Persistence Landscapes are Graded Persistence Diagrams

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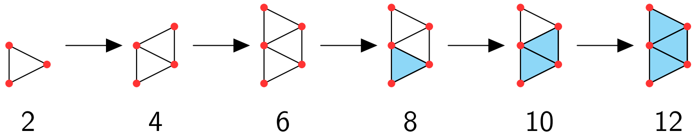
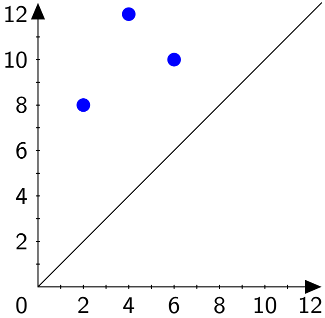
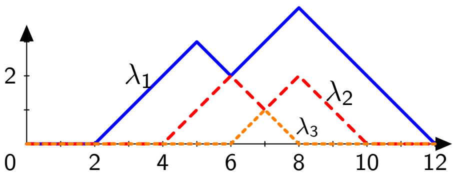
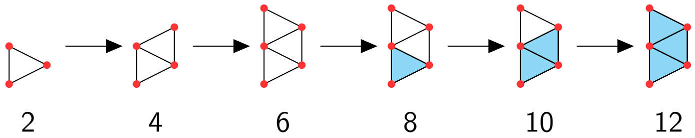
Motivation: Persistence diagrams and landscapes

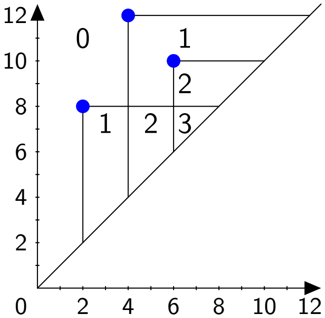






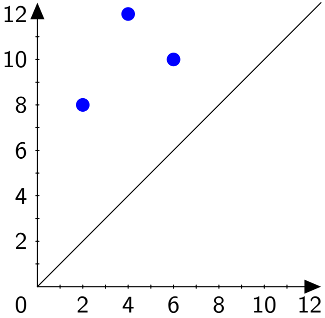
The rank function

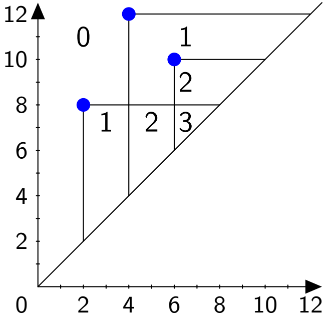




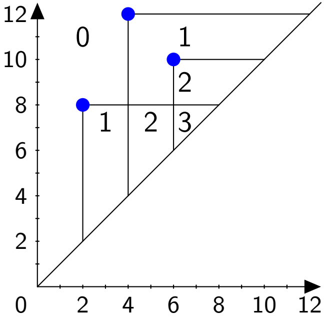
$$\mathrm{rank}(4,6) = 2$$
$$\mathrm{rank}(4,8) = 1$$
$$\mathrm{rank}(4,10) = 0$$

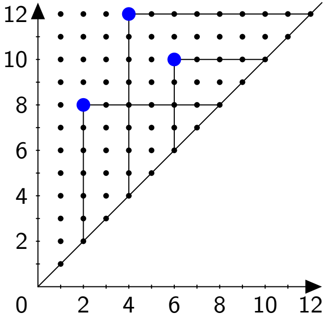
Persistence diagrams and the rank function





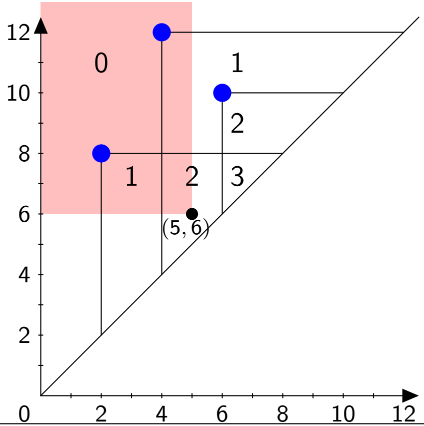
Discrete perspective





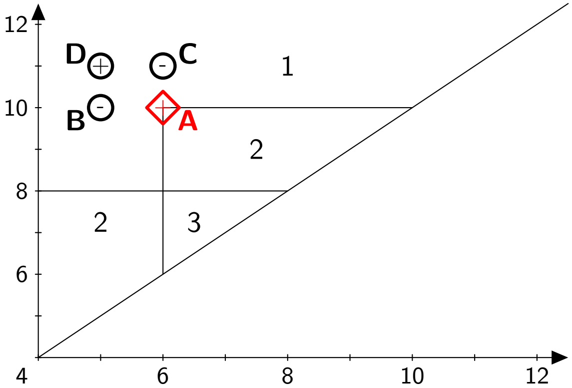
The rank function from the persistence diagram

$$\text{Rank function} \xleftarrow[\text{Upper Left Sum}]{} \text{Persistence diagram}$$



The persistence diagram from the rank function

$$\text{Rank function} \xrightarrow{\text{Möbius inversion}} \text{Persistence diagram}$$



$$\mathrm{PD}(\mathbf{A}) = \mathrm{rank}(\mathbf{A}) - \mathrm{rank}(\mathbf{B}) - \mathrm{rank}(\mathbf{C}) + \mathrm{rank}(\mathbf{D})$$

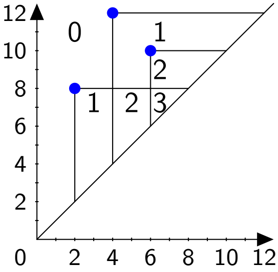
$$\mathrm{PD}(\mathbf{A}) = 2 - 1 - 1 + 1 = 1$$

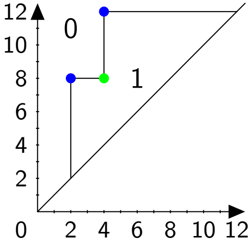
Definition

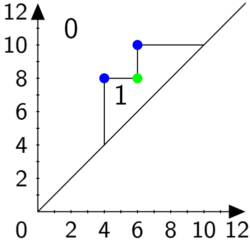
For $k \geqslant 1$, the k'th graded rank function is given by

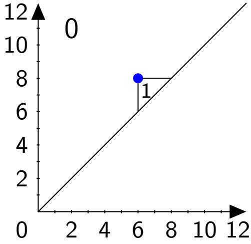
$$\mathrm{rank}_k(p) = \begin{cases} 1 & \mathrm{rank}(p) \geqslant k \\ 0 & \text{else} \end{cases}$$

Rank and graded rank



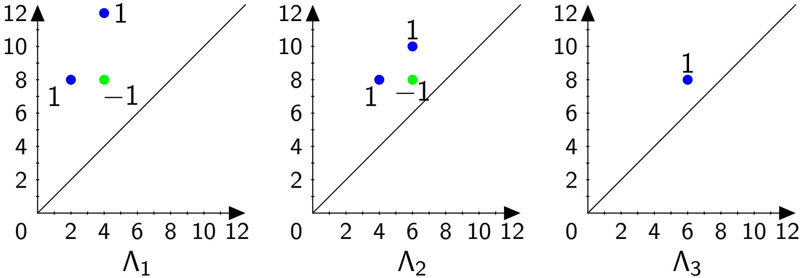






Definition

The k'th graded persistence diagram is the Möbius inversion of the k'th graded rank function.



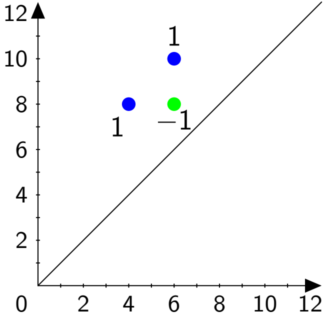
Consistency Theorem (Betthauser, Bubenik, E.)

The following diagram of transformations commutes:

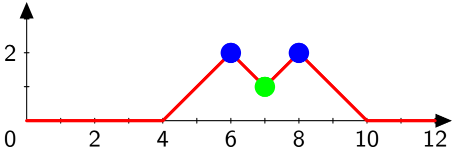
$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Rank function} \arrow[r, shift left, "\text{Möbius inversion}"] \arrow[d, shift right, swap, "\text{Grading}"] & \text{Persistence diagram} \arrow[l, shift left, "\text{Upper Left Sum}"] \\
\text{Graded rank} \arrow[r, shift left, "\text{Möbius inversion}"] \arrow[u, shift right, swap, "\text{Pointwise Sum}"] & \text{Graded persistence diagram} \arrow[l, shift left, "\text{Upper Left Sum}"] \arrow[u, swap, "\text{Pointwise Sum}"]
\end{tikzcd}$$

Correspondence Theorem (Betthauser, Bubenik, E.)

The k'th graded persistence diagram is equivalent to the k'th persistence landscape λ_k.



Λ_2



λ_2

Proof sketch

$\Lambda_k(A) = -1 \Rightarrow A$ is a local minimum

$$\Lambda_k(A) = \mathrm{rank}_k(A) - \mathrm{rank}_k(B) - \mathrm{rank}_k(C) + \mathrm{rank}_k(D) = -1$$

$$\begin{array}{cccc} + & & & - \\ & 0 & 1 & \\ & 0 & 0 & \\ - & & & + \end{array} \qquad\qquad \begin{array}{cccc} + & & & - \\ & 1 & 1 & \\ & 1 & 0 & \\ - & & & + \end{array}$$

Proof sketch

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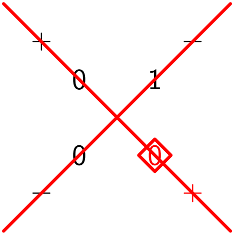

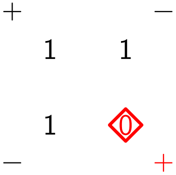

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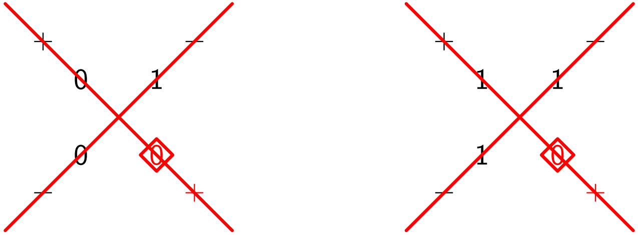

Proof sketch

$\Lambda_k(p) = -1 \Rightarrow p$ is a local minimum