ESTIMATES FOR $\overline{\partial}$ ON DOMAINS IN $\mathbb{C}^n$ AND $\mathbb{CP}^n$

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Abstract. In this paper we survey some results of the Cauchy-Riemann equation on domains in $\mathbb{C}^n$ and $\mathbb{CP}^n$ related to Sibony's work. We also raise many questions on estimates for $\overline{\partial}$ for smooth and non-smooth domains.

Dedicated to the memory of Nessim Sibony

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1. Introduction

In this paper, we survey some results on the estimates for $\overline{\partial}$ on pseudoconvex domains in $\mathbb{C}^n$ and $\mathbb{CP}^n$, where Sibony's work has profound influence. Our goal is to highlight some of his important work that has impacted the author's own research.

In Section 2 we first discuss some results in $\mathbb{C}^n$, where tremendous work has been done for the past few decades. For a general pseudoconvex domain with smooth boundary, Sibony's deep results provide many counter examples on function spaces other than L^2 or Sobolev spaces (see Theorems 2.3, 2.4, 2.6).

The theory for $\overline{\partial}$ is less developed for domains in the complex projective space $\mathbb{CP}^n$, which is not Stein. The Fubini-Study metric on $\mathbb{CP}^n$ has a positive holomorphic bisectional curvature which can be used to study these problems for $(0, q)$-forms, but not for (p, q)-forms when $p \neq 0$. In Section 3 we discuss some methods and results on L^2 and Sobolev estimates on pseudoconvex domains in $\mathbb{CP}^n$, where the work of Sibony-Ohsawa (Theorem 3.6) plays an important role.

We also discuss some results on the Hartogs triangles in $\mathbb{CP}^2$. These problems are important not only in complex analysis, but in complex foliation theory and complex dynamics. We raise some open questions at the end.

I first met Sibony when I was a graduate student at Princeton in the late seventies. However, it was at the Albany meeting in 1985 that his insight and enthusiasm in mathematics really made an impression on me. Some of my research was inspired by his work. I dedicate this article to his memory.

2. Estimates for $\overline{\partial}$ on pseudoconvex domains in $\mathbb{C}^n$

Since the fundamental work of Kohn ([27, 28]) for $\overline{\partial}$ on smooth pseudoconvex domains and that of Hörmander ([24]) on L^2-estimates on bounded pseudoconvex domains in $\mathbb{C}^n$, there has been tremendous progress on L^2-Sobolev theory of the $\overline{\partial}$-operator and the $\overline{\partial}$-Neumann problem for bounded pseudoconvex domains in $\mathbb{C}^n$ (see, for example, monographs [16, 25, 31, 10, 51] for expositions on the subject).

Theorem 2.1 (**Hörmander** [24]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then for any $f \in L^2_{p,q}(\Omega)$ with $\overline{\partial} f = 0$, there exists $u \in L^2_{p,q-1}(\Omega)$ such that $\overline{\partial} u = f$ in Ω with*

$$\|u\| \le \sqrt{\frac{e}{q}} R \|f\|$$

where R is the diameter of the domain Ω.

Let $H^{p,q}_{W^s}(\Omega)$ be the Dolbeault cohomology with Sobolev W^s coefficients defined by

$$H^{p,q}_{W^s}(\Omega) = \frac{\{f \in W^s_{p,q}(\Omega) \mid \overline{\partial} f = 0\}}{\{f \in W^s_{p,q}(\Omega) \mid f = \overline{\partial} u, u \in W^s_{p,q-1}(\Omega)\}}. \tag{2.1}$$

Let $H^{p,q}(\overline{\Omega})$ denote the Dolbeault cohomology group with $C^\infty(\overline{\Omega})$ coefficients where we substitute $W^s(\Omega)$ in (2.1) by $C^\infty(\overline{\Omega})$. Similarly, we denote the cohomology group $H^{p,q}_{L^p}(\Omega)$ with forms in $L^p(\Omega)$.

The Sobolev estimates and boundary regularity for $\overline{\partial}$ on bounded smooth pseudoconvex domains in $\mathbb{C}^n$ are well known.

Theorem 2.2 (**Kohn** [28]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. For every $0 \le p \le n$, $0 < q < n$ and $s \ge 0$,*

$$H^{p,q}_{W^s}(\Omega) = 0, \tag{2.2}$$

In particular, we have

$$H^{p,q}(\overline{\Omega}) = 0.$$

Let $C^{k,\alpha}$ be the Hölder space of order k, α where k is a nonnegative integer and $0 < \alpha < 1$. The classical Schauder theory states that $C^{k,\alpha}$ and $W^{k,p}$ are the appropriate spaces to study the interior regularity for elliptic equations. It is natural to ask if such estimates hold for $\overline{\partial}$ with L^p estimates , $1 \le p \le \infty$ when $p \ne 2$ or in Hölder spaces. This is true if the domain is strongly pseudoconvex with smooth boundary using integral kernels. Sibony shows that this is not true for general pseudoconvex domains.

Theorem 2.3 (**Sibony** [45]). *There exists a bounded smooth pseudoconvex domain Ω in $\mathbb{C}^3$ and a $\overline{\partial}$-closed (0,1)-form $f \in C(\overline{\Omega})$ such that every solution $\overline{\partial} u = f$ is unbounded.*

This result is the first of many counterexamples that Sibony and others obtain in various function spaces. In particular, Hölder and $C^{k,\alpha}$ estimates also fail on certain pseudoconvex smooth domains in $\mathbb{C}^3$ (see [47]).

Theorem 2.4 (**Fornaess-Sibony** [14, 15]). *There exists a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^2$ such that for each $p > 2$, $\overline{\partial} : L^p(\Omega) \to L^p_{0,1}(\Omega)$ does not have closed range.*

The sup-norm estimate for $\overline{\partial}$ is especially important since it is related to the Corona problem. It is also proved in [15] that the Corona problem fails on certain pseudoconvex domains with smooth boundary. The Corona problem for strongly pseudoconvex domains remains to be one of the most important open problems in several complex variables. It is not known even for the ball in $\mathbb{C}^n$ when $n \ge 2$.

Corollary 2.5. *Let Ω and p be be the same as in Theorem 2.4. Then $H^{0,1}_{L^p}(\Omega)$ is non-Hausdorff.*

Recall that the Hausdorff property of the quotient group $H^{1,0}_{L^p}(\Omega)$ is equivalent to the closed range property of $\overline{\partial}$. Not only can one not solve $\overline{\partial}$ with L^p estimates for some $(0,1)$-forms on such domain, but they fail badly. Thus L^2 is the only function space that $H^{0,1}_{L^2}(\Omega) = 0$ for any bounded pseudoconvex domain. These results show the importance of L^2 Hilbert space techniques for $\overline{\partial}$ in complex analysis. These counterexamples show that in order to to obtain L^p estimates for $\overline{\partial}$, certain conditions have to be imposed. This has been carried out by many people under various finite type conditions. We refer the reader the the excellent survey article by Sibony [46].

2.1. **The Hartogs triangle in $\mathbb{C}^2$.** We first recall some results in $\mathbb{C}^2$. Let T be the Hartogs triangle in $\mathbb{C}^2$ defined by

$$T = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < |z_2| < 1\}.$$

Since T is a pseudoconvex domain, we have $H^{0,1}(T) = 0$. Moreover, using Hörmander's theorem, we have

$$H^{p,1}_{L^2}(T) = 0, \qquad p = 0, 1, 2.$$

The Hartogs triangle T is not Lipschitz near 0 in the sense that the boundary is not the graph of a Lipschitz function near 0.

Theorem 2.6 (**Sibony**[44, 45])**.** *For any ζ in the bidisc $P = \Delta \times \Delta$ and $\zeta \in P \setminus \overline{T}$, there exists a C^∞-smooth, $\overline{\partial}$-closed $(0,1)$-form α_ζ defined in $\mathbb{C}^2 \setminus \{\zeta\}$ such that there does not exist any C^∞-smooth function β on $\overline{T}$ such that $\overline{\partial}\beta = \alpha_\zeta$.*

We also refer the reader to the paper [11]) for related problems. One has a strengthened version of Theorem 2.6 (see [34]).

Corollary 2.7. *The cohomology group $H^{0,1}(\overline{T})$ is non-Hausdorff.*

On the other hand, one has the following results.

Theorem 2.8 (**Chaumat-Chollet**[11])**.** *For each nonnegative integer k and $0 < \alpha < \infty$, we have $H^{0,1}_{C^{k,\alpha}}(T) = 0$.*

Remark. Notice that

$$\cap_k C^{k,\alpha}(T) = C^\infty(\overline{T}).$$

Theorem 2.8 is in sharp contrast with Theorem 2.6. One should also compare it with the results by Sibony in [47].

There have been numerous works on Hartogs triangles in recent years (see e.g. [34, 35]). One of the most basic question is to ask if the Hartogs triangle is an extension domain.

Definition 2.1. A domain $\Omega \subset \mathbb{R}^N$ is called a *(Sobolev) extension domain*, if for each $k \in \mathbf{N}$ and $1 \le p \le \infty$, there exists a bounded linear operator

$$\eta_k : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^N)$$

such that $\eta_k f\big|_\Omega = f$ for all $f \in W^{k,p}(\Omega)$.

It is well known by a theorem of Calderon-Stein that every bounded Lipschitz domain in $\mathbb{R}^N$ is an extension domain (see [50, 36]).

Theorem 2.9. *The Hartogs triangle T is a Sobolev extension domain.*

Proof. This is proved in the recent paper that the Hartogs triangle T is a uniform domain, hence an extension domain (see Theorem 2.12 in [2]). ☐

Now we consider the annulus domain $\Omega = B\setminus\overline{T}$, where B is the ball of radius 2 centered at 0. The following results are known for the annulus between the ball and the Hartogs triangle.

Theorem 2.10. *Let $\Omega = B\setminus\overline{T}$. Then*

(1) $H^{0,1}(\Omega)$ *is non-Hausdorff.*
(2) $H^{0,1}_{W^1}(\Omega)$ *is Hausdorff.*

(1) is proved in [33], while (2) is proved in the recent paper in [2].

Question. (1) Determine if $H^{0,1}_{L^2}(\Omega)$ is Hausdorff.

This question is equivalent to ask the following:

Question. (2) Determine if

$$H^{0,1}_{W^1}(T) = 0. \tag{2.3}$$

We refer the reader to the papers [2] and [33], where Question (1) and Question (2) are proved to be equivalent.

Remark. Question (1) is raised in [2] and is called the *Dolloar Bill* Problem since the shape of Ω appears in the U. S. dollar bill.

An earlier question when the domain is the annulus between the ball B and the the bidisk $P = \Delta\times\Delta$ is called the *Chinese Coin* Problem. It is proved in [9] that $H^{0,1}_{L^2}(B\setminus\overline{P})$ is Hausdorff. Hence $H^{0,1}_{W^1}(P) = 0$. In view of Theorem 2.8, it is plausible (2.3) could hold.

A more general question arises naturally.

Question. Let D be a bounded pseudoonvex domain $\mathbb{C}^n$ with Lipschitz boundary. Determine if

- $H^{0,1}_{W^1}(D) = 0$;
- $H^{0,1}(\overline{D}) = 0$.

We remark that if D has C^2 boundary, we have $H^{0,1}_{W^1}(D) = 0$. This is proved by Kohn [28] for domains with C^4 boundary and by Harrington [20]) for domains with C^2 boundary. The proof for $H^{0,1}_{W^1}(P) = 0$ in [9] is completely different from the proof of Kohn's theorem on Sobolev W^1 estimates for smooth domains (Theorem 2.2). There are many open questions related to the Hartogs triangle in $\mathbb{C}^2$. We refer the reader to [43, 2].

3. L^2 theory for $\overline{\partial}$ on pseudoconvex domains in $\mathbb{CP}^n$

Let Ω be a relatively compact domain in a complex Kähler manifold X with C^2-smooth boundary $b\Omega$ and Kähler metric ω. Let $\rho(z)$ be the signed distance function from z to $b\Omega$ such that $\rho(z) = -d(z, b\Omega)$ for $z\in\Omega$ and $\rho(z) = d(z, b\Omega)$ when $z\in X\setminus\Omega$. Let φ be a real-valued C^2 function on $\overline{\Omega}$. Let $L^2_{p,q}(\Omega, e^{-\varphi})$ be the space of (p,q)-forms u on Ω such that

$$\|u\|^2_\varphi = \int_\Omega |u|^2_\omega e^{-\varphi}dV < \infty.$$

We use $(\cdot,\ \cdot)_\varphi$ to denote the associated inner product. Let $\overline{\partial}^*_\varphi$ be the adjoint of the maximally defined $\overline{\partial}\colon L^2_{p,q}(\Omega, e^{-\varphi}) \to L^2_{p,q}(\Omega, e^{-\varphi})$.

Let Θ be the curvature term with respect to the Kähler metric ω and $|\overline{\nabla}u|^2 = \sum_{j=1}^n |\nabla_{\bar{L}_j}u|^2$, where $\{L_1,\dots,L_n\}$ is an orthonormal frame for $T^{1,0}(X)$. The following *Basic Identity* (see [53, 7]) is of fundamental importance in several complex variables and complex geometry.

Theorem 3.1 (**Bochner-Kodaira-Morrey-Kohn-Hörmander**). *Let Ω be a relatively compact domain in a Kähler manifold X with C^2-smooth boundary $b\Omega$. For any $u \in C^1_{p,q}(\overline{\Omega}) \cap \operatorname{dom}(\overline{\partial}^*)$, we have*

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi = \|\overline{\nabla} u\|^2_\varphi + (\Theta u,\ u)_\varphi + ((\partial\overline{\partial}\varphi)u,\ u)_\varphi + \int_{b\Omega} \langle(\partial\overline{\partial}\rho)u,\ u\rangle e^{-\varphi} dS \tag{3.1}$$

where dS is the induced surface element on $b\Omega$,

The Kähler form associated with the Fubini-Study metric ω in the complex projective space $\mathbb{CP}^n$ is given by

$$\omega = i\partial\overline{\partial}\log(1+|z|^2) \tag{3.2}$$

$$= i\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}(z)\, dz_\alpha \wedge d\bar{z}_\beta \tag{3.3}$$

in local inhomogeneous coordinates, where

$$g_{\alpha\bar{\beta}}(z) = \frac{\partial^2 \log(1+|z|^2)}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{(1+|z|^2)\delta_{\alpha\bar{\beta}} - \bar{z}_\alpha z_\beta}{(1+|z|^2)^2}. \tag{3.4}$$

The volume form is then

$$dV_\omega = \det(g_{\alpha\bar{\beta}}(z)) dV_{\mathrm{E}} = \frac{1}{(1+|z|^2)^{n+1}} dV_{\mathrm{E}} \tag{3.5}$$

where dV_{E} is the Euclidean volume form. The curvature tensor is then given by

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}} g_{\bar{\beta}\gamma}.$$

It follows that the complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric has constant holomorphic sectional curvature 2 and its holomorphic bisectional curvature is bounded between 1 and 2. Furthermore, we have that if u is a (p,q)-form on $\mathbb{CP}^n$ with $q \geq 1$, then

$$\langle \Theta u, u\rangle = 0, \quad \text{if} \quad p = n; \qquad \langle \Theta u, u\rangle \geq 0, \quad \text{if} \quad p \geq 1; \tag{3.6}$$

and

$$\langle \Theta u, u\rangle = q(2n+1)|u|^2 \quad \text{if} \quad p = 0. \tag{3.7}$$

For a proof of these results, see [53] or Proposition A.5 in the Appendix in [7].

3.1. L^2 **theory of $\overline{\partial}$ for $(0,q)$-forms.** The positive curvature gives the following estimates for $(0,q)$-forms.

Proposition 3.2. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with C^2 boundary and $1 \leq q \leq n-1$. Let φ be a plurisubharmonic function on Ω. Then*

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi \geq q(2n+1)\|u\|^2_\varphi \tag{3.8}$$

for any $(0,q)$-form $u \in \operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}^*_\varphi)$.

Proof. This is a direct consequence of the curvature property (3.1) and (3.1):

$$\begin{aligned} \|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi &= \|\overline{\nabla} u\|^2_\varphi + (\Theta u,\ u)_\varphi + ((\partial\overline{\partial}\varphi)u, u)_\varphi + \int_{b\Omega} \langle(\partial\overline{\partial}\rho)u,\ u\rangle e^{-\varphi} dS \\ &\geq (\Theta u,\ u)_\varphi \geq q(2n+1)\|u\|^2_\varphi. \end{aligned} \tag{3.9}$$

$\square$

Theorem 3.3. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ such that $\overline{\Omega} \neq \mathbb{CP}^n$ and $1 \leq q \leq n-1$. Let φ be a plurisubharmonic function on Ω. For any $\overline{\partial}$-closed $(0,q)$-form $f \in L^2_{0,q}(\Omega, e^{-\varphi})$, there exists a $(0,q-1)$-form $u \in L^2_{0,q-1}(\Omega, e^{-\varphi})$ such that $\overline{\partial} u = f$ with*

$$\|u\|^2_\varphi \leq \frac{1}{q(2n+1)} \|f\|^2_\varphi. \tag{3.10}$$

Proof. If Ω has C^2 boundary, estimate (3.10) is then a consequence of (3.9). The general case is then proved by exhausting Ω from inside by pseudoconvex domains with smooth boundaries. □

Corollary 3.4. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with the Euclidean diameter R, where $R = \sup_{z,z' \in \Omega} |z - z'|$. Then for any $f \in L^2_{p,q}(\Omega)$ with $\overline{\partial} f = 0$, there is a $(p, q-1)$-form $u \in L^2_{(p,q-1)}(\Omega)$ such that $\overline{\partial} u = f$ with*

$$\|u\|^2 \leq C_{n,q} R^2 \|f\|^2 \tag{3.11}$$

where $C_{n,q}$ is a constant depending only on n and q, but is independent of Ω.

Proof. Since $\Omega \in \mathbb{C}^n$ equipped with the Euclidean metric, we may assume that $p = 0$.

From (3.10) with $\varphi = 0$, we have

$$\int_\Omega |u|^2_\omega dV_\omega \leq \frac{1}{q(2n+1)} \int_\Omega |f|^2_\omega dV_\omega. \tag{3.12}$$

First we assume that $\Omega \subset \mathbb{C}^n$ with diameter $R < 1$. It follows from (3.5) that

$$\int_\Omega |u|^2 dV_E \leq C_n \int_\Omega |u|^2_\omega dV_\omega \leq \frac{C_n}{q(2n+1)} \int_\Omega |f|^2_\omega dV_\omega \leq C_{n,q} \int_\Omega |f|^2 dV_E \tag{3.13}$$

where $C_{n,q}$ is a constant depending only on n and q, but is independent of Ω.

For general bounded domain $\Omega \subset \mathbb{C}^n$ with diameter $1 \leq R < \infty$, the estimate (3.11) follows from scaling argument with $z \to z/R$.

□

We remark that Corollary 3.4 gives an alternative proof of the Hörmander's L^2 theorem (Theorem 2.1).

3.2. **Bounded plurisubharmonic exhaustion functions.** The classical Oka's lemma states that for a pseudoconvex domain in $\mathbb{C}^n$, any distance function δ for Ω satisfies

$$i\partial\overline{\partial}(-\log \delta) \geq 0. \tag{3.14}$$

Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$. We first recall the following theorem of Takeuchi [52] (see also [5] for an alternative approach).

Theorem 3.5 (**Takeuchi**). *Let $\Omega \subset\subset \mathbb{CP}^n$ be a pseudoconvex domain. Then the distance function δ satisfies*

$$i\partial\overline{\partial}(-\log \delta) \geq C_0 \omega \tag{3.15}$$

as currents where $C_0 > 0$ and ω is the Kähler form of the Fubini-Study metric on $\mathbb{CP}^n$.

Definition 3.1. A distance function δ satisfying (3.15) is said to satisfy the *strong Oka's lemma*.

In a Kähler manifold with strictly positive holomorphic curvature, Ohsawa-Sibony [38] showed that there exists a bounded plurisubharmonic exhaustion function for any pseudoconvex domain with C^2 boundary.

Theorem 3.6 (Ohsawa-Sibony). *Let $\Omega \subset\subset X$ be a pseudoconvex domain with C^2 boundary in a complete Kähler manifold M. Assume that the holomorphic curvature of M is strictly positive. let $\delta(x) = d(x, b\Omega)$ be the distance function to $b\Omega$ with the Kähler metric ω. Then there exists t with $0 < t \le 1$ and a constant $C > 0$ such that*

$$i\partial\overline{\partial}(-\delta^t) \ge C\delta^t\omega. \tag{3.16}$$

In particular, Theorem 3.6 holds for $\mathbb{CP}^n$. We will show that the strong Oka's lemma is equivalent to the eixistence of a Hölder continuous strictly plurisubharmonic exhaustion function for Ω. This follows from ideas used in [7, 6, 22]).

Proposition 3.7. *Let M be a complex hermitian manifold with metric ω and let $\Omega \subset\subset M$ be a pseudoconvex domain with C^2 boundary $b\Omega$. Let δ be a C^2 distance function for Ω. The following conditions are equivalent:*

(1) *The function δ satisfies*

$$i\partial\overline{\partial}(-\log\delta) \ge C_0\omega$$

for some constant $C_0 > 0$.

(2) *There exists a constant $0 < t \le 1$ such that*

$$i\partial\overline{\partial}(-\delta^t) \ge C_t\delta^t(\omega + i\frac{\partial\delta \wedge \overline{\partial}\delta}{\delta^2})$$

for some $C_t > 0$.

Proof. We first prove (1) implies (2).

From (1), we have

$$\langle \partial\overline{\partial}(-\log\delta), a \wedge \bar{a}\rangle \ge C_0|a|^2 \tag{3.17}$$

where a is any $(1,0)$-vector.

To prove (2), we first prove that there exists t_0 such that

$$i\partial\overline{\partial}(-\delta^{t_0}) \ge 0. \tag{3.18}$$

Observe that inequality (3.18) is equivalent to

$$i\frac{\partial\overline{\partial}(-\delta)}{\delta} + (1 - t_0)\frac{i\partial\delta \wedge \overline{\partial}\delta}{\delta^2} \ge 0. \tag{3.19}$$

Comparing (3.19) with

$$i\partial\overline{\partial}(-\log\delta) = i\frac{\partial\overline{\partial}(-\delta)}{\delta} + \frac{i\partial\delta \wedge \overline{\partial}\delta}{\delta^2}, \tag{3.20}$$

we see that (3.18) is equivalent to

$$i\partial\overline{\partial}(-\log\delta) \ge t_0\frac{i\partial\delta \wedge \overline{\partial}\delta}{\delta^2}. \tag{3.21}$$

We will show that (3.21) holds for some $0 < t_0 \le 1$.

Near a boundary point, we choose a special orthonormal basis $w_1, \cdots, w_n$ for $(1,0)$-forms such that $w_n = \sqrt{2}\partial(-\delta)$. Let $L_1, \cdots, L_n$ be its dual and let a be any $(1,0)$-vector. We decompose $a = a_\tau + a_\nu$ where $a_\nu = \langle a, L_n\rangle$ is the complex normal component and a_τ is the complex tangential component. We have

$$\begin{aligned}&\langle \partial\overline{\partial}(-\log\delta), a\wedge\bar a\rangle\\ &= \langle \frac{\partial\overline{\partial}(-\delta)}{\delta}, a_\tau\wedge\bar a_\tau\rangle + 2\,\mathrm{Re}\,\langle \frac{\partial\overline{\partial}(-\delta)}{\delta}, a_\tau\wedge\bar a_\nu\rangle\\ &+ \langle \frac{\partial\overline{\partial}(-\delta)}{\delta}, a_\nu\wedge\bar a_\nu\rangle + \frac{|a_\nu|^2}{\delta^2}.\end{aligned} \tag{3.22}$$

From (3.20) and (3.22), we have

$$\langle \partial\overline{\partial}(-\log\delta), a_\tau\wedge\bar a_\tau\rangle = \langle \frac{\partial\overline{\partial}(-\delta)}{\delta}, a_\tau\wedge\bar a_\tau\rangle \ge C_0|a_\tau|^2.$$

Thus from (3.22),

$$\begin{aligned}\langle \partial\overline{\partial}(-\log\delta), a\wedge\bar a\rangle \ge |a_\tau|^2 + \frac{|a_\nu|^2}{\delta^2} - 2|\langle \frac{\partial\overline{\partial}(-\delta)}{\delta}, a_\tau\wedge\bar a_\nu\rangle|\\ - |\langle \frac{\partial\overline{\partial}(-\delta)}{\delta}, a_\nu\wedge\bar a_\nu\rangle|.\end{aligned} \tag{3.23}$$

Since δ is a C^2 function, all second derivatives of δ are bounded. Thus for any $\epsilon > 0$, we have

$$|\langle \frac{\partial\overline{\partial}(-\delta)}{\delta}, a_\tau\wedge\bar a_\nu\rangle| \le C\left(\frac{1}{\epsilon}|a_\tau|^2 + \epsilon\frac{|a_\nu|^2}{\delta^2}\right), \tag{3.24}$$

and

$$|\langle \frac{\partial\overline{\partial}(-\delta)}{\delta}, a_\nu\wedge\bar a_\nu\rangle| \le C\frac{\epsilon}{\delta^2}|a_\nu|^2 \tag{3.25}$$

on a sufficiently small neighborhood U_ϵ near the boundary.

Substituting (3.24) and (3.25) into (3.21) and choosing ϵ sufficiently small, we have

$$\langle \partial\overline{\partial}(-\log\delta), a\wedge\bar a\rangle \ge \frac{1}{2}\frac{|a_\nu|^2}{\delta^2} - K|a_\tau|^2 \tag{3.26}$$

for some large constant K depending on ϵ.

Multiplying (3.17) by K/C_0 and adding it to (3.26), we have

$$(K+1)\langle \partial\overline{\partial}(-\log\delta), a\wedge\bar a\rangle \ge \frac{1}{2}\frac{|a_\nu|^2}{\delta^2}.$$

This proves (3.21) with $t_0 = C_0/2(K+1)$ near the boundary, or equivalently, (3.19) is proved near the boundary. Since Ω is Stein, on any relatively compact submanifold $\Omega' \subset\subset \Omega$, there exists a bounded strictly plurisubharmonic function on $\overline{\Omega}'$.

By standard arguments one can extend δ so that δ is the distance function near the boundary and δ satisfies (3.21) in Ω. Since (3.21) implies (3.18), we have proved that

$$i\frac{\partial\overline{\partial}(-\delta)}{\delta} + (1-t_0)\frac{i\partial\delta\wedge\overline{\partial}\delta}{\delta^2} \ge 0. \tag{3.27}$$

For any $0 < t < t_0$, we have from (3.27)

$$i\partial\overline{\partial}(-\delta^t) = it\delta^t\left(\frac{\partial\overline{\partial}(-\delta)}{\delta} + (1-t)\frac{\partial\delta\wedge\overline{\partial}\delta}{\delta^2}\right) \ge C_t\delta^t\frac{i\partial\delta\wedge\overline{\partial}\delta}{\delta^2} \tag{3.28}$$

for some $C_t = t(t_0 - t) > 0$. On the other hand,

$$\partial\overline{\partial}(-\log\delta) - t\frac{\partial\delta\wedge\overline{\partial}\delta}{\delta^2} \geq (1-\frac{t}{t_0})\partial\overline{\partial}(-\log\delta) \geq C_0(1-\frac{t}{t_0})\omega.$$

Thus

$$i\partial\overline{\partial}(-\delta^t) = it\delta^t(\partial\overline{\partial}(-\log\delta) - t\frac{\partial\delta\wedge\overline{\partial}\delta}{\delta^2}) \geq C_t\omega. \tag{3.29}$$

Combining (3.28) and (3.29), we have proved (2).

It is easy to see that (2) implies (1). Using (2) and (3.28), we have

$$i\partial\overline{\partial}(-\log\delta) \geq C_t(\omega + \frac{\partial\delta\wedge\overline{\partial}\delta}{\delta^2}) \tag{3.30}$$

which is even stronger than (1).

□

Remark. (1) If Ω is a pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary, let $\tilde{\delta} = \delta e^{-\lambda|z|^2}$ with $\lambda > 0$. Diederich-Fornaess [13] first showed that then (2) holds with δ substituted by $\tilde{\delta}$ for some large λ. For this reason, the best exponent t_0 is sometimes called the Diederich-Fornaess exponent.

From Proposition 3.7, we proved that actually any $\lambda > 0$ suffices. This follows from Oka's lemma (3.14) and

$$i\partial\overline{\partial}(-\log\tilde{\delta}) = i\partial\overline{\partial}(-\log\delta) + \lambda\omega_E \geq \lambda\omega_E$$

where ω_E is the Kähler form for $\mathbb{C}^n$.

(2) If Ω is a Lipschitz bounded pseudoconvex domain in $\mathbb{C}^n$ or a Stein manifold, Demailly [12] showed that there exists a bounded strictly plurisubharmonic function in Ω (see also Kerzman-Rosay [30] for the C^1 case).

(3) Theorem 3.6 also holds for pseudoconvex domains with Lipschitz boundary in $\mathbb{CP}^n$ (see Harrington [21]).

3.3. **Applications of the Ohsawa-Sibony Theorem.** Based on an earlier result of Berndtsson and Charpentier [3, Theorem 2.3] (see also [23, 7]), we obtain the following theorem.

Theorem 3.8. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary. Then the $\overline{\partial}$-Neumann Laplacian $\Box$ has a bounded inverse N on $L^2_{p,q}(\Omega)$ and for $u \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*)$,*

$$\frac{q\eta K}{4}\|u\|^2 \leq \|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2. \tag{3.31}$$

Furthermore, the operator N is bounded from $W^s_{p,q}(\Omega) \to W^s_{p,q}(\Omega)$ with

$$\|\overline{\partial}^*Nu\|^2_s \leq C_\eta\|u\|^2_s; \qquad \|\overline{\partial}Nu\|^2_s \leq C_\eta\|u\|^2_s. \tag{3.32}$$

for any $u \in W^s_{p,q}(\Omega)$ with $0 < s < t_0/2$.

The exponent t_0 in Theorem 3.6 is closely related to the Levi-flat hypersurfaces in $\mathbb{CP}^n$, see the articles by Fu-Shaw [17] and Adachi-Brinkschulte [1].

Theorem 3.9. *Let Ω be a bounded Stein domain with C^2 boundary in a complex manifold M of dimension n. If the Diederich-Fornæss index t_0 of Ω is greater than k/n, $1 \leq k \leq n-1$, then Ω has a boundary point at which the Levi form has rank $\geq k$.*

The proof of Theorem 3.9 is based on an earlier reuslt of Nemirovski [37], where $k = n$ and $t_0 = 1$.

4. Some Open Questions for $\overline{\partial}$ in $\mathbb{CP}^n$

4.1. **Non-vanishing Results for L^2 Dolbeault Cohomolgy.** The situation is different if the pseudoconvex domain Ω is not Lipschitz in $\mathbb{CP}^n$. Theorem 3.8 might not hold for (p, q)-forms, where $p \neq 0$.

We consider the Hartogs triangles in $\mathbb{CP}^2$. We denote the homogeneous coordinates by $[z_0, z_1, z_2]$. On the domain where $z_0 \neq 0$, we set $z = \frac{z_1}{z_0}$ and $w = \frac{z_2}{z_0}$. Define the domains $\mathbb{H}^+$ and $\mathbb{H}^-$ by

$$\mathbb{H}^+ = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| < |z_2|\}$$
$$\mathbb{H}^- = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| > |z_2|\}$$

then $\mathbb{H}^+ \cap \mathbb{H}^- = \emptyset$ and $\overline{\mathbb{H}}^+ \cup \overline{\mathbb{H}}^- = \mathbb{CP}^2$. These domains are called Hartogs' triangles in $\mathbb{CP}^2$. They are not Lipschitz domains at $[1, 0, 0]$. The Hartogs triangles are both pseudoconvex and pseudoconcave. They are especially important since they provide simple examples of singular Levi-flat hypersurfaces (see [8]).

Theorem 4.1. *The L^2 Dolbeault cohomology group $H^{2,1}_{L^2}(\mathbb{H}^-)$ (or $H^{2,1}_{L^2}(\mathbb{H}^+)$) is infinite dimensional.*

The proof of Theorem 4.1 essentially follows from [35] and the recent work [2]. In [35], it is proved that $H^{2,1}_{\overline{\partial}_s, L^2}(\mathbb{H}^-)$ is infinite dimensional where $\overline{\partial}_s$ is the $\overline{\partial}$ closure in the strong sense of Kohn. In [2], we show that the strong closure $\overline{\partial}_s$ is equal to the weak closure $\overline{\partial}$ (in Hörmander's sense). Combining these two results, Theorem 4.1 follows.

From Theorem 3.3, we have $H^{0,1}(\mathbb{H}^-) = 0$. The following question remains open.

Question. (1) Determine if $H^{2,1}_{L^2}(\mathbb{H}^-)$ is Hausdorff.

(2) Determine if $H^{1,1}_{L^2}(\mathbb{H}^-) = 0$.

We refer the reader to the papers [19, 26, 39, 41, 42] for related results.

4.2. **Sobolev Estimates for $\overline{\partial}$.** Suppose that Ω is a pseudoconvex domain in $\mathbb{CP}^n$. Not much is known for Sobolev estimates for $\overline{\partial}$ if Ω is a smooth pseudoconvex domain in $\mathbb{CP}^n$ except for small $s < \frac{1}{2}$ (see Theorem 3.8).

However, if we consider the complement of a pseudoconvex domain, *a pseudoconcave domain*, then the following results are known.

Theorem 4.2. *Let $\Omega^+ = \mathbb{CP}^n \setminus \Omega$, where Ω is a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary. Then*

- *$H^{p,q}_{W^1}(\Omega^+) = 0$ for all $0 \leq p \leq n$ and $p \neq q$;*
- *$H^{p,q}(\Omega^+)$ is Hausdorff for all $p = q$ or $q = n - 1$.*

Here we only need that $b\Omega^+$ to be Lipschitz (see [7, 6] and the recent paper [18]). We finish the paper by raising the following questions.

Question. Let Ω be a bounded pseudoconvex domain in $\mathbb{CP}^n$ with smooth boundary. For every $0 \leq p \leq n$, $0 < q < n$, determine for $k \in \mathbb{N}$,

$$H^{p,q}_{W^k}(\Omega) = 0. \tag{4.1}$$

It is especially interesting to know if (4.1) holds for $k = 1$ and $n = 2$. An affirmative answer to this question will complete the proof of the nonexistence of Levi-flat hypersurfaces in $\mathbb{CP}^2$ in [7]. For the nonexistence of Levi-flat hypersurfaces in $\mathbb{CP}^n$ with $n \geq 3$, see [32], [49], [7] and [6]). Sibony's most recent paper [48] is very closely related to this question. If (4.1) is not possible, perhaps one should ask a weaker result.

Question. Determine if

$$H^{p,q}_{W^{\frac{1}{2}}}(\Omega) = 0. \tag{4.2}$$

The Sobolev 1/2-estimates are related to the closed range of $\overline{\partial}_b$. If Ω is a domain in $\mathbb{C}^n$ with C^2 boundary, then we have L^2 existence for $\overline{\partial}_b$ (see [40, 4, 29] and [10]). We remark that Theorem 3.8 yields only Sobolev regularity for $s < \frac{1}{2}$. The lack of Sobolev estimates for $\overline{\partial}$ in the complex projective space $\mathbb{CP}^n$ is one of the most challenging open questions.

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References

[1] M. Adachi and J. Brinkschulte, *A global estimate for the Diederich-Fornaess index of weakly pseudoconvex domains*, Nagoya Math. J. **220** (2015), 67–80.

[2] A. Burchard, J. Flynn, G. Lu and M.-C. Shaw *Extendability and the $\bar{\partial}$ operator on the Hartogs triangle*, Math. Zeit. **301** (2022), 2771-2792.

[3] B. Berndtsson and Ph. Charpentier, *A Sobolev mapping property of the Bergman knernel*, Math. Z. **235** (2000), 1–10.

[4] H. Boas and M.-C. Shaw, *Sobolev Estimates for the Lewy Operator on Weakly pseudo-convex boundaries*, Math. Annalen **274** (1986), 221-231.

[5] J. Cao and M.-C. Shaw, *A new proof of the Takeuchi Theorem*, Lecture Notes of Seminario Interdisp. di Mate. **4** (2005), 65–72.

[6] J. Cao and M.-C. Shaw, *The $\overline{\partial}$–Cauchy problem and nonexistence of Lipschitz Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$*, Math. Z. **256** (2007), 175–192.

[7] J. Cao, M.-C. Shaw, and L. Wang, *Estimates for the $\overline{\partial}$-Neumann problem and nonexistence of C^2 Levi-flat hypersurfaces in $\mathbb{C}P^n$*, Math. Z. **248** (2004), 183–221. Erratum, 223–225.

[8] D. Chakrabarti and M.-C. Shaw, *L^2 Serre duality on domains in complex manifolds and applications*, Trans. Amer. Math. Society **3**64 (2012), 3529–3554.

[9] D. Chakrabarti, C. Laurent-Thiébaut and M.-C. Shaw, *On the L^2-Dolbeault cohomology of annuli*, Indiana Univ. Math. J. **67** (2018), 831–857.

[10] S.-C. Chen and M.-C. Shaw, Partial differential equations in several complex variables. AMS/IP Studies in Advanced Mathematics, vol. 19, International Press, 2001.

[11] J. Chaumat and A.-M. Chollet, *Régularité höldérienne de l'opérateur $\overline{\partial}$ sur le triangle de Hartogs*, Ann. Inst. Fourier (Grenoble), **41** (1991), 867–882.

[12] J.-P. Demailly, *Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète*, Ann. Sci. École Norm. Sup. **15** (1982), 457–511.

[13] K. Diederich and J. E. Fornæss, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), 129–141.

[14] E. Fornaess and N. Sibony, *On L^p estimates for $\overline{\partial}$*, Several Complex Variables and Complex Geometry, Part 3 (Santa Cruz, CA 1989), Proc. Symp. Pure Math. **52**, (1991), 129-163.

[15] E. Fornaess and N. Sibony, *Smooth pseudoconvex domains in $\mathbb{C}^2$ for which the corona theorem and L^p estimates for $\overline{\partial}$ fail*, Complex analysis and geometry, Univ. Ser. Math. Plenum, (1993), 209-222.

[16] G. B. Folland and J. J. Kohn, The Neumann problem for the Cauchy-Riemann complex, Princeton University Press, 1972.

[17] S. Fu and M.-C. Shaw, *The Diederich-Fornæss exponent and non-existence of Stein domains with Levi-flat boundaries*, J. Geom. Anal. **26** (2016), 220–230.

[18] S. Fu and M.-C. Shaw, *Sobolev estimates and duality for $\overline{\partial}$ on domains in $\mathbb{CP}^n$*, Pure Appl. Math. Q. **18** (2022), 503–529.

[19] S. Fu, C. Laurent-Thiébaut and M.-C. Shaw, *Hearing pseudoconvexity in Lipschitz domains with holes with $\bar{\partial}$*, Math. Zeit. **2**87 (2017), 1157-1181.

[20] P. S. Harrington, *Sobolev estimates for the Cauchy-Riemann complex on $\mathbb{C}^1$ pseudoconvex domains*, Math. Zeitschrift **262** (2009), 199–217.

[21] P. S. Harrington, *Bounded plurisubharmonic exhaustion functions for Lipschitz pseudoconvex domains in $\mathbb{CP}^n$*, J. Geom. Anal. (2017), 3404-3440.

[22] P. Harrington and M.-C. Shaw, *The Strong Oka's Lemma, bounded plurisubharmonic functions and the $\overline{\partial}$-Neumann problem*, Asian J. Math. **1**1 (2007), 127-140.

[23] Henkin, G., and Iordan, A.: *Regularity of $\overline{\partial}$ on pseudoconcave compacts and applications*, Asian J. Math. **4** (2000), no. 4, 855–883. Erratum: Asian J. Math. **7** (2003), 147–148.

[24] Hörmander, L.: *L^2 estimates and existence theorems for the $\overline{\partial}$ operator*, Acta Math. **113** (1965), 89–152.

[25] Hörmander, L.: *An introduction to complex analysis in several variables. 3rd ed.* North Holland, Amsterdam, 1990.

[26] Hörmander, L.: *The null space of the $\overline{\partial}$-Neumann operator*, Ann. Inst. Fourier (Grenoble) **54** (2004), 1305-1369.

[27] J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds, I*, Ann. of Math. (2) **78** (1963), 112–148.

[28] J. J. Kohn, *Global regularity for $\overline{\partial}$ on weakly pseudoconvex manifolds*, Trans. Amer. Math. Soc., **181** (1973) 273–292.

[29] Kohn, J. J. *The range of the tangential Cauchy-Riemann operator*, Duke Math. J., **53** (1986) 525–545.

[30] N. Kerzman and J.-P. Rosay, *Fonctions plurisousharmoniques d'exhaustion bornées et domaines taut*, Math. Ann. **257** (1981), 171–184.

[31] S. G. Krantz, Function Theory of Several Complex Variables, 2nd edition, Wadswort Belmont, California 1992.

[32] A. Lins Neto, *A note on projective Levi flats and minimal sets of algebraic foliations*, Ann. Inst. Fourier **49**(1999), 1369–1385.

[33] C. Laurent-Thiébaut and M.-C. Shaw, *On the Hausdorff property of some Dolbeault cohomology groups*, Math. Zeitschrift **274** (2013), 1165–1176.

[34] C. Laurent-Thiébaut and M.-C. Shaw, *Non-closed range property for the Cauchy-Riemann operator*. Analysis and Geometry, Springer Proc. Math. Stat. **127** (2015), 207–218.

[35] C. Laurent-Thiébaut and M.-C. Shaw, *Solving $\overline{\partial}$ with prescribed support on Hartogs triangles in $\mathbb{C}^2$ and $\mathbb{CP}^2$*, Trans. Amer. Math. Soc. **371**(2019), 6531-6546.

[36] J.-L. Lions and E. Magenes *Non-Homogeneous Boundary Value Problems and Applications, Volume I* Springer-Verlag, New York 1972.

[37] S. Nemirovskii, *Stein domains with Levi-plane boundaries on compact complex surfaces* (Russian) Mat. Zametki **66** (1999), 632–635; translation in Math. Notes **66** (1999), 522–525.

[38] T. Ohsawa and N. Sibony, *Bounded P.S.H functions and pseudoconvexity in Kähler manifolds*, Nagoya Math. J. **149** (1998), 1–8.

[39] M.-C. Shaw, *Global solvability and regularity for $\bar{\partial}$ on an annulus between two weakly pseudo-convex domains*, Trans. Amer. Math. Society **2**91 (1985), 255-267.

[40] M.-C. Shaw, *L^2 estimates and existence theorems for the tangential Cauchy-Riemann complex*, Invent. Math. **8**2 (1985), 133–150.

[41] M.-C. Shaw, *The closed range property for $\overline{\partial}$ on domains with pseudoconcave boundary*, Proceedings for the Fribourg conference, Trends in Mathematics, (2010), 307-320.

[42] M.-C. Shaw, *Duality Between Harmonic and Bergman Spaces*, Contemporary Mathematics, Proceedings of the conference on Several Complex Variables, Marrakech, (2011) 161-172.

[43] M.-C. Shaw, *The Hartogs triangle in complex analysis.* Geometry and topology of submanifolds and currents, 105–115. Contemp. Math., 646, Amer. Math. Soc., 2015.

[44] N. Sibony, *Prolongement des fonctions holomorphes bornées et métrique de Carathéodory*, Invent. Math. **29** (1975), 205–230.

[45] N. Sibony, *Un exemple de domaine pseudoconvexe régulier où l'équation $\overline{\partial}u = f$ n'admet pas de solution bornée pour f bornée*, Invent. Math. **62** (1980), 235-242.

[46] N. Sibony, *Some aspects of weakly pseudoconvex domains*, Several Complex Variables and Complex Geometry, Part I (Santa Cruz, CA 1989), Proc. Symp. Pure Math. **52**, (1991), 199-231.

[47] N. Sibony, *On Hölder estimates for* $\overline{\partial}$, Several Complex Variables, Math. Notes, Princeton Univ. Press **38**, (1993), 587-599.

[48] N. Sibony, *Pseudoconvex domains with smooth boundary in projective spaces*, Math. Zeit. **298** (2021), 625-637.

[49] Y.-T. Siu, *Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimension* ≥ 3, Ann. of Math. **151** (2000), 1217–1243.

[50] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Math. Series 30, Princeton University Press, Princeton, New Jersey, 1970.

[51] E. J. Straube, *Lectures on the* L^2*-Sobolev theory of the* $\overline{\partial}$*-Neumann problem*, ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2010.

[52] A. Takeuchi, *Domaines pseudoconvexes infinis et la métrique riemannienne dans un espace projectif*, J. Math. Soc. Japan **16** (1964), 159–181.

[53] H.-H. Wu, *The Bochner technique in differential geometry*, Math. Rep. **3** (1988), no. 2, i–xii and 289–538, Harwood Academic Publishers.

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