EXTENDABILITY AND THE $\overline{\partial}$ OPERATOR ON THE HARTOGS TRIANGLE

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Abstract. In this paper it is shown that the Hartogs triangle $\mathbf{T}$ in $\mathbf{C}^2$ is a uniform domain. This implies that the Hartogs triangle is a Sobolev extension domain. Furthermore, the weak and strong maximal extensions of the Cauchy-Riemann operator agree on the Hartogs triangle. These results have numerous applications. Among other things, they are used to study the Dolbeault cohomology groups with Sobolev coefficients on the complement of $\mathbf{T}$.

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1. Introduction

The Hartogs triangle $\mathbf{T} = \{(z, w) \in \mathbf{C}^2 \mid |z| < |w| < 1\}$ is an important example in several complex variables. It is biholomorphic to the product of the unit disc with the punctured disc, hence a pseudoconvex domain and also a domain of holomorphy. However, it admits neither a Stein neighborhood basis nor a bounded plurisubharmonic exhaustion function. The Hartogs triangle plays an important role in our understanding of function theory for pseudoconvex domains (see the survey paper [35]). There has been considerable interest in the Bergman projection (see [5, 35]) and the $\overline{\partial}$ problem on the Hartogs triangle (see e.g., [5, 7, 8, 10, 13, 26, 28, 36]), but many fundamental questions remain to be answered for this important model domain.

The Hartogs triangle is not a Lipschitz domain since it is not the graph of any function near $(0, 0)$. This presents a substantial obstacle to the study of function theory on $\mathbf{T}$. In this paper we show that $\mathbf{T}$ enjoys a number of properties generally associated with Lipschitz domains (see [37, 11]). Our first result is that the Hartogs triangle is an extension domain for Sobolev spaces (Theorem 2.12). Consequently, smooth functions on $\mathbf{C}^2$ are dense in the Sobolev spaces on $\mathbf{T}$, and the Sobolev embedding theorems hold.

Our main result concerns the Cauchy-Riemann operator on the Hartogs triangle. A fundamental tool for solving the $\overline{\partial}$-problem for forms on any pseudoconvex domain is Hörmander's L^2 existence theory, where the $\overline{\partial}$ operator is defined in the weak maximal sense. Specifically, for every weakly closed form f on $\mathbf{T}$ with coefficients in L^2, the equation $\overline{\partial} u = f$ admits a weak solution in L^2. The L^2-theory does not

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easily yield information about the regularity of u, even when f is smooth up to the boundary of the domain.

There is another closed extension of the Cauchy-Riemann operator, known as the strong maximal extension $\overline{\partial}_s$ (see Definition 3.2) which is the closure of forms smooth up to the boundary in the L^2 graph norm. The strong maximal extension was used by Kohn [23, 24] and Morrey [31] in their approach to the $\overline{\partial}$-Neumann problem. If the domain is smooth and strongly pseudoconvex, and f is $\overline{\partial}$-closed and smooth up to the boundary, the Morrey-Kohn approach yields solutions that are smooth up to the boundary. The strong extension $\overline{\partial}_s$ and its dual operator have many applications. For bounded domains with Lipschitz boundary, the equality of $\overline{\partial}$ and $\overline{\partial}_s$ was proved by Hörmander from the Friedrichs lemma (see [17, Chapter 1]).

In this paper we prove that the weak and strong maximal extensions agree on the Hartogs triangle. This is a first step towards understanding the regularity of solutions of the $\overline{\partial}$-problem on $\mathbf{T}$. Since the Hartogs triangle is not a Lipschitz domain, the classical Friedrichs lemma does not apply, and the relationship between the weak and strong extensions of $\overline{\partial}$ is more subtle. Our results are based on the L^2 Serre duality that relates the $\overline{\partial}$-Neumann problem to the $\overline{\partial}$-Cauchy problem.

The plan of the paper is as follows: In Section 2, we study the Sobolev space $W^1(\mathbf{T})$, consisting of L^2 functions with weak derivatives in L^2. In the past few decades, there has been tremendous progress in harmonic analysis on domains which are not Lipschitz, yet share some of their properties (see [20] or the more recent paper [1]). We show that the Hartogs triangle is such a domain. Specifically we prove (Theorem 2.4) that the Hartogs triangle is a uniform domain in the sense of [14, 29]. By a result of Jones [21], any uniform domain is a Sobolev extension domain. It follows that smooth functions are dense in the Sobolev spaces $W^k(\mathbf{T})$ (Corollary 2.14), which in turn yields the Rellich compactness lemma and the Poincaré inequality on $W^1(\mathbf{T})$. Furthermore, we show that the Hartogs triangle has Ahlfors-David regular boundary (Lemma 2.9), and the trace theorem holds (Corollary 2.15). These properties are used in our study of the $\overline{\partial}$ operator in later sections.

Section 3 is dedicated to the proof that $\overline{\partial} = \overline{\partial}_s$ on $\mathbf{T}$ (Theorem 3.13). We take advantage of a number of recent results on the properties of $\overline{\partial}_s$ on $\mathbf{T}$. Following [26], we use L^2 Serre duality to relate $\overline{\partial}$ and $\overline{\partial}_s$ to two other closures of $\overline{\partial}$, namely the strong minimal closure $\overline{\partial}_c$, and the weak minimal closure $\overline{\partial}_{\tilde{c}}$ (see Definition 3.3). It was proved in [26] that for a rectifiable domain, the dual of $\overline{\partial}_s$ is $\overline{\partial}_{\tilde{c}}$. Solving $\overline{\partial}_{\tilde{c}}$ amounts to solving $\overline{\partial}$ with prescribed support. This is known as the $\overline{\partial}$-Cauchy problem and has numerous applications (see [34, 9]).

The results from Section 2 are used to analyze the operators $\overline{\partial}_s$ and $\overline{\partial}_{\tilde{c}}$ on functions. It turns out that the kernel of $\overline{\partial}_s$ equals the kernel of $\overline{\partial}$, given by the Bergman space of square integrable harmonic functions on $\mathbf{T}$ (Proposition 3.6). In the proof, we explicitly estimate the terms in the Laurent expansion on the Bergman space from [5, 35]. Furthermore, $\overline{\partial}_c$ and $\overline{\partial}_{\tilde{c}}$ agree on functions (Proposition 3.7).

For $(0,1)$-forms, it is difficult to prove directly that the kernel of $\overline{\partial}$ and kernel $\overline{\partial}_s$ are the same since no simple Laurent expansion is available. Instead, we use duality to show that $\overline{\partial}_s$ has closed range (Theorem 3.11) and the relevant cohomology group is trivial. Using duality once more, we conclude that $\overline{\partial}_s = \overline{\partial}$ (see Theorem 3.13).

In Section 4, we study solutions of $\overline{\partial}$ on an annular domain between a pseudoconvex domain and the Hartogs triangle $\mathbf{T}$. Using the Sobolev extension theorem, we prove that the W^1 Dolbeault cohomology on the annulus is isomorphic to the Bergman space on $\mathbf{T}$ (see Theorem 4.3). This is in contrast with the non-Hausdorff property for the classical Dolbeault cohomology group for (0,1)-forms on the annulus between a pseudoconvex and the Hartogs triangle obtained earlier (see [38] or [26]).

There remain many open problems related to the Hartogs triangle which are yet to be explored. In Section 5 we present a number of problems. It is still not known if the L^2 Dolbeault cohomology on the annulus is Hausdorff (Problem 1). Using the Sobolev extension theorem, this problem is equivalent to asking if one can solve $\overline{\partial}$ in the Sobolev space $W^1(\mathbf{T})$. One can ask more generally if one can solve $\overline{\partial}$ in any Sobolev space $W^s(\mathbf{T})$ (Problem 2). The Hodge theorem for the de Rham complex d on $\mathbf{T}$ is also unknown for forms (Problem 3). Finally, one would also like to understand the spectrum of the $\overline{\partial}$-Laplacian (Problem 4). We also refer the reader to the many open problems for the Hartogs triangle in $\mathbf{CP}^2$ in [28]. The results in this paper are only the beginning of understanding function theory on non-smooth domains.

2. **Sobolev spaces on T**

In this section, we establish some basic facts about the Sobolev spaces $W^{1,p}$ on the Hartogs triangle

$$\mathbf{T} := \left\{ (z,w) \in \mathbf{C}^2 \;\middle|\; |z| < |w| < 1 \right\} . \tag{2.1}$$

We show that these spaces have many useful properties, including bounded extension and trace operators, smooth approximation, Sobolev embeddings, and the Poincaré inequality. The key to these properties are two geometric regularity results: $\mathbf{T}$ is a uniform domain (Theorem 2.4), whose boundary is Ahlfors-David regular (Lemma 2.9).

2.1. **Uniform domain.**

Definition 2.1. Let Ω be a domain in $\mathbf{R}^n$. The domain Ω is called an (ϵ, δ) *domain* if for every $p_1,\ p_2 \in \Omega$ and $|p_1 - p_2| < \delta$, there exists a rectifiable curve $\gamma \in \Omega$ joining x and y such that

$$\ell(\gamma) \le \frac{1}{\varepsilon}|p_1 - p_2|$$

and

$$\mathrm{dist}(p, b\Omega) \ge \frac{\varepsilon |p-p_1||p-p_2|}{|p_1 - p_2|} \quad \text{for all } p \in \gamma.$$

where $\ell(\gamma)$ denotes the Euclidean length of γ and dist $(p, b\Omega)$ denotes the distance from p to $b\Omega$.

When $\delta = \infty$, Ω is called a *uniform domain.*

Uniform domains were first introduced in [29] and [14], while the notion of a (ε, δ) domain was introduced in [21]. It turns out that for bounded domains, they are equivalent (see [40]).

We will prove by direct computation that the Hartogs triangle $\mathbf{T}$ is a uniform domain. Following [21], it suffices to show that there exists a constant $c > 0$ such that every pair of points $p_1 \neq p_2 \in \mathbf{T}$ can be joined by a rectifiable curve γ in $\mathbf{T}$ with

$$\ell(\gamma) \leq c|p_1 - p_2|\,, \tag{2.2}$$

such that

$$\min\{|p-p_1|, |p-p_2|\} \leq c\,\mathrm{dist}\,(p, b\mathbf{T}) \quad \text{for all } p \in \gamma\,. \tag{2.3}$$

We begin with an elementary inequality.

Lemma 2.2 (Distance in $\mathbf{C}^2$)**.** *Let $p_1, p_2 \in \mathbf{C}^2$ be given by $p_j = (r_j e^{i\alpha_j}, s_j e^{i\beta_j})$, where $r_j, s_j \geq 0$ and $\alpha_j, \beta_j \in \mathbf{R}$, j=1,2. If $|\alpha_1 - \alpha_2| \leq \pi$ and $|\beta_1 - \beta_2| \leq \pi$, then*

$$|r_1 - r_2| + |s_1 - s_2| + \min\{r_1, r_2\}|\alpha_1 - \alpha_2| + \min\{s_1, s_2\}|\beta_1 - \beta_2| \ \leq\ 3|p_1 - p_2|\,.$$

Proof. We compare the right hand side of the inequality to the squared distance

$$|p_1 - p_2|^2 = |r_1 - r_2|^2 + |s_1 - s_2|^2 + r_1 r_2 |e^{i\alpha_1} - e^{i\alpha_2}|^2 + s_1 s_2 |e^{i\beta_1} - e^{i\beta_2}|^2\,.$$

Since $|\alpha_1 - \alpha_2| \leq \pi$, their difference is comparable to the distance of the corresponding unit vectors in $\mathbf{C}$,

$$|\alpha_1 - \alpha_2| \ \leq\ \tfrac{\pi}{2}|e^{i\alpha_1} - e^{i\alpha_2}|\,,$$

and correspondingly for $|\beta_1 - \beta_2|$. By Schwarz' inequality,

$$\begin{aligned}
&\Big(|r_1 - r_2| + |s_1 - s_2| + \min\{r_1, r_2\}|\alpha_1 - \alpha_2| + \min\{s_1, s_2\}|\beta_1 - \beta_2|\Big)^2 \\
&\qquad \leq \Big(|r_1 - r_2| + |s_1 - s_2| + \tfrac{\pi}{2}\sqrt{r_1 r_2}|e^{i\alpha_1} - e^{i\alpha_2}| + \tfrac{\pi}{2}\sqrt{s_1 s_2}|e^{i\beta_1} - e^{-\beta_2}|\Big)^2 \\
&\qquad \leq \Big(2 + \tfrac{\pi^2}{2}\Big)\, |p_1 - p_2|^2\,.
\end{aligned}$$

Since $\sqrt{2 + \frac{\pi^2}{2}} < 3$, this proves the claim. $\square$

In order to understand the role of the singularity of $\mathbf{T}$ the origin, we consider the infinite Hartogs triangle

$$\mathbf{T}_\infty := \Big\{(z, w) \in \mathbf{C}^2 \ \Big|\ |z| < |w|\Big\}\,. \tag{2.4}$$

Lemma 2.3. $\mathbf{T}_\infty$ *is a uniform domain.*

Proof. We will join any given pair of points $p_1 \neq p_2 \in \mathbf{T}_\infty$ by a curve γ in $\mathbf{T}_\infty$ that satisfies Eqs. (2.2) and (2.3) with $c = 5 + 2\pi < 12$. The curve consists of an arc γ_0 that maintains a constant distance from the boundary, and a pair of line segments γ_1 and γ_2 attached at the ends.

For $j=1,2$, write the points in polar coordinates as $p_j=\big(r_je^{i\alpha_j},s_je^{i\beta_j}\big)$ with $0\le r_j<s_j$, $|\alpha_1-\alpha_2|\le\pi$, and $|\beta_1-\beta_2|\le\pi$. Choose γ_0 as the arc parametrized by $p=\big(r_*e^{i\alpha},s^*e^{i\beta}\big)$, where

$$r_*:=\min\{r_1,r_2\}\,,\qquad s^*:=\max\{s_1,s_2\}+|p_1-p_2|\,,\tag{2.5}$$

and the angles vary linearly from α_1,β_1 to α_2,β_2. Its endpoints

$$q_j:=\big(r_*e^{i\alpha_j},s^*e^{i\beta_j}\big)\,,\quad j=1,2\tag{2.6}$$

are joined to the corresponding points p_j by line segments γ_j.
Length of the curve. Since $s^*\le\min\{s_1,s_2\}+2|p_1-p_2|$, we obtain for the arc

$$\ell(\gamma_0)\ \le\ \min\{r_1,r_2\}|\alpha_1-\alpha_2|+\min\{s_1,s_2\}|\beta_1-\beta_2|+2\pi|p_1-p_2|\,.$$

The initial and final segments satisfy

$$\begin{aligned}\ell(\gamma_1)&\le\ (r_1-r_2)_++(s_1-s_2)_-+|p_1-p_2|\,,\\ \ell(\gamma_2)&\le\ (r_1-r_2)_-+(s_1-s_2)_++|p_1-p_2|\,,\end{aligned}$$

where x_+ and x_- denote, respectively, the positive and negative parts of a number $x\in\mathbf{R}$. Adding the three inequalities yields by Lemma 2.2 that $\ell(\gamma)\le\big(5+2\pi\big)|p_1-p_2|$.
Distance from the boundary. Let $p=(re^{i\alpha},se^{i\beta})\in\mathbf{T}_\infty$. Since the ball of radius $\frac{1}{\sqrt2}|r-s|$ about p is contained in $\mathbf{T}_\infty$, and its boundary meets $b\mathbf{T}_\infty$ in the single point $\frac{r+s}{2}(e^{i\alpha},e^{i\beta})$,

$$\operatorname{dist}(p,b\mathbf{T}_\infty)=\Big|p-\tfrac{r+s}{2}(e^{i\alpha},e^{i\beta})\Big|=\tfrac{1}{\sqrt2}|s-r|\,.$$

On the central arc, the distance from the boundary is constant,

$$\operatorname{dist}(p,b\mathbf{T}_\infty)\ =\ \tfrac{1}{\sqrt2}(s^*-r_*)\ \ge\ \tfrac{1}{\sqrt2}|p_1-p_2|\,,\quad p\in\gamma_0\,.$$

Since $\min\{|p-p_1|,|p-p_2|\}\le\frac12\ell(\gamma)$, it follows that

$$\frac{\min\{|p-p_1|,|p-p_2|\}}{\operatorname{dist}(p,b\mathbf{T}_\infty)}\ \le\ \frac{5+2\pi}{\sqrt2}\,,\quad p\in\gamma_0\,.$$

This yields Eq. (2.3) on γ_0. On the segments γ_j, the distance from the boundary increases linearly (in the arclength parametrization) from p_j to q_j, and therefore

$$\frac{|p-p_j|}{\operatorname{dist}(p,b\mathbf{T}_\infty)}\ \le\ \frac{|q_j-p_j|}{\operatorname{dist}(q_j,b\mathbf{T}_\infty)}\,,\qquad p\in\gamma_j,\ j=1,2\,.$$

Since q_1,q_2 lie on the arc γ_0, this completes the proof of Eq. (2.3). □

Theorem 2.4. *The Hartogs triangle* $\mathbf{T}$ *is a uniform domain.*

Proof. Given two points $p_1,p_2\in\mathbf{T}$, define the radii r_*, s^* by Eq. (2.5), and let $\gamma'=\gamma_0'\cup\gamma_1'\cup\gamma_2'$ be the curve constructed in the proof of Lemma 2.3. We rescale the arc to

$$\gamma_0:=\frac{1}{1{+}2|p_1-p_2|}\gamma_0'\,,$$

and then join its endpoints q_j to p_j by line segments γ_j, for $j=1,2$. Since $r_*<s^*<1{+}|p_1-p_2|$, the curve $\gamma:=\gamma_0\cup\gamma_1\cup\gamma_2$ lies in $\mathbf{T}$. We will show that γ satisfies Eqs. (2.2) and (2.3) with $c=(1+4\sqrt2)(5+2\pi+4\sqrt2)/\sqrt2<80$.

Length of the curve. By construction, $\ell(\gamma_0) \le \ell(\gamma_0')$. For $j=1,2$, we write the endpoints of γ_0 as $q_j = (1+2|p_1-p_2|)^{-1}q_j'$, where q_j' is given by Eq. (2.6). Since

$$|q_j' - q_j| \le 2|q_j|\,|p_1-p_2| \quad \text{and} \quad |q_j| \le \sqrt{2}\,,$$

by the triangle inequality the lengths of the line segments satisfy

$$\ell(\gamma_j) \le |p_j - q_j'| + |q_j' - q_j| \le \ell(\gamma') + 2\sqrt{2}|p_1-p_2|\,, \quad j=1,2\,.$$

Adding these inequalities yields, by Lemma 2.3,

$$\ell(\gamma) \le \ell(\gamma') + 4\sqrt{2}|p_1-p_2| \le (5+2\pi+4\sqrt{2})\,|p_1-p_2|\,.$$

Distance from the boundary. For $p=(re^{i\alpha}, se^{i\beta}) \in \mathbf{T}$, we have

$$\operatorname{dist}(p, b\mathbf{T}) = \min\left\{\tfrac{1}{\sqrt{2}}(s-r), 1-s\right\}, \quad p \in \mathbf{T}\,.$$

On the central arc γ_0, this distance constant,

$$\begin{aligned}\operatorname{dist}(p, b\mathbf{T}) &= \min\left\{\frac{1}{\sqrt{2}}\left(\frac{s^*-r_*}{1+2|p_1-p_2|}\right), 1-\frac{s^*}{1+2|p_1-p_2|}\right\}\\ &\ge \frac{|p_1-p_2|}{\sqrt{2}(1+4\sqrt{2})}\,, \qquad p \in \gamma_0\,.\end{aligned}$$

In the second step we have used that $|p_1-p_2| \le \operatorname{diam}(\mathbf{T}) = 2\sqrt{2}$ to bound the denominators from above, and that $s^* < 1+|p_1-p_2|$ to bound the second fraction from below.

Since $\min\{|p-p_1|, |p-p_2|\} \le \frac{1}{2}\ell(\gamma)$ for every $p \in \gamma$, it follows that

$$\frac{\min\{|p-p_1|, |p-p_2|\}}{\operatorname{dist}(p, b\mathbf{T})} \le \frac{(1+4\sqrt{2})(5+2\pi+4\sqrt{2})}{\sqrt{2}}\,, \quad p \in \gamma_0\,.$$

This yields Eq. (2.3) on γ_0. On γ_j, we argue as in the proof of Lemma 2.3 that the distances from the parts of the boundary at $\{z=w\}$ and $|w|=1$ change linearly along the segments to obtain

$$\frac{|p-p_j|}{\operatorname{dist}(p, b\mathbf{T})} \le \frac{|q_j-p_j|}{\operatorname{dist}(q_j, b\mathbf{T})}\,, \qquad p \in \gamma_j,\ j=1,2\,.$$

Since $q_1, q_2 \in \gamma_0$, this completes the proof. □

For later use, we briefly discuss domains in the complement of $\mathbf{T}$.

Lemma 2.5. *Let $\Omega \subset \mathbf{C}^2$ be a bounded Lipschitz domain with $\Omega \supset \overline{\mathbf{T}}$. Then $\Omega \setminus \overline{\mathbf{T}}$ is a uniform domain.*

Proof. It suffices to prove that there exist constants $\delta > 0$ and $c > 0$ such that any pair of points $p_1, p_2 \in \Omega \setminus \mathbf{T}$ with $|p_1-p_2| < \delta$ can be joined by a curve γ in $\Omega \setminus \mathbf{T}$ that satisfies Eqs. (2.2) and (2.3).

For $\rho > 0$, denote by $D_\rho^2 := \{(z,w) \in \mathbf{C}^2 : |z| < \rho, |w| < \rho\}$ the bidisk of radius ρ. Using that Ω contains the origin, we choose $\rho \in (0,1)$ so small that $D_\rho^2 \subset \Omega$, and write $\Omega = O_1 \cup O_2$, where

$$O_1 := D_\rho^2\,, \quad O_2 := \Omega \setminus \overline{D_{\rho/2}^2}\,.$$

Then $O_1 \setminus \overline{\mathbf{T}} = \{(z,w) \in \mathbf{C}^2 \bigm| |w| < |z| < \rho\}$ is a scaled copy of $\mathbf{T}$, and $O_2 \setminus \overline{\mathbf{T}}$ has Lipschitz boundary since $bD^2_{\rho/2}$ intersects $b\mathbf{T}$ transversally. Thus both are uniform domains. For $\delta > 0$ sufficiently small, any pair of points in $\Omega \setminus \overline{T}$ with $|p_1 - p_2| < \delta$ is contained in one of the sets Ω_j, in such a way that the distance of both points from $b\mathbf{T}$ is comparable to the distance from $\Omega \setminus \overline{\mathbf{T}}$. Therefore, a connecting curve satisfying Eq. (2.3) can be constructed in either $O_1 \setminus \overline{\mathbf{T}}$ or $O_2 \setminus \overline{\mathbf{T}}$. □

2.2. **Boundary regularity.**

Definition 2.6. Let σ denote the $(d-1)$-dimensional Hausdorff measure on $\mathbf{R}^d$. A closed subset $S \subset \mathbf{R}^d$ is *Ahlfors-David regular*, if there exists a constant $c > 0$ such that

$$c^{-1}\rho^{d-1} \ \le\ \sigma\big(B_\rho(p) \cap S)\big) \ \le\ c\rho^{d-1} \tag{2.7}$$

for all $p \in S$ and all $\rho < \operatorname{diam} S$. Here, $B_\rho(p)$ denotes the ball of radius ρ about p.

We first consider the unbounded Hartogs triangle from Eq. (2.4). Clearly, $b\mathbf{T}_\infty$ is rectifiable with respect to the three-dimensional Hausdorff measure, σ. In particular, the restriction of σ to $b\mathbf{T}_\infty$ agrees with the natural surface measure.

Lemma 2.7. $\sigma\big(B_\rho(p) \cap b\mathbf{T}_\infty)\big)$ *depends continuously on* $\rho \ge 0$ *and* $p \in b\mathbf{T}_\infty$.

Proof. Consider first the dependence on the radius. For every fixed $p \in b\mathbf{T}_\infty$, the function $g(\rho) := \sigma\big(B_\rho(p) \cap b\mathbf{T}_\infty)\big) = 0$ is non-decreasing in ρ. Hence g has at most countably many discontinuities, given by jumps. Since g is left continuous, the size of the jump at ρ equals

$$g(\rho_+) - g(\rho) = \sigma\big(bB_\rho(p) \cap b\mathbf{T}_\infty\big)\,.$$

For $p \in b\mathbf{T}_\infty$ and $\rho \ne |p|$, the sphere $bB_\rho(p)$ intersects $b\mathbf{T}_\infty$ transversally. Indeed, if a sphere were to touch $b\mathbf{T}_\infty$ at a point $q = (z,w) \ne 0$, then its center would lie on the line through q normal to the surface. This normal line has the form $\{(z,w) + s(z,-w) : s \in \mathbf{R}\}$, which does not intersect $b\mathbf{T}_\infty$ again. By the Implicit Function Theorem, the transversal intersection $bB_\rho(p) \cap b\mathbf{T}_\infty$ is a submanifold of dimension 2. On the other hand, for $\rho = |p| > 0$ the intersection $bB_\rho(p) \cap b\mathbf{T}_\infty$ contains the singular point at the origin; away from the origin the intersection is again transversal. In any case, the three-dimensional Hausdorff measure of the intersection vanishes, establishing continuity in ρ.

We turn to the dependence on p. For $q \in b\mathbf{T}_\infty$ with $|p - q| < \delta$, we have that

$$B_{\rho-\delta}(p) \ \subset\ B_\rho(q) \ \subset\ B_{\rho+\delta}(p)\,,$$

and hence, by the positivity of σ,

$$\sigma(B_{\rho-\delta}(p) \cap b\mathbf{T}_\infty) \ \le\ \sigma(B_\rho(q) \cap b\mathbf{T}_\infty) \ \le\ \sigma(B_{\rho+\delta}(p) \cap b\mathbf{T}_\infty)\,.$$

Continuity in p thus follows from continuity in ρ. □

Lemma 2.8. $\mathbf{T}_\infty$ *has Ahlfors-David regular boundary.*

Proof. We parametrize the boundary $b\mathbf{T}_\infty$ by

$$p(r,\alpha,\beta) := \tfrac{1}{\sqrt{2}}(re^{i\alpha}, re^{i\beta})\,, \qquad r\ge 0\,, \alpha,\beta\in(-\pi,\pi]\,.$$

The Jacobian of the parametrization equals $\frac12 r^2$.

For $t\ge 0$, let

$$f(t) := \ \sigma\big(B_1(p(t,0,0))\cap b\mathbf{T}_\infty\big)$$

be the surface measure of the intersection of the boundary with a ball of unit radius centered at $\frac{1}{\sqrt{2}}(t,t)\in b\mathbf{T}_\infty$. By the rotation and dilation invariance of $\mathbf{T}_\infty$,

$$\sigma\big(B_\rho(p)\cap b\mathbf{T}_\infty)\big) = \rho^3 f\left(\rho^{-1}|p|\right)\,, \qquad \rho>0\,, p\in b\mathbf{T}_\infty\,.$$

In the chosen parametrization,

$$f(t) \ = \ \frac12\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^\infty \mathbf{1}_{B_1(p(t,0,0))}\big(p(r,\alpha,\beta)\big)\, r^2\, dr d\alpha d\beta\,.$$

The intersection is described by the inequality

$$p(r,\alpha,\beta)\in B_1(p(t,0,0)) \ \Leftrightarrow\ (r-t)^2 + 2rt(\sin^2\tfrac{\alpha}{2} + \sin^2\tfrac{\beta}{2}) < 1\,.$$

For $t=0$, the condition simplifies to $r^2<1$, and we find that

$$f(0) = \frac12\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1 r^2\, dr d\alpha d\beta = \frac{2\pi^2}{3}\,.$$

For larger t, the condition is $\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2} < \frac{1-(r-t)^2}{2tr}$, and hence $\alpha,\beta = O(t^{-1})$ as $t\to\infty$. Taylor expansion of the sine yields

$$\alpha^2+\beta^2 \le \frac{2(1-(r-t)^2)}{tr} + O(t^{-4})\,.$$

For any $r>0$ with $|r-t|<1$, this inequality defines an approximate disk in the α-β variables. After integrating out these variables, we are left with

$$f(t) \ = \ \pi\int_{t-1}^{t+1}\frac{1-(r-t)^2}{tr} r^2\, dr + O(t^{-2}) \quad \to \quad \frac{4\pi}{3} \qquad (t\to\infty)\,.$$

In the last step, we have evaluated the integral explicitly. Since f is continuous by Lemma 2.7, as well as strictly positive, it follows that

$$\inf_{t\ge 0} f(t) > 0\,, \qquad \sup_{t\ge 0} f(t) < \infty\,,$$

proving the claim. □

Lemma 2.9. *The Hartogs triangle* $\mathbf{T}$ *has Ahlfors-David regular boundary.*

Proof. We need to verify Eq. (2.7) for all $p\in b\mathbf{T}$ and all $\rho\in(0,2]$. By rotational symmetry, we may assume that $p=(r,s)$ with $0\le r\le s\le 1$.

We partition the boundary as

$$b\mathbf{T} \ = \ \big(b\mathbf{T}_\infty\cap\{|w|\le 1\}\big) \ \cup \ \big(\mathbf{T}_\infty\cap\{|w|=1\}\big)\,.$$

For the upper bound on the first term, if $B_\rho(p)$ meets $b\mathbf{T}_\infty$ in a point $q\in b\mathbf{T}_\infty$ with $|p-q|<\rho$, then $B_\rho(p)\cap b\mathbf{T}_\infty$ is contained in $B_{2\rho}(q)\cap b\mathbf{T}_\infty$. The measure of this ball satisfies the desired bound by Lemma 2.8. Similarly, if $q\in B_\rho(p)\cap\{|w|=1\}$, then we use that $B_\rho(p) \cap \{|w| = 1\} \subset B_{2\rho}(q) \cap \{|w| = 1\}$, which is contained in the product of a disk of radius 2ρ with a spherical arc of diameter at most 2ρ. The surface measure of this intersection is at most comparable to ρ^3.

For the lower bound, we distinguish two cases. If $p = (r, r)$ with $r \leq 1$, then

$$B_\rho(p) \cap b\mathbf{T} \supset \left\{(z,w) \in b\mathbf{T}_\infty \;\middle|\; r - \tfrac{\rho}{\sqrt{2}} < |z| = |w| < r\right\}.$$

The right hand side is homogeneous of order three, as well as continuous and strictly positive. As in the proof of Lemma 2.8, this implies a lower bound of order ρ^3. If, on the other hand, $p = (r, 1)$ with $r < 1$ then

$$\begin{aligned} B_\rho(p) \cap b\mathbf{T} &\supset \left\{(z,w) \in \mathbf{T}_\infty \;\middle|\; |w| = 1, |z-r|^2 + |w-1|^2 < \rho^2\right\} \\ &\supset \left\{|z| < 1, |z-r| < \tfrac{\rho}{\sqrt{2}}\right\} \times \left\{|w| = 1, |w-1| < \tfrac{\rho}{\sqrt{2}}\right\}. \end{aligned}$$

Its surface measure is bounded from below by a constant multiple of ρ^3. $\square$

Our results can be summarized as follows.

Corollary 2.10. *The Hartogs triangle* $\mathbf{T}$ *is a chord-arc domain.*

Proof. Both $\mathbf{T}$ and $\mathbf{C}^2 \setminus \overline{\mathbf{T}}$ are uniform domains, and $b\mathbf{T}$ is Ahlfors-David regular. $\square$

2.3. **Sobolev theorems.** Let Ω be a domain in $\mathbf{R}^d$. For $k \in \mathbf{N}$ and $1 \leq p \leq \infty$, denote by $W^{k,p}(\Omega)$ the Sobolev space of functions whose weak derivatives of order up to k lie in L^p. When $p = 2$, we also use $W^k(\Omega)$ to denote $W^{k,2}(\Omega)$.

Definition 2.11. A domain $\Omega \subset \mathbf{R}^d$ is called a *(Sobolev) extension domain*, if for each $k \in \mathbf{N}$ and $1 \leq p \leq \infty$, there exists a bounded linear operator

$$\eta_k : W^{k,p}(\Omega) \to W^{k,p}(\mathbf{R}^d)$$

such that $\eta_k f\big|_\Omega = f$ for all $f \in W^{k,p}(\Omega)$.

It is well known that every bounded Lipschitz domain in $\mathbf{R}^d$ is an extension domain. Our main result in this section is the following:

Theorem 2.12. *The Hartogs triangle* $\mathbf{T}$ *is a Sobolev extension domain.*

Proof. By Theorem 2.4, $\mathbf{T}$ is a uniform domain. Hence $\mathbf{T}$ is an (ε, δ) domain with $\delta = \infty$. Using [21, Theorem 1], every (ε, δ) domain is an extension domain. $\square$

For later use, we state the corresponding result for the complement of $\mathbf{T}$.

Lemma 2.13 (Extension from the complement of $\mathbf{T}$)**.** *Let* $\Omega \subset \mathbf{C}^2$ *be a domain with* $\Omega \supset \overline{\mathbf{T}}$*. There exists a bounded linear operator* $\eta : W^1(\Omega \setminus \overline{\mathbf{T}}) \to W^1(\Omega)$ *such that* $\eta f\big|_{\Omega\setminus\overline{\mathbf{T}}} = f$*. If, moreover,* Ω *is a Lipschitz domain, then* $\Omega \setminus \overline{\mathbf{T}}$ *is an extension domain. In particular,* $C^\infty(\mathbf{C}^2)$ *is dense in* $W^1(\Omega \setminus \overline{\mathbf{T}})$*.*

Proof. If Ω is a bounded Lipschitz domain, then $\Omega \setminus \overline{\mathbf{T}}$ is a uniform domain by Lemma 2.5, and hence a Sobolev extension domain.

For an arbitrary domain $\Omega \supset \overline{\mathbf{T}}$, we choose a Lipschitz domain Ω' with $\Omega \supset \overline{\Omega'}$ such that $\Omega' \supset \overline{\mathbf{T}}$. By the first part of the proof, there exists a bounded linear extension operator $\eta' : W^1(\Omega' \setminus \overline{\mathbf{T}}) \to W^1(\Omega')$. For $f \in W^1(\Omega \setminus \overline{\mathbf{T}})$, we define ηf on $\overline{\mathbf{T}}$ by first restricting f to $\Omega' \setminus \mathbf{T}$ and then applying η'. □

From Theorem 2.12, we have the following results easily.

Corollary 2.14. *Let $W^1(\mathbf{T})$ denote the Sobolev space of L^2-functions on $\mathbf{T}$ with weak first-order derivatives in L^2. Then the following statements hold:*

(1) (Smooth approximation). *$C^\infty(\overline{\mathbf{T}})$ is dense in $W^1(\mathbf{T})$.*
(2) (Sobolev embedding). *$W^1(\mathbf{T}) \subset L^4(\mathbf{T})$, and the inclusion map is bounded.*
(3) (Rellich lemma). *The inclusion $W^1(\mathbf{T}) \subset L^2(\mathbf{T})$ is compact.*

Proof. Let $\eta : W^1(\mathbf{T}) \to W^1(\mathbf{R}^4)$ be the bounded linear extension operator provided by Theorem 2.12. Given $f \in W^1(\mathbf{T})$, set $f_0 := \eta f \in W^1(\mathbf{C}^2)$. We regularize f by convolution $f_\varepsilon = f_0 * \phi_\varepsilon$, where $\{\phi_\varepsilon\}$ is an approximate identity such that each ϕ_ε is a smooth function of compact support with $\int \phi_\varepsilon = 1$. The restrictions of the smooth functions f_ε to $\mathbf{T}$ converge to f in $W^1(\mathbf{T})$, proving the first claim. By the Sobolev inequality on $\mathbf{C}^2$, $\|\eta f\|_{L^4(\mathbf{C}^2)} \le C\|\eta f\|_{W^1(\mathbf{C}^2)}$, where C is the Sobolev constant. Since ηf agrees with f on $\mathbf{T}$, and $\eta : W^1(\mathbf{T}) \to W^1(\mathbf{C}^2)$ is bounded, this implies the second claim. Similarly, the Rellich lemma holds on $\mathbf{T}$ because it holds on a ball $\Omega_0 \supset \mathbf{T}$. □

Corollary 2.15 (Trace). *There exists a bounded linear operator $\tau : W^1(\mathbf{T}) \to L^2(b\mathbf{T})$ with the property that $\tau f = f\big|_{b\mathbf{T}}$ for every continuous function f on $\overline{\mathbf{T}}$.*

Proof. Since $\mathbf{T}$ is a uniform domain with Ahlfors-David regular boundary, the existence of the trace operator follows from [22, Theorem 3]. □

Corollary 2.16 (Poincaré inequality). *There exists a constant $C > 0$ such that*

$$\|f\|^2 \le C\|df\|^2$$

for all $f \in W^1(\mathbf{T})$ with $(f, 1) = 0$, where $\|\ \|$ denotes the L^2-norm on T.

Proof. The proof is the same as for smooth domains. Let

$$\lambda := \inf\left\{\|du\|^2 \;\middle|\; u \in W^1(\mathbf{T}), \|u\| = 1, (u,1) = 0\right\}$$

and consider a minimizing sequence $\{u_\nu\}$. By the Rellich lemma, we may assume (after passing to a subsequence) that u_ν converges strongly in $L^2(\mathbf{T})$, as well as weakly in $W^1(\mathbf{T})$, to some limit $v \in W^1(\mathbf{T})$. The function v is non-constant, because

$$\|v\| = \lim\|u_\nu\| = 1\,, \qquad (v,1) = \lim(u_\nu, 1) = 0\,.$$

Since $\|dv\|^2 \le \lim\|du_\nu\|^2 = \lambda$ by the weak lower semicontinuity of the Dirichlet integral, v is a minimizer. Therefore $\lambda = \|dv\|^2 > 0$, and the Poincaré inequality holds with $C = \lambda^{-1}$. □

3. Identity of weak and strong extensions of $\overline{\partial}$

We collect some basic facts on unbounded operators in Hilbert spaces which will be used later.

3.1. **Basic facts from functional analysis.** We recall the definition of an unbounded linear operator from a Hilbert space to another. By an *operator* A from a Hilbert space H_1 to another Hilbert space H_2 we mean a $\mathbf{C}$-linear map from a linear subspace $\mathrm{Dom}\,(A)$ of H_1 into H_2. We use the notation

$$A : H_1 \dashrightarrow H_2,$$

to denote the fact that A is defined on a subspace of H_1. Recall that an operator is said to be *closed* if its graph is closed as a subspace of the product Hilbert space $H_1 \times H_2$. Suppose that A is defined on all of H_1, then we write $A : H_1 \to H_2$. Notice that if A is defined on the whole Hilbert space H_1, then A has to be a bounded operator from the closed graph theorem.

Let A be a closed densely defined operator from H_1 to H_2. Let $A^* : H_2 \dashrightarrow H_1$ be the (Hilbert space) adjoint of A, defined as follows: An element $g \in \mathrm{Dom}\,(A^*)$ if and only if there exists an element $g^* \in H_1$ such that

$$(Af, g) = (f, g^*) \qquad \text{for all } f \in \mathrm{Dom}\,(A)\,.$$

In this case, $A^*g := g^*$. Then A^* is also a densely defined closed operator, and $A^{**} = A$.

We refer the reader to the book of Riesz-Nagy for Hilbert space adjoints (see page 305 in [32]). We will also need the following lemma (see [17, Theorem 1.1.1]).

Lemma 3.1. *Let A be a closed densely defined operator from one Hilbert space H_1 to another H_2. Then the following conditions are equivalent:*

(1) *The range of A is closed.*
(2) *The range of A^* is closed.*
(3) $H_1 = \mathrm{Range}\,(A^*) \oplus \mathrm{Ker}\,(A)$.
(4) $H_2 = \mathrm{Range}\,(A) \oplus \mathrm{Ker}\,(A^*)$.

Proof. For a proof that (1) and (2) are equivalent, see [17]. By definition of the adjoint operator, the range of A is the closure of the orthogonal complement of the kernel of A^*. Thus if the range of A is closed, then it is the orthogonal complement of the kernel of A^*. Thus (1) and (4) are equivalent. Similarly, (2) and (3) are equivalent. □

3.2. **Maximal extensions and minimal closures of $\overline{\partial}$.** Let Ω be a bounded domain in $\mathbf{C}^n$. Let

$$\overline{\partial} : C^\infty_{p,q-1}(\overline{\Omega}) \to C^\infty_{p,q}(\overline{\Omega})\,, \qquad 0 \le p \le n, 1 \le q \le n \tag{3.1}$$

be the classical Cauchy-Riemann operator on smooth forms. We will use the same symbol, $\overline{\partial}$, to denote the weak Cauchy-Riemann operator acting on currents. Since the index p plays no role on $\mathbf{C}^n$, we will make convenient choices for the value of p in our arguments.

Let $L^2_{p,q}(\Omega)$ be the space of square-integrable (p,q)-forms on Ω. The classical $\overline{\partial}$ operator on smooth forms can be extended to a closed densely defined unbounded operator from $L^2_{p,q-1}(\Omega)$ to $L^2_{p,q}(\Omega)$ in several different ways.

Definition 3.2. Let Ω be a bounded domain in $\mathbf{C}^n$.

(1) The *weak maximal* extension of $\overline{\partial}$, denoted by

$$\overline{\partial} : L^2_{p,q-1}(\Omega) \dashrightarrow L^2_{p,q}(\Omega)\,,$$

is defined by $f \in \mathrm{Dom}\,(\overline{\partial}) \cap L^2_{p,q-1}(\Omega)$ if and only if $\overline{\partial} f \in L^2_{p,q}(\Omega)$ in the distribution sense.

(2) The *strong maximal extension* of $\overline{\partial}$, denoted by

$$\overline{\partial}_s : L^2_{p,q-1}(\Omega) \dashrightarrow L^2_{p,q}(\Omega)\,,$$

is the closure in the graph norm of the restriction of $\overline{\partial}$ to the smooth forms in $C^\infty(\overline{\Omega})$. In other words, $f \in \mathrm{Dom}\,(\overline{\partial}_s)$ if and only if there exists a sequence of smooth forms f_ν in $C^\infty_{p,q-1}(\overline{\Omega})$ such that $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in L^2.

It is easy to check (by smooth approximation) that if $f \in \mathrm{Dom}\,(\overline{\partial}_s)$, then $f \in \mathrm{Dom}\,(\overline{\partial})$ and $\overline{\partial} f = \overline{\partial}_s f$. Hence $\overline{\partial}$ is a closed extension of $\overline{\partial}_s$. On any bounded Lipschitz domain Ω, the Friedrichs lemma implies that $\overline{\partial} = \overline{\partial}_s$, see Hörmander [16, 17] (or Lemma 4.3.2 in the book by Chen-Shaw [9]).

In addition to the maximal extensions, we will consider the following minimal closures of the Cauchy-Riemann operator.

Definition 3.3. Let Ω be a bounded domain in $\mathbf{C}^n$.

(3) The *strong minimal* closure of $\overline{\partial}$, denoted by

$$\overline{\partial}_c : L^2_{p,q-1}(\Omega) \dashrightarrow L^2_{p,q}(\Omega)$$

is the closure in the graph norm of the restriction of $\overline{\partial}$ to the smooth compactly supported forms. In other words, $f \in \mathrm{Dom}\,(\overline{\partial}_c)$ if and only if there is a sequence of smooth forms f_ν in $C^\infty_{p,q-1}(\Omega)$ compactly supported in Ω, such that $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in L^2.

(4) The *weak minimal* closure, denoted by

$$\overline{\partial}_{\tilde{c}} : L^2_{p,q-1}(\Omega) \dashrightarrow L^2_{p,q}(\Omega)$$

is defined by $f \in \mathrm{Dom}\,(\overline{\partial}_{\tilde{c}}) \cap L^2_{p,q-1}(\Omega)$ if and only if $\overline{\partial} f^0$ is in $L^2_{p,q-1}(\mathbf{C}^2)$, where f^0 is the extension of f to zero outside Ω and $\overline{\partial} f^0$ is defined in sense of distribution on $\mathbf{C}^2$.

As above, it is easy to check that $\overline{\partial}_{\tilde{c}}$ is a closed extension of $\overline{\partial}_c$. In fact,

$$\mathrm{Dom}\,(\overline{\partial}) \ \supset\ \mathrm{Dom}\,(\overline{\partial}_s) \ \supsetneq\ \mathrm{Dom}\,(\overline{\partial}_{\tilde{c}}) \ \supset\ \mathrm{Dom}\,(\overline{\partial}_c)\,,$$

and the corresponding inclusions hold for the ranges and the kernels. The middle inclusion is proper, because non-zero constant functions lie in the domain of $\overline{\partial}_s$, but not in the domain of $\overline{\partial}_{\tilde{c}}$.

3.3. **The operators $\overline{\partial}_s$ and $\overline{\partial}_{\tilde{c}}$ on functions.** We start with some simple observations about the Sobolev space W^1. Since $\overline{\partial}$ is a first-order differential operator, $\mathrm{Dom}\,(\overline{\partial}) \supset W^1$, and $\mathrm{Dom}\,(\overline{\partial}_c) \supset W^1_0$, the closure of $C^\infty_0(\Omega)$ in W^1.

Lemma 3.4. *For any bounded domain $\Omega \subset \mathbf{C}^n$, we have* $\mathrm{Dom}\,(\overline{\partial}_{\tilde{c}}) \subset W^1(\Omega)$. *If, moreover, Ω is a Sobolev extension domain, then* $\mathrm{Dom}\,(\overline{\partial}_s) \supset W^1(\Omega)$.

Proof. Let $f \in \mathrm{Dom}\,(\overline{\partial}_{\tilde{c}})$, and let f^0 be its trivial extension. By definition, $f \in L^2(\Omega)$ and $\overline{\partial} f^0 \in L^2_{0,1}(\mathbf{C}^n)$ in the sense of distributions. Let $\{\phi_\varepsilon\}$ be an approximate identity such that $\phi_\varepsilon \in C_0^\infty(B_\varepsilon(0))$, $\phi_\varepsilon \geq 0$ and $\int \phi_\varepsilon = 1$. We regularize f by convolution $f_\varepsilon = f^0 * \phi_\varepsilon$. Then $f_\varepsilon \in C_0^\infty(\mathbf{C}^2)$, and

$$f_\varepsilon \to f^0 \,, \quad \overline{\partial} f_\varepsilon \to \overline{\partial} f^0 \qquad \text{in } L^2(\mathbf{C}^2). \tag{3.2}$$

Using integration by parts, we have

$$(\overline{\partial} f_\varepsilon, \overline{\partial} f_\varepsilon) = -\sum_{j=1}^n \left(\frac{\partial^2 f_\varepsilon}{\partial z_j \partial \overline{z}_j}, f_\varepsilon \right) = (\partial f_\varepsilon, \partial f_\varepsilon). \tag{3.3}$$

It follows that $f_\varepsilon \to f^0$ in $W^1(\mathbf{C}^2)$, and hence $f \in W^1(\Omega)$.

For the second claim, let $f \in W^1(\Omega)$, where Ω is an extension domain. Then there exists a sequence of smooth functions f_ν on $\overline{\Omega}$ such that f_ν converges to f in $W^1(\Omega)$. In particular, $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in $L^2(\Omega)$, that is, $f \in \mathrm{Dom}\,(\overline{\partial}_s)$ and $\overline{\partial}_s f = \overline{\partial} f$. □

Definition 3.5. The *Bergman space* $\mathcal{H}(\mathbf{T})$ is the closed subspace of $L^2(\mathbf{T})$ consisting of the holomorphic functions on $\mathbf{T}$, i.e., $\mathcal{H}(\mathbf{T}) = \mathrm{Ker}\,(\overline{\partial})$. The orthogonal projection

$$B : L^2(\mathbf{T}) \to \mathcal{H}(\mathbf{T})$$

is called the *Bergman projection*.

We next analyze the kernel of $\overline{\partial}_s$. Any function $f \in \mathcal{H}(\mathbf{T})$ admits a Laurent expansion of the form

$$f = \sum_{j=0}^\infty \sum_{k=-1}^\infty a_{jk} \left(\frac{z}{w}\right)^j w^k \tag{3.4}$$

that converges in $L^2(\mathbf{T})$. The functions

$$v_{jk}(z,w) := \left(\frac{z}{w}\right)^j w^k \,, \quad j \geq 0, k \geq -1 \tag{3.5}$$

that appear in the expansion are pairwise orthogonal, since their restrictions to any torus $\{|z| = r, |w| = s\}$ agree (up to re-labeling and multiplication by constants) with a subset of the standard Fourier basis $e^{i(\ell\alpha + m\beta)}$. By Eq. (3.4) they form a complete orthogonal system for $\mathrm{Ker}\,(\overline{\partial})$.

Proposition 3.6. *On* $\mathbf{T}$*, we have* $\mathrm{Ker}\,(\overline{\partial}_s) = \mathrm{Ker}\,(\overline{\partial})$ *on functions.*

Proof. We will show that $\mathrm{Ker}\,(\overline{\partial}_s)$ contains the functions v_{jk} from Eq. (3.5). Since $v_{jk} \in W^1(\mathbf{T})$ for $j, k \geq 0$, Lemma 3.4 implies that

$$v_{jk} \in \mathrm{Ker}\,(\overline{\partial}) \cap W^1(\mathbf{T}) \subset \mathrm{Ker}\,(\overline{\partial}_s) \,, \qquad j \geq 0, k \geq 0 \,.$$

For $k = -1$, fix $j \geq 0$ and set $u := v_{j,-1}$. Given $0 < \delta \leq 1$, consider the subdomain

$$\mathbf{T}_\delta := \left\{ (z,w) \in \mathbf{T} \;\middle|\; |z| < |w| < \delta \right\} \,,$$

and define the function

$$u_\delta = \begin{cases} \left(\frac{|w|}{\delta}\right)^\delta u, & \text{on } \mathbf{T}_\delta, \\ u, & \text{on } \mathbf{T} \setminus \mathbf{T}_\delta. \end{cases} \tag{3.6}$$

Clearly, $|u_\delta| \le |u|$, and $u_\delta \to u$ in $L^2(\mathbf{T})$ by dominated convergence.

By construction, u_δ is piecewise C^1. Its first-order partial derivatives are pointwise bounded by $C|w|^{-2+\delta}$, where C depends on j. For $\delta > 0$ this is square integrable, and $u_\delta \in W^1(\mathbf{T})$. Therefore $u_\delta \in \mathrm{Dom}\,(\overline{\partial}_s)$ and $\overline{\partial}_s u_\delta = \overline{\partial} u_\delta$. Since $\overline{\partial} u = 0$, we see that $\overline{\partial}_s u_\delta = u\,\overline{\partial}\left(\frac{|w|}{\delta}\right)^\delta$ on $\mathbf{T}_\delta$, and vanishes on the complement. By scaling,

$$\|\overline{\partial}_s u_\delta\|_{L^2(\mathbf{T})} = \|\overline{\partial} u_\delta\|_{L^2(\mathbf{T}_\delta)} = \delta \|\overline{\partial} u_1\|_{L^2(\mathbf{T})} \to 0$$

as $\delta \to 0$. Hence $u \in \mathrm{Dom}\,(\overline{\partial}_s)$ and $\overline{\partial}_s u = 0$.

Thus $\mathrm{Ker}\,(\overline{\partial}_s)$ contains an orthonormal basis of $\mathrm{Ker}\,(\overline{\partial})$. Since $\mathrm{Ker}\,(\overline{\partial}_s)$ is a closed subspace of $\mathrm{Ker}\,(\overline{\partial})$, the two spaces agree. □

Proposition 3.7. *On $\mathbf{T}$, we have $\overline{\partial}_{\tilde{c}} = \overline{\partial}_c$ on functions.*

Proof. Since $\overline{\partial}_{\tilde{c}}$ is an extension of $\overline{\partial}$, we have that $\mathrm{Dom}\,(\overline{\partial}_c) \subset \mathrm{Dom}\,(\overline{\partial}_{\tilde{c}})$. We now establish the reverse inclusion. By Lemma 3.4 and the Sobolev Embedding Theorem,

$$\mathrm{Dom}\,(\overline{\partial}_{\tilde{c}}) \subset W^1(\mathbf{T}) \subset L^4(\mathbf{T}) \tag{3.7}$$

Given $f \in \mathrm{Dom}\,(\overline{\partial}_{\tilde{c}})$, we approximate f by a function that vanishes near the singular point at the origin. Let χ_δ be a smooth cut-off function such that $\chi_\delta = 1$ outside the ball $B_{2\delta}(0)$, χ_δ vanishes on $B_\delta(0)$, and its differential satisfy the pointwise bound $|d\chi_\delta| \le C\delta^{-1}$ where C is a constant independent of δ.

By the chain rule,

$$\overline{\partial}(\chi_\delta f) = (\overline{\partial}\chi_\delta) f + \chi_\delta \overline{\partial} f\,. \tag{3.8}$$

It is clear that $\chi_\delta f \to f$ and $\chi_\delta \overline{\partial} f \to \overline{\partial} f$ in L^2 as $\delta \to 0$.

It remains to show that $(\overline{\partial}\chi_\delta) f \to 0$. By the Cauchy-Schwarz inequality, we have

$$\int_{\mathbf{T}} |\overline{\partial}(\chi_\delta) f|^2 dV \le \left(\int_{B_{2\delta}(0)\cap\mathbf{T}} |\overline{\partial}\chi_\delta|^4 \, dV \right)^{\frac{1}{2}} \left(\int_{B_{2\delta}(0)\cap\mathbf{T}} |f|^4 dV \right)^{\frac{1}{2}}.$$

The first factor is bounded independently of δ. Since $f \in L^4(\Omega)$, it follows that

$$\|\overline{\partial}(\chi_\delta) f\|^2 \le \tilde{C} \left(\int_{B_{2\delta}(0)\cap\mathbf{T}} |f|^4 dV \right)^{\frac{1}{2}} \to 0\,.$$

Therefore $\overline{\partial}(\chi_\delta f) \to \overline{\partial} f$ as $\delta \to 0$.

We have approximated $f \in \mathrm{Dom}\,(\overline{\partial}_{\tilde{c}})$ in the graph norm of $\overline{\partial}$ by $\chi_\delta f$. Since $\chi_\delta f$ is supported in the bounded Lipschitz domain $\mathbf{T} \setminus \overline{B_\delta(0)}$, it can be further approximated by compactly supported functions in $\mathbf{T}$. This proves that $f \in \mathrm{Dom}\,(\overline{\partial}_c)$. □

3.4. **Weak equals strong.** We need two more tools, Serre duality and Dolbeault cohomology. L^2 Serre duality establishes a relation between $\overline{\partial}$ and $\overline{\partial}_c$, and correspondingly between $\overline{\partial}_s$ and $\overline{\partial}_{\tilde{c}}$. Denote by $\star : L^2_{p,q}(\Omega) \to L^2_{n-p,n-q}(\Omega)$ the Hodge star operator.

Lemma 3.8. *Let Ω be a bounded domain in $\mathbf{C}^n$. Then $\overline{\partial}_c = -\star \overline{\partial}^* \star$.*

Proof. See [4, Proposition 1] or [27, Lemma 2.2]. □

Lemma 3.9. *On $\mathbf{T}$, we have $\overline{\partial}_{\tilde{c}} = -\star \overline{\partial}_s^* \star$.*

Proof. Since the boundary of $\mathbf{T}$ is rectifiable, the weak minimal closure $\overline{\partial}_{\tilde{c}}$ is dual to the strong maximal extension $\overline{\partial}_s$ (see [26]). □

Definition 3.10. For $0 \le p \le n$ and $0 \le q \le n$, the L^2 *Dolbeault cohomology groups* are defined by

$$H^{p,q}_{L^2,\overline{\partial}}(\Omega) = \frac{\{f \in L^2_{p,q}(\Omega) \mid \overline{\partial} f = 0\}}{\{f \in L^2_{p,q}(\Omega) \mid f = \overline{\partial} u \text{ for some } u \in L^2_{p.q-1}(\Omega)\}}.$$

Similarly, we define $H^{p,q}_{L^2,\overline{\partial}_s}(\Omega)$ by substituting $\overline{\partial}$ with $\overline{\partial}_s$.

When Ω is a bounded pseudoconvex domain in $\mathbf{C}^n$, the L^2 theory for $\overline{\partial}$ is completely known from Hörmander's L^2 theorem for $\overline{\partial}$ (see [17]). The key result is that

$$H^{p,q}_{L^2,\overline{\partial}}(\Omega) = 0\,, \qquad 1 \le p \le n, 1 \le q < n\,.$$

For the strong maximal extension $\overline{\partial}_s$ on a pseudoconvex domain with rectifiable boundary, it was proved in [26] that either $H^{0,1}_{L^2,\overline{\partial}_s}(\Omega) = 0$, or $H^{0,1}_{L^2,\overline{\partial}_s}(\Omega)$ is not Hausdorff.

Theorem 3.11. *On $\mathbf{T}$, the strong maximal extension $\overline{\partial}_s$ of the Cauchy-Riemann operator has closed range.*

Proof. We will show that $\overline{\partial}_s : L^2_{p,q-1}(\mathbf{T}) \dashrightarrow L^2_{p,q}(\mathbf{T})$ has closed range for $p = 0, 1, 2$ and $q = 1, 2$. As noted above, the value of p plays no role here.

q=2: Take $p = 2$. By Proposition 3.7, $\overline{\partial}_{\tilde{c}} = \overline{\partial}_c$ on functions. By Lemmas 3.8 and 3.9, this is equivalent to

$$\overline{\partial}_s = \overline{\partial} : L^2_{2,1}(\mathbf{T}) \dashrightarrow L^2_{2,2}(\mathbf{T})\,. \tag{3.9}$$

In particular, Range $(\overline{\partial}_s) =$ Range $(\overline{\partial}) = L^2_{2,2}(\mathbf{T})$, which is closed.

q=1: Take $p = 0$, and consider $\overline{\partial}_s : L^2(\mathbf{T}) \dashrightarrow L^2_{0,1}(\mathbf{T})$. By combining Proposition 3.6 with Lemma 3.1, we see that

$$\text{Range}\,(\overline{\partial}_s^*) \subset (\text{Ker}\,(\overline{\partial}_s))^\perp = (\text{Ker}\,(\overline{\partial}))^\perp = \text{Range}\,(\overline{\partial}^*)\,,$$

where we have used that Range $(\overline{\partial}^*) \subset L^2(\mathbf{T})$ is closed by Hörmander's L^2-theory. Since $\overline{\partial}$ is an extension of $\overline{\partial}_s$, we also have the reverse inclusion

$$\text{Range}\,(\overline{\partial}_s^*) \supset \text{Range}\,(\overline{\partial}^*)\,.$$

Therefore Range $(\overline{\partial}_s^*) =$ Range $(\overline{\partial}^*) \subset L^2(\mathbf{T})$. By Lemma 3.1, $\overline{\partial}_s : L^2(\mathbf{T}) \dashrightarrow L^2_{0,1}(\mathbf{T})$ has closed range as well. □

Proposition 3.12. $H^{p,1}_{L^2,\overline{\partial}_s}(\mathbf{T}) = 0$ *for* $0 \le p \le 2$.

Proof. Take $p = 0$. Since $\overline{\partial}_s : L^2(\mathbf{T}) \dashrightarrow L^2_{0,1}(\mathbf{T})$ has closed range by Theorem 3.11, the corresponding cohomology group $H^{0,1}_{L^2,\overline{\partial}_s}(\mathbf{T})$ is Hausdorff (see [39, Proposition 4.5]). It follows from [26, Theorem 3.2 (iv)] that $H^{0,1}_{L^2,\overline{\partial}_s}(\mathbf{T}) = 0$. □

Theorem 3.13. *On* $\mathbf{T}$*, the strong maximal extension* $\overline{\partial}_s$ *of the Cauchy-Riemann operator equals the weak maximal extension* $\overline{\partial}$*.*

Proof. q=2: By Eq. (3.9), we have that $\overline{\partial}_s = \overline{\partial}$ on $(p,1)$-forms for $p = 0, 1, 2$.

q=1: Take $p = 0$ and consider $\overline{\partial}_s : L^2(\mathbf{T}) \dashrightarrow L^2_{0,1}(\mathbf{T})$. By Proposition 3.6

$$\mathrm{Ker}\,(\overline{\partial}_s) = \mathrm{Ker}\,(\overline{\partial}) = \mathcal{H}(\mathbf{T})$$

on functions. Since $\overline{\partial} = \overline{\partial}_s : L^2_{0,1} \dashrightarrow L^2_{0,2}$ by the first part of the proof, we have that $\mathrm{Ker}\,(\overline{\partial}_s) = \mathrm{Ker}\,(\overline{\partial}) \subset L^2_{0,1}(\mathbf{T})$. By Proposition 3.12 and Hörmander's L^2 results, $H^{0,1}_{L^2,\overline{\partial}_s} = H^{0,1}_{L^2,\overline{\partial}} = 0$, which means by the definition of the cohomology groups that

$$\mathrm{Range}\,(\overline{\partial}_s) = \mathrm{Range}\,(\overline{\partial}) \subset L^2_{0,1}(\mathbf{T}).$$

Since $\overline{\partial}$ is a closed extension of the densely defined operator $\overline{\partial}_s$, with the same kernel and range, $\overline{\partial} = \overline{\partial}_s$ on functions. □

Corollary 3.14 (**Bergman projection**). *Let* $B_s : L^2(\mathbf{T}) \to \mathcal{H}(\mathbf{T})$ *be the Bergman projection with respect to* $\overline{\partial}_s$ *on* $\mathbf{T}$*. Then* $B = B_s$*. Moreover, for any* $f \in L^2(\mathbf{T})$*, the complementary projection satisfies*

$$f - Bf \;=\; \overline{\partial}^*_s \overline{\partial}_s N_0 f \;=\; \overline{\partial}^*_s N_1 \overline{\partial}_s f\,,$$

where N_0 *is the* $\overline{\partial}$*-Neumann operator on functions, and* N_1 *is the* $\overline{\partial}$*-Neumann operator on* $(0,1)$*-forms.*

Proof. Since $\mathrm{Ker}\,(\overline{\partial}_s) = \mathcal{H}(\mathbf{T}) = \mathrm{Ker}\,(\overline{\partial})$, either $\overline{\partial}_s$ or $\overline{\partial}$ can be used to define the Bergman projection. The formulas for the orthogonal projection hold for $\overline{\partial}$ by Hörmander's theory and by Kohn's formula for the Bergman projection (see Theorem 4.4.3 and Corollary 4.4.4 in [9]). Since $\overline{\partial} = \overline{\partial}_s$, the corollary follows. □

4. Dolbeault cohomology on the complement of T

In this section, we study the Dolbeault cohomology groups on an annulus between a pseudoconvex domain and the Hartogs triangle $\mathbf{T}$.

Definition 4.1. Let Ω be a bounded domain in $\mathbf{C}^n$. Let $W^k(\Omega)$ be the Sobolev space of order $k \in \mathbf{N} \cup \{0\}$. We denote by $H^{p,q}_{W^k}(\Omega)$ the associated cohomology group defined by

$$H^{p,q}_{W^k}(\Omega) = \frac{\{f \in W^k_{p,q}(\Omega) \mid \overline{\partial} f = 0\}}{\{f \in W^k_{p,q}(\Omega) \mid f = \overline{\partial} u \text{ for some } u \in W^k_{p.q-1}(\Omega)\}}.$$

When $k = 0$, we also use the notation $H^{p,q}_{L^2}(\Omega)$ to denote the L^2 Dolbeault cohomology groups with respect to $\overline{\partial}$.

We will need the following result from the book of Chen-Shaw ([9, Theorem 9.1.3]).

Lemma 4.2. *Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$, $n \geq 2$. For any $f \in L^2_{p,q}(\mathbf{C}^n)$, $0 \leq p \leq n$, $1 \leq q \leq n$, such that f is supported in $\overline{\Omega}$ and*

$$\int_\Omega f \wedge \phi = 0, \quad \phi \in L^2_{2-p,0}(\Omega) \cap \mathrm{Ker}\,(\overline{\partial})$$

there exists $u \in L^2_{p,q-1}(\mathbf{C}^n)$ such that $\overline{\partial}_{\tilde{c}} u = f$.

Consider an annular domain

$$\Omega = \Omega_1 \setminus \overline{\mathbf{T}} \tag{4.1}$$

where Ω_1 is a pseudoconvex domain in $\mathbf{C}^2$ containing $\overline{\mathbf{T}}$.

Theorem 4.3. *Let Ω be given by Eq. (4.1), where $\Omega_1 \subset \mathbf{C}^2$ is a bounded pseudoconvex domain with C^2-boundary, and $\overline{\mathbf{T}} \subset \Omega_1$. Then*

$$H^{p,1}_{W^1}(\Omega) \cong \mathcal{H}(\mathbf{T})\,, \qquad 0 \leq p \leq 2\,,$$

where $\mathcal{H}(\mathbf{T})$ is the Bergman space of $\mathbf{T}$. In particular, $H^{p,1}_{W^1}(\Omega)$ is Hausdorff and infinite-dimensional.

Proof. We will prove that $H^{p,1}_{W^1}(\Omega) \cong (\mathcal{H}(\mathbf{T}))'$, the space of bounded linear forms on the Bergman space. It suffices to consider $p = 2$.

By Lemma 2.13, there is a bounded linear extension operator $\eta : W^1(\Omega) \to W^1(\Omega_1)$. For $f \in W^1_{2,1}(\Omega)$, define $\ell_f \in (\mathcal{H}(\mathbf{T}))'$ by

$$\ell_f(h) := \int_{\mathbf{T}} \overline{\partial}(\eta f) \wedge h, \quad h \in \mathcal{H}(\mathbf{T})\,. \tag{4.2}$$

Clearly, $\ell_h(f)$ is bilinear and jointly continuous in $f \in W^1_{2,1}(\mathbf{T})$ and $h \in \mathcal{H}$.

Smooth approximation. Let $h \in \mathcal{H}(\mathbf{T})$. Since $\mathcal{H}(\mathbf{T}) = \mathrm{Ker}\,(\overline{\partial}_s)$ by Proposition 3.6, there is a sequence $\{h_\nu\}$ in $C^\infty(\overline{\mathbf{T}})$ such that

$$\begin{cases} h_\nu \to h, & \text{in } L^2(\mathbf{T}), \\ \overline{\partial} h_\nu \to 0 & \text{in } L^2_{0,1}(\mathbf{T}). \end{cases} \tag{4.3}$$

Since $\overline{\partial}(\eta f) \wedge h = \overline{\partial}(\eta f \wedge h)$, Stokes' theorem implies that

$$\ell_f(h) = \lim_{\nu\to\infty} \int_{\mathbf{T}} \overline{\partial}(\eta f \wedge h_\nu) = \lim_{\nu\to\infty} \int_{b\mathbf{T}} \tau f \wedge h_\nu\,, \tag{4.4}$$

where τf is the trace of f. We have used that $\tau f \in L^2(b\mathbf{T})$ by Corollary 2.15. It is apparent from Eq. (4.4) that ℓ_f does not depend on the choice of the extension η.

ℓ_f is determined by the cohomology class $[f]$. Suppose that $f = \overline{\partial} u$ for some $u \in W^1_{2,0}(\Omega)$. Let ηu be the extension of u to $W^1(\Omega_1)$. Since Ω_1 is a Lipschitz domain, there exists a sequence $u_j \in C^\infty(\mathbf{C}^2)$ with $u_j \to \eta u$ in $W^1(\Omega_1)$. By Eq. (4.4) and two more applications of Stokes' theorem,

$$\ell_{\overline{\partial} u_j}(h_\nu) = \int_{b\mathbf{T}} (\overline{\partial} u_j) \wedge h_\nu = \int_{b\mathbf{T}} u_j \wedge \overline{\partial} h_\nu = \int_{\mathbf{T}} \overline{\partial} u_j \wedge \overline{\partial} h_\nu\,.$$

Taking first $j \to \infty$ and then $\nu \to \infty$, we arrive at $\ell_{\overline{\partial}u}(h) = 0$. Thus the map $[f] \mapsto \ell_f$ is well-defined from $H^{0,1}_{W^1}(\Omega)$ to $(\mathcal{H}(\mathbf{T}))'$.

$[f] \mapsto \ell_f$ *is injective.* Suppose that ℓ_f vanishes on $\mathcal{H}(\mathbf{T})$. By Lemma 4.2, there exists $g \in L^2_{2,1}(\mathbf{T})$ such that $\overline{\partial}_{\tilde{c}} g = \overline{\partial}(\eta f)$ on $\mathbf{T}$. In fact, the trivial extension g^0 of g lies in $W^1_{2,1}(\mathbf{C}^2)$. Set $F = \eta f - g^0 \in W^1(\Omega_1)$.

By construction, $\overline{\partial} F = 0$ on Ω. Since Ω_1 has C^2 boundary, we can solve $\overline{\partial} u = F$ for some function $u \in W^1_{2,0}(\Omega_1)$ (see [25] and [15]). In particular, $\overline{\partial} u = f$ on Ω.

$[f] \mapsto \ell_f$ *is surjective.* Let $\ell \in (\mathcal{H}(\mathbf{T}))'$. Since $\mathcal{H}(\mathbf{T})$ is a Hilbert space, ℓ can be represented by some holomorphic function $g \in \mathcal{H}(\mathbf{T})$. Let $g^0 \in L^2(\Omega_1)$ be the trivial extension of g, and let $\star g^0$ be the dual $(2,2)$-form on Ω_1. Since a top degree form is always $\overline{\partial}$-exact, there exists $v \in W^1_{2,1}(\Omega_1)$ that solves $\overline{\partial} v = \star g^0$ on Ω_1. By construction,

$$\ell(h) = (g, h) = \int_{b\mathbf{T}} \overline{\partial} v \wedge h\,, \qquad h \in \mathcal{H}(\mathbf{T})\,.$$

Let f be the restriction of v to Ω. Then $f \in W^1_{2,1}(\Omega)$, and v is an extension of f to Ω_1. Since the extension does not matter, $\ell = \ell_f$.

We conclude that $[f] \mapsto \ell_f$ is a linear isomorphism from $H^{2,1}_{W^1}(\Omega)$ to $\mathcal{H}(\mathbf{T})'$. Since $\mathcal{H}$ is a Hilbert space, the theorem is proved. □

5. **Some open questions**

Let Ω_1 and Ω_2 be two bounded pseudoconvex domains in $\mathbf{C}^n$ and let $\overline{\Omega}_2 \subset \Omega_1$. Let Ω be the annulus between the two pseudoconvex domains with

$$\Omega = \Omega_1 \setminus \overline{\Omega}_2.$$

It is known for $\Omega = \Omega_1 \setminus \overline{\mathbf{T}}$ that the classical Dolbeault cohomology with smooth coefficients on Ω,

$$H^{0,1}(\Omega) := \frac{\{f \in C^\infty_{0,1}(\Omega) \mid \overline{\partial} f = 0\}}{\{f \in C^\infty_{0,1}(\Omega) \mid f = \overline{\partial} u \text{ for some } u \in C^\infty(\Omega)\}}$$

is non-Hausdorff (see [26, Corollary 4.6]). This is in sharp contrast to Theorem 4.3.

Theorem 4.3 is a generalization of a result by Hörmander for the case when Ω is the annulus between two concentric balls in $\mathbf{C}^n$ (see [19]. In that case, $H^{0,n-1}_{L^2}(\Omega)$ is Hausdorff, and one can realize the space $H^{0,n-1}_{L^2}(\Omega)$ explicitly as the Bergman space of the inner domain.

When Ω_2 is a pseudoconvex domain with C^3 boundary and $0 < q < n-1$, the L^2 and Sobolev cohomology groups for $\overline{\partial}$ on the annulus were studied much earlier in [33]. For general pseudoconvex domains with C^2 boundary, the Hausdorff property for the critical degree $q = n-1$ is proved in [34]. The necessary conditions for the Hausdorff properties for the Dolbeault cohomology group for $\overline{\partial}$ on annuli is proved in [12].

In view of Theorem 4.3 and the remarks above, it is natural to ask the following question.

Problem 1. Let $\Omega = \Omega_1 \setminus \overline{\mathbf{T}}$. Determine if $H^{0,1}_{L^2}(\Omega)$ is Hausdorff.

Without loss of generality, we can take the outer domain Ω_1 in Problem 1 to be the ball of radius $r \geq 2$ centered at 0 (see [2] for a discussion on this). Problem 1 can be called the *Dollar Bill* problem since the shape is featured on the reverse of the American one-dollar bill.

When the inner domain is the bidisk D^2, the corresponding problem for $\Omega = B \backslash D^2$, is called the *Chinese Coin* problem since it has the shape of an ancient Chinese coin. The Chinese coin problem is solved in [2]. Problem 1 has an equivalent formulation in terms of the W^1 Dolbeault cohomology of $\mathbf{T}$:

Proposition 5.1. *Let $\Omega = \Omega_1 \setminus \overline{\mathbf{T}}$, where Ω_1 is a bounded pseudoconvex domain in $\mathbf{C}^2$ with $\overline{\mathbf{T}} \subset \Omega_1$. Then the following are equivalent:*

(1) *$H^{0,1}_{L^2}(\Omega)$ is Hausdorff;*
(2) *$H^{0,1}_{W^1}(\mathbf{T}) = 0$.*

The proof of Proposition 5.1 is the same as for Lipschitz domains, as given in [26, Corollary 4.8]. The key points are the L^2-duality between $\overline{\partial}_s$ and $\overline{\partial}_{\tilde{c}}$ and the extension property (Lemma 2.13).

This leads to a more general question.

Problem 2. Determine if $H^{0,1}_{W^s}(\mathbf{T}) = 0$, where $s > 0$.

We remark that if Ω is a bounded pseudoconvex domain with smooth boundary in $\mathbf{C}^n$, it follows from [25] that $H^{0,1}_{W^s}(\Omega) = 0$ for all $s > 0$. Not much is known about Sobolev estimates for solutions of $\overline{\partial}$ for the Hartogs triangle. But there has been a lot of work for $\overline{\partial}$ in other function spaces.

It is proved in [36] that there is a form $f \in C^\infty_{(0,1)}(\overline{\mathbf{T}})$ with $\overline{\partial} f = 0$ such that the equation $\overline{\partial} u = f$ has no solution $u \in C^\infty(\overline{\mathbf{T}})$. Furthermore, it is proved in [27] that the Dolbeault cohomology with smooth coefficients on $\mathbf{T}$ is non-Hausdorff.

On the other hand, since $\mathbf{T}$ is pseudoconvex, we have from the Dolbeault theorem that

$$H^{0,1}(\mathbf{T}) = 0 \tag{5.1}$$

where $H^{0,1}(\mathbf{T})$ denotes the Dolbeault cohomology with smooth $C^\infty(\mathbf{T})$ coefficients. Furthermore, there do exist *almost* smooth solutions to the $\overline{\partial}$ problem on the Hartogs triangle: For every $k \in \mathbf{N}$ and $0 < \alpha < 1$, let $C^{k,\alpha}(\mathbf{T})$ denote the Hölder continuous function spaces of order k, α. Let $H^{p,q}_{C^{k,\alpha}}(\mathbf{T})$ denote the Dolbeault cohomology of (p,q)-forms with $C^{k,\alpha}(\mathbf{T})$ coefficients. Using the integral kernel method, it is proved in [10] that

$$H^{0,1}_{C^{k,\alpha}}(\mathbf{T}) = 0.$$

Notice that the intersection $\cap_k C^{k,\alpha}(\mathbf{T}) = C^\infty(\overline{\mathbf{T}})$. These results show the subtlety of such problems on the Hartogs triangle.

We can also consider the de Rham complex d on $\mathbf{T}$ instead of $\overline{\partial}$. Let d and d_s denote the weak and strong maximal extensions from $L^2_q(\mathbf{T})$ to $L^2_{q+1}(\mathbf{T})$. Consider the d-Laplacian

$$\Delta = dd^* + d^*d : L^2_q(\mathbf{T}) \dashrightarrow L^2_q(\mathbf{T}),$$

where $0 \le q \le 4$. Similarly, we can consider $\Delta_s = d_s d_s^* + d_s^* d_s$. We refer to the paper by Hörmander (see [18]) for a historical overview of the Hodge theorem for domains with smooth boundary. The Hodge theorem on Lipschitz domains in $\mathbf{R}^n$ was studied in [30]

Problem 3. On the Hartogs triangle $\mathbf{T}$, determine

- if the Hodge theorem holds for $\triangle$ (or $\triangle_s$);
- if the spectrum of $\triangle$ (or $\triangle_s$) on forms is discrete;
- if $d = d_s$.

Notice that on functions we have $d = d_s : L^2(\mathbf{T}) \dashrightarrow L^2_1(\mathbf{T})$, since smooth functions are dense in $\mathrm{Dom}(d) = W^1(\mathbf{T})$ by Corollary 2.14. We can also show that $d = d_s : L^2_3(\mathbf{T}) \dashrightarrow L^2_4(\mathbf{T})$ by using arguments similar to the proof of Proposition 3.7. It is not known if $d = d_s$ for other degrees. We refer the reader to [17] for the identity of weak and strong extensions of general systems of first-order differential operators on Lipschitz domains.

The Neumann problem is the natural boundary value problem for $\Delta : L^2(\mathbf{T}) \dashrightarrow L^2(\mathbf{T})$ on functions, where $\Delta = d^* d$. By definition, $u \in Dom(\Delta)$ if and only if $du \in \mathrm{Dom}\,(d^*)$, i.e., if there exists some $f \in L^2(\mathbf{T})$ such that

$$(du, dv) = (f, v) \qquad \text{for all } v \in W^1(\mathbf{T}). \tag{5.2}$$

By taking v to be a smooth test function on $\mathbf{T}$, we see that $\Delta u = f$ in the sense of distributions. Moreover, any $f \in \mathrm{Range}\,(\Delta)$ satisfies $(f, 1) = 0$.

Corollary 2.16 directly yields the solution of the Neumann problem, by providing for each $f \in L^2(\mathbf{T})$ with $(f, 1) = 0$ a unique $u \in W^1(\mathbf{T})$ such that Eq. (5.2) holds.

To see this, consider the closed subspace

$$V := \left\{ v \in W^1(\mathbf{T}) \;\middle|\; (v, 1) = 0 \right\} ,$$

equipped with the inner product $Q(u, v) := (du, dv)$. By the Poincaré inequality (Corollary 2.16), Q is positive definite, hence an inner product on V, and the resulting norm $Q(v, v)^{\frac{1}{2}}$ is equivalent to the W^1-norm. The map $v \mapsto (f, v)$ defines a continuous linear form on V. Since V is a Hilbert space, there exists a unique $u \in V$ such that

$$(f, v) = Q(u, v) = (du, dv), \quad v \in V .$$

Since $(f, 1) = 0$ by assumption, this holds also for $v = 1$, proving Eq. (5.2).

For $f \in L^2(\Omega)$, let f_a be the average of f over $\mathbf{T}$. The operator $G_N : L^2(\mathbf{T}) \to L^2(\mathbf{T})$ that maps f to the unique solution of the Neumann problem $\Delta u = (f - f_a)$ is called the *Neumann operator* on $L^2(\mathbf{T})$. Since $\mathrm{Range}\,(G_N) \subset W^1(\mathbf{T})$, the Rellich lemma implies that G_N is compact. Its spectrum consists of a sequence of eigenvalues μ_j of finite multiplicity, with $\mu_j \to 0$. Its the eigenvalues are positive (except for the simple eigenvalue at zero), and $L^2(\mathbf{T})$ has an orthonormal basis of eigenvectors. This implies that Δ has discrete spectrum $\lambda_j = \frac{1}{\mu_j} \to \infty$ on $L^2(\mathbf{T})$.

We also know that $\Delta = dd^*$ on the top degree ($q = 4$) has discrete spectrum since it corresponds to the Dirichlet problem. However, it is not known if $\Delta = dd^* + d^* d$ on $L^2_q(\mathbf{T})$ has closed range when $1 \le q \le 3$, and if the de Rham cohomology is represented by the harmonic forms.

Problem 4. Determine the spectrum of the $\overline{\partial}$-Neumann operator

$$N_1 : L^2_{0,1}(\mathbf{T}) \to L^2_{0,1}(\mathbf{T})\,.$$

The operator N_1 is not compact on $L^2_{0,1}(\mathbf{T})$, since $\mathbf{T}$ is biholomorphic to a product domain (see [3]). It is not known whether the spectrum consists of a sequence of eigenvalues (of possibly infinite multiplicity), or if continuous spectrum may be present. Since we can express N_0 by the formula (see [9, Theorem 4.4.3])

$$N_0 = \overline{\partial}^* N_1^2 \overline{\partial}, \tag{5.3}$$

we have that $N_0 : L^2(\mathbf{T}) \to L^2(\mathbf{T})$ is also non-compact on the orthogonal complement of the Bergman space.

We note that for $q = 2$, the operator $N_2 : L^2_{0,2}(\mathbf{T}) \to L^2_{0,2}(\mathbf{T})$ is compact since it corresponds to the Dirichlet problem. Thus the spectrum of N_2 is discrete.

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