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Frege's Critique of Modern Axioms

Abstract.[1] Axioms, as Frege understands them, are the fundamental principles of a science. The sense in which the axioms ground the rest of a theory is one from which it follows that a number of philosophically-important characteristics of theories can be read off from the corresponding characteristics of the axioms. The purpose of this talk is to explain some critical ways in which Frege's conception differs from a more modern conception, one familiar from the work of Hilbert, Dedekind, and Tarski. Frege's objections to the modern conception of axioms are, it is argued here, important: when we move to the new conception of axioms and leave Frege's conception behind, we gain a good deal of mathematical tractability, but we also lose a good deal of philosophical content.

Introduction

Axioms, we are taught from an early age, are the fundamental principles of a science. They are the truths on which the rest of a given theory hangs, and they determine the subject-matter and the scope of the theory. They are the principles that govern a specific collection of objects, relations and functions, or equivalently that ground the theory of those objects, relations and functions.

This, which we might call the "old-fashioned" view of axioms, is familiar from Euclidean geometry, at least as that geometry was understood by Euclid. The axioms of geometry, from this point of view, are about such objects as points, lines, planes, and spheres, and the relations that obtain between these objects. As axioms, they are both the most basic truths of the science, and statements of the most-important features of the objects and relations that make up its subject-matter.

Since the end of the nineteenth century, a quite different conception of axioms has arisen. In what we will call the "modern" conception of axioms, an axiom is a sentence of a formal language, a sentence that has no determinate subject-matter, but is instead susceptible to a wide variety of different interpretations. On this understanding, the axioms of geometry

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are only about points, lines, and planes when considered under one of their many available interpretations; they themselves have no specific subject-matter, and are not confined to a specific application.

Axioms in the modern sense play a central role in modern mathematical investigations, especially in investigations of the logical structure of theories and their subject-matter. Rigorous proofs of the consistency of a theory, and of the independence of a given axiom or theorem from others, require the modern understanding of axioms, since these proofs turn on the existence of alternative interpretations. The all-important modern notion of categoricity, too, as well as various kinds of completeness, also apply most straightforwardly to axioms in the modern sense, i.e. to reinterpretable formulas and sets thereof.

In comparison with the older, Euclidean conception, the new and streamlined conception of axioms stands out for its rigor, its tractability and especially for its fruitfulness. Once axioms are treated "formally," as we might put it, the proof of theorems from axioms achieves a transparent precision. And once we couple the formal axioms with rigorous definitions of satisfaction on structures, we achieve crisp definitions of, and proof-techniques for, the central notions just mentioned, those of consistency, independence, and categoricity.

The move from axioms as understood by Euclid to axioms as understood by e.g. Dedekind, Hilbert and Tarski might seem to leave the fundamental role of axioms intact: axioms are still the deductive starting-point for a theory, and an axiomatization is still a way of distilling the content of a theory to a tractable core. The fundamental change, as it might seem, has simply to do with rigor: axioms as newly-understood within the confines of a formal theory are considerably better-defined, with clearer logical properties and relations, than were their Euclidean predecessors.

This idea, that axioms as conceived by Hilbert and Tarski are merely a cleaned-up version of the kinds of things taken in earlier centuries as the fundamental principles of a science, is a view that is resoundingly rejected by Frege. In the move from an earlier conception of axioms, one shared by Frege, to the modern conception, we take a step backwards, from Frege's point of view. As Frege sees it, the things newly called "axioms" are fundamentally the wrong kinds of things to take to be the building-blocks of a science: they are not, and cannot play the role of, the things that Frege himself calls "axioms." The dispute is not merely terminological: on the view championed by Frege, the kinds of questions we can ask, and answer, about theories and axioms as newly-conceived are radically different from the kinds of questions we can make sense of with respect to theories and axioms of the old kind. And, most importantly, from Frege's point of view there are significant theoretical questions to ask about e.g. an axiomatization of geometry or of arithmetic – questions having to do with independence and consistency – that cannot be answered by the techniques that come hand-in-hand with the new view of axioms.

If Frege is right, then the move at the end of the nineteenth century to a conception of axioms, and hence of theories, as sets of formulas of a reinterpretable language brought with it not just gains in tractability, but also real losses. The purpose of this talk to spell out Frege's reasons for this view, and in so doing to make it apparent that he was right.

Frege on Proof and Conceptual Analysis

In *Grundlagen*, Frege lays out the role of proof as follows:

> The aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another. After we have convinced ourselves that a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely? The further we pursue these enquiries, the fewer become the primitive truths to which we reduce everything; and this simplification is in itself a goal worth pursuing. - *Grundlagen* sec 2

The idea, in short, is that by proving the truths of a science from the simplest principles we can manage, we learn what "supports" that theory; we learn, of those simpler principles, that they suffice to ground the theory. As he puts it some years later,

> Because there are no gaps in the chains of inference, every 'axiom,' every 'assumption,' 'hypothesis,' or whatever you wish to call it, upon which a proof is based is brought to light, and in this way we gain a basis upon which to judge the epistemological nature of the law that is proved. - *Grundgesetze* I p. 3

One crucial aspect of Frege's view of axioms is that they are not formulas, but are the nonlinguistic items expressed by formulas. Known in his mature period as thoughts (*Gedanken*), these nonlinguistic items are, generally speaking, the things of which theories are made, and the things that stand as the premises and conclusion of proofs. As he puts it in 1906:

> When one uses the phrase 'prove a proposition' in mathematics, then by the word 'proposition' one clearly means not a sequence of words or a group of signs, but a thought; something of which one can say that it is true. - Frege [1906] p 332

Though Frege is widely known for having developed the first rigorous formal systems for the expression of proofs, it is important to note that this "formal" aspect of Frege's work does not carry with it the idea that the items demonstrated in a mathematical proof are formulas. The relationship between the formulas of a formal system and the thoughts with whose proof we are concerned is the straightforward one of expression: each formula in a Fregean formal deduction expresses a specific thought. As we might say, the *proof* (a series of thoughts) is expressed by a *deduction* (a series of formulas). The final sentence of a deduction expresses the thought proven.

A further essential feature of Frege's conception of proof is that, on this conception, proof bears a close relationship to conceptual analysis. As Frege describes the connection in 1884:

> [T]he fundamental propositions of arithmetic should be proved, if in any way possible, with the utmost rigor . . . If we now try to meet this demand, we very soon come to propositions which cannot be proved so long as we do not succeed in analyzing concepts which occur in them into simpler concepts or in reducing them to something of greater generality. Now here it is above all Number which has to be either defined or recognized as indefinable. This is the point which the present work is meant to settle. - *Grundlagen* p. 5

Frege's idea here is clarified by his practice in both *Grundlagen* and *Grundgesetze*. In both of these works, the attempt to provide rigorous proofs of the truths of arithmetic involves an essential analytic step: we begin by breaking down some of the central concepts involved in those truths into complexes of simpler or more-general components. We in this way reveal a more highly-articulated structure of the thought in question, which thought we can then go on to prove.

To choose an example: in order to demonstrate that

(E) Every cardinal number has a successor

is grounded in pure logic, Frege first provides an analysis of the notions of cardinal number and of successor. On the basis of this analysis, we achieve a more clearly-articulated thought (E*) that bears to (E) the relation of analysans to analysandum. The subsequent proof of (E*) from principles of pure logic suffices, as Frege sees it, to demonstrate the purely-logical grounding of the original (E).

As he himself puts the point in 1914,

> In the development of science it can ... happen that one has used a word, a sign, an expression, over a long period under the impression that its sense is simple until one succeeds in analysing it into simpler logical constituents. By means of such an analysis, we may hope to reduce the number of axioms; for it may not be possible to prove a truth containing a complex constituent so long as that constituent remains unanalysed; but it may be possible, given an analysis, to prove it from truths in which the elements of the analysis occur. - Frege [1914] p 209

Frege's general picture of proof and analysis, then, might be summed up as follows: Given a thought τ and a set P of premise-thoughts, we can demonstrate that τ is logically entailed by P in the following way: we first give conceptual analyses of the thoughts in question, yielding the (set of) analysans-thoughts τ* and P*; and then proceed to give a rigorous proof of τ* from P*. Success in such a two-step process establishes the original claim of logical entailment.

Frege on Independence and Independence-Demonstrations

The difference between Frege's conception of axioms and the conception familiar to all of us in the 21^{st} century is drawn most starkly in Frege's reaction to David Hilbert's 1899 monograph, *Foundations of Geometry.* Hilbert's goal in *Foundations* is to present an economical axiomatization of Euclidean geometry, and to provide a number of consistency and independence-demonstrations for various collections of axioms and theorems. Frege's response to this work of Hilbert's is entirely negative: he claims that Hilbert's attempts to prove consistency and independence are failures, and that the things Hilbert calls "axioms" are the wrong kinds of entities to bear that name.

Axioms, for Hilbert, are sentences. In the case of geometry, the sentences include standard geometric terms, including for example the terms "point," "line," "between," and so on. The technique Hilbert uses for demonstrating consistency and independence is similar to the standard technique in use today, that of constructing "models." The procedure, as employed by Hilbert, can be explained by means of the following schematic example.

Suppose we want to prove the consistency of a set $\{A_1 \ldots A_n\}$ of axioms. This set will typically consist of sentences not all of which express truths of Euclidean geometry. (If each sentence expresses an acknowledged truth, then its consistency is already acknowledged and not in need of demonstration.) It might contain, for example, n-1 axioms of Euclid, together with the negation of an axiom of Euclid. We demonstrate its consistency via a two-step method: First, we provide a new interpretation of the geometric terms appearing in the axioms. Following Hilbert, we might for example interpret the term "point" as standing for pairs of real numbers drawn from a specified domain, "line" for triples of ratios of such numbers, "lies on" for an algebraic relation between such pairs and triples, and so on. The second step is the demonstration that, when the terms are thus interpreted, each member of the set $\{A_1 \ldots A_n\}$ expresses a theorem of a background theory B (here, a theory of real numbers), which theory is assumed to be consistent.

It follows from this demonstration that a contradiction is derivable from the set $\{A_1 \ldots A_n\}$ only if a contradiction is derivable from the theorems of B, and hence that $\{A_1 \ldots A_n\}$ is inconsistent (in the sense of permitting the derivation of a contradiction) only if B is. The procedure, then, provides a relative consistency proof: the set in question is consistent if the background theory B is consistent. The same procedure is used by Hilbert to demonstrate independence: a sentence A_n is independent of a set $\{A_1 \ldots A_{n-1}\}$ iff the sets $\{A_1 \ldots A_{n-1}, \neg A_n\}$ and $\{A_1 \ldots A_{n-1}, A_n\}$ are both consistent. Here, the "independence" of A_n from the set in question is a matter of there being no derivation of A_n, and no derivation of $\neg A_n$, from that set; this is immediately demonstrated by the procedure just outlined, again assuming the consistency of the background theory B.

The Differences and their Import

The contrast between the two conceptions of axioms is stark. For Frege, an axiom is a determinate thought. And thoughts, as Frege sees it, are the kinds of entities with respect to which questions of consistency and independence make sense. Axiom-sentences, from this point of view, are important only as vehicles for the expression of axiom-thoughts.

For Hilbert on the other hand, the sentences are important not as vehicles for the expression of determinate thoughts, but as a means of laying down general conditions satisfiable by various collections of objects, functions, and relations. Sentences as so understood are on Hilbert's view the kinds of things about which we raise questions of consistency and independence. From this point of view, but not from Frege's, it is of the essence of axioms that their non-logical (here, geometrical) terminology is susceptible of varying interpretations.

The importance of this distinction is most significant when coupled with Frege's view that the important logical properties of thoughts - for example their provability from a given set of premises - can often be determined only after a thorough conceptual analysis of those thoughts and of their components. This means that there is for Frege an important gap, in principle, between the relation of deducibility (a relation between sentences) and the relation of provability (a relation between thoughts). Given a well-designed formal system, a sentence S is deducible in that system from a set P of sentences only if the thought τ(S) expressed by S is in fact logically entailed by the thoughts τ(P) expressed by the members of P. But the converse, from Frege's point of view, is often false. That S is not deducible from P does not guarantee that τ(S) fails to be logically entailed by τ(P). When the thought τ(S) can be subjected to deeper conceptual analysis, on the basis of which the resulting analysans-thought is expressible via the more-complex sentence S*, we can find, via a deduction of the new S* from P (or from new sentences P* achieved similarly from P via conceptual analysis), that the original thought τ(S) is in fact provable from, and hence logically entailed by, the original premise-thoughts τ(P). This is Frege's point when he notes that, as quoted above, "it may not be possible to prove a truth containing a complex constituent so long as that constituent remains unanalysed; but it may be possible, given an analysis, to prove it from truths in which the elements of the analysis occur." ([1914] p. 209). Failure of deducibility, in short, does not entail independence.

From Frege's point of view, Hilbert's reinterpretation of the geometric axiom-sentences involves an illicit shift from one set of thoughts (the geometric ones) to a new set (those concerned with real numbers). While the original questions of consistency and independence concern, as Frege sees it, the original thoughts concerning points and lines, Hilbert's reinterpretation of the axiom-sentences marks a shift of attention to a new set, one that has nothing to do with the points and lines of geometry. And because of Frege's view of the connection between conceptual analysis and logical entailment, the fact that the two sets of thoughts are expressible via the same set of sentences is no guarantee that from the consistency of one we can infer the consistency of the other. The change in the objects, functions and relations under discussion when we move from thoughts about geometry to thoughts about real numbers or vice-versa can (and indeed often will, as Frege sees it) bring with it changes in relations of entailment between the thoughts in question. Hence the inference from the consistency of a set of thoughts expressed by a set Σ of sentences to the consistency of a different set of thoughts expressed by Σ under a re-interpretation is, as Frege puts it, "a fallacy."

Frege further recognizes that, were we to take each set of Hilbert's axiom-sentences as providing an implicit definition of an n-place higher-level relation (where n is the number of undefined geometric terms appearing in the members of that set), then Hilbert's interpretation does show an important result: that the relation so defined is satisfiable, and in that sense consistent. But, says Frege, the consistency of such a relation is no guarantee of the consistency of the thoughts that are obtained via any particular instance of it. Referring to Hilbert's axiom-sentences when so understood as 'pseudo-axioms,' Frege remarks:

> Mr. Hilbert's independence-proofs simply are not about the real axioms, the axioms in the Euclidean sense, for these, surely, are thoughts. ... Mr. Hilbert appears to transfer the independence putatively proved of his pseudo-axioms to the axioms proper ... This would seem to constitute a considerable fallacy. - [1906], 402

That the inference he takes Hilbert to make is fallacious from Frege's point of view is again a consequence of the Fregean view that the consistency and independence in question have to do with logical relations between thoughts that can, in principle, turn on what's expressed by such terms as "point," "between," and so on. And indeed, on that understanding of consistency and independence, the inference is in fact fallacious. Where Hilbert understands the "consistency" of a set Σ of axiom-sentences to mean either the non-deducibility of a contradiction from Σ or the satisfiability of the higher-level relation defined by Σ, the inference from the consistency in Hilbert's sense of Σ to the consistency in Frege's sense of $\tau(\Sigma)$ is unwarranted. Similarly for independence.

It is worth pointing out at this juncture that though Frege takes Hilbert to make a fallacious inference, Hilbert in fact does nothing of the sort. Hilbert is simply not interested, here, in the kinds of consistency and independence on which Frege focuses. For he is not concerned with the entities Frege calls "axioms." In short, while Hilbert's technique is unsuited to the demonstration of what Frege calls "consistency" and "independence," the technique is conclusive in the demonstration of the weaker notions intended by Hilbert.

Axioms as Definitions

As noted above, each set of Hilbert's axiom-sentences defines a complex relation, or, as we might now put it, a structure-type. An interpretation that satisfies each sentence in such a set constitutes a structure, a particular organization of objects under specified orderings and relations. The axiom-sentences of Euclidean geometry, viewed from this perspective, define a structure-type that is variously satisfiable via the usual constellation of points and lines, under the usual incidence and order relations, and also via infinitely many other structures, some geometrical and some not.

The richness of this modern conception of axioms (i.e. as definitions of structure-types) is perhaps most clearly seen in the late 19th century in the work of Richard Dedekind. Dedekind's axiomatic treatment of the natural numbers in *Was sind und was sollen die Zahlen* provides a strikingly simple and fruitful application of the new axiomatic method. This treatment turns on the definition of a structure-type each instance of which is called a "simply infinite system." For Dedekind, a simply infinite system is any set S satisfying the conditions that, for some relation f and object α,

- f is a 1-1 function;
- α is not in the range of f;
- S is the closure of $\{\alpha\}$ under f.

That these conditions completely characterize the type is shown via Dedekind's demonstration that all structures satisfying these conditions are isomorphic. This categoricity result establishes that, if the purpose of a set of axioms is to characterize a type of structure as completely as possible, then this particular collection of axioms is entirely successful.

The natural numbers, for Dedekind, are the members of that ordered collection of objects whose only properties are given by the axioms just listed. They are, as we might put it, the items that form the "minimal" simply infinite system. This makes Dedekind's treatment of the numbers dramatically different from Frege's. For Frege, the truths of arithmetic are thoughts about determinate objects, concepts, and relations; the thoughts and their components are "rich" in the sense that they are in principle susceptible to fruitful conceptual analysis. And there is, for Frege, nothing stipulative about the axiom-thoughts that ground that collection of truths. For Dedekind, on the other hand, the axioms are simply stipulations; as long as they are consistent, they define a condition on structures, and it is this condition in which we are interested. That a consistent defined condition is uniquely satisfiable up to isomorphism is all we need in order to take the condition to be in fact satisfied by some "minimal" structure, whose further properties we can then investigate. This investigation, from Dedekind's point of view, is what we are engaged in when doing e.g. number theory.

Hilbert's brand of consistency- and independence-questions are the natural ones to ask from the point of view of a Dedekind-style treatment of mathematics. Even if we take the axiom-sentences to express determinate truths, as Dedekind does, those truths are not about concepts, objects, and relations whose nature might yield an as-yet undiscovered conceptual richness when subjected to conceptual analysis. Those expressed truths are instead about concepts, objects and relations whose whole nature is given by the structure-defining conditions explicitly laid out, i.e. by the axiom-sentences. Hence the questions Hilbert asks, and answers, concerning deducibility-relations amongst sentences and the satisfiability of implicitly-defined conditions, are exactly the right ones to ask when enquiring about the consistency and independence of axioms as understood by Dedekind.

From Dedekind's point of view, there is a sense in which structure is everything: there is nothing more to the natural numbers than the fact that they (under less-than) instantiate a particular canonical type of structure; there is nothing more to the reals (similarly) than that they instantiate their own characteristic structure-type, and so on. On this conception, the idea of axioms as expressing the foundation of a theory is straightforwardly cashed out as the requirement that axioms provide categorical characterizations of the type of structure in question: once the set of axioms of theory T is rich enough to constrain its models up to isomorphism, that set of axioms has said everything there is to say about T. It has done so, that is, if T is the theory of a collection of objects and relations whose nature is exhausted by the abstract structure that they instantiate.

We can now draw the distinction between Frege and the new tradition even more clearly. For Frege, in contrast to Dedekind, there is a great deal more to the natural numbers than the fact that they, under less-than, form an ω-sequence. Notice that the points on a line segment extending infinitely to the right similarly form, under the ordering "one unit to the left of," such a sequence. But for Frege, it is essential to ask *in virtue of what* the objects under that relation form that sequence. Crucially, the question of what it is that grounds the infinity of the objects must be answered if one is to answer the question of the foundations of the science. In the case of the sequence of points on a line, the infinity is grounded in the structure of space, something given us (as Frege sees it, following Kant) via pure intuition. But the grounding of the infinity of the series of natural numbers is entirely different in Frege's view: because of the nature of the objects in question, and of the relation under which they are ordered, the infinity of the series is guaranteed by purely logical truths. The distinction between ω-sequences whose existence and ordering is guaranteed via principles of logic and those whose existence and ordering is guaranteed via something else, e.g. via the structure of straight lines in space, is of the essence of Frege's logicist project: the crucial claim Frege makes here, and the one on which he spends his life's work, is that the natural numbers are of the first kind and not the second. This distinction is by contrast of no significance from the point of view occupied by Dedekind: that the truths of arithmetic hold solely in virtue of their instantiation of the type *simply infinite system* is crucial, and forms the heart of Dedekind's logicism, but the question of what it is in virtue of which they instantiate this structure is one whose answer plays no role in that logicism.

From Frege's point of view, axiom-sentences that define structure-types are perfectly legitimate objects of investigation. But the structure-types they define are merely the shell of a science. Until we are have a particular collection of objects and relations that together satisfy that structure-type, we have no science at all. And until we know the nature of that collection, and of the principles in virtue of which it exists and satisfies the structure-type, we do not know the grounds of the science. And it is these principles, those that ground the existence and the ordering of the specific objects in question, that form the ultimate foundational truths, the true axioms of the science, in Frege's view.

Retrospective

The modern conception of axioms, that conception of which we have taken the work of Hilbert and of Dedekind as exemplars, is now the everyday, standard conception in foundational work. We prove consistency by the construction of models, and we take categoricity to be an important criterion of axiomatic success. The fruitfulness of this approach is undeniable: it is only against the backdrop of the modern conception that we have any systematic means at all of proving consistency and independence. The Fregean conception of axioms by contrast, on which the contents of individual terms might always, in principle, give rise on conceptual analysis to as-yet unrecognized sources of logical entailment or contradiction between axioms, and on which there is no rigorous means of demonstrating consistency (aside from a demonstration of truth) leaves foundational work always just a little bit, in principle, in peril.

Nevertheless, the streamlined and mathematically-tractable modern conception of axioms does leave out of account some important aspects of the foundations of scientific theories. One way to characterize Frege's view is as, in large part, the insistence that these aspects of theories and of their foundations are important, and that the failure of modern approaches to take such features seriously is a significant failing.

The difference between the modern and the Fregean conception of axioms is perhaps most clearly seen when comparing the sense in which, on each conception, the axioms of a theory provide the "grounds" of that theory.

Axioms, as Frege sees it, ground a theory in the sense of being its fundamental truths. The grounding is transitive: if we want to know what ultimately guarantees the truth of a theory T, we ask what guarantees the truth of those axioms. If some of a theory's axioms are grounded in the structure of space, or in contingent empirical truths, then so too is the theory; if all of the axioms are analytic, then so too is the theory. The question of whether the theory carries a commitment to objects of a given kind is the question of whether the axioms do; the question of whether a theory is knowable a priori is the question of whether its axioms are, and so on. It is only when we ask such questions, those of the nature of the axiom-thoughts, that we can answer the kinds of questions that are central to Frege, the kinds of questions that animated much work in the epistemology of mathematics up to his day.

On the modern conception, on the other hand, axioms are not truths at all, but are instead either partially-interpreted sentences or the structure-defining conditions stipulated by such sentences. Hence the sense in which an axiomatized theory is "grounded" in its axioms is quite different. The theorems themselves are, in a sense, "consequences" of the axioms: a theorem-sentence is always deducible from (or otherwise entailed by) the axiom-sentences, and the structural condition it expresses is satisfied by any structure that satisfies the conditions defined by the axioms. But we cannot ask what grounds the axioms: they are stipulations, not truths, and have no grounds. The question of whether a theory is analytic or synthetic, empirical or otherwise, cannot be answered by examining the nature of its axioms, if those axioms are of a modern mathematical kind. Indeed, there is no sense in which the theory itself is either analytic or synthetic, empirical or non-empirical, when consistently viewed as an axiomatic theory in the modern sense. For in this sense, the theory is a collection of stipulations and consequences thereof.

The adoption of the modern conception of axioms therefore involves giving up on some theses that were of cardinal importance to Frege. If number theory is, as Frege took it to be, a collection of truths about a particular collection of objects under a specific ordering, then we can meaningfully ask, and answer, questions about the metaphysical and epistemological status of that theory. We can ask whether its truths turn on anything outside of logic, whether its objects can be known *a priori* to exist, whether its theorems require any contingent truths for their grounding, and so on. Shifting to the modern conception of number theory means shifting to a perspective from which the theory itself, at least as this is given by its axioms, is not a collection of truths about a particular series of objects. And if there is, as for example Dedekind holds, a particular collection of objects characterizable as "the numbers," these are objects whose existence is not entailed by, and whose nature is not described by, the axioms; their existence falls outside the purview of the axioms – and hence, strictly speaking, of the theory – altogether. Whether or not one takes there to be, in this sense, a "canonical" model of the axioms, the fundamental difference with Frege is that, on this modern conception, the central Fregean questions are ill-formed: because a theory on the modern view is not a body of truths, but rather a body of stipulations and their consequences, it is neither analytic nor synthetic; it is not knowable either *a priori* or *a posteriori*, and its axioms cannot be interrogated for specific existential commitments.

In short, the idea that a mathematical theory is, like an empirical theory, a body of truths about a specific domain is given up in the move to the modern conception of axioms, and with it the foundational questions that Frege took to be central to the philosophy of mathematics. And while it is (perhaps) a coherent philosophical question to hold that mathematics *is* in some sense a matter of stipulation, so that the Fregean questions are somehow ill-formed, it is important to notice that the adoption of the modern perspective has not been accompanied with anything like a compelling argument for this position; it has instead been ushered in on a wave of fruitful techniques that leave this central philosophical issue unaddressed.

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