Contents

1 Basic Properties of Holomorphic Functions **13**

1.0 Notation and complex numbers 14

1.1 Complex differentiation and holomorphic functions 17

1.2 Real and complex differentiation 24

1.3 The chain rule and inverse functions 32

1.4 Conformal mappings 37

1.5 Linear fractional transformations 43

1.6 Notes on Chapter 1 54

2 Complex Integration **57**

2.1 Complex line integrals 58

2.2 Cauchy's theorem and Cauchy's integral formula 68

2.3 Cauchy's integral formula for derivatives 73

2.4 Cauchy's estimates and applications 77

2.5 Sequences of holomorphic functions 82

2.6 Power series 85

2.7 Holomorphic functions as power series 89

2.8 Notes on Chapter 2 93

3 Singularities and the Residue Theorem **95**

3.1 Zeros of holomorphic functions 95
3.2 The Laurent series 99
3.3 Poles and essential singularities 104
3.4 The residue theorem and applications 107
3.5 Notes on Chapter 3 123

4 Holomorphic Mappings and Simply Connected Domains 125

4.1 The mean value property and maximum principle 126
4.2 The argument principle and applications 129
4.3 The Schwarz lemma and automorphisms of the unit disk 138
4.4 Function theory on simply connected domains 142
4.5 Normal families and Montel's theorem 148
4.6 The Riemann mapping theorem 151
4.7 Notes on Chapter 4 154

5 Prescribing Zeros and Poles 157

5.1 Prescribing poles on $\mathbf{C}$ 157
5.2 Prescribing zeros on $\mathbf{C}$ 163
5.3 Runge's theorem 175
5.4 Prescribing zeros and poles on general domains 181
5.5 Notes on Chapter 5 186

6 The Inhomogeneous Cauchy-Riemann Equation 187

6.1 The generalized Cauchy integral formula 188
6.2 The inhomogeneous Cauchy-Riemann equation 192
6.3 Applications of the $\bar{\partial}$-equation 199
6.4 The Plemelj jump formula 209
6.5 The Cauchy problem for $\bar{\partial}$ 214
6.6 Notes on Chapter 6 219

7 Riemann Surfaces **221**

7.1 Riemann surfaces and holomorphic maps 222

7.2 More examples of Riemann surfaces 226

7.3 Multiple-valued functions 233

7.4 Local properties of holomorphic maps 242

7.5 Holomorphic maps on compact Riemann surfaces 249

7.6 Topology of compact Riemannn surfaces 258

7.7 Notes on Chapter 7 266

8 Harmonic Functions and the Dirichlet Problem **267**

8.1 Harmonic functions 268

8.2 The Poisson equation and Newton potentials 282

8.3 The Dirichlet problem 291

8.4 Green's function 304

8.5 Weyl's lemma and the Dirichlet principle 312

8.6 Notes on Chapter 8 322

9 The Dirichlet Problem: L^2 Techniques **325**

9.1 The energy method 326

9.2 Sobolev spaces and the Rellich lemma 336

9.3 The Dirichlet norm and Poincaré's inequality 338

9.4 L^2 Theory for $\bar{\partial}$ 348

9.5 Deeper properties of the Green's operator 353

9.6 Notes for Chapter 9 365

10 Function Theory on Compact Riemann Surfaces **367**

10.1 The Cauchy-Riemann equation 368

10.2 Meromorphic functions and forms 373

10.3 The Riemann-Roch theorem 383

10.4 Finite dimensionality of $\mathcal{O}^1(X)$ 393
10.5 Harmonic theory on compact Riemann surfaces 402
10.6 The de Rham-Hodge theorem 415
10.7 Notes on Chapter 10 421

11 The Uniformization Theorem **423**
11.1 The uniformization theorem 424
11.2 The Dirichlet problem: Perron's method 427
11.3 The Dirichlet problem: L^2 methods 431
11.4 Normal families 436
11.5 Uniformization for non-compact surfaces 438
11.6 Uniformization from a geometric point of view 443
11.7 Notes on Chapter 11 450

12 An Introduction to Several Complex Variables **453**
12.1 Holomorphic functions in several variables 453
12.2 The inhomogeneous Cauchy-Riemann equations 462
12.3 Domains of holomorphy 466
12.4 Pseudoconvexity 470
12.5 The Levi problem 476
12.6 Notes on Chapter 12 483

Appendices **484**

A Compactness in Function Spaces **485**
A.1 Compactness in $\mathbf{R}^n$ 485
A.2 Uniform convergence 485
A.3 Compactness in function spaces 486

B Hilbert Spaces **491**

B.1 Basic properties 491
B.2 Projections in Hilbert spaces 494
B.3 The Riesz representation theorem 496
B.4 Linear operators on Hilbert spaces 497

C Integral Kernels and Fourier Series 503
C.1 Fourier series 503
C.2 Separation of variables 504
C.3 The Poisson and Cauchy kernels 510
C.4 Integral kernels for the upper half plane 512

D Fourier Transforms and the Rellich Lemma 515
D.1 Fourier transforms 515
D.2 The Rellich lemma 517

E Distributions and Fundamental Solutions 521
E.1 Test functions 522
E.2 Distributions 525
E.3 Differentiation of distributions 528
E.4 Convolutions 531
E.5 Fundamental solutions 533

F Cohomology on a Riemann Surface 539
F.1 Partitions of unity 539
F.2 The d operator 540
F.3 The $\bar{\partial}$ operator 543
F.4 The de Rham theorem 545
F.5 The Dolbeault theorem 548

G Analytic continuation and the monodromy theorem 553

G.1 Analytic continuation in a Riemann surface 553
G.2 The monodromy theorem 557
G.3 Applications of the monodromy theorem 559

Bibliography **562**

Index **568**