Frege on Mathematical Progress[1]

Patricia Blanchette

This is a not-quite-final version of the essay that appears in Early Analytic Philosophy: New Perspectives on the Tradition*, Sorin Costreie (ed); Springer 2016, 3-19.*

1. Introduction

One of the central motivations behind Frege's concern with *thoughts* is his concern with the communal nature of science. The fact that people separated by gulfs of time, of space, and of language can share a common science is, from Frege's point of view, due to the fact that the substance of a given science is not a collection of sentences or of ideas, but of thoughts, the kinds of things that can be expressed by sentences of different languages, and can be conveyed from person to person despite differences in ideas or contingent circumstance. As Frege himself puts it,

> Can the same thought be expressed in different languages? Without a doubt, so far as the logical kernel is concerned; for otherwise it would not be possible for human beings to share a common intellectual life.[2]

In addition to the *communal* nature of science, Frege is also importantly concerned with its developmental side, i.e. with the fact that sciences, mathematical ones in particular, experience significant conceptual refinement over time. This circumstance is especially important from the Fregean point of view in mathematics,

[1] Versions of this essay were presented at the 2014 "Frege@Stirling" workshop at Stirling University, and at the 2014 Logic Colloquium in Vienna. Many thanks to the organizers and audience members, especially to Philip Ebert, Bob Hale, and Rob Trueman for helpful comments.

[2] Frege (1979) 6; Frege (1983) 6, from the "Logic" notes, undated. See also Frege (1892a) 29 (160/146), and Frege (1892b) 196 note (185/170) for similar sentiments about the sharing of thoughts as the ground of common science.

since he takes his own project, one that involves highly non-trivial reconceptualizations of central mathematical notions, to be of a piece with the history of conceptual development in mathematics generally. On the importance of conceptual development in mathematics, Frege says at the beginning of *Grundlagen*:

> After deserting for a time the old Euclidean standards of rigor, mathematics is now returning to them, and even making efforts to go beyond them. … The discovery of higher analysis only served to confirm this tendency; for considerable, almost insuperable, difficulties stood in the way of any rigorous treatment of these subjects … The concepts of function, of continuity, of limit and of infinity have been shown to stand in need of sharper definition. Negative and irrational numbers, which had long since been admitted into science, have had to submit to a closer scrutiny of their credentials. [Frege (1884) §1]

And as to the connection between his own work and this tradition:

> In all directions these same ideals can be seen at work – rigour of proof, precise delimitation of the concept of validity, and as a means to this, sharp definition of concepts. (*…die Begriffe scharf zu fassen*.)
>
> Proceeding along these lines, we are bound eventually to come to the concept of Number, and to the simplest propositions holding of positive whole numbers… [Frege (1884) §§1-2]

The last-mentioned project, that of providing a deeper analysis of the concept of Number, and of "the simplest propositions holding of positive whole numbers," is the central work of *Grundlagen*.

One of the crucial features of conceptual analysis in mathematics, as Frege sees it, is that it is often highly non-trivial:

> Often it is only after immense intellectual effort, which may have continued over centuries, that humanity at last succeeds in achieving knowledge of a concept in its pure form, in stripping off the irrelevant accretions which veil it from the eyes of the mind. [Frege (1884) p. vii]

But now we seem to face a real difficulty. Fregean *thoughts* are not obviously the kinds of things that can survive the sort of significant conceptual development of which the history of mathematics consists. And if they cannot do so, then Frege's fundamental way of understanding the nature and the continuity of mathematical sciences is in tension with his conception of mathematical progress. Our central question in what follows is that of how we are to understand this tension in Frege, and of whether there is a plausible Fregean account of the nature of mathematics that makes sense both of continuity and of significant conceptual change over time.

2. Sense vs. Conventional Significance

One way of trying to clarify Frege's conception of the *sense* of an expression is by means of what a speaker of the language is aware of when, and in virtue of which, he or she is competent with respect to that expression. If this is the correct way to understand sense, then the tension between Frege's view of continuity and his view of mathematical progress is stark: there is no sense in which, for example, a speaker's linguistic competence in the mid-18th century with the term "continuous function" requires any inkling of the content of its 19th-century analysans.

But as Tyler Burge has argued, the identification of Frege's notion of sense with linguistic meaning is a mistake.[3] Because the sense of a sentence is the

[3] See Burge [1979].

fundamental truth-bearer, it is determined by the world in ways that can often outstrip the thin collection of information awareness of which is required for linguistic competence. As a corollary, the sense of an individual term can often be considerably richer than the collection of information that would be conveyed by a good dictionary. And as Burge has further argued, the separation of the "conventional significance" of a word - i.e. the material whose grasp constitutes linguistic competence - from the sense of that word offers a straightforward route to understanding how sense can outstrip what even expert speakers associate with a term or sentence.[4] In cases in which the conventional significance of a word is insufficiently precise to pin down a particular reference, the sense of that word, as determiner of reference, must go beyond that ordinary significance. As Burge sees it, the "extra" input is delivered, in the case of mathematics, by the mathematical facts themselves, those facts in whose systematization and explanation the term plays a central role. Concerning the example of the term "Number" as used prior to Frege's work, and hence whose conventional significance involves in Frege's view a good deal of imprecision, Burge asks:

> How could the term 'Number' indicate a definite "concept" when all current mathematical understanding and usage failed to determine a sense or concept? [Burge (1984) 10]

And replies:

> To say, as Frege says, that 'Number' *does* denote a concept and *does* express a sense is to say that the ultimate foundation and *justification* of mathematical practice supplements current usage and understanding of the term in such a way as to attach it to a concept and a sense. From this point of view, vague

[4] See Burge [1984].

usage and understanding do not entail vague sense-expression. [Burge (1984) 11]

As we might put it, the rich body of mathematical facts that underpins our mathematical practice serves, together with the sometimes-incomplete information conventionally associated with mathematical terms, to fix determinate sense and reference on those terms. On this way of understanding the project, it is clear why the conceptual analysis so essential to mathematical progress requires non-trivial mathematical work: to gain clarity about the nature of the objects and concepts we have all along been referring to is, in part, to gain clarity about their mathematical properties and relations to one another. In gaining this clarity, we make it clear what it is in virtue of which it is *this* rather than *that* function or object to which we have been referring all along.

But while this clarification of the connection between conventional significance, mathematical facts, and sense provides an answer to Burge's question of how vagueness of conventional significance can be compatible with expression of determinate sense, it will not solve the whole of the difficulty sketched above, the difficulty of reconciling Frege's view of theoretical continuity as requiring thought-identity with the non-trivial nature of conceptual development. For it is not always the case that the mathematical facts underlying a given mathematical practice are sufficiently rich to disambiguate its terms in the context of later development. Suppose we have two mathematicians separated by a time-span in which there has been significant conceptual change, so that the later mathematician uses a mathematical term t whose sense and reference are fixed by a precise definition given in the interval. Burge's account gives us a way of understanding what has happened if the earlier mathematician uses the term *t* in such a way that her practice and the underlying mathematical facts together fix just that sense and reference, so that the conceptual work in the interval has consisted in clarifying a sense that was already determinate. But conceptual progress in mathematics is not always so simple. Consider for example the case of *continuous function*, a notion familiar to mathematicians before the precision instituted in the 19^{th} century, and a notion open then to multiple non-equivalent precisifications. A plausible reading of the history is one on which nothing about eighteenth- or early-nineteenth-century practice served to pick out precisely one of the later-disentangled notions of *epsilon-delta continuity, uniform continuity,* and *differentiability.* Similarly for *cardinality*: pre-Cantorian practice with finite sets fails to pin down a notion of cardinality on which the natural numbers and the evens are of the same cardinality in virtue of the existence of a bijection, as opposed to a notion on which they are *not* of the same cardinality because one is a proper subset of the other. There are of course often good reasons for choosing one coherent precisification over another, reasons having to do with overall economy, tractability, fruitfulness, and so on. The important point here is that earlier practice in such cases does not pin down a collection of mathematical facts sufficient to dis-ambiguate the central terms. Instead, the reference-fixing is accomplished in part by straightforward decision: we stipulate that, in future, *this* is what we will mean by "continuous," by "infinite," by "of the same cardinality as," and so on. In these cases, one cannot say that later mathematicians expressed senses or referred to functions and objects that had been determinately pinned down by the combination of earlier usage and underlying mathematical facts.

Frege's own work towards conceptual clarification would seem to provide examples of just this kind. The reference of the numeral "2," as Frege presents it in *Grundgesetze*, is the extension of a first-level function (under which fall extensions of other first-level functions), while its reference as presented in *Grundlagen* is the extension of a second-level function (under which fall first-level concepts). Similarly for all other numerals. The change from the *Grundlagen* to the *Grundgesetze* numbers is not due to a change of view on Frege's part: the two treatments succeed for his purposes in just the same way. The change would seem to be driven by reasons of technical convenience. What this means, though, is that the pre-Fregean use of the numerals, despite arguably fixing everything that matters from the point of view of pure arithmetic, does not fix enough to settle whether the numerals refer to the extensions of first-level functions. No arithmetical facts determine whether the ordinary 2 is the *Grundlagen*'s 2, the *Grundgesetze*'s 2, or something else altogether. Just as in the cases of ordinary mathematical development noted above, the reference of the terms in the mature version of the science is determined in part by *fiat*, and not just by plumbing the depths of those mathematical facts that have, all along, grounded the original practice.

3. Domain Expansion

The Fregean case just mentioned is arguably an example of a phenomenon that arises whenever we expand the domain of a mathematical theory. Having added extensions (or value-ranges) to the domain of discourse, we can frame new sentences, e.g. identity-sentences involving one term from the old theory and one from the expansion zone, whose truth-value is not fixed by anything that has gone before. No arithmetical facts determine whether zero is a value-range, and if so precisely which one. No facts about the cardinal numbers and the rationals determine whether the cardinal 2 is identical with the rational 2; similarly for expansions to reals and to complex numbers. This is of course as it should be: it makes no mathematical difference how we answer these "outlying" questions, and it would be absurd to expect that the domain of underlying mathematical facts has any bearing here. But thoughts are determinately true or false. If nothing about mathematical practice or about the facts to which we advert when carrying out that practice determines whether "$2_{card} = 2_{rat}$" expresses a truth or a falsehood, then nothing about that practice or about those facts determines what thought is expressed by that sentence. Similarly, nothing about that practice or about those facts determines whether two sentences differing just in the replacement of one such term for the other express the same thought, so that the indeterminacy would seem to affect not just such inconsequential sentences as "$2_{card} = 2_{rat}$," but virtually all sentences of the language.

The difficulty for Frege's view of the nature of mathematical discourse and of scientific continuity now seems to have deepened. Because later, cleaned-up versions of fundamental concepts often arise not just as the result of analyzing content, but in part as a result of making arbitrary decisions in the face of newly-recognized ambiguity, the idea that the original terminology had determinate reference seems to have been undermined. If there is no fact of the matter whether the ordinary "2" refers to a given extension, or whether early uses of "continuous" refer to Weierstrass's notion, then it would seem that these terms have no determinate reference – which is to say that they have no reference. The difficulty about thoughts is now not just the subtle question of whether one can make sense of thought-identity across significant conceptual development, but of whether one can make sense of the idea that ordinary mathematical discourse involves the expression of thoughts at all, in the face of this degree of ambiguity about reference.

4. Frege on Domain-Expansion

Frege recognizes two kinds of domain-expansion in mathematical theories: those in which the "added" objects are of a not strictly-mathematical kind, and so give rise to identity-statements linking e.g. numerical and non-numerical terms (for example, "$2 = \{\{\emptyset\}\}$"), and those in which the "added" objects are from an enlarged but already-mathematical domain (e.g. "$2_{card} = 2_{rat}$"). In what follows, we examine his discussions of these cases, with an eye toward understanding to what extent the Fregean account of theoretical unity is undermined by domain-expansion. As we'll see, the difficulties for Frege are not negligible, but they are not as stark as has been suggested above.

In *Grundgesetze*, Frege introduces two kinds of singular terms: sentences (which, recall, are singular terms whose references are truth-values), and value-ranges. The truth-conditions of identity-sentences linking two value-range terms are given immediately by Law V, according to which the value-range of F = the value-range of G iff F and G give the same value for every argument. The truth-conditions of identity-sentences each of whose terms is a sentence are similarly straightforward: such identities are true if and only if the sentences on each side have (of course) the same reference, which is to say that they have the same truth-value. Left indeterminate by these factors, however, are the truth-conditions of identity-sentences in which the identity-sign is flanked by a sentence on one side, and a value-range term on the other. Such sentences will play no role in Frege's development of arithmetic, and hence, barring inconsistency, it does not matter how one fixes truth-conditions on them. But in keeping with Frege's insistence that every well-formed sentence of *Grundgesetze* have a determinate truth-value, it is essential that such sentences are fitted out with truth-conditions of some kind. Frege's way of meeting this requirement is simply to stipulate that all true sentences will refer to the value-range of any function under which exactly the True falls, and that every false sentence will refer to the value-range of any function under which exactly the False falls. The stipulation is arbitrary, in the sense that alternative stipulations could easily and unproblematically have been made in its place; the important point is simply that some coherent stipulation be made. Frege's remark about this stipulation is as follows:

> We have hereby determined the *value-ranges* as far as is possible here. Only when the further issue arises of introducing a function that is not completely reducible to the functions already known will we be able to stipulate what values it should have for value-ranges as arguments; and this can then be viewed as a determination of the value-ranges as well as of that function. [Frege (1893) §10]

The interest of this passage is that it undermines what one might call a “naïve platonist” reading of Frege’s understanding of the objects to which his singular terms refer. If we on a later occasion expand the language of *Grundgesetze* so as to make it suitable, say, for use in proofs about mechanics, we will introduce, amongst other things, new singular terms. The new “cross-category” identity sentences, i.e. those identifying a value-range and a new object, will have truth-conditions and hence truth-values only after a further arbitrary stipulation is made, as described by Frege above. Hence there is no fact of the matter, prior to the stipulation, whether the terms in question co-refer. And this is not due to a failure of determinate reference on the part of the introduced terms; the indeterminacy of the identity-sentences obtains even when the newly-introduced terms are those of a fixed, determinate science, one whose claim to the expression of truth is as robust as possible.

The same holds not just for identity-sentences but, as Frege remarks above, for sentences that express the application of a function from the old theory to an object (or function, or n-tuple) from the new: prior to the imposition of some arbitrary stipulations, such sentences will frequently not have truth-values fixed either by the linguistic meanings of the terms, or by the underlying mathematical or other facts. That the stipulation needed in *Grundgesetze* §10 applies merely to identity-sentences is an artifact of the very simple language of that formal system.

The indeterminacy of cross-theory sentences is just what one should expect in the normal course of events: that the merging of two self-standing theories, or the simple expansion of a single theory, will give rise to cross-theory sentences whose truth-conditions aren't determined by any of the facts with which either theory (or: the original theory) is concerned is the standard case. But it's a situation that does not square well with a certain conception of what it is for the terms of the original theory or theories to have determinate reference. If one takes it that e.g. a function-term $f(x)$ and a singular term t both have determinate reference only if $f(t)$ does, and hence that singular terms t_1 and t_2 have determinate reference only if the identity-sentence $t_1=t_2$ has a determinate truth-value, then the situation just described can only be understood as one in which the original theory or theories in question, no matter their long usefulness and success, have no terms with determinate reference. Given the possible (indeed, probable) expansion of mathematical terminology, and hence the possible (indeed probable) introduction into our vocabulary of novel cross-theory sentences of the kind just discussed, this conception is one on which none of our terms ever has determinate reference. So much the worse, of course, for the view that determinate reference in mathematics requires the kind of cross-theory determinacy just described.

It may seem, and indeed does seem to many, that Frege endorsed the platonic requirement on referentiality just discussed: the idea that determinacy of reference on the part of terms taken from different theories requires the determinacy of reference or truth-conditions for all syntactically-permissible combinations of those terms. The central reason one might have for attributing such a view to Frege is that one takes it to be an immediate consequence of his often-repeated claim that all functions are in some sense "total." But this, I take it, is a mistake. Frege's many and varied discussions of the requirement of totality for functions, which is to say the requirement that functions be defined for all arguments, are in every case discussions that apply to a single language: they are discussions of, indeed arguments for, the conclusion that rigor in formal systems requires that every function referred to in such a system is defined over every argument referred to in that system. This is, in short, the requirement of "linguistic completeness," the requirement that every well-formed expression of a formal system has a determinate reference. I have argued elsewhere, so won't go into details here, that Frege's commitment to the requirement of linguistic completeness is absolute for formal languages, that he holds no such requirement for languages of ordinary discourse, and that he does not hold the considerably stronger requirement that the functions referred to in a given system be defined over arguments from outlying areas.[5] Frege is not, in short, a platonist in the above sense about reference.

Regarding the broadening of the sense and reference of such terms as "function" and "sum," Frege remarks as follows:

> Now how has the meaning of the word 'function' been extended by the progress of science? We can distinguish two directions in which this has happened.
>
> In the first place, the field of mathematical operations that serve for constructing functions has been extended. Besides addition, multiplication, exponentiation, and their converses, the various means of transition to the limit have been introduced…
>
> Secondly, the field of possible arguments and values for functions has been extended by the admission of complex numbers. In conjunction with this, the sense of the expressions 'sum,' 'product,' etc. had to be defined more widely. [Frege (1891) 12/ Frege (1984) 144]

Specifically regarding the addition-function, we find:

[5] See Blanchette (2012a), (2012b).

> After thus extending the field of things that may be taken as arguments, we must get more exact specifications as to what is meant by the signs already in use. So long as the objects dealt with in arithmetic are the integers, the plus-sign need be defined only between integers. Every widening of the field to which the objects indicated by *a* and *b* belong obliges us to give a new definition of the plus-sign. [Frege (1891) 19 / Frege (1984) 148]

Frege has, in short, the ordinary mathematician's view about the development of a given theory into a new domain: that the widening of the objects dealt with requires a widening of the domains of the relevant functions, but that this ever-present possibility is no hindrance to perfectly determinate reference on the part of the original terms in their old settings.

The widening of the domain of a function is strictly speaking a matter of dealing with a new function; as a consequence, Frege takes it that strict logical rigor mandates in such cases the use of a new term. As he puts it in the second volume of *Grundgesetze*,

> *§56.* A definition of a concept (a possible predicate) must be complete; it has to determine unambiguously for every object whether it falls under the concept or not…
>
> *§57.* From this now follows the inadmissibility of piecemeal definition, which is so popular in mathematics. This consists in providing a definition for a special case – for example, for the positive whole numbers – and putting it to use and then, after various theorems, following it up with a second explanation for a different case – for example, for the negative whole numbers and for Zero – at which point, all too often, the mistake is committed of once again making determinations for the case already dealt with. …
>
> *§58.* To be sure, we have to grant that the development of the science which occurred in the conquest of ever wider domains of numbers almost inevitably demands such a practice; and this demand could be used as an apology. Indeed, it would be possible to replace the old signs and notations by new ones, and actually, this is what logic requires; but this is a decision that is hard to make. …
>
> *§60:* It is, moreover, very easy to avoid multiple explanations of the same sign. Instead of first explaining it for a restricted domain and then using it to explain itself for a wider domain, that is, instead of employing the same

> sign twice over, one need only choose different signs and to confine the reference of the first to the restricted domain once and for all, so that the first definition is now also complete and draws sharp boundaries. [Frege (1903) §§56-60]

The earlier sign, with restricted domain, has unproblematic reference despite remaining undefined over objects that lie outside the bounds of its theory. Frege makes the same point in his lectures of the summer of 1914, if Carnap's notes are accurate:

> In the development of mathematics one does, however reach certain points where one wants to expand the system. But then one has to begin from scratch again. In any case, there always has to be a complete *system* at hand that is logically unproblematic. E.g. one would have to proceed as follows: as long as the plus sign + is used only for positive whole numbers, one chooses a different sign for it, e.g., ¬. [Reck & Awodey (2004) p. 155]

In short: the difficulty most recently mentioned, i.e. that Frege's requirement of total definition for functions makes impossible the recognition of cross-theory sentences whose terms each have determinate reference in their original setting, but whose own truth-conditions are settled only by stipulation, is ill-founded. Frege is in this sense a perfectly ordinary mathematician, one who takes it that the new sentences yielded by an expansion of the domain of a mathematical theory will include some whose truth-value is determined by the mathematical facts, and some whose truth-value can be fixed only by arbitrary stipulation.

5. Reference

The platonist conception of reference, to which we have contrasted Frege's, is a conception on which the determinacy of reference in their own distinct settings of a function-term *f(x)* and a singular term *t* requires that the cross-theoretical sentence *f(t)* have a determinate truth-value. As we have seen, this conception is neither plausible as a constraint on reference in mathematics, nor plausibly attributed to Frege. But to say that Frege is not a platonist about reference is not to say that he lacks stringent requirements on referential terms within rigorous theories. We turn here to a brief account of Frege's requirements on reference in *Grundgesetze*.

In *Grundgesetze* I §29, Frege gives the following sufficient conditions for reference:

- A one-place first-level function-name has a reference if every result of filling its argument-place with a referring proper name has a reference.
- A proper name has a reference if the result of using it to fill an argument-place of a referring first-level function-name itself always has reference.
- And so on

Taking for granted that some simple sentences express truths or falsehoods, sections 30-32 contain a rigorous proof that all well-formed names (including sentences) of *Grundgesetze* have determinate reference.

In these seceions of *Grundgesetze*, we get a clear picture of exactly what, according to Frege, is required in order for a piece of language in a mathematical theory to count as having determinate reference. The requirement is very strictly theory-bound: what's required is that, as proven in §§30-32, each function-term provides a determinate value when given as argument any object in the domain of the theory (which in *Grundgesetze* includes just the two truth-values and value-ranges of first-level functions; recall that numbers are value-ranges). The clear, and rigorously-demonstrated view that every well-formed piece of *Grundgesetze* notation has a determinate reference is not undermined by the similarly clear, and clearly-acknowledged fact that the functions in question are not defined over "outlying" objects. The referentiality of a term *t* of *Grundgesetze* by no means requires that identity-sentences linking *t* with terms from outside the theory have truth-values or truth-conditions. Finally, it is worth recalling that, as Frege understands it, the well-formed terms of *Grundgesetze*, including its sentences, have been shown by the reasoning in sections 30-32 not just to have determinate reference, but to have sense as well.[6]

Frege's fundamental picture of reference as it applies to a mathematical theory is that a mathematical term has sense and reference if our understanding of that term (supplemented if necessary by stipulations), together with mathematical facts, fix the truth-values of sentences formulable in the theory. That the terms of a theory have determinate sense and reference is compatible with two kinds of ignorance on the part of its users. It is compatible with intratheoretical ignorance, i.e. ignorance of answers to questions formulable in the theory, as long as those answers are determined by mathematical facts. We can, for example, use the terms of number theory entirely competently while remaining in ignorance of the truth-value of Goldbach's conjecture. The second kind of ignorance compatible with the competent use of the theory is extratheoretical, i.e. ignorance of answers to

[6] "Thus it is shown that our eight primitive names have a reference and thereby that the same applies to all names correctly formed out of them. However, not only a reference but also a sense belongs to all names correctly formed from our signs. Every such name of a truth-value *expresses* a sense, a *thought*." [Frege (1893) §32]

questions not formulated in the theory as it stands. Indeed, it is no part of Frege's theory to suppose that the latter kinds of question have answers at all.

6. Thought-Identity

As argued above, Frege's conception of sense and reference is one on which the kinds of domain-expansion that go along with scientific progress are compatible with the possession of determinate sense and reference by the terms of early-stage theories. The pressing question, now, is whether scientific continuity in the face of progress is compatible with Frege's idea of continuity as involving the preservation of thought.

To get an idea of Frege's understanding of the sense in which theoretical continuity is possible in the face of conceptual progress, we will look at his own project, that of providing a newly-rigorous and clarified version of arithmetic in *Grundlagen* and *Grundgesetze*. We should note at the outset that the move from arithmetic as ordinarily understood to Frege's rigorous new account of it involves both of the kinds of development discussed above: the new theory has a broader domain (including, in the case of *Grundgesetze*, infinitely many value-ranges), and its sentences are not easily-recognizable synonyms of their original counterparts.

We begin with *Grundlagen*. The central question is this: what, exactly, does Frege take his well-developed theory of *Grundlagen* to "preserve" with respect to ordinary arithmetic? We note first that Frege is centrally concerned in *Grundlagen* with biconditionals of the following forms:

(i) The number of F’s = the number of G’s iff there’s a bijection of the F’s onto the G’s;

(ii) The number of F’s = 0 iff $\forall x \sim Fx$;

(iii) The number of F’s = n+1 iff $\exists a$(Fa & the number of (F’s other than a) = n).

Much of *Grundlagen* is taken up with arguing that the left-hand side of (i), as ordinarily understood, expresses essentially what its right-hand side expresses. Similarly for (ii) and (iii). Frege’s suggestion is that in each case, the right-hand side provides an adequate analysis of the ordinary content found on the left.[7]

If one grants Frege this non-trivial claim of analytic adequacy, the rest of the *Grundlagen* project flows smoothly (or would have, if not for the difficulty about extensions). Terms of the form “the number of F’s” refer, in the theory of *Grundlagen*, to extensions of concepts in such a way that the left-hand sides of (i) – (iii), so understood, are immediately logically equivalent with the respective right-hand sides of the biconditionals, as ordinarily understood.[8] This means that the connection between statements of the form “The number of F’s = the number of G’s” as ordinarily understood, and those statements as understood via the new *Grundlagen* account, is straightforward: in each case, the latter is logically equivalent with a good analysis of the former. Similarly for (ii) and (iii). If we focus just on sentences of the form (i) – (iii), what is “preserved” in the move from the early

[7] This account of Frege’s project as one of conceptual analysis is argued for in Blanchette (2012a).

[8] The logical equivalence here requires the faulty principle about extensions, assumed by Frege in *Grundlagen*, that the extension of F = the extension of G iff $\forall x$(Fx iff Gx).

theory of ordinary arithmetic to its development as *Grundlagen* arithmetic is quite significant: each sentence of the latter camp is obviously logically equivalent with its counterpart from the former. Because Frege develops the whole of the arithmetic of the finite cardinals from these sentences, the result is that the arithmetical claims expressible in the new theory have the same grounds as do claims expressible in the old: if Frege's project of proving *Grundlagen* arithmetic from principles of logic had succeeded, he would arguably, modulo agreement about his original analysis, have succeeded in demonstrating that the claims of *ordinary* arithmetic are, themselves, grounded in pure logic.

This picture of the relationship between the earlier theory, that of ordinary arithmetic, and its later development in a more sophisticated framework is exhibited again in Frege's mature development of the theory in *Grundgesetze*. Here too we find an account of terms of the form "the number of F's", and of "0" and "successor" that suffice to deliver the result that: if all had gone well, the thoughts expressed in the new theory, thoughts about value-ranges of functions, would have been clearly logically equivalent with good analyses of their ordinary counterparts. There is no question, in Frege's development, of trying to assign to the numerals of the new theory, taken in isolation, the same sense or reference as is had by the numerals of ordinary arithmetic. The question of whether the "2" of *Grundgesetze* co-refers with the "2" of ordinary arithmetic is, as above, not a well-formed question from the Fregean point of view. But what *is* preserved is what matters for a foundational project: had all gone well with the value-ranges, the truths of the newly-constructed arithmetic would have been self-evident logical equivalents of their ordinary counterparts.

In sum: Frege's strategy in both *Grundlagen* and *Grundgesetze* is to provide an accurate account of central notions – e.g. "cardinal number," "0," "successor," "finite cardinal" by providing accurate accounts of the core statements of the theory of these notions, including centrally our (i) – (iii) above. The accuracy of the accounts is judged via the question of whether these core sentences and their analysantia are sufficiently similar, semantically, that a proof of the latter from a given collection of premises will suffice to establish the logical entailment of the former by those premises. The success of *Grundlagen* and *Grundgesetze* in this attempt was to have turned on (a) the success of the original analyses, as discussed above in connection with biconditionals (i) – (iii), and (b) the correctness of Frege's assumption that Law V (or its counterpart with respect to extensions) was a law of logic.

Do the sentences of *Grundgesetze* express the same thoughts as do their counterparts in ordinary arithmetic? Here, the only thing to say is that there is no clear answer. Frege's guiding ideas that sentences express thoughts, and that thoughts are the constituents of theories, do not come along with rigorous criteria of thought-identity. To answer our question, one would have to know whether in Frege's view a successful analysis, of the kind linking "the number of F's = the number of G's" and "there is a bijection of the F's onto the G's" delivers pairs of sentences that express the same thought. To this question, Frege simply gives no answer. But perhaps most relevant to the questions raised at the outset of this essay regarding the consistency of Frege's view of theories with the phenomena of mathematical progress are the following points that have now become clear. First of all, the fact that the sentences of *Grundgesetze* differ in cognitive significance from those of ordinary arithmetic is no reason to conclude that the thoughts expressed thereby are different. Similarly, the fact that "cross-theory" identity statements, those linking terms of ordinary arithmetic with terms of *Grundgesetze* notation, lack truth-conditions is no barrier to the determinacy of reference and sense on the part of those terms. Neither, finally, is the indeterminacy of cross-theory identities a barrier to the idea that corresponding sentences of the two theories express the same thought. That no aspect of the ordinary use of "the number of even primes" determines that this term in its ordinary use co-refers with any term of *Grundgesetze* is not by itself a barrier to the expression of the same thought by the ordinary "the number of even primes = the number of positive square roots of 9" and its *Grundgesetze* counterpart. The question of whether they do express the same thought turns on the questions of whether good conceptual analysis preserves thought, and of whether the two sides of an instance of Law V express the same thought.

Perhaps most importantly: the theoretical continuity essential for Frege's project is not, in the end, that sentences from the old and the new theories express the *same* thought, but whether the thoughts they express are sufficiently similar for the purposes of the logicist project. Essential here are the fact that new and old counterparts have the same truth-conditions, and even more robustly that any premises providing a logical ground of one will suffice to ground the other. Whether the relation between ordinary and *Grundgesetze* sentence is understood as thought-identity, in line with a very coarse-grained account of the identity-conditions of thoughts, or instead as a broader kind of content-similarity, in line with a more fine-grained such account, is not determined by Frege's general views about mathematics or about thoughts. This is, one might say, as it should be, since it makes no difference to Frege's project or to his understanding of mathematics how one decides to describe the situation.

Similarly for mathematical progress generally. The move from earlier theories whose concepts are relatively vague to those with more sharply-defined notions and perhaps with larger domains is one that requires a certain recognizable similarity between the thoughts expressed, and not merely, for example, mere similarity in syntactic or proof-theoretic structure. Frege's work gives no algorithm for the precise similarity necessary for theoretical continuity, just as it gives no precise criterion of thought-identity. The former is presumably what one should expect, since the precise kind and degree of similarity required for continuity will vary from field to field and era to era. Frege therefore simplifies and over-states the case when describing theoretical continuity in terms of thought-identity. But from what we have seen so far, his fundamental conception of mathematical progress and continuity is not threatened by the phenomena that at first seemed problematic: the non-trivial conceptual clarification and the domain expansions that go along with mathematical progress.

7. Fleshing Out The Positive Picture

We turn, finally, to one particular aspect of Frege's conception of theoretical continuity in mathematics that helps to fill out his positive view of the similarity required between earlier and later theories. As a description of his own project, Frege says that:

> "[F]or every object there is one type of proposition which must have a sense, namely the recognition-statement, which in the case of numbers is called an identity. Statements of number too are, we saw, to be considered as identities. The problem, therefore, was to fix the sense of a numerical identity … " [Frege (1884) §106]

Similarly,

> [F]rom our previous treatment of the positive whole numbers, [we] have seen that it is possible to avoid all importation of external things and geometrical intuitions into arithmetic, without, for all that, falling into the error of the formalists. Here, just as there, <u>it is a matter of fixing the content of a recognition-judgment</u>. Once suppose this everywhere accomplished, and numbers of every kind, whether negative, fractional, irrational, or complex, are revealed as no more mysterious than the positive whole numbers … [Frege (1884) §109; emphasis added]

And with respect to later expansions to larger classes of numbers, we find again the same fundamental ideas: that the analysis of the theory of those numbers is to turn on the analysis of a collection of core sentences, and that those core sentences are identity-sentences involving the numbers in question:

> How are complex numbers to be given to us then, and fractions and irrational numbers? If we turn for assistance to intuition, we import something foreign into arithmetic … .
>
> [*review of Grundlagen's account of Number … - PB*]

> In the same way with the definitions of fractions, complex numbers and the rest, everything will in the end come down to the search for a judgment-content which can be transformed into an identity whose sides precisely are the new numbers. [Frege (1884) §104]

It is worth noting that if, as Frege holds, the core to be preserved in the development of a theory includes (or, as above, is composed of) identity sentences, then it is essential that the content of the identity sign not be re-defined as we move from theory to theory. This is not a trivial point, and would not have been uncontroversial to Frege's readers: it was, indeed still is, common to take a certain form of definition as involving the redefinition of identity over a given domain: consider for example the "identification" of diametrically opposed points in a spherical model of geometry, the "identification" of multiple pairs of integers as the same rational, and so on. Because Frege's understanding of what's preserved across developments of a given subject-matter includes the content of core identity-sentences, it is essential (in order that this requirement be non-trivial) that the identity-sign in question expresses the real identity-relation, and not some simulacrum, across all of the theories in question. We see this requirement in operation in Frege's response to Peano. First, the relevant passage from Peano, as quoted by Frege:

> [G]iven equality between integers, one defines equality between rationals, between imaginary numbers, etc. In geometry one is used to defining the equality of two areas, of two volumes, the equality of two vectors, etc. … The various references have properties in common; but I do not see how they suffice to specify all the possible references of equality. … Moreover, the opinions of the various authors concerning the concept of equality are very diverse…
> [Peano (1898) , as quoted by Frege in Frege (1903) §58 n]

Frege's predictable reply is as follows:

> If mathematicians' opinions about equality diverge, then this means nothing less than that mathematicians disagree with respect to the content of their science; and if one views the essence of the science as being thoughts, rather than words or signs, then this means that there is no one united mathematical science, that mathematicians do not, in fact, understand each other. For the sense of nearly all arithmetical propositions and of many geometrical propositions depends, directly or indirectly, on the sense of the word 'equal.' [Frege (1903) §58 fn]

This returns us essentially to our starting-point. Two mathematicians are engaged in the same science only if each of them expresses thoughts that are appropriately related to the other's. For a science is simply a body of thoughts.

The idea that theoretical continuity turns on the successful treatment of a handful of core sentences means that, from the Fregean perspective, the question of continuity is at least in some cases a relatively tractable one: a proposed development of a theory counts as merely changing the subject if it fails to get the core sentences right – i.e. if it assigns to those sentences thoughts insufficiently similar to the thoughts they ordinarily express. This is the criticism leveled by Frege in *Grundlagen* at those accounts of arithmetic that construe the core sentences as expressing claims about geometry or about ideas, and it is his perennial criticism of the so-called "formalist" accounts of arithmetic.

Once we get the core sentences right, Frege seems to say, we count as thereby having remained faithful to the central concepts of the science. Amongst the general requirements we have seen, above, for doing so is the further requirement that the developed theory maintains the original relation of identity: no relation that, for example, holds between distinct objects can be taken as the reference of the identity-sign, at the risk of entirely undermining the required similarity between the core identity sentences and their developed counterparts.

8. Conclusion

We have seen that Frege's requirements for theoretical continuity are not what one might have taken them to be. Theoretical continuity does not require preservation of conventional significance. It also does not require preservation of reference at the level of individual terms. It requires instead a certain imprecise but often recognizable similarity of thought expressed by each of the pairs of core sentences. Whether this relationship between the sentences is to be understood as the expression of the *same* thought, as opposed to that of recognizably-similar thoughts, has turned out to be unimportant, neither entailed nor contradicted by Frege's central views. Finally, Frege's guiding principle, according to which theoretical continuity requires the kind of thought-similarity discussed above, though less than precise, is a forceful view, ruling out for example all of those attempts to develop arithmetic on geometric, formalist or idealist grounds, at least without supplementary argument to the effect that the core sentences of arithmetic have, as ordinarily understood, a geometric, formalist or idealist content.

References

Blanchette, Patricia. 2012a. *Frege's Conception of Logic*. New York: Oxford University Press.

Blanchette, Patricia. 2012b. Frege on Shared Belief and Total Functions. *The Journal of Philosophy* CIX 1/2: 9-39.

Burge, Tyler. 1979. Sinning Against Frege. *The Philosophical Review* 88: 398-432. Reprinted in Burge (2005) 213-239.

Burge, Tyler. 1984. Frege on Extensions of Concepts, From 1884 to 1903, *The Philosophical Review* 93: 3-34. Reprinted in Burge (2005) 273-298.

Burge, Tyler. 2005. *Truth, Thought, Reason: Essays on Frege*. Oxford: Clarendon Press.

Frege, Gottlob. 1884. *Die Grundlagen der Arithmetik.* Breslau: William Koebner. English edition: Frege, Gottlob 1978. *The Foundations of Arithmetic*. (trans: Austin, J. L.). Evanston, Ill.: Northwestern University Press.

Frege, Gottlob. 1891. *Funktion und Begriff*. Jena: Hermann Pohle. Reprinted in Frege (1967): 125-142. English edition: Frege, Gottlob 1984a. Function and Concept. Frege (1984): 137-156 (trans: Geach, P.)

Frege, Gottlob. 1892a. Über Sinn und Bedeutung. *Zeitschrift für Philosophie und Philosophische Kritik* 100: 25-50. Reprinted in Frege (1967): 143-162. English edition: Frege, Gottlob (1892). On Sense and Reference. Frege (1984): 157-177 (trans: Black, M.).

Frege, Gottlob. 1892b. Über Begriff und Gegenstand. *Vierteljahrsschrift für wissenschaftliche Philosophie* 16: 192-205. Reprinted in Frege, Gottlob (1967): 167-178. English edition: Frege, Gottlob. 1984b. On Concept and Object. Frege (1984): 182-194 (trans: Geach, Peter).

Frege, Gottlob. 1893. *Grundgesetze der Arithmetik* Vol I. Jena: Hermann Pohle. English edition: Frege, Gottlob 2013. *Basic Laws of Arithmetic* (trans: Ebert, P. and Rossberg, M.): Oxford: Oxford University Press.

Frege, Gottlob. 1903. *Grundgesetze der Arithmetik* Vol II. Jena: Hermann Pohle. English edition: Frege, Gottlob 2013. *Basic Laws of Arithmetic* (trans: Ebert, P. and Rossberg, M.): Oxford: Oxford University Press.

Frege, Gottlob. 1967. *Kleine Schriften*. Ed. Ignacio Angelelli. Olms: Hildesheim.

Frege, Gottlob. 1979. *Posthumous Writings*. Ed. H. Hermes, F. Kambartel, F. Kaulbach. Trans. Long and White. Chicago: University of Chicago Press.

Frege, Gottlob. 1983. *Nachgelassene Schriften*. Ed. Hans Hermes, Friedrich Kambartel, Friedrich Kaulbach. Hamburg: Felix Meiner Verlag.

Frege, Gottlob. 1984. *Collected Papers on Mathematics, Logic, and Philosophy*. Ed. Brian McGuinness. Oxford: Blackwell.

Peano, Giuseppe. 1898. *Riposta*. *Rivista di Matematica* 6: 60-61.

Reck, Erich and Awodey, Steve, eds. 2004. *Frege's Lectures on Logic: Carnap's Student Notes, 1910-1914*, Chicago: Open Court.