

L^2 -SOBOLEV THEORY FOR $\bar{\partial}$ ON DOMAINS IN $\mathbb{CP}^n$

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ABSTRACT. In this article, we study the range of the Cauchy-Riemann operator $\bar{\partial}$ on domains in the complex projective space $\mathbb{CP}^n$. In particular, we show that $\bar{\partial}$ does not have closed range in L^2 for (2,1)-forms on the Hartogs triangle in $\mathbb{CP}^2$. We also study the $\bar{\partial}$ -Cauchy problem on pseudoconvex domains and use it to prove the Sobolev estimates for $\bar{\partial}$ on pseudoconcave domains in $\mathbb{CP}^n$.

Dedicated to the memory of Professor Joseph J. Kohn

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1. INTRODUCTION

Since the fundamental work of Kohn for the $\bar{\partial}$ -Neumann problem on smooth bounded strongly pseudoconvex domains in $\mathbb{C}^n$, there has been tremendous progress on L^2 -Sobolev theory of the $\bar{\partial}$ -operator and the $\bar{\partial}$ -Neumann problem for bounded pseudoconvex domains in $\mathbb{C}^n$. In particular, Kohn proved the following landmark results (see [29, 30]).

Theorem 1.1 (Kohn 1963). *Let Ω be bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, $n \geq 2$. Then the $\bar{\partial}$ -Neumann operator*

$$N : L_{p,q}^2(\Omega) \rightarrow L_{p,q}^2(\Omega)$$

exists on Ω . Furthermore, the following sub-elliptic estimates hold for any $s \geq 0$

$$\|Nf\|_{s+1} \leq C\|f\|_s$$

and

$$\|\bar{\partial}^* Nf\|_{s+\frac{1}{2}} \leq C\|f\|_s$$

where $\|\cdot\|_s$ denotes the Sobolev s -norm $W^s(\Omega)$.

The solution $\bar{\partial}^* Nf$ is called the *canonical solution* (or Kohn's solution), since it is the energy minimizing solution.

Corollary 1.2. *Let $f \in C_{p,q}^\infty(\bar{\Omega})$ with $\bar{\partial}f = 0$ in Ω , where $0 \leq p \leq n$ and $1 \leq q \leq n$. There exists $u = \bar{\partial}^* Nf \in C_{p,q-1}^\infty(\bar{\Omega})$ satisfying $\bar{\partial}u = f$ in Ω .*

Another important result for $\bar{\partial}$ is the global regularity for $\bar{\partial}$ proved later by Kohn based on the weighted $\bar{\partial}$ -Neumann problem (see [31]).

Theorem 1.3 (Kohn 1973). *Let Ω be bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, $n \geq 2$. Let $f \in W_{p,q}^s(\Omega)$ with $\bar{\partial}f = 0$ in Ω , where $0 \leq p \leq n$ and $1 \leq q \leq n$ and $s > 0$, there exists $u_s \in W_{p,q-1}^s(\Omega)$ satisfying $\bar{\partial}u_s = f$ in Ω .*

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Corollary 1.4. *Let Ω be bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, $n \geq 2$. Let $f \in C_{p,q}^\infty(\bar{\Omega})$ with $\bar{\partial}f = 0$ in Ω , where $0 \leq p \leq n$ and $1 \leq q \leq n$. Then there exists $u \in C_{p,q-1}^\infty(\bar{\Omega})$ satisfying $\bar{\partial}u = f$ in Ω .*

For $s \geq 0$, let $H_{W^s}^{p,q}(\Omega)$ be the Dolbeault cohomology with Sobolev W^s coefficients defined by

$$H_{W^s}^{p,q}(\Omega) = \frac{\{f \in W_{p,q}^s(\Omega) \mid \bar{\partial}f = 0\}}{\{f \in W_{p,q}^s(\Omega) \mid f = \bar{\partial}u, u \in W_{p,q-1}^s(\Omega)\}}.$$

When $s = 0$, we also use the notation $H_{L^2}^{p,q}(\Omega)$ for the L^2 Dolbeault cohomology.

Similarly, we use $H^{p,q}(\Omega)$ and $H^{p,q}(\bar{\Omega})$ to denote the Dolbeault cohomology group for (p, q) -forms with $C^\infty(\Omega)$ and $C^\infty(\bar{\Omega})$ coefficients respectively. Using these notation, Theorem 1.3 can be formulated as

$$(1.1) \quad H_{W^s}^{p,q}(\Omega) = 0, \quad s > 0.$$

Corollary 1.4 can be written as

$$(1.2) \quad H^{p,q}(\bar{\Omega}) = 0$$

When $s = 0$, L^2 existence for $\bar{\partial}$ was proved by Hörmander [26] for bounded pseudoconvex convex domains, not necessarily with smooth boundary.

Theorem 1.5 (Hörmander 65). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then*

$$H_{L^2}^{p,q}(\Omega) = 0, \quad q > 0.$$

Similar results also hold for domains in a Stein manifold. Both L^2 and Sobolev regularity for $\bar{\partial}$ have numerous applications. Though the L^2 and Sobolev theory for $\bar{\partial}$ has been studied extensively for domains in $\mathbb{C}^n$, (see monographs [14, 27, 11, 47] for expositions on the subject), much less is known for $\bar{\partial}$ on domains in complex manifolds which are not Stein.

In this paper we present some recent results of the L^2 and Sobolev theory for $\bar{\partial}$ on domains in the complex projective space $\mathbb{CP}^n$. There are many known results on L^2 existence theorems for $\bar{\partial}$ on pseudoconvex domains in $\mathbb{CP}^n$. In particular, we have

$$H_{L^2}^{p,q}(\Omega) = 0 \quad \text{for all } 0 \leq p \leq n, \ 1 \leq q \leq n$$

for any pseudoconvex domain $\Omega \subset \mathbb{CP}^n$ with Lipschitz boundary $b\Omega$ (see Theorem 2.9).

One of the main results in this paper is to show that $\bar{\partial}$ might not have closed range on some pseudoconvex domain in $\mathbb{CP}^2$ if the Lipschitz condition is dropped. In particular, Theorem 1.5 does not hold for arbitrary pseudoconvex domains in $\mathbb{CP}^n$. The examples are given by the Hartogs triangles in $\mathbb{CP}^2$ (see Theorem 5.1 and Corollary 5.2).

The Hartogs triangles in $\mathbb{C}^2$ and $\mathbb{CP}^2$ are important examples of domains which are not Lipschitz. Hartogs triangles in $\mathbb{CP}^2$ are also interesting examples in complex foliation theory. They can be viewed as Levi-flat hypersurfaces with singularities since they are both pseudoconvex and pseudoconcave. Non-closed range properties for $\bar{\partial}$ on domains in complex manifolds with smooth Levi-flat boundaries have been obtained in [9] (see also [35]).

The plan of the paper is as follows. In section 2 we summarize some known results for $\bar{\partial}$ in L^2 for pseudoconvex domains in $\mathbb{CP}^n$ and give an alternative proof for Theorem 1.5 (see Theorem 2.6). In section 3, we prove the Sobolev estimates for $\bar{\partial}$ on pseudoconcave domains in $\mathbb{CP}^n$ using the $\bar{\partial}$ -Cauchy problem with weights. In section 4, we give some basic properties of holomorphic functions and forms on the Hartogs triangles. The non-closed

range property for $\bar{\partial}$ for $(2,1)$ -forms with L^2 coefficients on Hartogs triangles in $\mathbb{CP}^2$ is proved in section 5. Theorem 5.1 is a stronger assertion of the earlier results proved in [36] and [2]. The dimension of $H_{L^2}^{2,1}(\mathbb{H}^+)$ is not only infinite, but is uncountable.

We collect several open problems which are related to the L^2 or Sobolev estimates for $\bar{\partial}$ on domains in $\mathbb{C}^n$ or $\mathbb{CP}^n$ in section 6. Since Hartogs triangles in $\mathbb{CP}^2$ are both pseudoconvex and pseudconcave, Theorem 5.1 also provide examples that $\bar{\partial}$ does not have closed L^2 range on some *pseudoconcave* domains in $\mathbb{CP}^2$. We remark that L^2 theory for even pseudoconcave domains with smooth boundary remains an open question (see Problem 2). The missing ingredient is exactly the lack of Kohn's type Sobolev estimates (1.1) for pseudoconvex domains in $\mathbb{CP}^n$ (see Problem 1).

2. L^2 THEORY FOR $\bar{\partial}$ ON PSEUDOCONVEX DOMAINS IN $\mathbb{CP}^n$

In this section, we review some known results on pseudoconvex domains in $\mathbb{CP}^n$. Let ω be the Kähler form associated with the Fubini-Study metric in $\mathbb{CP}^n$. Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ such that $\bar{\Omega} \neq \mathbb{CP}^n$ and Ω has C^2 -smooth boundary $b\Omega$. Let δ be the distance function from z to $b\Omega$.

Let $\delta(z) = \text{dist}(z, b\Omega)$ be the distance, with respect to the Fubini-Study metric, from z to the boundary $b\Omega$. Let $\Omega_\varepsilon = \{z \in \Omega \mid \delta(z) > \varepsilon\}$. It then follows from Takeuchi's theorem [48] that there exists a universal constant $K_0 > 0$ such that

$$(2.1) \quad i\partial\bar{\partial}(-\log \delta) \geq K_0\omega$$

on Ω . In particular, there exists $\epsilon_0 > 0$ such that

$$(2.2) \quad \partial\bar{\partial}(-\delta)(\zeta, \bar{\zeta}) \geq K_0\epsilon|\zeta|_\omega^2$$

for all $\zeta \in T_x^{1,0}(b\Omega_\epsilon)$ for $0 \leq \epsilon \leq \epsilon_0$. (See also [19, 6] for different proofs of Takeuchi's theorem.)

Using (2.2), we have the following theorem.

Theorem 2.1 (Takeuchi). *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ such that $\bar{\Omega} \neq \mathbb{CP}^n$. Then the Dolbeault cohomology group $H^{p,q}(\Omega) = 0$ for all $q > 0$.*

2.1. L^2 existence theorems for $(0, q)$ -forms. Let $L_{p,q}^2(\Omega)$ be the space of (p, q) -forms u on Ω with respect to the Fubini-Study metric ω such that

$$\|u\|_\omega^2 = \int_\Omega |u|_\omega^2 dV_\omega < \infty.$$

We will also use $(\cdot, \cdot)_\omega$ to denote the associated inner product. Let $\bar{\partial}: L_{p,q-1}^2(\Omega) \rightarrow L_{p,q}^2(\Omega)$ be the weak maximal closure of $\bar{\partial}$ and let $\bar{\partial}_\omega^*$ be the Hilbert space adjoint of $\bar{\partial}$. We now recall an integration by parts formula.

Let $L_1, \dots, L_n$ be a local orthonormal frame field of type $(1,0)$ and $\phi^1, \dots, \phi^n$ be the coframe field. For a (p, q) -form u , we set

$$\langle \Theta u, u \rangle_\omega = \sum_{j,k=1}^n \langle \bar{\phi}^j \wedge (\bar{L}_k \lrcorner R(L_j, \bar{L}_k)u), u \rangle_\omega,$$

where R is the curvature operator on (p, q) -forms with respect to the Fubini-Study metric and $\lrcorner$ is the usual contraction operator (see e.g [5, 18]). We have that if u is a (p, q) -form on $\mathbb{CP}^n$ with $q \geq 1$, then

$$(2.3) \quad \langle \Theta u, u \rangle_\omega = 0, \quad \text{if } p = n; \quad \langle \Theta u, u \rangle_\omega \geq 0, \quad \text{if } p \geq 1;$$

and

$$(2.4) \quad \langle \Theta u, u \rangle_\omega = q(2n+1)|u|^2 \quad \text{if } p = 0.$$

With the above notations, we can now state the following *Basic Identity* (see [50, 45, 5]).

Theorem 2.2 (Bochner-Kodaira-Morrey-Kohn). *Let Ω be a domain with C^2 boundary $b\Omega$ in $\mathbb{CP}^n$ and let ω be the Fubini-Study metric. For any $u \in C_{p,q}^1(\bar{\Omega}) \cap \text{dom}(\bar{\partial}_\omega^*)$, we have*

$$(2.5) \quad \|\bar{\partial}u\|_\omega^2 + \|\bar{\partial}_\omega^*u\|_\omega^2 = \|\bar{\nabla}u\|_\omega^2 + (\Theta u, u)_\omega + \int_{b\Omega} \langle (\partial\bar{\partial}\rho)u, u \rangle_\omega dS_\omega$$

where dS_ω is the induced surface element on $b\Omega$ and $\|\bar{\nabla}u\|_\omega^2 = \sum_{j=1}^n \|\nabla_{\bar{L}_j}u\|_\omega^2$.

For a proof of these results, see [50] or Proposition A.5 in [5]. We first exploit the positivity of the curvature Θ for $(0, q)$ -forms. When $p = 0$, we have the following proposition using (2.4).

Proposition 2.3. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with C^2 boundary and $1 \leq q \leq n-1$. Then*

$$(2.6) \quad \|\bar{\partial}u\|_\omega^2 + \|\bar{\partial}_\omega^*u\|_\omega^2 \geq q(2n+1)\|u\|_\omega^2$$

for any $(0, q)$ -form $u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}_\omega^*)$.

Theorem 2.4 (L^2 existence for $(0, q)$ -Forms). *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ such that $\bar{\Omega} \neq \mathbb{CP}^n$ and $1 \leq q \leq n-1$. For any $\bar{\partial}$ -closed $(0, q)$ -form $f \in L_{0,q}^2(\Omega)$, there exists a $(0, q-1)$ -form $u \in L_{0,q-1}^2(\Omega)$ such that $\bar{\partial}u = f$ with*

$$(2.7) \quad \|u\|_\omega^2 \leq \frac{1}{q(2n+1)}\|f\|_\omega^2.$$

Proof. If Ω has C^2 boundary, estimate (2.7) is then a consequence of (2.6). The general case is then proved by exhausting Ω from inside by pseudoconvex domains with smooth boundaries. $\square$

Notice that the constant $1/q(2n+1)$ in (2.7) is independent of the diameter of the domain in $\mathbb{CP}^n$ with respect to the Fubini-Study metric.

Corollary 2.5. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ such that $\bar{\Omega} \neq \mathbb{CP}^n$. We have*

$$H_{L^2}^{0,q}(\Omega) = 0 \quad \text{for every } q > 0.$$

Theorem 2.4 gives an alternative proof of Hörmander's L^2 existence for $\bar{\partial}$ for bounded pseudoconvex domains in $\mathbb{C}^n$.

Theorem 2.6 (Hörmander's Theorem Revisited). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with diameter d , where $d = \sup_{z, z' \in \Omega} |z - z'|$. Then for any $f \in L_{p,q}^2(\Omega)$ with $\bar{\partial}f = 0$, there is a $(p, q-1)$ -form $u \in L_{p,q-1}^2(\Omega)$ such that $\bar{\partial}u = f$ with*

$$(2.8) \quad \|u\|^2 \leq c_{n,q}d^2\|f\|^2$$

where $\|\cdot\|$ is the Euclidean norm and

$$(2.9) \quad c_{n,q} = \frac{e(n+2q+1)}{4q(2n+1)}.$$

Proof. Since the domain Ω is in $\mathbb{C}^n$, p plays no role. We may assume that $p = 0$. We first embed $\mathbb{C}^n$ in $\mathbb{CP}^n$ and view Ω as a domain in $\mathbb{CP}^n$ endowed with the Fubini-Study metric. Using Theorem 2.4, there exists $u \in L^2_{0,q-1}(\Omega)$ such that

$$(2.10) \quad \|u\|_\omega^2 \leq \frac{1}{q(2n+1)} \|f\|_\omega^2.$$

Next we compare norm $\|\cdot\|_\omega$ with the Euclidean norm $\|\cdot\|$. The Kähler form ω is given by

$$(2.11) \quad \omega = i\partial\bar{\partial}\log(1+|z|^2)$$

$$(2.12) \quad = i \sum_{\alpha,\beta=1}^n g_{\alpha\bar{\beta}}(z) dz_\alpha \wedge d\bar{z}_\beta$$

where

$$(2.13) \quad g_{\alpha\bar{\beta}}(z) = \frac{\partial^2 \log(1+|z|^2)}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{(1+|z|^2)\delta_{\alpha\bar{\beta}} - \bar{z}_\alpha z_\beta}{(1+|z|^2)^2}.$$

The volume form dV_ω with respect to ω is

$$(2.14) \quad dV_\omega = \det(g_{\alpha\bar{\beta}}(z)) dV_E = \frac{1}{(1+|z|^2)^{n+1}} dV_E$$

where dV_E is the Euclidean volume form. Furthermore, we have

$$(2.15) \quad \frac{1}{(1+|z|^2)^2} \omega_E \leq \omega \leq \frac{1}{1+|z|^2} \omega_E.$$

Assume that Ω has diameter $d = 2\epsilon$ in $\mathbb{C}^n$ with respect to the Euclidean norm for some $\epsilon > 0$. Without loss of generality, we may assume that $\Omega \subset B_\epsilon(0) = \{z \in \mathbb{C}^n \mid |z| < \epsilon\}$. Then using (2.14), we have

$$\frac{1}{(1+\epsilon^2)^{n+1}} dV_E \leq dV_\omega \leq dV_E.$$

Using (2.15), we have

$$\frac{1}{(1+\epsilon^2)^2} \omega_E \leq \omega \leq \omega_E$$

and

$$(2.16) \quad 1 \leq |dz_j|_\omega \leq (1+|z|^2).$$

Let $f = \sum_K f_K d\bar{z}_K$, where K is a multi-index with $|K| = q$. It follows from (2.14) and (2.16) that

$$|f|_\omega^2 dV_\omega \leq |f|^2 (1+|z|^2)^{2q} dV_\omega \leq (1+\epsilon^2)^{2q} |f|^2 dV_0,$$

and

$$(2.17) \quad \|f\|_\omega^2 \leq (1+\epsilon^2)^{2q} \|f\|^2.$$

Similarly, we have

$$|u|_\omega^2 dV_\omega \geq |u|^2 dV_\omega \geq \frac{1}{(1+\epsilon^2)^{n+1}} |u|^2 dV_0,$$

and

$$(2.18) \quad \|u\|_\omega^2 \geq \frac{1}{(1+\epsilon^2)^{n+1}} \|u\|^2.$$

It follows from (2.10), (2.17) and (2.18) that we have

$$(2.19) \quad \|u\|^2 \leq \frac{(1 + \epsilon^2)^{n+1+2q}}{q(2n+1)} \|f\|^2,$$

when we assume that Ω has diameter 2ϵ .

Suppose that Ω lies in $B_1(0)$, the ball of radius 1. By scaling, we have for any $\epsilon > 0$,

$$(2.20) \quad \|u\|^2 \leq \frac{1}{q(2n+1)} \frac{(1 + \epsilon^2)^{n+1+2q}}{\epsilon^2} \|f\|^2.$$

Since $\epsilon > 0$ is arbitrary, we see that the function

$$\phi(\epsilon) = \frac{(1 + \epsilon^2)^{n+1+2q}}{\epsilon^2}$$

achieves its minimum m at $\epsilon = 1/\sqrt{n+2q}$ with

$$m = (n+2q+1) \left(1 + \frac{1}{n+2q}\right)^{n+2q}.$$

Since $(1 + 1/(n+2q))^{n+2q} \nearrow e$, let $c_{n,q}$ be defined by (2.9). Then it follows from (2.20) that

$$\|u\|^2 \leq c_{n,q} 2^2 \|f\|^2.$$

when Ω has diameter 2.

Suppose the domain Ω in $\mathbb{C}^n$ is with arbitrary diameter d . Notice that the Euclidean metric admits a dilation. Thus (2.8) follows easily from a scaling argument. $\square$

Remark. Theorem 2.6 is an alternative proof of the Hörmander's L^2 theory (see Theorem 1.5 and [26]). Hörmander's method is to use the weight function $\varphi = t|z|^2$ to obtain the L^2 existence with estimate (2.8) with $c_{n,q} = e/q$. In comparison, the proof here uses the positive curvature of the Fubini-Study metric and the constant $c_{n,q}$ given by (2.9) is comparable up to a factor to Hörmander's results.

2.2. The $\bar{\partial}$ -Neumann operator with weights. When $p > 0$, L^2 theory for $\bar{\partial}$ on a domain Ω in $\mathbb{CP}^n$ requires more work since the curvature Θ is only nonnegative. Let Ω be a pseudoconvex domain with Lipschitz boundary $b\Omega$ in $\mathbb{CP}^n$. We may assume that there exists a Lipschitz defining function $\rho = -\delta$ such that

$$(2.21) \quad i\partial\bar{\partial}(-\log \delta) \geq C\omega$$

for some $C > 0$. Let $t > 0$ and let $\phi_t = -t \log \delta$. Then ϕ_t is a strictly plurisubharmonic function on Ω . Using ϕ_t as the weight function in Hörmander's L^2 methods with the weight function ϕ_t , we have

$$e^{-\phi_t} = e^{t \log \delta} = \delta^t.$$

We use $\|\cdot\|_t$ to denote the L^2 norm with weight under the Fubini-Study metric.

Let

$$\bar{\partial} : L^2_{p,q-1}(\Omega, \delta^t) \rightarrow L^2_{p,q}(\Omega, \delta^t)$$

be the weak maximal L^2 closure of $\bar{\partial}$ and its Hilbert space adjoint is denoted by $\bar{\partial}_t^*$ such that $\bar{\partial}_t^* : L^2_{p,q}(\Omega, \delta^t) \rightarrow L^2_{p,q-1}(\Omega, \delta^t)$ and if $v \in \text{Dom}(\bar{\partial}_t^*)$ if and only if

$$(\bar{\partial}u, v)_t = (u, \bar{\partial}_t^*v)_t \quad \text{for every } u \in \text{Dom}(\bar{\partial}).$$

Now we use the Hörmander's L^2 methods with the weight function ϕ_t combined with the Bochner-Kodaira-Morrey-Kohn formula. Using the same notation as before, but now we

suppress the dependence of the Fubini-Study metric ω and emphasize on the weighted norm with respect to ϕ_t .

Theorem 2.7 (Bochner-Kodaira-Morrey-Kohn-Hörmander). *Let Ω be a domain in $\mathbb{CP}^n$ with C^2 boundary $b\Omega$. For any $u \in C_{p,q}^1(\overline{\Omega}) \cap \text{dom}(\bar{\partial}_t^*)$, we have*

$$(2.22) \quad \|\bar{\partial}u\|_t^2 + \|\bar{\partial}_t^*u\|_t^2 = \|\bar{\nabla}u\|_t^2 + (\Theta u, u)_t + ((i\bar{\partial}\bar{\partial}\phi_t)u, u)_t + \int_{b\Omega} \langle (\bar{\partial}\bar{\partial}\rho)u, u \rangle_t e^{-\phi_t} dS$$

where dS is the induced surface element on $b\Omega$ and $\|\bar{\nabla}u\|_t^2 = \sum_{j=1}^n \|\nabla_{\bar{L}_j} u\|_t^2$.

Corollary 2.8 (Weighted L^2 Existence for $\bar{\partial}$). *Let Ω be a domain in $\mathbb{CP}^n$ with Lipschitz boundary $b\Omega$. For any $0 \leq p \leq n-1$, $1 \leq q \leq n$ and $f \in L_{p,q}^2(\Omega, \delta^t)$ such that f is $\bar{\partial}$ -closed, there exists a $(p, q-1)$ -form $u \in L_{p,q-1}^2(\Omega, \delta^t)$ such that $\bar{\partial}u = f$ with*

$$(2.23) \quad Ct\|u\|_t^2 \leq \|f\|_t^2$$

where C is the same constant as in (2.21).

Proof. Suppose Ω is a domain with C^2 boundary, this follows from (2.22) since

$$\|\bar{\partial}u\|_t^2 + \|\bar{\partial}_t^*u\|_t^2 \geq Ct\|u\|_t^2.$$

For a domain with Lipschitz boundary, we can use an exhaustion argument. $\square$

Suppose that Ω is a bounded domain in $\mathbb{C}^n$ with C^2 boundary. Diederich and Fornaess [13] proved that there exists a defining function ρ and an exponent $0 < \eta < 1$ such that $-(-\rho)^\eta$ is strictly plurisubharmonic in Ω . Based on Takeuchi's theorem, Ohsawa and Sibony [38] generalized the Diederich-Fornaess results to domains in $\mathbb{CP}^n$. In fact, they showed that one can take $\rho = -\delta$ where δ is the distance function from $z \in \Omega$ to $b\Omega$. Ohsawa-Sibony results have been extended to domains with Lipschitz boundary by Harrington in [24], where he proved that there exists a Lipschitz defining function ρ and $0 < \eta < 1$ such that

$$(2.24) \quad i\bar{\partial}\bar{\partial}(-\delta^\eta) \geq C\eta\delta^\eta\omega$$

in the sense of currents for some constant $C > 0$.

Using (2.24) and a technique of Berndtsson-Charpentier [3] have the following L^2 existence theorem without weights.

Theorem 2.9 (L^2 Existence for (p, q) -Forms). *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary. Then*

$$(2.25) \quad H_{L^2}^{p,q}(\Omega) = 0 \quad \text{for every } q > 0.$$

Furthermore, for any $s < \frac{\eta}{2}$, where η is the exponent in (2.24), we have

$$(2.26) \quad H_{W^s}^{p,q}(\Omega) = 0 \quad \text{for every } q > 0.$$

Proof. We refer the reader to [3, 25, 18] for a proof of this theorem. $\square$

The following proposition is a consequence of the above L^2 -theory for $\bar{\partial}$ on $\mathbb{CP}^n$. Its proof follows the same lines of arguments as those in [20, 12, 25, 5] when the boundary is C^2 -smooth.

Proposition 2.10. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary. Then the L^2 holomorphic $(n, 0)$ -forms in $L_{n,0}^2(\Omega) \neq \{0\}$. Furthermore, L^2 Holomorphic $(n, 0)$ -forms separate points.*

Notice that in Theorem 2.9, we need the assumption of Lipschitz boundary for Ω when $p > 0$. We will prove that Theorem 2.9 does not hold if the Lipschitz condition is dropped. In contrast, if $p = 0$, we do not need any regularity for Ω in Theorem 2.4.

Remark. Bounded plurisubharmonic exhaustion functions have been studied in $\mathbb{C}^n$ and $\mathbb{CP}^n$ extensively. The Diederich-Fornaess theorem has been extended to pseudoconvex domains in $\mathbb{C}^n$ with Lipschitz boundary (see Demailly [12]). The Diederich-Fornaess exponent is the supremum of $0 < \eta < 1$ such that (2.24) holds. It is related to the nonexistence of Levi-flat hypersurfaces in complex manifolds (see [1, 16, 17]).

2.3. The $\bar{\partial}$ -Cauchy problem with weights. For fixed $t \geq 0$, let

$$\bar{\partial}_c : L_{p,q-1}^2(\Omega, \delta^{-t}) \rightarrow L_{p,q}^2(\Omega, \delta^{-t})$$

be the minimal (strong) closure of $\bar{\partial}$. By this we mean that $f \in \text{Dom}(\bar{\partial}_c)$ if and only if there exists a sequence of smooth compactly supported forms f_ν in $C_{p,n-1}^\infty(\Omega)$ such that $f_\nu \rightarrow f$ and $\bar{\partial}f_\nu \rightarrow \bar{\partial}f$ in $L^2(\Omega, \delta^{-t})$.

Lemma 2.11. *The following conditions are equivalent:*

- (1) $\bar{\partial} : L_{p,q-1}^2(\Omega, \delta^t) \rightarrow L_{p,q}^2(\Omega, \delta^t)$ has closed range.
- (2) $\bar{\partial}_t^* : L_{p,q}^2(\Omega, \delta^t) \rightarrow L_{p,q-1}^2(\Omega, \delta^t)$ has closed range.
- (3) $\bar{\partial}_c : L_{n-p,n-q}^2(\Omega, \delta^{-t}) \rightarrow L_{n-p,n-q+1}^2(\Omega, \delta^{-t})$ has closed range.

Proof. It is well-known that $\bar{\partial}$ has closed range if and only if $\bar{\partial}_t^*$ has closed range (see [26] or Lemma 4.1.1 in [11]). By using the Hodge star operator, we have that (1) and (3) are equivalent (see [8, 34]). $\square$

Theorem 2.12 (L^2 Serre Duality with Weights). *Let Ω be a pseudoconvex domain with Lipschitz boundary in $\mathbb{CP}^n$. We have for any $t \geq 0$,*

$$H_{L^2}^{p,q}(\Omega, \delta^t) \cong H_{\bar{\partial}_c, L^2}^{n-p, n-q}(\Omega, \delta^{-t}) = \{0\}, \quad q \neq 0.$$

Proof. Using Corollary 2.8, $\bar{\partial}$ has closed range in $L_{p,q}^2(\Omega, \delta^t)$ for all degrees and $t \geq 0$. Thus using Lemma 2.11 and the L^2 Serre duality (see [8]), the theorem follows. $\square$

Corollary 2.13 ($\bar{\partial}$ -Cauchy Problem in L^2 Spaces with Weights). *Let Ω be a pseudoconvex domain with Lipschitz boundary in $\mathbb{CP}^n$, $n \geq 2$. Suppose that $f \in L_{p,q}^2(\Omega, \delta^{-t})$ where $t \geq 0$, $0 \leq p \leq n$ and $1 \leq q < n$. Assuming that $\bar{\partial}f = 0$ in $\mathbb{CP}^n$ with $f = 0$ outside Ω . Then there exists $u \in L_{p,q-1}^2(\Omega, \delta^{-t})$ with $u = 0$ outside Ω satisfying $\bar{\partial}u = f$ in the distribution sense in $\mathbb{CP}^n$.*

For $q = n$, if f satisfies the compatibility condition

$$(2.27) \quad \int_{\Omega} f \wedge \phi = 0, \quad \phi \in L_{n-p,0}^2(\Omega, \delta^t) \cap \text{Ker}(\bar{\partial}),$$

then the same conclusion holds.

Proof. Since the boundary is Lipschitz, we have that solving $\bar{\partial}_c$ is the same as solving $\bar{\partial}$ with prescribed support in $\bar{\Omega}$ (see Lemma 2.3 in [34]). $\square$

3. SOBOLEV ESTIMATES FOR $\bar{\partial}$ ON PSEUDOCONCAVE DOMAINS IN $\mathbb{CP}^n$

Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary, where $n \geq 2$. We always assume that $\bar{\Omega} \neq \mathbb{CP}^n$. Let Ω^+ be the complement of $\bar{\Omega}$ defined by

$$\Omega^+ = \mathbb{CP}^n \setminus \bar{\Omega}.$$

Then Ω^+ is a pseudoconcave domain with Lipschitz boundary. Estimates for the $\bar{\partial}$ -equation in Sobolev spaces $W^k(\Omega^+)$ have been obtained for $k = 1$ in earlier papers using the $\bar{\partial}$ -Cauchy problem (see [5, 7]. For $k \geq 2$, it is proved in Henkin-Iordan [25]) under the condition that the boundary $b\Omega^+$ is C^2 . Here we will give a streamlined proof of Sobolev estimates for $k \geq 1$ for pseudoconcave domains with Lipschitz boundary using the $\bar{\partial}$ -Cauchy problem with weights.

Let $W_0^k(\Omega)$ be the Sobolev space which is the completion of $C_0^\infty(\Omega)$ under the $W^k(\Omega)$ norm. We have the following characterization of the space $W_0^k(\Omega)$.

Lemma 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{CP}^n$. Let $\delta(z)$ be the distance function from $z \in \Omega$ to $b\Omega$. Then for $k \geq 1$, $g \in W_0^k(\Omega)$ if and only if $g \in W^k(\Omega)$ and*

$$(3.1) \quad \delta^{-s+|\beta|} D^\beta g \in L^2(\Omega) \quad \text{for all } |\beta| < k.$$

For a proof of this lemma, see Theorem 11.8 in Lions-Magenes [37], where theorem is stated for smooth domains. Similar proof can be applied to domains with Lipschitz boundary (see also Grisvard [22]).

Theorem 3.2. *Let Ω^+ be a pseudoconcave domain in $\mathbb{CP}^n$ with Lipschitz boundary, $n \geq 2$. Let $k \in \mathbb{N}$. For any $\bar{\partial}$ -closed $f \in W_{p,q}^k(\Omega^+)$, where $0 \leq p \leq n$, $0 \leq q < n-1$, there exists $F \in W_{p,q}^{k-1}(\mathbb{CP}^n)$ with $F|_{\Omega^+} = f$ and $\bar{\partial}F = 0$ in $\mathbb{CP}^n$ in the distribution sense.*

Proof. The $W^1(\Omega^+)$ estimates have already been proved earlier (see [5, 7, 18, 25]). We will show that the proof can be modified for $k \geq 2$. Since Ω^+ has Lipschitz boundary, there exists a bounded extension operator from $W^k(\Omega^+)$ to $W^k(\mathbb{CP}^n)$ (see, e.g., [46]). Let $\tilde{f} \in W_{p,q}^k(\mathbb{CP}^n)$ be the extension of f so that $\tilde{f}|_{\Omega^+} = f$ with $\|\tilde{f}\|_{W^k(\mathbb{CP}^n)} \leq C\|f\|_{W^k(\Omega^+)}$. We have $\bar{\partial}\tilde{f} \in L_{p,q+1}^2(\Omega, \delta^{-2k+2})$, where $\Omega = \mathbb{CP}^n \setminus \Omega^+$.

From Corollary 2.13, there exists $u_c \in L_{p,q}^2(\Omega, \delta^{-2k+2})$ such that $\bar{\partial}u_c = \bar{\partial}\tilde{f}$. Extending u_c to be zero outside $\bar{\Omega}$, we have

$$\bar{\partial}u_c = \bar{\partial}\tilde{f} \quad \text{in } \mathbb{CP}^n.$$

Since u_c satisfies an elliptic system, we have that $u_c \in L_{p,q}^2(\Omega, \delta^{-2k+2})$ implies that $u_c \in W_{0,p,q}^{k-1}(\Omega)$. Define

$$(3.2) \quad F = \tilde{f} - u_c.$$

Then $F \in W_{p,q}^{k-1}(\mathbb{CP}^n)$ and F is a $\bar{\partial}$ -closed extension of f . □

Corollary 3.3. *Let Ω^+ be a pseudoconcave domain in $\mathbb{CP}^n$ with Lipschitz boundary, where $n \geq 2$. Then $W_{p,0}^1(\Omega^+) \cap \text{Ker}(\bar{\partial}) = \{0\}$ for every $1 \leq p \leq n$ and $W^1(\Omega^+) \cap \text{Ker}(\bar{\partial}) = \mathbb{C}$.*

Proof. Using Theorem 3.2 for $q = 0$, we have that any holomorphic $(p, 0)$ -form on Ω^+ extends to be a holomorphic $(p, 0)$ in $\mathbb{CP}^n$, which are zero (when $p > 0$) or constants (when $p = 0$). □

Theorem 3.4. *Let Ω^+ be a pseudoconcave domain in $\mathbb{CP}^n$ with Lipschitz boundary, where $n \geq 3$. For any $\bar{\partial}$ -closed $f \in W_{p,q}^k(\Omega^+)$, where $0 \leq p \leq n$, $1 \leq q < n-1$, $p \neq q$ and $k \in \mathbb{N}$, there exists $u \in W_{p,q-1}^k(\Omega^+)$ with $\bar{\partial}u = f$ in Ω^+ .*

Proof. Let $F \in W_{p,q}^{k-1}(\mathbb{CP}^n)$ be the $\bar{\partial}$ -closed extension of f from Ω to $\mathbb{CP}^n$. Since

$$H_{W^{k-1}}^{p,q}(\mathbb{CP}^n) = \{0\},$$

there exists $u \in W_{p,q-1}^k(\Omega)$ such that $\bar{\partial}u = F$ on $\mathbb{CP}^n$. By the elliptic theory of the $\bar{\partial}$ -complex on compact complex manifolds, one can choose such a solution $u \in W_{p,q-1}^k(\mathbb{CP}^n)$. $\square$

For $q = n-1$, there is an additional compatibility condition for the $\bar{\partial}$ -closed extension of $(p, n-1)$ -forms from Ω^+ to the whole space $\mathbb{CP}^n$. This case differs from the others since the cohomology group does not vanish in general (see [15]). We first derive the compatibility condition for the extension of $\bar{\partial}$ -closed forms when $q = n-1$.

Lemma 3.5. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary and let $\Omega^+ = \mathbb{CP}^n \setminus \bar{\Omega}$. For any $f \in W_{p,n-1}^k(\Omega^+)$, $k \in \mathbb{N}$ and $\phi \in L_{n-p,0}^2(\Omega, \delta^{2k-2}) \cap \text{Ker}(\bar{\partial})$, the pairing*

$$(3.3) \quad \int_{b\Omega^+} f \wedge \phi$$

is well-defined.

Proof. Since the boundary is Lipschitz, any function in $W^k(\Omega^+)$ has a trace in $W^{k-\frac{1}{2}}(b\Omega^+)$. Also holomorphic functions or forms in $L^2(\Omega, \delta^{2k-2})$ have trace in $W^{-k+\frac{1}{2}}(b\Omega)$. The pairing (3.3) is well-defined follows from these known facts on Lipschitz domains. The rest of the proof of the lemma is exactly the same as in [42] and we give a sketch of the arguments.

Since the boundary is Lipschitz, for any $\bar{\partial}$ -closed (holomorphic) $(n-p, 0)$ -form ϕ with $L^2(\Omega, \delta^{2k-2})$ coefficients, there exists a sequence $\phi_\nu \in C_{n-p,0}^\infty(\bar{\Omega})$ such that $\phi_\nu \rightarrow \phi$ and $\bar{\partial}\phi_\nu \rightarrow 0$ in $L^2(\Omega, \delta^{2k-2})$ -norm. This implies that $\phi_\nu \rightarrow \phi$ in $W^{-k+1}(\Omega)$ norm since ϕ is holomorphic.

Let $\tilde{f} \in W_{p,n-1}^k(\mathbb{CP}^n)$ be a bounded extension of f . We have

$$(3.4) \quad \int_{b\Omega} f \wedge \phi_\nu = \int_{\Omega} \bar{\partial}(\tilde{f} \wedge \phi_\nu) = \int \bar{\partial}\tilde{f} \wedge \phi_\nu \pm \int \tilde{f} \wedge \bar{\partial}\phi_\nu \rightarrow \int \bar{\partial}\tilde{f} \wedge \phi.$$

Thus the limit on the left-hand-side of (3.4) exists and is independent of the approximating sequence $\{\phi_\nu\}$ that we choose. It is also independent of the extension function $\tilde{f}$. Hence the pairing (3.3) is well-defined. $\square$

Theorem 3.6. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary and let $\Omega^+ = \mathbb{CP}^n \setminus \bar{\Omega}$. For $\bar{\partial}$ -closed $f \in W_{p,n-1}^k(\Omega^+)$, where $k \geq 1$, $0 \leq p \leq n$ and $p \neq n-1$, the following conditions are equivalent:*

- (1) *The restriction of f to $b\Omega^+$ satisfies the compatibility condition*

$$(3.5) \quad \int_{b\Omega^+} f \wedge \phi = 0, \quad \phi \in L_{n-p,0}^2(\Omega, \delta^{2k-2}) \cap \text{Ker}(\bar{\partial}).$$

- (2) *There exists $F \in W_{p,n-1}^{k-1}(\mathbb{CP}^n)$ such that $F|_{\Omega} = f$ in Ω^+ and $\bar{\partial}F = 0$ in $\mathbb{CP}^n$ in the sense of distribution.*

- (3) *There exists $u \in W_{p,n-2}^k(\Omega^+)$ satisfying $\bar{\partial}u = f$ in Ω^+ .*

Corollary 3.7. *Let Ω^+ be the same as in Theorem 3.6. Then $\bar{\partial} : W_{p,n-2}^k(\Omega^+) \rightarrow W_{p,n-1}^k(\Omega^+)$ has closed range, where $k \geq 1$ and $0 \leq p \leq n$.*

Proof. Let f be a $\bar{\partial}$ -closed $(p, n-1)$ -form in $W_{p,n-1}^k(\Omega^+)$. Suppose that f is in the closure of the range of $\bar{\partial} : W_{p,n-2}^k(\Omega^+) \rightarrow W_{p,n-1}^k(\Omega^+)$. There exists a sequence $u_\nu \in W_{p,n-2}^k(\Omega^+)$ such that $\bar{\partial}u_\nu \rightarrow f$ in $W_{p,n-1}^k(\Omega^+)$. It suffices to show that there exists $u \in W_{p,n-2}^k(\Omega^+)$ such that $\bar{\partial}u = f$.

From Theorem 3.6, it suffices to show that the condition (3.5) is satisfied for every $\phi \in L_{n-p,0}^2(\Omega, \delta^{2k-2}) \cap \text{Ker}(\bar{\partial})$. This follows from

$$(3.6) \quad \int_{b\Omega^+} f \wedge \phi = \lim_{\nu \rightarrow \infty} \int_{b\Omega^+} \bar{\partial}u_\nu \wedge \phi = \lim_{\nu \rightarrow \infty} (-1)^{p+n-2} \int_{b\Omega^+} u_\nu \wedge \bar{\partial}\phi = 0.$$

Thus $f = \bar{\partial}u$ for some $u \in W_{p,n-2}^k(\Omega^+)$. Thus the range of $\bar{\partial}$ is closed in $W_{p,n-1}^k(\Omega^+)$. $\square$

Combining the results, we have proved the following theorem.

Theorem 3.8. *Let Ω^+ be the same as in Theorem 3.6. Then for any $k \in \mathbb{N}$,*

- $H_{W_k}^{p,q}(\Omega^+) = 0$, if $0 \leq q < n-1$ and $p \neq q$;
- $H_{W_k}^{p,n-1}(\Omega^+)$ is Hausdorff and infinite dimensional, if $p \neq n-1$.

Remark. It is still an open question if Theorems 3.4 and 3.6 hold for $k = 0$ (see Problem 2). The missing ingredient is the lack of W^1 -estimates with pseudoconvex domains in $\mathbb{CP}^n$.

When the domain Ω is a bounded domain with smooth boundary in $\mathbb{C}^n$, there has been a lot of results obtained earlier. The space of L^2 harmonic forms for the critical degree $q = n-1$ on an annulus between two concentric balls or strongly pseudoconvex domains in $\mathbb{C}^n$ has been computed in [28]. This has been generalized to annulus between two pseudoconvex domains in $\mathbb{C}^n$ in [42, 43]. We also remark that the conditions on the cohomology groups can be used to characterize domains with holes with Lipschitz boundary in $\mathbb{C}^n$ (see [15]). All these results depend on the Sobolev estimates for $\bar{\partial}$ proved by Kohn (see Theorem 1.3).

4. PROPERTIES OF HOLOMORPHIC FUNCTIONS AND FORMS ON THE HARTOGS TRIANGLES

We denote the homogeneous coordinates in $\mathbb{CP}^2$ by $[Z_0, Z_1, Z_2]$. Let $\mathbb{H}^+$ and $\mathbb{H}^-$ be the Hartogs triangles defined by

$$\mathbb{H}^+ = \{[Z_0 : Z_1 : Z_2] \in \mathbb{CP}^2 \mid |Z_1| < |Z_2|\}$$

$$\mathbb{H}^- = \{[Z_0 : Z_1 : Z_2] \in \mathbb{CP}^2 \mid |Z_1| > |Z_2|\}$$

then $\mathbb{H}^+ \cap \mathbb{H}^- = \emptyset$ and $\overline{\mathbb{H}^+} \cup \overline{\mathbb{H}^-} = \mathbb{CP}^2$.

Let $U_j = \{[Z_0, Z_1, Z_2] \mid Z_j \neq 0\}$, $j = 0, 1, 2$. Then $\mathbb{H}^+ \subset U_2$. In local coordinates,

$$\mathbb{H}^+ = \{(z, w) \in \mathbb{C}^2 \mid |w| < 1\}.$$

Thus $\mathbb{H}^+$ is the product $\mathbb{C} \times D$, where D is the unit disk. Hence $\mathbb{H}^+$ is pseudoconvex.

In this section, we first recall some known results on the Hartogs triangles. The Hartogs triangles are not Lipschitz. However, some function properties for the Hartogs triangles still hold. Recall that a domain $\Omega \subset \mathbb{CP}^n$ is called a Sobolev extension domain if for any $f \in W^s(\Omega)$, there exists $\tilde{f} \in W^s(\mathbb{CP}^n)$ such that $\tilde{f} = f$ on Ω .

Lemma 4.1. *The Hartogs triangles $\mathbb{H}^+$ and $\mathbb{H}^-$ are Sobolev extension domains.*

Proof. The Hartogs triangle $\mathbb{H}^+$ is smooth except at the point $[1, 0, 0]$. If we set $z = z_1/z_0$ and $w = z_2/z_0$, then the domain $\mathbb{H}^+$ is defined by the inhomogeneous coordinates (z, w) by

$$\mathbb{H}^+ = \{z, w\} \in \mathbb{C}^2 \mid |z| < |w|\}.$$

The Hartogs triangle $\mathbb{H}^+$ and $\mathbb{H}^-$ are not Lipschitz at $(0, 0)$. At $(0, 0)$, the singularity of $\mathbb{H}^+$ and $\mathbb{H}^-$ are the same as the Hartogs triangle

$$T = \{(z, w) \in \mathbb{C}^2 \mid |z| < |w| < 1\}.$$

It is proved in [2] that T is an extension domain. Thus the lemma follows from the same proof. $\square$

It is also proved in Theorem 3.13 in [2] that the weak and strong extensions of $\bar{\partial}$ are the same. Define the L^2 Dolbeault cohomology group with respect to $\bar{\partial}_c$ as follows:

$$H_{\bar{\partial}_c, L^2}^{p,q}(\mathbb{H}^-) = \frac{\text{Ker}(\bar{\partial}_c)}{\text{Range}(\bar{\partial}_c)}.$$

The following lemma is proved in Proposition 6 in [8].

Lemma 4.2. *Let $\mathbb{H}^+ \subset \mathbb{CP}^2$ be the Hartogs' triangle. Then we have the following:*

- (1) *The Bergman space of L^2 holomorphic functions $L^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ on the domain $\mathbb{H}^+$ separates points in $\mathbb{H}^+$.*
- (2) *There exist non-constant functions in the space $W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$. However, this space does not separate points in $\mathbb{H}^+$ and is not dense in the Bergman space $L^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$.*
- (3) *Let $f \in W^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ be a holomorphic function on $\mathbb{H}^+$ which is in the Sobolev space $W^2(\mathbb{H}^+)$. Then f is a constant.*

Lemma 4.3. *The following results hold:*

- (1) $H_{L^2}^{2,0}(\mathbb{H}^+) = 0$;
- (2) $H_{L^2}^{0,1}(\mathbb{H}^+) = 0$;
- (3) $H_{L^2}^{2,1}(\mathbb{H}^\pm)$ is infinite dimensional.

Proof. Let $z = z_0/z_2$ and $w = z_1/z_2$. Then $\mathbb{H}^-$ is biholomorphic to $\mathbb{C} \times D$. Let $\phi = f dz \wedge dw$, where f is holomorphic in $\mathbb{C} \times D$. Since ϕ is a $(2, 0)$ -form, its L^2 -norm is metric independent. We can just use the Euclidean metric on $\mathbb{C} \times D$. If $f \in L^2(\mathbb{C} \times D)$, then f is L^2 on the leaf $(\cdot, w)$ a.e., where $w \in D$. Since f is holomorphic, this implies that $f = 0$ on $\mathbb{C} \times D$. Thus $\phi = 0$ on $\mathbb{H}^+$. This proves (1).

(2) is already proved in Corollary 2.5. The proof of (3) uses Lemma 4.2. It follows from [36] combining with the results in [2] (see also [44]). $\square$

5. NON-CLOSED RANGE PROPERTY FOR $\bar{\partial}$ ON HARTOGS TRIANGLES $\mathbb{CP}^2$

We now state and prove the main result in this paper.

Theorem 5.1. $\bar{\partial} : L_{2,0}^2(\mathbb{H}^+) \rightarrow L_{2,1}^2(\mathbb{H}^+)$ does not have closed range.

Corollary 5.2. $H_{L^2}^{2,1}(\mathbb{H}^+)$ is non-Hausdorff.

Proof. We will show that the corollary follows easily from the theorem. It is well-known that $\bar{\partial} : L_{2,0}^2(\mathbb{H}^+) \rightarrow L_{2,1}^2(\mathbb{H}^+)$ has closed range if and only if $H_{L^2}^{2,1}(\mathbb{H}^+)$ is Hausdorff (see e.g., Treves [49]). $\square$

It remains to prove Theorem 5.1. To prove the theorems, we need two lemmas.

Lemma 5.3. *The following are equivalent:*

- (1) $\bar{\partial} : L_{2,0}^2(\mathbb{H}^+) \rightarrow L_{2,1}^2(\mathbb{H}^+)$ has closed range.
- (2) $\bar{\partial}_c : L_{0,1}^2(\mathbb{H}^+) \rightarrow L_{0,2}^2(\mathbb{H}^+)$ has closed range and the range is $L_{0,2}^2(\mathbb{H}^+)$.
- (3) $H_{W^1}^{0,1}(\mathbb{H}^-) = 0$.

Proof. It follows from Lemma 2.11 that $\bar{\partial}_c : L_{0,1}^2(\mathbb{H}^+) \rightarrow L_{0,2}^2(\mathbb{H}^+)$ has closed range if and only if $\bar{\partial} : L_{2,0}^2(\mathbb{H}^+) \rightarrow L_{2,1}^2(\mathbb{H}^+)$ has closed range. Using (1) in Lemma 4.3 and L^2 Serre duality, we have and

$$(5.1) \quad H_{\bar{\partial}_c, L^2}^{0,2}(\mathbb{H}^+) \cong H_{L^2}^{2,0}(\mathbb{H}^+) = 0.$$

This proves that the range of $\bar{\partial}_c$ is $L_{0,2}^2(\mathbb{H}^+)$. We have proved that (1) implies (2). Thus (1) and (2) are equivalent.

Next we prove that (2) implies (3). Since $\mathbb{H}^-$ is an extension domain by Lemma 4.1, let $\tilde{f}$ be an extension of f to $W_{0,1}^1(\mathbb{CP}^2)$. Let $f_c = \bar{\partial}\tilde{f}$. Then $f_c \in L_{0,2}^2(\mathbb{H}^+)$. Using (2), there exists $u_c \in L_{0,1}^2(\mathbb{H}^+)$ such that $\bar{\partial}_c u_c = f_c$ in $\mathbb{CP}^2$. Letting $F = \tilde{f} - u_c$. Then $F \in L_{0,1}^2(\mathbb{CP}^2)$, $\bar{\partial}F = 0$ in $\mathbb{CP}^2$ and $F = f$ in Ω^+ . Any $\bar{\partial}$ -closed $f \in W_{0,1}^1(\Omega^+)$ extends to be an L^2 $\bar{\partial}$ -closed form F in $\mathbb{CP}^2$. So $F = \bar{\partial}U$ with U in $W^1(\mathbb{CP}^2)$. Letting $u = U|_{\Omega^-}$, then $u \in W^1(\mathbb{H}^-)$ and $\bar{\partial}u = f$. We have proved (3).

Finally, we prove that (3) implies (2). Let $f \in L_{0,2}^2(\mathbb{H}^+)$. Then there exists $V \in W_{0,1}^1(\mathbb{CP}^2)$ such that $\bar{\partial}V = f$ in $\mathbb{CP}^2$. We set $v = V|_{\mathbb{H}^-}$. Using (3), we there exists $u \in W^1(\mathbb{H}^-)$ such that $\bar{\partial}u = v$ on $\mathbb{H}^-$. Let $\tilde{u} \in W^1(\mathbb{CP}^2)$ be an extension of u .

We define

$$u_c = V - \bar{\partial}\tilde{u}.$$

Then $\bar{\partial}u_c = f$ in $\mathbb{CP}^2$ and $u_c = 0$ on $\mathbb{H}^-$. This proves (2). The lemma is proved. $\square$

Lemma 5.4. *Suppose $H_{W^1}^{0,1}(\mathbb{H}^+) = 0$. Then functions in $W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ separate points in $\mathbb{H}^+$.*

Proof. The proof is similar to the proof of Proposition 4.3 in [25]. Let a and b be two distinct points in $\mathbb{H}^+$ and let $c \in \mathbb{H}^-$. There exists a Riemann surface S of degree 2 passing through a, b, c . The Riemann surface is given by $S = \{[Z] = [Z_1, Z_2, Z_3] \mid P(Z) = 0\}$, where $P(Z)$ is a homogeneous polynomial of degree 2 in $[Z]$. We can choose c such that $dP \neq 0$ on S . Let $\tilde{\mathbb{H}}^+$ be an open neighborhood of $\mathbb{H}^+$ and $c \notin \tilde{\mathbb{H}}^+$. The Riemann surface $S \cap \tilde{\mathbb{H}}^+$ is Stein and there exists a Stein neighborhood U of $S \cap \tilde{\mathbb{H}}^+$ such that U is Stein (see [21]). Let h be a holomorphic function in U such that $h(a) \neq h(b)$.

Let χ be a function in $C_0^\infty(U)$ such that $\chi = 1$ in a neighborhood of $S \cap \tilde{\mathbb{H}}^+$. Consider the $(0, 1)$ -form

$$\alpha = \frac{(\bar{\partial}\chi)h}{P}.$$

Then $\alpha \in C_{0,1}^\infty(\tilde{\mathbb{H}}^+)$. Using the assumption that $H_{W^1}^{0,1}(\mathbb{H}^+) = 0$, there exists $u \in W^1(\mathbb{H}^+)$ such that $\bar{\partial}u = \alpha$ in $\mathbb{H}^+$. Let

$$F = \chi h - Pu.$$

Then $F \in W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$. Furthermore, $F(a) = h(a) \neq h(b) = F(b)$. The lemma is proved. $\square$

5.1. Proof of Theorem 5.1. Suppose that $\bar{\partial} : L^2_{2,0}(\mathbb{H}^+) \rightarrow L^2_{2,1}(\mathbb{H}^+)$ has closed range. Using Lemma 5.3,

$$H^{0,1}_{W^1}(\mathbb{H}^+) = 0.$$

It follows from Lemma 5.4 that holomorphic functions in $W^1(\mathbb{H}^+)$ separate points. This is a contradiction to (2) in Lemma 4.2. Theorem 5.1 is proved. $\square$

We have also proved the following corollary.

Corollary 5.5. $H^{0,1}_{W^1}(\mathbb{H}^+) \neq 0$.

6. OPEN PROBLEMS

There are numerous open problems concerning $\bar{\partial}$ on domains in $\mathbb{C}^n$ and $\mathbb{CP}^n$. In this section we list only a few open problems which are related to this article.

Problem 1 (Sobolev Estimates for $\bar{\partial}$ on Pseudoconvex domains $\mathbb{CP}^n$). Let Ω be a bounded pseudoconvex domain in $\mathbb{CP}^n$ with smooth boundary. Can one solve $\bar{\partial}$ with W^s estimates for all $s > 0$? In other words, prove (or disprove)

$$(6.1) \quad H^{p,q}_{W^s}(\Omega) = 0.$$

We remark that by Corollary 5.5, some smoothness of Ω must be assumed. When Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, this is exactly Kohn's theorem (Theorem 1.3).

Problem 2 (L^2 Existence for $\bar{\partial}$ on Pseudoconcave domains $\mathbb{CP}^n$). Let Ω be a bounded pseudoconvex domain in $\mathbb{CP}^n$ with smooth (or Lipschitz) boundary and let $\Omega^+ = \mathbb{CP}^n \setminus \bar{\Omega}$. Prove (or disprove) that

$$H^{p,q}_{L^2}(\Omega^+) = 0$$

if $p \neq q$ and $q < n - 1$.

Problem 3. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary. Determine if

$$H^{0,1}_{W^1}(\Omega) = 0.$$

In other words, can one extend Kohn's results for $s = 1$ (Theorem 1.3) to Lipschitz pseudoconvex domains? This question has been raised many years ago. It has been shown that W^1 estimates hold for pseudoconvex domains with C^2 boundary (see [23]). But it remains unsolved for general Lipschitz domains. When the domain is the bidisk $D \times D$, this is proved rather recently (see [10]). One can also ask similar questions for the Hartogs triangle in $\mathbb{C}^2$ (see Problem 6 and Problem 7).

Let Ω be a bounded domain in $\mathbb{CP}^n$ with smooth boundary. Let $0 \leq p \leq n$ and $1 \leq q \leq n - 1$. Consider the induced operator $\bar{\partial}_b : L^2_{p,q-1}(b\Omega) \rightarrow L^2_{p,q}(b\Omega)$, the tangential Cauchy-Riemann operator on $b\Omega$.

Let $\vartheta_b : L^2_{p,q}(b\Omega) \rightarrow L^2_{p,q-1}(b\Omega)$ be the adjoint operator of $\bar{\partial}_b$ with respect to the Fubini-Study metric. Let

$$\square_b = \bar{\partial}_b \vartheta_b + \vartheta_b \bar{\partial}_b : L^2_{p,q}(b\Omega) \rightarrow L^2_{p,q}(b\Omega)$$

be the $\bar{\partial}_b$ -Laplacian (or Kohn-Rossi operator).

Problem 4 (Kohn-Rossi Cohomology). Let Ω be a bounded pseudoconvex domain in $\mathbb{CP}^n$ with smooth boundary. Does $\bar{\partial}_b : L^2_{p,q-1}(b\Omega) \rightarrow L^2_{p,q}(b\Omega)$ have closed range where $0 \leq p \leq n$ and $1 \leq q \leq n-1$?

If $\bar{\partial}_b$ has closed range for all degrees, show that for $1 \leq q < n-1$, show that the dimension of the Kohn-Rossi cohomology vanishes, i.e.,

$$\dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}_b, L^2}(b\Omega) = \dim_{\mathbb{C}} \text{Ker}(\square_b^{p,q}) = 0.$$

If not, what is the dimension of the Kohn-Rossi cohomology $\dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}_b, L^2}(b\Omega)$?

We remark the following results are known: If Ω is strongly pseudoconvex, then we have that $\bar{\partial}_b$ has closed range following the results of Kohn-Rossi [33].

If Ω is a bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, the closed range property and L^2 existence for $\bar{\partial}_b$ is proved in [40] for $q < n-1$ and in [4, 32] for $q = n-1$. In this case, we have that the Kohn-Rossi cohomology vanishes for $q < n-1$.

Problem 5. Determine if the L^2 Dolbeault cohomology on the Hartogs triangles for $(1,1)$ -form $H^{1,1}_{L^2}(\mathbb{H}^{\pm})$ satisfies

- (1) $H^{1,1}_{L^2}(\mathbb{H}^{\pm})$ is Hausdorff,
- (2) $H^{1,1}_{L^2}(\mathbb{H}^{\pm}) = 0$.

We have proved that $H^{2,1}_{L^2}(\mathbb{H}^{\pm})$ is non-Hausdorff in Theorem 5.1. We also know that $H^{0,1}_{L^2}(\mathbb{H}^{\pm}) = 0$. It remains to investigate the closed range property for $(1,1)$ -form.

Problem 6. Let T be the Hartogs triangle in $\mathbb{C}^2$.

- (1) Let B be a ball of radius 2 centered at 0. Determine if $H^{0,1}(B \setminus \bar{T})$ is Hausdorff.
- (2) Determine the spectrum of the $\bar{\partial}$ -Neumann operator on T ,
- (3) Determine the spectrum of the d -Neumann operator,

This problem is raised in [2]. Since T is pseudoconvex and bounded, Hörmander's L^2 existence theorem holds for T . We have

$$H^{p,1}_{L^2}(T) = 0 \quad \text{for all } 0 \leq p \leq 2.$$

Notice that $H^{0,1}(B \setminus \bar{T})$ is Hausdorff is equivalent to $H^{2,1}(B \setminus \bar{T})$ is Hausdorff. This question is equivalent to the following question (see [2] and also [34]).

Problem 7. Determine if

$$H^{0,1}_{W^1}(T) = 0.$$

It is proved in a recent paper [39] that

$$H^{0,1}_{W^{k,p}}(T) = 0, \quad k \in \mathbb{N}, p > 4.$$

However, it is still not known if this holds for $W^{1,2}(T) = W^1(T)$. By Corollary 5.5, we have

$$H^{0,1}_{W^1}(\mathbb{H}^+) \neq 0.$$

There are numerous other interesting problems on $\bar{\partial}$ which are yet to be understood. We list only these few problems to highlight the importance of understanding the L^2 -Sobolev theory for $\bar{\partial}$ on domains in complex manifolds.

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