighborhood U of p. Let $\omega_1, \cdots, \omega_n$ be an orthonormal basis of $(1,0)$-forms on U with $\omega_n = \sqrt{2}\partial\rho$. Let $L_1, \cdots, L_n$ be the dual frames of $\{\omega_i\}_{i=1}^n$ on U. Then U is called a special boundary chart. Any (p,q)-form Φ on $U \cap \overline{\Omega}$ can be expressed as follows:

$$\Phi = {\sum}'_{|I|=p,|J|=q} \Phi_{IJ}\omega^I \wedge \overline{\omega^J}. \tag{3.1}$$

We say Φ satisfies $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$ if $\Phi_{IJ}|_{\partial\Omega_2} = 0$ when $n \notin J$. We say Φ satisfies $\overline{\partial}$-Neumann condition on $\partial\Omega_1$ if $\Phi_{IJ}|_{\partial\Omega_1} = 0$ when $n \in J$. Let $\mathcal{B}^2_{(p,q)}(\Omega)$ denote the space of (p,q)-forms which are C^2-smooth in a neighborhood of $\overline{\Omega}$ and satisfies $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$ and $\overline{\partial}$-Neumann condition on $\partial\Omega_1$. Then by Friederichs' lemma, we have the following density lemma.

Lemma 3.1. *For every* $u \in \mathrm{Dom}(\overline{\partial}_{\mathrm{mix}}) \cap \mathrm{Dom}(\overline{\partial}^*_{\mathrm{mix}})$, *there exists* $\{u_n\}_{n=1}^{\infty} \subset \mathcal{B}^2_{(p,q)}(\Omega)$ *such that*

$$\|u_n - u\| + \|\overline{\partial}_{\mathrm{mix}} u_n - \overline{\partial}_{\mathrm{mix}} u\| + \|\overline{\partial}^*_{\mathrm{mix}} u_n - \overline{\partial}^*_{\mathrm{mix}} u\| \to 0$$

For the proof of Lemma 3.1, we refer the reader to Proposition 2 in [5] and Lemma 4.3.2 in [4].

Let $e^{-\varphi}$ denote the Hermitian metric of the trivial line bundle E over Ω. We denote by $L^2_{(p,q)}(\Omega, \varphi)$ of square integrable forms with respect to the weight function φ. Let Φ and Ψ be smooth (p,q)-forms on Ω. The pointwise inner product with respect to the weight function φ of the forms Φ and Ψ is given by $< \Phi, \Psi >_x e^{-\varphi}$ at each point x. We can extend the pointwise inner product to $C^{\infty}_{(*,*)}(\Omega)$. Let $\star_{\varphi}$ denote the Hodge-star operater which is defined as follows:

$$< \Phi, \Psi > e^{-\varphi} dV = \Phi \wedge \star_{\varphi} \Psi, \tag{3.2}$$

where dV is the volume form on Ω induced by the Euclidean metric on $\mathbb{C}^n$.

Lemma 3.2.

$$< \star_{\varphi}\Phi, \star_{\varphi}\Psi > e^{\varphi} dV =< \Psi, \Phi > e^{-\varphi} dV. \tag{3.3}$$

Proof. Since

$$< \Phi, \Psi > e^{-\varphi} dV = \Phi \wedge \star_{\varphi} \Psi =< \Phi, \Psi e^{-\varphi} > dV = \Phi \wedge \star(e^{-\varphi}\Psi),$$

we have

$$\star_{\varphi}(\Psi) = \star(e^{-\varphi}\Psi) = e^{-\varphi} \star \Psi,$$

where the operator $\star$ is the Hodge-star operator with respect to the Euclidean metric on $\mathbb{C}^n$.

$$\begin{aligned} < \star_{\varphi}\Phi, \star_{\varphi}\Psi > e^{\varphi} &=< e^{-\varphi} \star \Phi, e^{-\varphi} \star \Psi > e^{\varphi} dV \\ &= e^{-\varphi} < \star\Phi, \star\Psi > dV \qquad (3.4) \\ &= e^{-\varphi} < \Psi, \Phi > dV. \end{aligned}$$

$\square$

From Lemma 3.2, we have

$$\| \star_\varphi \Phi\|_{-\varphi}^2 = \|\Phi\|_\varphi^2.$$

Now let $\varphi_t = t(|z|^2 - \tau\rho^2)$ near $\partial\Omega_2$ and $\varphi_t = t|z|^2$ near $\partial\Omega_1$, where t and τ are positive constants which will be determined later. Let $L^2(\Omega, \varphi_t)$ denote the space of functions on Ω which are square integrable with respect to the weight function φ_t and the norm of $g \in L^2(\Omega, \varphi_t)$ is defined by $\|g\|_{\varphi_t}$. Similarly, we denote by $L^2_{(p,q)}(\Omega, \varphi_t)$ the space of (p,q)-forms with coefficients in $L^2(\Omega, \varphi_t)$. We denote by $\overline{\partial}^*_{\mathrm{mix,t}}$ the Hilbert-space adjoint of $\overline{\partial}_{\mathrm{mix}}$ in $L^2_{(p,q)}(\Omega, \varphi_t)$.

Theorem 3.3. *Let $0 \le p \le n, 2 \le q \le n-1$. There exists a constant $t_0 > 0$ and a compact subset $K \subset\subset \Omega$ such that for any $t \ge t_0$, $g \in \mathrm{Dom}(\overline{\partial}_{\mathrm{mix}}) \cap \mathrm{Dom}(\overline{\partial}^*_{\mathrm{mix}}) \cap L^2_{(p,q)}(\Omega, \varphi_t)$, we have*

$$\|g\|^2_{\varphi_t} \le \|\overline{\partial}_{\mathrm{mix}} g\|^2_{\varphi_t} + \|\overline{\partial}^*_{\mathrm{mix,t}} g\|^2_{\varphi_t} + \int_K |g|^2 e^{-\varphi_t} dV \tag{3.5}$$

Proof. By Lemma 3.1, we can assume $g \in \mathcal{B}^2_{(p,q)}(\Omega)$. Using a partition of unity we first assume g is supported in a small neighborhood U of $p \in \partial\Omega$. Since g satisfies $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$, then $\star_{\varphi_t} g$ satisfies $\overline{\partial}$-Neumann condition on $\partial\Omega_2$. Now $2 \le q \le n-1$, then $1 \le n-q \le n-2$. Thus by a similar argument of Proposition 3.1 in [16], we have

$$t\| \star_{\varphi_t} g\|^2_{-\varphi_t} \le \|\overline{\partial} \star_{\varphi_t} g\|^2_{-\varphi_t} + \|\overline{\partial}^*_{-\varphi_t} \star_{\varphi_t} g\|^2_{-\varphi_t} \tag{3.6}$$

when t is sufficiently large. Since

$$\begin{aligned} \| \star_{\varphi_t} g\|^2_{-\varphi_t} &= \|g\|^2_{\varphi_t}, \\ \|\overline{\partial} \star_{\varphi_t} g\|^2_{-\varphi_t} &= \| \star_{\varphi_t} \overline{\partial} \star_{\varphi_t} g\|^2_{\varphi_t} = \|\overline{\partial}'_{\varphi_t} g\|^2_{\varphi_t} = \|\overline{\partial}^*_{\mathrm{mix,t}} g\|^2_{\varphi_t}, \\ \|\overline{\partial}^*_{-\varphi_t} \star_{\varphi_t} g\|^2_{-\varphi_t} &= \| \star_{\varphi_t} \overline{\partial}^*_{-\varphi_t} \star_{\varphi_t} g\|^2_{\varphi_t} = \|\overline{\partial} g\|^2_{\varphi_t} = \|\overline{\partial}_{\mathrm{mix}} g\|^2_{\varphi_t}, \end{aligned} \tag{3.7}$$

it follows that

$$t\|g\|^2_{\varphi_t} \le \|\overline{\partial}_{\mathrm{mix}} g\|^2_{\varphi_t} + \|\overline{\partial}^*_{\mathrm{mix,t}} g\|^2_{\varphi_t}. \tag{3.8}$$

When g is supported in a neighborhood U of $p \in \partial\Omega_1$, by the standard argument of Hörmander, we have

$$t\|g\|^2_{\varphi_t} \le \|\overline{\partial}_{\mathrm{mix}} g\|^2_{\varphi_t} + \|\overline{\partial}^*_{\mathrm{mix,t}} g\|^2_{\varphi_t}. \tag{3.9}$$

Then using a partition unity and when t is sufficiently large, we get the basic estimate as in (3.5). □

We denote by

$$\square_{\mathrm{mix,t}} = \overline{\partial}_{\mathrm{mix}} \overline{\partial}^*_{\mathrm{mix,t}} + \overline{\partial}^*_{\mathrm{mix,t}} \overline{\partial}_{\mathrm{mix}} \tag{3.10}$$

the complex Laplace operator associated with the operator $\overline{\partial}_{\mathrm{mix}}$. A form $f \in \mathrm{Dom}(\square_{\mathrm{mix,t}})$ if and only if $f \in \mathrm{Dom}(\overline{\partial}_{\mathrm{mix}}) \cap \mathrm{Dom}(\overline{\partial}^*_{\mathrm{mix,t}})$, $\overline{\partial}_{\mathrm{mix}} f \in \mathrm{Dom}(\overline{\partial}^*_{\mathrm{mix,t}})$ and $\overline{\partial}^*_{\mathrm{mix,t}} f \in \mathrm{Dom}(\overline{\partial}_{\mathrm{mix}})$. From the basic estimate in (3.5), we have

Corollary 3.4. *We assume $0 \le p \le n, 2 \le q \le n-1$. When t is sufficiently large we have that there exists a bounded operator $N_{\mathrm{mix,t}} : L^2_{(p,q)}(\Omega, \varphi_t) \to \mathrm{Dom}(\square_{\mathrm{mix,t}})$ such that*

$$\square_{\mathrm{mix,t}} N_{\mathrm{mix,t}} = I.$$

Proof. From the basic estimate in (3.5), we have

$$\dim H^{(p,q)}_{\overline{\partial}_{\mathrm{mix}}, L^2}(\Omega, \varphi_t) < \infty. \tag{3.11}$$

In particular, $\mathrm{Range}(\overline{\partial}_{\mathrm{mix}})$ is closed in $L^2_{(p,q)}(\Omega, \varphi_t)$. This follows the same arguments as in [16]. But if we further use another parameter τ such that τ is sufficiently large (a technique first communicated to the second author by Zampieri (see also [1]), we can obtain that

$$\dim H^{(p,q)}_{\overline{\partial}_{\mathrm{mix}}, L^2}(\Omega, \varphi_t) = 0. \tag{3.12}$$

But this is already known from our results in Section 2 since the L^2 space is defined independent of the parameter t. In any way, we can concluded that

$$\dim H^{(p,q)}_{\overline{\partial}_{\mathrm{mix}}, L^2}(\Omega, \varphi_t) = \dim H^{(p,q)}_{\overline{\partial}_{\mathrm{mix}}, L^2}(\Omega) = 0. \tag{3.13}$$

The corollary follows from the the basic estimate. □

Let $\mathcal{H}^{(p,q)}_{\mathrm{mix,t}}$ denote the Harmonic space associated with the complex Laplace operator $\Box_{\mathrm{mix,t}}$. Then from the basic estimate in (3.5) and (3.12), we have that for any $0 \le p \le n, 2 \le q \le n-1$, the harmonic space

$$\mathcal{H}^{(p,q)}_{\mathrm{mix,t}} = 0.$$

Corollary 3.5. *Let $0 \le p \le n, 2 \le q \le n-1$. There exists a constant $c_1 > 0$ such that for any $f \in L^2_{(p,q)}(\Omega, \varphi_t) \cap \mathrm{Dom}(\overline{\partial}_{\mathrm{mix}}) \cap \mathrm{Dom}(\overline{\partial}^*_{\mathrm{mix,t}})$, we have*

$$\|f\|^2_{\varphi_t} \le c_1 \left(\|\overline{\partial}_{\mathrm{mix}} f\|^2_{\varphi_t} + \|\overline{\partial}^*_{\mathrm{mix,t}} f\|^2_{\varphi_t} \right). \tag{3.14}$$

Corollary 3.6. *We assume that $0 \le p \le n, 2 \le q \le n-1$ and the boundary of Ω is smooth. Let $N_{\mathrm{mix,t}}$ be the weighted $\overline{\partial}_{\mathrm{mix}}$-Neumann operator. For every $k \ge 0$, there exists S_k such that when $t \ge S_k$ we have that $N_{\mathrm{mix,t}}, \overline{\partial}_{\mathrm{mix}} N_{\mathrm{mix,t}}, \overline{\partial}^*_{\mathrm{mix,t}} N_{\mathrm{mix,t}}, \overline{\partial}^*_{\mathrm{mix,t}} N_{\mathrm{mix,t}} \overline{\partial}_{\mathrm{mix}}$ are exactly regular on $W^k_{(p,q)}(\Omega)$.*

Proof. When $f \in C^\infty_{(p,q)}(\overline{\Omega}) \cap \mathrm{Dom}(\Box_{\mathrm{mix,t}})$ and $\mathrm{supp} f \subset\subset U \cap \overline{\Omega}$, where U is a special boundary chart, then from the estimate in (3.8) and (3.9), we have that

$$t\|f\|^2_{\varphi_t} \le \|\Box_{\mathrm{mix,t}} f\|^2_{\varphi_t}. \tag{3.15}$$

When $f \in C^\infty_{(p,q)}(\Omega)$ with $\mathrm{supp} f$ a compact subset in Ω, we have the following Gårding's inequality

$$\|f\|^2_1 \le \|\overline{\partial}_{\mathrm{mix}} f\|^2_{\varphi_t} + \|\overline{\partial}^*_{\mathrm{mix,t}} f\|^2_{\varphi_t} + C_t \|f\|^2_{\varphi_t}, \tag{3.16}$$

where $\|f\|^2_k = \sum_{0 \le |\alpha| \le k} \|D^\alpha f\|^2_{\varphi_t}$. Combining (3.15) and (3.16) and with a similar argument as in Kohn [14] or Theorem 6.1.4 in [4], we get the conclusion of Corollary 3.6. $\square$

Corollary 3.7. *Suppose $\overline{\partial}_{\mathrm{mix}} f = 0, f \in L^2_{(p,q)}(\Omega)$, where $0 \le p \le n, 2 \le q \le n-1$.Then for each $k > 0$, there exists $f_n \in W^k_{(p,q)}(\Omega)$ with f_n satisfying $\overline{\partial}$-Dirichlet condition on $\partial\Omega_2$ such that $f_n \to f$ in $L^2_{(p,q)}(\Omega)$ and $\overline{\partial}_{\mathrm{mix}} f_n = 0$.*

Corollary 3.8. *Suppose $\overline{\partial}_{\mathrm{mix}} f = 0, f \in C^\infty_{(p,q)}(\overline{\Omega})$, where $0 \le p \le n, 2 \le q \le n-1$. Then there exists $u \in C^\infty_{(p,q-1)}(\overline{\Omega}) \cap \mathrm{Dom}(\overline{\partial}_{\mathrm{mix}})$ with $\overline{\partial} u = f$ in Ω.*

The proof of these corollaries follows from the regularity results for $N_{\text{mix,t}}$.

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