Exercises on $D$-modules

1. Let $f$ be a homogeneous polynomial in $R := \mathbb{C}[x_1, \ldots, x_n]$. Consider the Euler operator

$$E := x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}.$$ 

Verify that $E(f) = (\deg f)f$.

2. Let $R$ be a commutative ring; recall that $D_R^k$ denotes the differential operators on $R$ of order up to $k$. Show that

$$D_R^k \circ D_R^l \subseteq D_R^{k+l}.$$ 

It follows that $D_R := \bigcup_{k \geq 0} D_R^k$ is a ring!

3. Let $R := \mathbb{C}[x]$. Express the following elements of $D_{\mathbb{R}[x]}$ in terms of the PBW basis:

(a) $\partial^2 \circ x$
(b) $\partial \circ f$, where $f \in R$
(c) $\partial^2 \circ f$, where $f \in R$

4. Let $D_{\mathbb{R}[x]}$ be the Weyl algebra, where $R := \mathbb{C}[x_1, \ldots, x_n]$. Show that the center of $D_{\mathbb{R}[x]}$ is $\mathbb{C}$ as follows:

(a) If $P \in D_{\mathbb{R}[x]}$ is central, then it belongs to $\text{Hom}_R(R, R) \cong R$.
(b) For a polynomial $P \in R$, one has $[\partial_i, P] = \partial P/\partial x_i$.
(c) Conclude that the center of $D_{\mathbb{R}[x]}$ is $\mathbb{C}$.

5. Let $D_{\mathbb{R}[x]}$ be the Weyl algebra, where $R := \mathbb{C}[x_1, \ldots, x_n]$. Let $P$ be a nonzero element on $D_{\mathbb{R}[x]}$.

(a) If $\partial_i$ occurs in $P$ when $P$ is expressed in terms of the PBW basis, prove that $[P, x_i] \neq 0$.
(b) If $x_i$ occurs in $P$ when $P$ is expressed in terms of the PBW basis, prove that $[P, \partial_i] \neq 0$.
(c) Conclude (yes, once again!) that the center of $D_{\mathbb{R}[x]}$ is $\mathbb{C}$.

6. Let $f$ be an element of $\mathbb{C}[x]$. Prove that in $D_{\mathbb{R}[x]}$ one has

$$\frac{\partial^k}{\partial x^k} \circ f = \sum_{i+j=k} \binom{k}{i} \left( \frac{\partial^i f}{\partial x^i} \right) \frac{\partial^j}{\partial x^j}.$$ 

7. Let $\mathcal{F}_\circ$ denote the Bernstein filtration on the Weyl algebra $D_{\mathbb{R}[x]}$. Prove that

$$[\mathcal{F}_t, \mathcal{F}_j] \subseteq \mathcal{F}_{t+j-2}.$$ 

8. Let $\mathcal{F}_\circ$ denote the Bernstein filtration on the Weyl algebra $D_{\mathbb{R}[x]}$. Take $M$ to be $D_{\mathbb{R}[x]}$ and define $\mathcal{F}_\circ$ on $M$ by $\mathcal{G}_t := M$ for all $t \geq 0$. Is $\text{gr}M$ finitely generated over $\text{gr}D_{\mathbb{R}[x]}$?

9. Let $R := \mathbb{C}[x]$ and let $\mathcal{F}_\circ$ denote the Bernstein filtration on $D_{\mathbb{R}[x]}$. Consider the induced filtration on $R_x = R[1/x]$; specify a basis for

$$\mathcal{F}_t \cdot \frac{1}{x}$$ 

for each $t \geq 0$.

Use this to compute the multiplicity of $R_x$ as a $D_{\mathbb{R}[x]}$-module.

10. Let $R := \mathbb{C}[x]$. Fix $\lambda \in \mathbb{C}$, and consider the natural action of $D_{\mathbb{R}[x]}$ on $M := \bigoplus_{i \in \mathbb{Z}} \mathbb{C} x^{\lambda+i}$.

(a) Compute $e(M)$, i.e., the multiplicity of $M$.
(b) Prove that $M$ is a simple $D_{\mathbb{R}[x]}$-module if and only if $\lambda \notin \mathbb{Z}$.

11. Let $R := \mathbb{C}[x_1, \ldots, x_n]$. For $k \leq n$, determine the multiplicity of $R_{x_1 \cdots x_k}$ as a $D_{\mathbb{R}[x]}$-module.
12. (Nuking a mosquito) Using the above, and the Čech complex \( \check{\mathcal{C}}^*(x_1, \ldots, x_n; R) \), prove that
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^k = (-1)^n.
\]

With conventional weapons, one could set \( x := 2 \) in the binomial expansion of \((1 - x)^n\).

13. Prove that every holonomic module \( M \) over the Weyl algebra \( D := D_{R[C]} \) is cyclic as follows:
   (a) Recall that \( \ell(M) \) is finite; by induction, reduce to the case \( M = Du + Dv \), where \( Dv \) is simple.
   (b) Since \( Du \) has finite length, there exists a nonzero \( P \) in \( D \) with \( Pu = 0 \).
   (c) Since \( DPD = D \), one has \( DPDv \neq 0 \), so there exists \( Q \in D \) with \( PQv \neq 0 \).
   (d) Show that \( u + Qv \) generates \( M \).

14. Let \( R := \mathbb{C}[x] \) and set \( D := D_{R[C]} \). Construct an isomorphism of left \( D \)-modules
\[
\frac{D}{Dx^2} \xrightarrow{\cong} \frac{D}{Dx} \oplus \frac{D}{Dx}.
\]
If you are having fun, go for
\[
\frac{D}{Dx^3} \xrightarrow{\cong} \frac{D}{Dx} \oplus \frac{D}{Dx} \oplus \frac{D}{Dx}.
\]

15. Consider a \( 2 \times n \) matrix of indeterminates
\[
Z := \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix},
\]
and the \( \mathbb{C} \)-linear action of \( G := SL_2(\mathbb{C}) \) on the polynomial ring \( R := \mathbb{C}[Z] \), where \( M \in G \) acts as
\[
M : Z \mapsto MZ.
\]
The goal is to show that the invariant ring \( R^G \) is \( S := \mathbb{C}[\Delta_{ij} : 1 \leq i < j \leq n] \), where \( \Delta_{ij} := x_iy_j - x_jy_i \). Set
\[
E_{ij} := x_i \frac{\partial}{\partial x_j} + y_i \frac{\partial}{\partial y_j} \quad \text{and} \quad D_{ij} := \det \begin{bmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial y_j} \\ \frac{\partial}{\partial x_j} & \frac{\partial}{\partial y_i} \end{bmatrix}.
\]

(a) If \( n = 1 \), prove that \( R^G = \mathbb{C} \).
(b) Show that each \( E_{ij} \) acts on \( S \).
(c) Show that \( E_{ij} \circ g = g \circ E_{ij} \) for each \( g \in G \).
(d) Show that each \( E_{ij} \) acts on \( R^G \).
(e) Show that each \( D_{ij} \) acts on \( R^G \).
(f) Prove Capelli’s identity:
\[
\det \begin{bmatrix} E_{ii} + 1 & E_{ij} \\ E_{ji} & E_{jj} \end{bmatrix} = \det \begin{bmatrix} x_i & x_j \\ y_i & y_j \end{bmatrix} \circ \det \begin{bmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial y_j} \\ \frac{\partial}{\partial x_j} & \frac{\partial}{\partial y_i} \end{bmatrix},
\]
for \( i \neq j \), where determinants are read left to right; in other words prove that
\[(E_{ii} + 1)E_{jj} - E_{ii}E_{jj} = \Delta_{ij}D_{ij}.
\]
(g) Take the \( \mathbb{N}^2 \)-grading on \( R \) with \( \deg x_i = \deg y_i = e_i \), the \( i \)-th basis vector; show that \( R^G \) inherits a grading.
(h) Prove that \( R^G = S \) as follows: if not, choose a homogeneous \( f \) in \( R^G \setminus S \) of degree \( (d_1, \ldots, d_n) \) such that \( \sum d_i \) is minimal, and that, amongst such \( f \), the entry \( d_1 \) maximal. Then \( d_j \neq 0 \) for some \( j \neq 1 \) by (a). Consider
\[(E_{ii} + 1)E_{jj}(f) = E_{jj}E_{ij}(f) + \Delta_{ij}D_{ij}(f).
\]

16. Let \( D_{R[C]} \) be the Weyl algebra, where \( R := \mathbb{C}[x_1, \ldots, x_n] \). Suppose \( M \) is a \( D_{R[C]} \)-module with a filtration \( \mathcal{F} \), for which there exist \( c, m \) such that
\[
\text{rank}_C \mathcal{F}_i \leq ct^m \quad \text{for all } t \gg 0.
\]
Does \( M \) need to be finitely generated? (We saw that this is true if \( m = n \).)
Hint: Take \( n = 1 \) and consider the \( D_{R[C]} \)-module
\[M := R \oplus R \oplus R \oplus \cdots \]
with the filtration
\[\mathcal{F}_i := [R]_{\leq i-1} \oplus [R]_{\leq i-2} \oplus [R]_{\leq i-3} \oplus \cdots.
\]
17. (Symmetry of the Weyl algebra) Recall that for a ring, the opposite ring $A^{\text{op}}$ consists of $A$ as an abelian group, with multiplication in “reverse order.” More precisely, 

$$A^{\text{op}} := \{ a^{\text{op}} \mid a \in A \},$$

with $a^{\text{op}} + b^{\text{op}} = (a+b)^{\text{op}}$, and $a^{\text{op}} b^{\text{op}} = (b a)^{\text{op}}$. Let $D_{R|C}$ be the Weyl algebra, where $R := \mathbb{C}[x_1, \ldots, x_n]$.

(a) Show that the $\mathbb{C}$-algebra map with $x_i \mapsto x_i^{\text{op}}$ and $\partial_i \mapsto - \partial_i^{\text{op}}$ gives an isomorphism $D_{R|C} \rightarrow D_{R|C}^{\text{op}}$.

(b) Note that right $D_{R|C}$-modules correspond to left modules over $D_{R|C}^{\text{op}}$. Using the fact that $D_{R|C}$ is left noetherian, conclude that the ring $D_{R|C}$ is also right Noetherian.

18. Set $R := \mathbb{F}_p[w, x, y, z]$ and $f := wx - yz$. Construct a differential operator $P \in D_R|\mathbb{F}_p$ such that $P(1/f) = 1/f^p$.

19. Let $R := \mathbb{F}_p[x]$. Recall that $D_t := 1/t! \partial_x^t$ for $t \geq 1$.

(a) Prove that $D_t^p = 0$.

(b) Prove that $[D_q, x^p] = 1$ for each integer $q = p^r$.

Hint: For $q$ as above, and $m \in \mathbb{N}$, a theorem of Lucas implies that 

$$\left(\frac{m+q}{q}\right) - \left(\frac{m}{q}\right) \equiv 1 \mod p.$$ 

20. For $R$ and $D_t$ as above. Prove that $D_1$ belongs to the $\mathbb{F}_p$-algebra generated by $x$ and $D_{p-1}$.

21. (An application of differential operators to computing F-thresholds) Let $f$ be a homogeneous cubic polynomial in $\mathbb{F}_p[x, y, z]$ for which the Jacobian ideal $J := (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ is primary to the homogeneous maximal ideal $m := \langle x, y, z \rangle$. The goal is to show that

$$f^{p-2} \not\in m^p := \langle x^p, y^p, z^p \rangle.$$

(a) Let $k$ be least such that $f^k \in m^p$. If $k < p$, show that $f^{k-1} J \subseteq m^p$.

(b) Prove that $m^4 \not\subseteq J$.

(c) Prove that $(m^p : m^4) = (m^p : m^3 p - 6)$.

(d) Conclude that $f^{k-1} \in (m^p : m^3 p - 6)$, and hence that deg $f^{k-1} \geq 3p - 6$.

(e) Conclude that $k \geq p - 1$.

22. For $p$ a prime integer, set $W$ to be $\mathbb{F}_p\langle x, y \rangle / \langle [x, y] - 1 \rangle$.

(a) Prove that $x^p$ is in the center of $W$.

(b) Prove that $R$ is not a simple ring, i.e., find a two-sided proper ideal.

23. The goal is to compute the center of the Weyl algebra in positive characteristic; let $R := \mathbb{F}_p[x_1, \ldots, x_n]$ and consider the Weyl algebra

$$W := \mathbb{F}_p\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle / \langle [x_i, x_j], [\partial_i, \partial_j], [\partial_i, x_j] - \delta_{ij} \rangle.$$

(a) Show that there is an $\mathbb{F}_p$-algebra homomorphism $W \rightarrow D_{R|\mathbb{F}_p}$. Show that it fails to be injective, and also fails to be surjective. Characterize the image in terms of the level filtration $D^{(c)} := \text{Hom}_{\mathbb{F}_p} (R, R)$.

(b) Show that each $x_i^p$ and $\partial_i^p$ is in $Z(W)$, i.e., the center of $W$.

(c) If $A \rightarrow B$ is a ring homomorphism, show that $Z(A)$ need not map to $Z(B)$. However, show that $Z(A)$ must map to $Z(B)$ when $A \rightarrow B$ is surjective. Using this, and your answer to (a), show that

$$Z(W) = k[x_1^p, \ldots, x_n^p, \partial_1^p, \ldots, \partial_n^p].$$
24. Set \( R := \mathbb{Z}[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1y_1 + x_2y_2 + x_3y_3) \) and \( a := (x_1, x_2, x_3)R \). The goal is to show that \( H^3_\wp(R) \) has infinitely many associated primes. Let \( p \) be an arbitrary prime integer; consider the cohomology class

\[
\eta_p := \left[ \frac{(x_1y_1)^p + (x_2y_2)^p + (x_3y_3)^p}{p(x_1x_2x_3)^p} \right] \quad \text{in} \quad H^3_\wp(R) = \frac{R_{x_1x_2x_3}}{R_{x_1x_2} + R_{x_1x_3} + R_{x_2x_3}}.
\]

(a) Check that the fraction \( \left( (x_1y_1)^p + (x_2y_2)^p + (x_3y_3)^p \right)/p \) is indeed an element of \( R \).

(b) Verify that \( p\eta_p = 0 \).

Prove that \( \eta_p \) is nonzero as follows: if \( \eta_p = 0 \), then there exists an integer \( k \) and elements \( c_i \) in \( R \) with

\[
\frac{(x_1y_1)^p + (x_2y_2)^p + (x_3y_3)^p}{p} = c_1x_1^{p+k} + c_2x_2^{p+k} + c_3x_3^{p+k}.
\]

Consider the \( \mathbb{N}^3 \)-grading on \( R \) with \( \deg x_i = e_i \) and \( \deg y_i = -e_i \), where \( e_i \) is the \( i \)-th basis vector.

(c) Without loss of generality, the \( c_i \) are homogeneous; determine the degree of each \( c_i \).

(d) Conclude that \( c_1 \) is a scalar multiple of \( y_1^p x_1^k x_2^k \), and draw similar conclusions for \( c_2 \) and \( c_3 \).

(e) Rewrite equation (1) using these observations; divide through by \( (x_1x_2x_3)^k \), then specialize each \( y_i \to 1 \), and \( x_3 \to -(x_1 + x_2) \), to obtain

\[
\frac{x_1^p + x_2^p + (-x_1 - x_2)^p}{p} \in (p, x_1^p, x_2^p)\mathbb{Z}[x_1, x_2].
\]

(f) Prove that the above is false, so as to obtain a contradiction.

References: expository


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References: papers


