Sampling real algebraic varieties for topological data analysis

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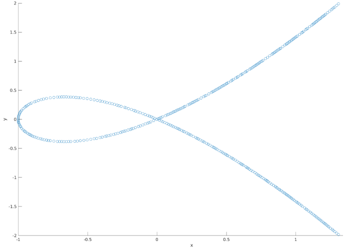
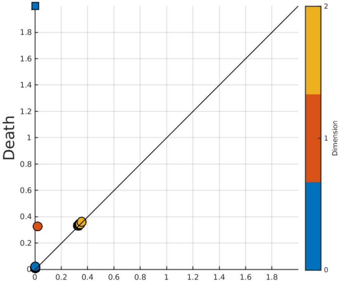
AG19, July 2019

Sampling real varieties

$$y^2 - x^2(x+1) = 0$$





Topological data analysis

Persistent homology overview

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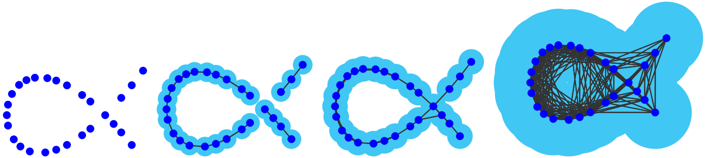

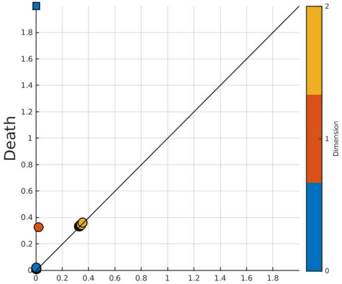

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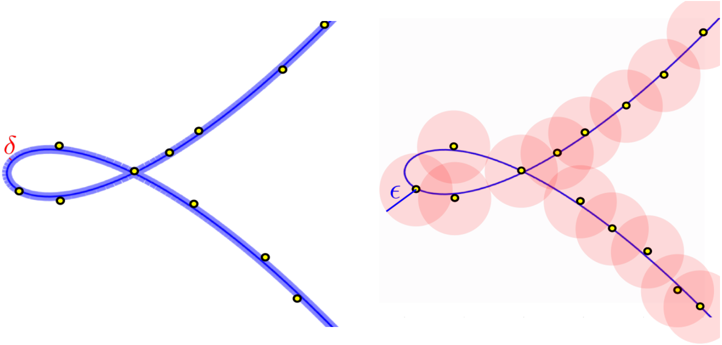

Figure: A (δ, ϵ) sample of an algebraic curve.

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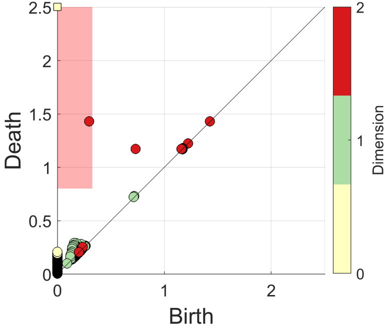
Theorem (Cohen-Steiner et al. 2007, Chazal and Lieutier 2005)

Let $X \subseteq \mathbb{R}^N$ be compact with reach$(X) > 2(\epsilon + \delta)$. β_p is the number of points in above and to the left of $(\epsilon, 2\epsilon + \delta)$ in the Čech diagram for a (δ, ϵ) sample of X.



Input

- A system of polynomial equations $f : \mathbb{C}^N \to \mathbb{C}^{N-d}$, $f = (f_1, \ldots, f_{N-d})$
- A density goal $\epsilon > 0$
- A rectangular region $R = [a_1, b_1] \times \cdots \times [a_N, b_N]$ to search
- A homotopy continuation error bound $0 \leqslant \delta < \epsilon$.

Output

A (δ, ϵ)-sample of $V_{\mathbb{R}}(f) \cap R$ that has as few points in the sample as possible.

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Sampling algebraic sets from equations

- Reduction methods: Sherbrooke and Patrikalakis, 1993. Produces (ϵ, ϵ)-sample.

Sampling from a distribution

- Sampling from the uniform distribution on real algebraic manifolds: Breiding and Marigliano, 2018.

Computing homology for semialgebraic sets

- Computation in weak exponential time: Bürgisser, Cucker, Tonelli-Cueto, 2019.

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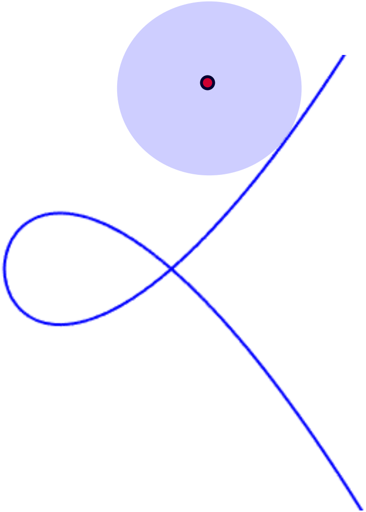
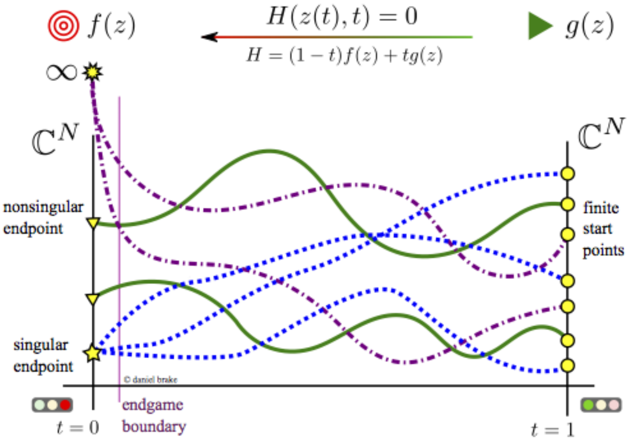


Figure: The minimum distance problem for a curve and point in $\mathbb{R}^2$.

Tools from numerical algebraic geometry

Homotopy continuation

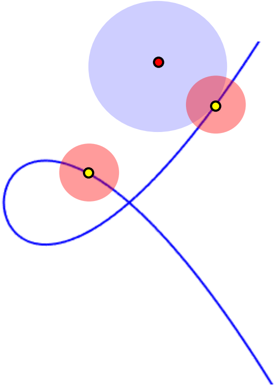


Theorem (J. Hauenstein, 2012)

For $y \in \mathbb{R}^N$ the minimum distance problem can be solved with homotopy continuation. The homotopy is as follows.

$$H(x, \lambda_0, \ldots, \lambda_{N-d}, t) = \begin{pmatrix} f(x) - t\gamma \\ \lambda_0(x - y) + \sum_{i=1}^{N-d} \lambda_i \nabla f_i(x) \\ \sum_{i=0}^{N-d} \alpha_i \lambda_i - 1 \end{pmatrix}$$

Sampling Algorithm



- Pick a point and find the critical points of the minimal distance equations with the variety.
- Record sample points, plus exclusion zone around these new sample points and the test point.

Sampling Algorithm

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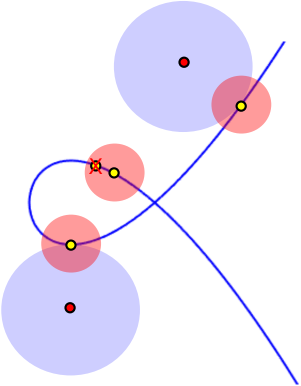

- Pick a point and find the critical points of the minimal distance equations with the variety.
- Record sample points, plus exclusion zone around these new sample points and the test point.
- Pick another test point do the same. Repeat until sample and exclusion balls cover the space.

Sampling algorithm

Termination

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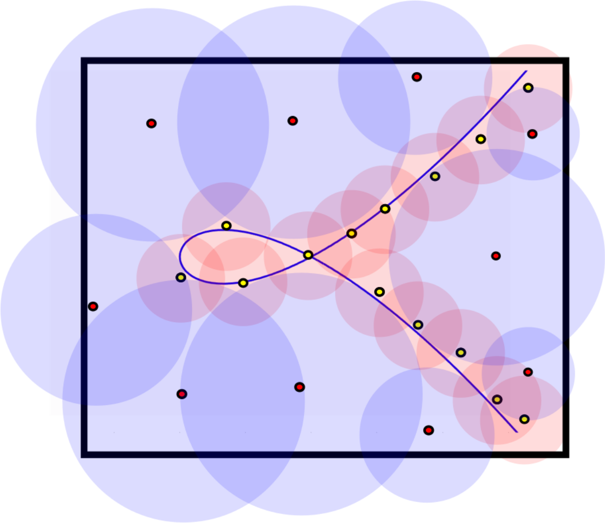

Example: quartic surfaces in $\mathbb{R}^3$

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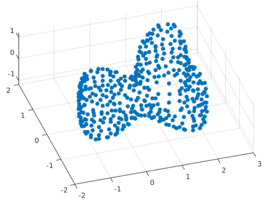
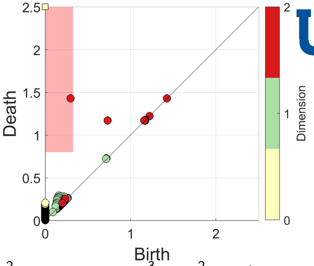
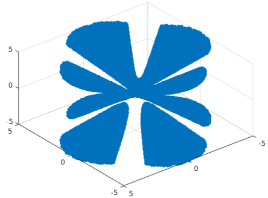
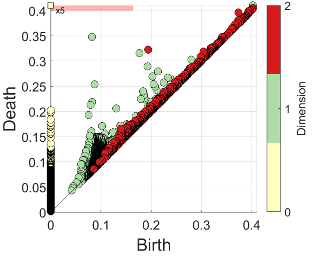


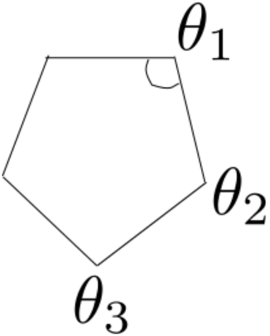


$$V_1 = V_{\mathbb{R}}(4x^4 + 7y^4 + 3z^4 - 3 - 8x^3 + 2x^2y - 4x^2 - 8xy^2 - 5xy + 8x - 6y^3 + 8y^2 + 4y)$$

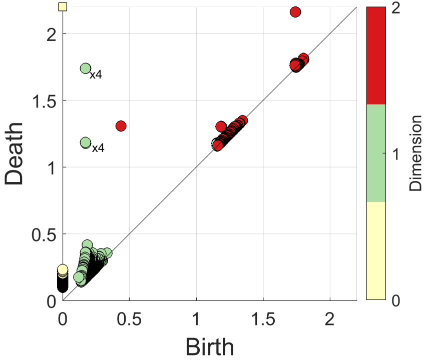




$$V_2 = V_{\mathbb{R}}(144x^4 + 144y^4 - 225(x^2 + y^2)z^2 + 350x^2y^2 + 81z^4 + x^3 + 7x^2y + 3x^2 + 3xy^2 - 4x - 5y^3 + 5y^2 + 5y)$$



θ_1
θ_2
θ_3



Death
Birth
Dimension
x4
x4
0
0.5
1
1.5
2

Preprint

Sampling real algebraic varieties for topological data analysis
https://arxiv.org/abs/1802.07716

Software

https://github.com/P-Edwards/tdasampling