Frege on Shared Belief and Total Functions[1]

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Frege's work, and especially his realism about thoughts, is anchored in the view that shared belief is best understood in terms of a relationship borne by believers to abstract objects. The fact that thinkers from vastly different contexts can nevertheless agree or disagree, can prove the same theorems, and can raise questions about each others' views has to do, on the Fregean picture, with the fact that belief, disbelief, proof, and so on are all essentially about timeless and language-independent thoughts. As Frege puts it in the unpublished "Logic" manuscript:

> Can the same thought be expressed in different languages? Without a doubt, so far as the logical kernel is concerned; for otherwise it would not be possible for human beings to share a common intellectual life.[2]

Similarly, in 1891:

> [F]or all the multiplicity of languages, mankind has a common stock of thoughts.[3]

And in 1892:

> [O]ne can hardly deny that mankind has a common store of thoughts which is transmitted from one generation to another.[4]

The importance of Frege's commitment to language- and believer-independent belief contents for the development of post-Fregean philosophy can hardly be overestimated. There are, to be sure, serious potential difficulties surrounding the view of belief as a relation to abstract contents, not least of which is skepticism about the existence of and the possibility of access to such things. The Fregean has things to say in response to such apparent difficulties. In brief: to those who claim that shared belief requires no such questionable entities as Fregean thoughts, but merely the obtaining of various hard-to-specify similarity relations between token mental states or utterances, the Fregean replies that to affirm the same thought as another just is to have states or to make utterances that stand in such relations. To those who take sameness of belief to require merely something like the identity of the objects and relations thought about, the Fregean gives the familiar reply that such an account flies in the face of the fact that sometimes a mode of presentation matters. And so on. The sorting out of the relative advantages of the

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[1] A version of this essay was presented to the logic group at the Ludwig-Maximilians-Universität, Munich. Many thanks to members of the audience, especially to Prof. Dr. Matthias Schirn, for helpful comments.

[2] PW 6 / NS 6. This manuscript is dated by the editors at 1879-1891. Given the relatively-sophisticated discussion of thoughts, it is presumably composed toward the end of this period.

[3] C&O 196 note (CP 185 / KS 170)

[4] S&R 29 (CP 160 / KS 146)

original Fregean position, of the contrary views that press the kinds of difficulties just noted, and of the Fregean replies has, it seems fair to say, animated much of the philosophy of language and related fields over the last hundred years.

The fruitfulness of Frege's view of belief has a good deal to do with the multiple roles played in his theory by thoughts. The simple picture on which the same entities serve as the contents of belief, the things preserved in good translation, the truth-bearers, the semantic values of utterances, and the bearers of such logical relations as entailment and incompatibility, enables the Fregean to give a *prima facie* compelling account of the internal connections between belief, language, truth, and entailment. But the attempt to cast the Fregean thought into these multiple roles is the source not just of much of the richness of Frege's theory and of its enduring influence, but also of many of its potential difficulties. As a contribution to this collection of essays about the enduring legacy of Frege's work, what follows is a discussion of one of the sources of tension in Frege's work, a tension that arises, arguably, at the intersection of his views about thoughts as the binding agents in shared belief, and as the bearers of truth-value. My hope is that this discussion will help to clarify some of the connections between strands of Frege's work that have left their mark across a broad spectrum of modern philosophical work.

1. Introduction

The fact that Isaac Newton and Euclid believed some of the same things – e.g. that there are infinitely many prime numbers – is on Frege's account simply a matter of their affirming the same thought, the thought that we would express via that English sentence. This is a nice picture as far as it goes. It means that Newton and Euclid needn't share a language in order to believe the same thing. It means that one belief is true if and only if the other is, and that Euclid's implies the existence of numbers if and only if Newton's does. It means that disagreement is straightforwardly described, and consists in one person's affirming just the thought that the other denies.

Nevertheless, to say that shared belief consists in the affirmation of the same thought is still not to say very much. It is not to answer any hard questions about what such agreement requires: it doesn't take a stand on whether e.g. an historical connection is required between the believers, whether their degrees of confusion about closely-related topics would have to be similar, and so on. In order to answer these questions, given the Fregean starting-point, we would need to know whether e.g. historical connections or shared confusions are necessary to the identity of thoughts located via their relationship to people who affirm them. The Fregean account of belief in terms of the affirmation of thoughts merely rules out some potential accounts of shared belief; it doesn't by itself constitute a thoroughgoing account. But it is, for the kinds of reasons just sketched, an elegant and helpful starting-point.

It seems, though, that Frege's picture of shared belief, especially in its most interesting application to the sharing of belief across large distances of time and/or of language, is in direct conflict with another important part of Frege's view of language and thought, his view of how predicate- and function-terms contribute both to the thoughts expressed by, and to the truth-values of, sentences. Frege's mature theory takes predicates, including open sentences of all kinds, to express parts of the thoughts expressed by the sentences in which they're embedded, and to refer to what he calls "functions." The most familiar case is that of a one-place first-level predicate, in which case the kind of function referred to is what Frege calls a "concept." Concepts are functions from objects to truth-values; the concept referred to by "… is red" is a function mapping red objects to the true, and non-red ones to the false. Objects are just the kinds of things that singular terms refer to. The sense of a piece of language is that piece's contribution to the thought expressed by sentences in which it appears; the sense of a sub-sentential piece of language is a part of a thought (indeed, of infinitely many thoughts), while the sense of a whole sentence is a thought. Frege's view of the connection between sense and reference includes the view that a sentence refers to the truth-value *true* iff the thought expressed by that sentence is, as we would casually put it, itself true. False thoughts are expressed by sentences that refer to the *false*. If a part of a sentence lacks reference, then no piece of language involving that part has reference, and in particular the sentence itself lacks a truth-value. Such sentences can express thoughts, but on the Fregean view these are thoughts which have no truth-value. So if Newton and Euclid share a true belief about the infinity of the prime numbers, then the thought they both affirm must be expressible only by sentences all of whose parts have reference.[5]

Frege says, repeatedly over the course of many years, that all functions are total. In 1891, for example, he insists on the

> requirement, as regards concepts, that, for any argument, they shall have a truth-value as their value; that it shall be determinate, for any object, whether it falls under that concept or not,

and notes that this

> requirement of the sharp delimitation of concepts … carries along with it the requirement for functions in general that they must have a value for every argument.[6]

In a circa-1897 criticism of Peano, Frege claims that

> [E]very concept must have sharp boundaries, so that it is determined for every object whether it falls under the concept or not.[7]

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[5] That is to say: their singular terms and predicates must all refer.
[6] F&C 20 (CP 148 / KS 135). The implication can be seen via Frege's example: that the concepts referred to by "x + 1 = 10" and similar predicates be defined for every object requires that the function referred to by "… + …" be defined for every pair of objects.
[7] "The Argument for my Stricter Canons of Definition" (PW 155 / NS 168)

With respect to functions more generally, Frege characterizes the requirement in *Grundgesetze* as follows:

> [A] first-level function of one argument must always be such as to yield an object as its value, whatever object we may take as its argument… [8]

and

> … every first-level function of two arguments must have an object as its value for any one object as its first argument and any other object as its second…[9]

And similarly for functions of higher level.[10] Earlier in *Grundgesetze*, we find:

> A definition of a concept (of a possible predicate) must be complete; it must unambiguously determine, as regards any object, whether or not it falls under the concept (whether or not the predicate is truly ascribable to it). … We may express this metaphorically as follows: the concept must have a sharp boundary. … [A] concept that is not sharply defined is wrongly termed a concept.[11]

And here's the difficulty. If every function is total, then a predicate phrase which fails to refer to a total function fails *a fortiori* to refer to a function, and hence on the Fregean scheme fails to refer. Hence any sentence in which it appears fails to refer to a truth-value, and the thought expressed by that sentence (if any) is neither true nor false. Consider the relation-term

[D]   "… divided by … = …".

If the sentence

[S]   "8 divided by 2 = 4"

expresses a true thought, then [D] refers to a total function f_D. The total nature of f_D means that it's defined not only over such pairs as <8, 2> and <3+2i, 9>, but also over e.g. <the Taj Mahal, Indiana>. And both of the last two are problematic. To say that f_D is defined over the last of these is to say (because sense determines reference) that whatever a competent speaker grasps in understanding [S] is sufficient to determine a function that answers the question

(Q)   What is the result of dividing the Taj Mahal by Indiana?

And this is implausible. The linguistic and arithmetical competence required in order to grasp the sense of [S] involves, presumably, knowledge of quite a few simple arithmetical facts, but it doesn't involve knowledge of anything that determines an answer to (Q).[12]

The two options for the law-abiding Fregean seem to be (a) to agree that [S] doesn't express a true thought, and (b) alternatively to hold that the sense of [D] does determine an answer to (Q), and hence that the sense of a phrase is determined by facts that no competent

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[8] *Grundgesetze* II §63; Geach & Black p. 146.
[9] *Grundgesetze* II §64; Geach & Black p. 148.
[10] See *Grundgesetze* I §31.
[11] *Grundgesetze* II §56; Geach & Black p. 139.
[12] The difficulty is not due to the strong (and mistaken) requirement that the grasp of a sense must carry with it comprehensive knowledge of that sense or of the reference it presents. The point is rather that no answer to (Q) is determined by any facts whose (even dim) apprehension is involved in the ordinary understanding and use of the function-expression in [D].

speaker can ever be in a position to know.[13] Both options would seem to undermine the view that the sense of a sentence is the content of the belief affirmed via the assertion of that sentence, and that we routinely have true or false beliefs.

The difficulty is broad, since virtually none of our predicate-phrases has a sense that determines a total function, at least if senses are tied to what competent speakers grasp and what believers believe. And it doesn't arise just in the kinds of "boundary-crossing" cases just discussed, in which we bring two very different kinds of discourse (here, arithmetical and non-) together in an unusual way. Consider the second example above, in which we ask about the result of dividing a complex number by a natural number. Because the mathematician working in 300 BCE has no inkling of complex numbers, it's arguably the case that nothing he grasps determines an answer to such questions. The constraint that senses be tied to the graspable and believable means then that no mathematician in 300 BCE refers by [D] to a function defined over the complex numbers, and hence no such mathematician can express a truth or a falsehood by any sentence that involves this predicate-phrase. This is especially bad for the idea that Euclid and Newton grasped the same thoughts, given the expansion of the known mathematical universe in the intervening years. It's also bad for the idea that we, today, grasp true or false thoughts when we do mathematics, given, as one might suppose, the expansions yet to come.

One might simply decide, if one is attracted to the broadly Fregean account of belief and its expression in terms of thoughts, to reject the (apparent) Fregean requirement of totality for functions. And in the end, this is roughly speaking what will be recommended here. But it is not an unilluminating task to figure out why Frege says the things he says about the totality of functions, and to see how this requirement, whatever it amounts to, fits (or fails to fit) with the rest of Frege's picture of logic, belief, and language. In the end, I'll suggest that Frege does not in fact take all functions to be total, so that the rejection just suggested is not a departure from Frege's own views. Part of what follows will be an attempt to clarify what he did mean, and why.

2. Total functions and Linguistic completeness

To say that a function is *total*, in the sense used here, is to say that it delivers a value for every argument or n-tuple of arguments of appropriate logical type. A 2-place function from pairs of objects to objects, for example, is total iff for every pair of objects, it determines a value.

A related requirement is one which is defined for languages, and which we'll call the requirement of *linguistic completeness*. A language L satisfies the requirement of linguistic completeness if each syntactically well-formed concatenation of signs of that language has a determinate reference. Of particular interest for our purposes is the fact that if a language is complete in this sense, then for any first-level function-sign f of n arguments and any n-tuple of

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[13] The first of these options is attributed to Frege by Joan Weiner, who holds that for Frege ordinary arithmetical discourse, and indeed ordinary discourse more generally, typically doesn't involve the apprehension or expression of truths or falsehoods. See Weiner [2007] p. 702.

singular terms $t_1 \ldots t_n$, the sentence $ft_1 \ldots t_n$ has a reference. When $ft_1 \ldots t_n$ is a sentence, then, on Frege's views about sentential reference, the requirement is that it refers to a truth-value.

The requirement of linguistic completeness is one which Frege takes to be crucial for languages that are intended for the expression of rigorous arguments, and especially for those like his own concept-script, with respect to which the conditions of rigorous proof are to be laid down syntactically. For the presence in a language of well-formed phrases lacking reference is an invitation to fallacious reasoning. As Frege puts it,

> It seems to be demanded by scientific rigor that we should have provisos against an expression's possibly coming to have no reference; we must see to it that we never perform calculations with empty signs in the belief that we are dealing with objects. People have in the past carried out invalid procedures with divergent infinite series.[14]

Not just mathematical calculations, but ordinary reasoning as well can be derailed by the presence of non-referring parts of language. Even the law of excluded middle, as Frege points out, will lead us astray if we rely on it in contexts in which the application of a concept-expression to an object-name gives no definite result. Particularly important to Frege, given his interest in induction, are the Sorites-style fallacies that are invited by the use of concept-expressions that fail to deliver a truth-value on some completions by object-names.[15] In short, a language which fails the requirement of linguistic completeness is one that's not suited for maximum argumentative clarity and rigor.

With respect to the formal language of *Grundgesetze*, satisfaction of the criterion of linguistic completeness is crucial, in virtue of the *Grundgesetze* ideal of the purely-syntactic specification of the rules of deduction. It is essential to the satisfaction of this goal that no well-formed sentence of the formal language lack a truth-value, since any such non-referring sentence will serve as the basis for a counterexample to the central claim that all of the principles of deduction are truth-preserving. This requirement in turn is satisfiable only if every well-formed concatenation of signs receives a determinate reference. As Frege puts it at *Grundgesetze* I §28: "Correctly-formed names must always denote something."[16]

The requirement of linguistic completeness is generally weaker than the requirement that function-expressions refer to total functions. In particular, the former requirement has nothing to say about arguments not referred to in the language: a language expressly restricted e.g. to integers will satisfy the linguistic requirement if all of its function-expressions refer to functions

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[14] F&C 19 (CP 148 / KS 135)

[15] See e.g. the letter to Peano of 29 September 1896, esp PMC pp 114-115; "The Argument for my Stricter Canons of Definition" (PW 152-156, esp. p. 155 / KS 164-170, esp pp. 168-9.)

[16] Grundgesetze I §28. See also *Grundgesetze* I, Introduction (p. 9 of Furth):

> After one has reached the end [of *Grundgesetze*]…, he may reread the Exposition of the Begriffsschrift [i.e. the first Part] as a connected whole, keeping in mind that the stipulations that are not made use of later and hence seem superfluous serve to carry out the basic principle that every correctly-formed name is to denote something, a principle that is essential for full rigor.

defined just on the integers. In general, with respect to a language especially formulated for a given science, one which is expressly designed to talk about the objects, functions and relations of that science, the satisfaction of the requirement of linguistic completeness can be a straightforward matter, and will typically be consistent with the reference, on the part of its function-expressions, to partial functions.

The first thing to note about the passages in which Frege seems to espouse the strong view that every function is total is that every one of them appears in the midst of a discussion of the requirements for scientific and proof-theoretic rigor. Each one occurs in a discussion either of a particular formal or mathematical language (e.g., Frege's own *Grundgesetze* language or the language of Peano's *Formulaire de Mathématiques*), or in an account of the general conditions on rigor that are to be met by any language suitable for rigorous proof. In discussions of the functioning of ordinary language, and of the necessary conditions for ordinary communication, Frege never mentions the requirement that functions be total.

Two questions which arise regarding Frege's "sharp boundaries" requirement, then, are these. First: is the requirement, however precisely one understands it, one that he takes to apply, as he indeed sometimes says, to all functions, or merely – as he also sometimes says – to all of the functions referred to in the context of rigorous proof? Here, part of the question is that of what grounds this requirement: is it a concern for the general metaphysical underpinning of semantics, or is it a concern with strict argumentative rigor? Secondly: is Frege's requirement the requirement that the functions in question be strictly speaking total, or is it that they be defined over all of the arguments treated in the science in question?

We'll look at the first question first. The evidence here is slim, but the following passages give some reason to suppose that Frege's considered view is that the sharp-boundaries requirement is one that has to do specifically with argumentative rigor, and not with the expression of truth-evaluable thought in general. In an 1896 letter to Peano, Frege explains his reasons for rejecting conditional definitions for concept-expressions, i.e. definitions which determine only amongst an antecedently-specified collection of objects which ones do and which do not satisfy the concept-expression being defined:

> A conditional definition of the sign for a concept decides only for some cases, not for all, whether an object falls under the concept or not; it does not therefore delimit the concept completely and sharply. But logic can only recognize sharply delimited concepts. Only under this presupposition can it set up precise laws. The logical law that there is no third case besides
>
> > a is b
>
> and
>
> > a is not b
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> is really only another way of expressing our requirement that a concept (b) must be sharply delimited. The fallacy known by the name of 'Acervus' rests on this, that words like 'heap' are treated as if they designated a sharply delimited concept whereas this is not the case. Just as it would be impossible for geometry to set up precise laws if it tried to recognize threads as lines and knots in threads as points, so logic must demand sharp limits of what it will recognize as a concept unless it wants to renounce all precision and

certainty. Thus a sign for a concept whose content does not satisfy this requirement is to be regarded as meaningless from the logical point of view. It can be objected that such words are used thousands of times in the language of life. Yes; but our vernacular languages are also not made for conducting proofs. And it is precisely the defects that spring from this that have been my main reason for setting up a conceptual notation.[17]

The sentiment here would seem to be that rigorous proof, not ordinary communication, demands sharply-defined concepts. A similar account appears in 1897, in a general discussion of the requirement of sharp boundaries:

 We can also argue for this requirement on the ground that the law of the excluded middle must hold. For that implies that every concept must have sharp boundaries, so that it is determined for every object whether it falls under the concept or not. Were this not so, there would be a third case besides just the two cases 'a falls under the concept F' and 'a does not fall under the concept F' namely, the case where this is undecided. The fallacy known as the 'Sorites' depends on something (e.g. a heap) being treated as a concept which cannot be acknowledged as such by logic because it is not properly circumscribed.
        The following consideration also gives the same result. Inference from two premises very often, if not always, depends on a concept being common to both of them. If a fallacy is to be avoided, not only must the sign for the concept be the same, it must also have the same reference. It must have a reference independent of the context and not first come to acquire one in context, which is no doubt what very often happens in speech.[18]

What "very often happens in speech," presumably, is that the concept-word plays its usual role in the expression of thought in ordinary contexts although its behavior in outlying contexts isn't thereby determined. In the case of concept- or function-expressions, the point is that no total concept or function is referred to.

When Frege says, as he often does, that "logic demands" that functions be total, the question we've raised is that of whether Frege means that contexts of rigorous logical argument demand total functions, or more generally that "logic" in some sense dictates that *all* functions be total. What we've seen so far is some evidence that Frege's is the former view.

The second thing to note about the passages in which Frege seems to hold the strong view is that they generally appear in the context of a discussion of the requirement of linguistic completeness. The criticism of Peano noted above is explicitly directed at what Frege views as the important defect of Peano's mathematical language, that various substitutions of argument-phrases into function-expressions are not provided via Peano's stipulations with any reference. The target here is not function-expressions which refer to partial functions, but rather those that aren't defined over arguments named in the formalism. Frege's insistence on what looks like the strong condition of totality in the remarks from *Grundgesetze*, similarly, appears in the context of a very pointed discussion of the linguistic requirement; the objects at issue here are all ones that

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[17] Letter to Peano of 29 September 1896 (PMC 114-15)
[18] "The Argument for my Stricter Canons of Definition" (PW 155 / NS 168) (translation slightly amended).

can be explicitly referred to in the formalism. The discussion of 1891, again, is explicitly concerned with the requirement of rigor that all well-formed strings of symbols have a reference, not that concepts be defined over outlying objects. The passage from "Function and Concept" quoted above comes just as Frege has pointed out the need to provide a reference for every combination of signs. The passage, quoting slightly more extensively, reads as follows:

> What rules we lay down is a matter of comparative indifference; but it is essential that we should do so – that 'a + b' should always have a reference, whatever signs for definite objects may be inserted in place of 'a' and 'b'. This involves the requirement as regards concepts, that, for any argument, they shall have a truth-value as their value; that it shall be determinate, for any object, whether it falls under the concept or not. … The requirement of the sharp delimitation of concepts … carries with it this requirement for functions in general that they must have a value for every argument.[19]

Note that the first sentence here simply lays out an instance of the requirement for which Frege has clear reasons, namely the requirement of linguistic completeness. But the claim that this requirement "involves" the requirement that every function have a value for every argument is just wrong, unless we read the talk of "every argument" (and of "any object") here as an inexact way of talking about the arguments and objects referred to in the formalism. This passage has it in common with several others on this topic that in it Frege seems to slide back and forth between the explicitly-motivated requirement of linguistic completeness and the much stronger requirement that every function be total.[20] With respect to all of these passages, the interpretive options each involve attributing a certain carelessness to Frege. Frege's requirement that the functions in question be defined on "all" arguments is in each case a non-sequitur if he means to include in this generalization arguments alien to the science or the formalism in question. If on the other hand Frege's talk of "all" arguments is a careless way of mentioning all of what one might call the "relevant" arguments – i.e those about which one can speak within the confines of the science in question - then the passages are argumentatively unproblematic, if less than perspicuous.

What we have seen in the texts so far is this: First of all, Frege's requirement that functions take a value on "all arguments" is discussed only in the context of a discussion of the conditions necessary for argumentative rigor, as carried out in a language expressly designed for proof within a given science. Though some of Frege's statements of the requirement in these contexts are highly general, and seem to include *all* functions, not just those referred to in a rigorous scientific language, others of his comments explicitly restrict the requirement to languages and contexts with respect to which argumentative rigor is essential, exempting ordinary, non-rigorous discourse. Secondly: Focusing just on the contexts of rigorous science and

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[19] F&C 19-20 (CP 148 / KS 135)
[20] See "On Schoenflies" (PW 176-183, esp.180 / NS 191-199, esp. p. 195); *Grundgesetze* I §5, footnote 15; *Grundgesetze* I §8. This is of course not a "slide" if Frege takes all discourse to be in some sense "about" everything, including as-yet unrecognized mathematical objects. See below for reasons to reject this possibility.

the languages suitable thereto, the requirement is generally discussed alongside, and typically not distinguished from, the independently-motivated weak requirement of linguistic completeness. Frege's apparent endorsement of the strong requirement of totality in these passages can be taken at face value, in which case his justification for the view is bad (or perhaps just nonexistent); it can alternatively be viewed as a carelessly-phrased statement of the requirement of linguistic completeness. We turn now to two contexts in which Frege's considered view is arguably more easily discernible.

3. Piecemeal Definition

Perhaps the most important context in which Frege discusses what looks like the requirement of totality, certainly the context in which the requirement is raised in the most sustained and most forceful way, concerns Frege's prohibition against what he calls "piecemeal definition." What he means by this, and the reasons for prohibiting it in a good scientific system, are laid out by Frege in *Grundgesetze* Vol II as follows:

> §56: A definition of a concept (of a possible predicate) must be complete; it must unambiguously determine, as regards any object, whether or not it falls under the concept (whether or not the predicate is truly ascribable to it). … We may express this metaphorically as follows: the concept must have a sharp boundary. … a concept that is not sharply defined is wrongly termed a concept. Such quasi-conceptual constructions cannot be recognized as concepts by logic; it is impossible to lay down precise laws for them. The law of excluded middle is really just another form of the requirement that the concept should have a sharp boundary.
> §57. Now from this it follows that the mathematicians' favorite procedure, piecemeal definition, is inadmissible. The procedure is this: First they give the definition for a particular case – e.g. for positive integers – and make use of it; then, many theorems later, there follows a second definition for another case – e.g. for negative integers and zero. Here they often commit the further mistake of making specifications all over again for the case they have already dealt with. Even if in fact they avoid contradictions, in principle their method does not rule them out. What is more, as a rule they do not attain to completeness, but leave over some cases, as to which they make no specification; and many are naïve enough to employ the word or symbol for these cases too, as if they had assigned a meaning to it. … But the chief mistake is that they are already using the symbol or word in theorems before it has been completely defined – often, indeed, with a view to further development of the definition itself. So long as it is not completely defined, or known in some other way, what a word or symbol means, it may not be used in an exact science – least of all with a view to further development of its own definition.[21]

The dangers associated with piecemeal definition, then, include (i) the possibility of introducing a contradiction by providing later stipulations incompatible with earlier ones, and (ii) the possibility of failing to cover some cases, with the result that some combinations of symbols are left without reference.

The immediate protest one might be inclined to make to Frege at this point is that to seriously rule out piecemeal definition is to rule out virtually all of what has gone on in the

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[21] English translation in G&B pp 139-140

historical development of mathematics. Because e.g. the addition-sign was in familiar use prior to the recognition of complex numbers, the introduction of those numbers into the mathematical canon required an extension of the definition of that sign. Every extension of the realm of recognized mathematical objects has, similarly, required an extension of the definition of familiar function-signs, which Frege seems implausibly to rule out as illegitimate.

Frege's reply to this objection is straightforward:

> Now, of course it must be admitted that scientific progress, which has been effected by conquering wider and wider domains of numbers, made such a procedure almost inevitably necessary; and this necessity might serve as an excuse. [Here appears a lengthy footnote directed at Peano.] It would indeed have been possible to replace the old symbols and terms by new ones, and logic really demands this; but that is a hard decision to make. And this horror over the introduction of new symbols or words is the cause of many obscurities in mathematics. The old definitions likewise could have been rejected as unsound, and new ones used, in order to set up the science over again from the beginning; but such a clean cut was never made… In this way people have got used to piecemeal definition; and what was originally an awkward makeshift became customary, and was admitted as one of the legitimate methods of science. The result is that nowadays hardly anybody is shocked when a symbol is first defined for a limited domain and then used in order to define the same symbol once more for a wider domain; for general custom has a power of justifying what is done, just as fashion can give the cachet of beauty to the most detestable mode. It is all the more necessary to emphasize that logic cannot recognize as concepts quasi-conceptual constructions that are still fluid and have not yet been given definitive and sharp boundaries, and that therefore logic must reject all piecemeal definition.[22]

The proper procedure to use when extending the mathematical domain to new objects, as Frege sees it, is simply to use a new sign for the newly-expanded function. This is, as he puts it, "what logic really demands." We leave e.g. the old addition-sign behind, moving to a new sign defined all at once for the whole of the new domain of objects. We in this way entirely avoid the central dangers associated with piecemeal definition, those of introducing inconsistency via overlapping stipulations, and of leaving some combinations of signs meaningless via gappy stipulations.

Notice that the difficulty with which Frege is concerned when new mathematical objects are introduced is a difficulty regarding the definitions of the function-terms of the newly-expanded science: the difficulty is that, if we simply add new stipulations to the previously-recognized definitions of the original function-symbols, we run the danger of contradicting ourselves or of leaving some combinations of symbols out of account. We run the danger, in short, of bringing into our system new combinations of symbols which have no determinate reference. The difficulty pointed out by the introduction of new elements to mathematics is importantly *not* what one might have expected, given some of the ways in which Frege presents the condition of sharp boundaries. The difficulty is not that, having introduced the new elements, we are forced to recognize a logical flaw in the old science – the flaw, namely, that its function-terms refer only to partial functions, ones not defined on the newly-recognized objects. Frege makes no such claim.

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[22] *Grundgesetze* II §58; G&B pp 140-142

The problematic system is the one obtained by flawed (inconsistent or gappy) expansion of the old system, and not the old, partial system itself.

The repeated expansion of the domain of mathematical objects is as Frege sees it a series of episodes in which mathematicians recognize the existence of objects about which they had previously been ignorant – but which had, nevertheless, always existed. Mathematical activity is a process of discovery, not of creation. At each historical stage, then, the functions referred to in mathematics are only partial, having no determinate values for the as-yet undiscovered objects. Frege's view about this circumstance is not that there is anything wrong with these partial functions, nor that logic must eschew them. His claim is rather the considerably more-moderate one that the introduction of new objects into the realm of arguments under consideration must, in the interests of logical rigor, be accompanied not by piecemeal add-ons to the definitions of function-signs already in use, but rather by the introduction of new function-signs defined in one fell swoop over all of the objects of the new domain. There is, importantly, nothing wrong with the old function-signs as restricted to their original domain.

> …it is so easy to avoid a plurality of definitions for one and the same symbol. Instead of first defining a symbol for a limited domain and then using it for the purpose of defining itself in regard to a wider domain, we need only choose different signs, and confine the meaning of the first, once for all, to the narrower domain..[23]

There is nothing wrong with an addition-sign that refers to a function defined only on the integers, and nothing illegitimate about that function itself. We are only warned against using such a sign, or referring to such a function, in a science whose domain is wider than that of the integers. A similar sentiment appears in "Function and Concept":

> So long as the only objects dealt with in arithmetic are the integers, the letters a and b in '$a + b$' indicate only integers; the plus-sign need be defined only between integers.[24]

Frege goes on in this passage to point out that if we introduce a sign for a non-integer, e.g. the sign '$\oplus$' for the sun, then we must "lay down rules from which it follows e.g. what '$\oplus + 1$' is to mean, if '$\oplus$' means the sun." The general requirement is explained as follows, and here we repeat a passage quoted above:

> What rules we lay down is a matter of comparative indifference; but it is essential that we should do so – that '$a + b$' should always have a reference, whatever signs for definite objects may be inserted in place of 'a' and 'b'. This involves the requirement as regards concepts, that, for any argument, they shall have a truth-value as their value; that it shall be determinate, for any object, whether it falls under the concept or not. … The requirement of the sharp delimitation of concepts … carries with it this requirement for functions in general that they must have a value for every argument.[25]

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[23] *Grundgesetze* II §60 (G&B p. 144)
[24] F&C 19 (CP 148 / KS 135)
[25] F&C 19-20 (CP 148 / KS 135)

As we can now see, the context of this discussion makes it abundantly clear which of the interpretive options noted above is the right one. Frege cannot be taken here to be requiring that every function take a value on *all* arguments, where this generalization includes objects alien to the science in question. Not only (as above) does reading him in this way convict him of an embarrassing argumentative lapse (and one which he makes repeatedly, on this reading), but more importantly as we can now see, this reading has Frege flatly contradicting himself within the space of a single paragraph. The only time, says Frege, that the plus-sign needs to be "defined" on a domain larger than that of the integers is when it appears in a science whose domain is broader than that of the integers. When Frege demands that our function-signs refer to functions that have a value for "all arguments", his point is simply that they must have a value for all of those objects with which the science is concerned. The "arguments" here are simply the objects which, in the course of the science in question, will be under discussion. In the context of a language explicitly devoted to a given science, the requirement is that of linguistic completeness.

If Carnap's notes from the summer of 1914 are accurate, Frege says in a lecture course of that term that

> In the development of mathematics one does, however reach certain points where one wants to expand the system. But then one has to begin from scratch again. In any case, there always has to be a complete *system* at hand that is logically unproblematic. E.g. one would have to proceed as follows: as long as the plus sign + is used only for positive whole numbers, one chooses a different sign for it, e.g., ¬.[26]

With respect to a function referred to in a rigorous science whose domain is clearly circumscribed, the *arguments* Frege refers to are the objects and lower-level functions in the domain. The Fregean requirement is, as we might put it, that the functions involved in such a science be defined on all arguments from the domain of discourse.

Frege is notoriously cavalier about the demarcation of the domain of a science: he simply takes it as obvious that there is some realm of things under discussion in a given science, as for example in the case of the arithmetic of the integers. He never engages the question of what, exactly, marks off the boundaries of that domain. Nevertheless, it is clear that Frege conceives of the sciences and scientific languages he discusses as each having a definite area of application, one whose objects and level-n functions form the domain of arguments for the first-level and level n-+-1 functions, respectively, of that science. The requirement of scientific rigor is the requirement that each function of that science be defined over every argument of that science.

Frege muddies the waters considerably in his more hyperbolic discussions of the requirement, in which he seems to claim absurdly that the function-signs of mathematics must be defined over clearly non-mathematical objects. Thus for example in *Grundgesetze* II §§63-4, Frege seems to claim, contrary to his position in "Function and Concept," that the addition-sign

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[26] Reck and Awodey p. 155, from the summer semester 1914.

must be defined over the moon, and can't be restricted to a particular domain, e.g. that of the numbers. But it's important to see that this is not his point. The passage reads as follows:

§63 … [A] first-level function of one argument must always be such as to yield an object as value, whatever object we may take as its argument – whatever object we may use to 'saturate' the function.
§64 We must make the corresponding requirement as regards functions with two arguments. The expression
'the sum of one object and another object'
purports to be the name of such a function. Here too, then, it must be determinate, as regards any one object and any other object, which object is the sum of the one and the other; and there must always be such an object. If that is not the case, then it is likewise indeterminate which object gives the result one when added to itself. In that case, therefore, the words 'something that gives the result one when added to itself' do not mean any concept with sharp boundaries, i.e. anything that can be used in logic. …
But can we not stipulate that the expression 'the sum of one object and another object' is to have meaning only when both objects are numbers? In that case, you may well think, the concept *something that gives the result one when added to itself* is one with sharp boundaries; for now we know that no object that is not a number falls under it. E.g. the Moon does not fall under it, since the sum of the Moon and the Moon is not one. This is wrong. On the present view, the sentence 'the sum of the Moon and the Moon is one' is neither true nor false; for in either case the words 'the sum of the Moon and the Moon' would have to mean something, and this was expressly denied by the suggested stipulation. Our sentence would be comparable, say, to the sentence 'Scylla had six dragon necks.' This sentence likewise is neither true nor false, but fiction, for the proper name 'Scylla' designates nothing.[27]

Frege's argument here is directed against the suggestion that we define the addition-sign via a stipulation that terms of the form "… + …" are to "have meaning only when both objects are numbers." And the objection to this view is that it will leave terms like "the sum of the Moon and the Moon" without reference, and will therefore have the result that the concept *something that gives the result one when added to itself* fails to have sharp boundaries. It is important to note that the failure of what Frege calls the "sharp boundaries" requirement is *not* demonstrated here by pointing to an object which neither determinately falls under nor determinately fails to fall under the concept in question. For under the proposal in question, there is no object denoted by the term "the sum of the Moon and the Moon." The failure of the sharp-boundaries requirement is instead demonstrated by the fact that the concept-phrase, when saturated by this singular term, gives no truth-value. That is to say, the failure is explicitly the failure of linguistic completeness. The problem Frege points out here is that the proposed stipulation does not provide the concept-phrase "x + x = 1" with a reference for each of its saturations by terms of the language. It's a problem that arises only in languages containing singular terms for non-numbers.

That the problem is essentially one having to do with linguistic completeness is reinforced as Frege continues the discussion by suggesting and rejecting another way around the difficulty, one which involves a restriction on the formation of concept-phrases:

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[27] G&B 146-7

If our sentence 'the sum of the Moon and the Moon is not one' were a scientific one, then it would assert that the words 'the sum of the Moon and the Moon' and the word 'one' did not coincide in meaning; but with the stipulation suggested above, the former words would not have any meaning; accordingly we could not truly assert either that their meaning did coincide with the meaning of the word 'one' or that it did not coincide with it. Thus it would be impossible to answer the question whether the sum of the Moon and the Moon is one, or whether the Moon falls under the concept *something that gives the result one when added to itself*. In other words, what we have just called a concept would not be a genuine concept at all, since it would lack sharp boundaries. But when once we have introduced the expression '*a* added to *b* gives the result *c*,' we can no longer stop the construction of a concept-name like 'something that gives the result one when added to itself.' If people would actually try to lay down laws that stopped the formation of such concept-names as this, which, though linguistically possible, are inadmissible, they would soon find the task exceedingly difficult and probably impracticable. The only way left open is to give to the words 'sum,' 'addition,' etc., if one means to use them at all, such definitions that the concept-names constructed out of the words in a linguistically correct manner stand for concepts with sharp boundaries and are thus admissible.[28]

The "exceedingly difficult and probably impracticable" strategy of restricting the formation of concept-names is a potential way of maintaining linguistic completeness in the face of the problematic stipulation; it is not even in the ballpark of a helpful strategy if the goal is that of ensuring that all functions are total.

Despite the incautious way in which he sometimes puts the requirement of sharp boundaries, a way which when taken in isolation makes it appear that Frege requires e.g. all first-level mathematical functions to be defined on all (n-tuples of) objects, mathematical or otherwise, the texts just don't bear this out. As he says in "Function and Concept," an arithmetic that deals just with the integers can quite happily incorporate an addition-sign defined just over the integers. What's crucial for Frege is that, within a rigorous theory, no grammatically-acceptable concatenation of function-sign and object-sign(s) fail to have reference, which is to say that function-signs must refer to functions that are defined over all of the objects under discussion. If our language has a name for, or quantifiers ranging over, the Moon, then the only acceptable addition-sign is one whose reference is defined over the pair <the Moon, the Moon>. But this is the requirement of linguistic completeness.

§4. Restricted Domains and *Grundgesetze*

One further textual issue concerns a passage which might easily be taken to be inconsistent with Frege's claim that e.g. the addition-sign as it appears in a scientific discussion of the arithmetic of the integers can legitimately be defined just over the integers. At *Grundgesetze* II §65, we find:

As soon as people aim at generality in propositions they will need in arithmetical formulae not only symbols for definite objects – e.g. the proper name '2' – but also letters that only indicate and do not designate [i.e., bound variables –PB]; and this already leads them,

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[28] *Grundgesetze* II §64 (G&B pp. 147-8)

quite unawares, beyond the domain within which they have defined their symbols. One may try to avoid the dangers thus arising by not making the letters indicate objects in general (as I did), but only those of a domain with fixed boundaries. Let us suppose for once that the concept *number* has been sharply defined; let it be laid down that italic letters are to indicate only numbers; and let the sign of addition be defined only for numbers. Then in the proposition 'a+b = b+a" we must mentally add the conditions that *a* and *b* are numbers; and these conditions, not being expressed, are easily forgotten. But let us deliberately not forget them for once! By a well-known law of logic, the proposition

'If a is a number and b is a number then a+b = b+a'

can be transformed into the proposition

'if a+b is not equal to b+a, and a is a number, then b is not a number'

and here it is impossible to maintain the restriction to the domain of numbers.[29]

Frege might be taken here to reject the view that the range of values of a given variable can be restricted to a specific domain, e.g. to numbers. But it is important to see that this is not his point. Frege's argument here is intended to expose the difficulty involved in restricting the definition of function-signs, e.g. of the plus-sign, to a proper part of the collection of objects already under discussion. As Frege puts it in the previous paragraph,

Such a restriction would have to be incorporated into the definition, which would thus take some such form as: 'If a and b are numbers, then a+b means…' We should have a conditional definition. But the sign of addition has not been defined unless every possible complex symbol of the form 'a+b' has a definite meaning, whatever meaningful proper names may take the places of 'a' and 'b'.[30]

We can't coherently restrict the functions referred to by our function-signs to those arguments that satisfy some predicate, since as Frege illustrates above, this strategy leads immediately to well-formed combinations of symbols that lack reference. It leaves out not only those singular terms formed by concatenating the function-sign with argument-signs whose referents don't satisfy the delimiting predicate, but also those quantified sentences that involve the function-sign and the negation of the predicate. The point, again, is not that our functions must be total, but rather that they must be defined on all of the arguments over which we quantify in the science in question.

The clearest indication of Frege's understanding of the requirements for function-expressions comes in his treatment of his own *Grundgesetze* language. At *Grundgesetze* I §10, in the course of laying down the references of his terms, Frege discusses the fact that the stipulations given to this point are not yet sufficient to satisfy the condition of linguistic completeness. Particularly instructive is his means of remedying this defect.[31]

Although we have laid it down that the combination of signs

"ε'Φ(ε) = α'Ψ(α)"

has the same denotation as

"∀x(Φ(x) = Ψ(x)),"

this by no means fixes completely the denotation of a name like "ε'Φ(ε)." We have only a means of always recognizing a course-of-values if it is designated by a name like "ε'Φ(ε),"

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[29] *Gg* II §65; G&B 149.

[30] *Ibid.*

[31] Or, as one should more carefully put it: his *attempt* to remedy this defect.

by which it is already recognizable as a course-of-values. But we can neither decide, so far, whether an object is a course-of-values that is not given us as such, and to what function it may correspond, nor decide in general whether a given course-of-values has a given property unless we know that this property is connected with a property of the corresponding function.

…

How may this indefiniteness be overcome? By its being determined for every function when it is introduced, what values it takes on for courses-of-values as arguments, just as for all other arguments. Let us do this for the following functions considered up to this point.[32]

At this point, Frege lists the three functions considered up to now in *Grundgesetze*, and notes that two of them are reducible to the third, i.e. to the identity-function.

Since in this way everything reduces to consideration of the function $\xi=\zeta$, we ask what value this has if a course-of-values occurs as argument. Since up to now we have introduced only the truth-values and courses-of-values as objects, it can only be a question of whether one of the truth-values can perhaps be a course-of-values.[33]

Consider a particular course-of-values, say $\varepsilon'(\varepsilon=\varepsilon)$. The indeterminacy in question is, with respect to this case, the fact that sentences of the form "___ $= \varepsilon'(\varepsilon=\varepsilon)$" and "$\varepsilon'(\varepsilon=\varepsilon) = $ ___" have no truth-value in cases in which the blank is filled in by anything other than a term of the form "$\alpha'\Phi(\alpha)$." One might think of the difficulty as consisting in the fact that the term "$\varepsilon'(\varepsilon=\varepsilon)$" does not have a determinate reference, since a vast number of identity-questions regarding its purported reference have been left unanswered. But that this would be a misleading way of construing the difficulty is made clear by Frege's means of addressing it. Frege does not solve the problem by providing a means of answering those unanswered questions in general. He does not bring it about that the concept-phrase "___ $= \varepsilon'(\varepsilon=\varepsilon)$" determinately holds or fails to hold of each object. He instead brings it about that every way of completing this concept-phrase with a singular term of the language, i.e. every sentence formed in this way, has a determinate truth-value. As he says, since the only objects that have been "introduced" up to this point (i.e. the only ones to which one can refer in the language) are truth-values and courses of value, and since the identity-questions about courses of value have been settled already, the only question left to be settled is that of whether the concept-phrase "___ $= \varepsilon'(\varepsilon=\varepsilon)$" is satisfied by terms for either of the truth-values. Similarly for each of the course-of-value terms: for each such term $\varepsilon'(\Phi(\varepsilon))$, it must be determined whether "___ $= \varepsilon'(\Phi(\varepsilon))$" is satisfied by terms for either of the truth-values.

Frege settles the issue by a simple arbitrary stipulation: "[L]et us lay it down that $\varepsilon'(\text{---}\varepsilon)$ is to be the True and $\varepsilon'(\varepsilon = \sim(\forall x)x=x)$ is to be the False."[34] This neatly solves the problem: every identity-sentence of the formal language now has a determinate truth-value. The stipulation does not, of course, have the result that the concept-phrases of the form "___ $= \varepsilon'(\Phi(\varepsilon))$" refer to total

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[32] *Gg* I §10. I have substituted modern universal-quantification notation for Frege's own.
[33] *Ibid.*
[34] *Ibid.*

concepts: nothing has been said to determine whether any such phrase is satisfied by objects not named in the language. Similarly, nothing has been said to determine whether or not the courses of value in question fall under or fail to fall under concepts not yet referred to in the language. Frege's remarks on this circumstance are simply that:

> With this we have determined the courses-of-value so far as is here possible. As soon as there is a further question of introducing a function that is not completely reducible to functions known already, we can stipulate what value it is to have for courses-of-values as arguments; and this can then be regarded as much as a further determination of the courses-of-values as of that function.[35]

Frege may seem here to be violating his own prohibition against piecemeal definitions. But presumably his point is simply this: that when the concept-script is to be applied in a wider domain or in conjunction with an expanded collection of primitive terms, as Frege envisions its future use, one will build each new such system from the ground up, incorporating stipulations sufficient to guarantee that each function is defined on the whole range of arguments of that system.

The fact that the concepts referred to by *Grundgesetze* concept-expressions are very clearly not total does not, as Frege sees it, stand in the way of the entirely unproblematic functioning of the language of *Grundgesetze*. Indeed, these failures of totality do not even stand in the way of the expression of determinate, truth-evaluable thoughts by the formulas of that language. As Frege puts it in §32, having argued that each of the ways of combining function- and argument-expressions of the very limited language has a well-defined reference (in the sense that no formula constructed out of them lacks a truth-value):

> In this way it is shown that our eight primitive names have denotation, and thereby that the same holds good for all names correctly compounded out of these. However, not only a denotation, but also a sense, appertains to all names correctly formed from our signs. Every such name of a truth-value *expresses* a sense, a *thought*. Namely, by our stipulations it is determined under what conditions the name denotes the True. The sense of this name – the *thought* – is the thought that these conditions are fulfilled.
> …
> …we have in every correctly-formed proposition of Begriffsschrift [here Frege refers to a sentence prefixed with the judgment-stroke] a judgment that a thought is true; and here a thought certainly cannot be lacking.
> …
> The names, whether simple or themselves composite, of which the name of a truth-value consists, contribute to the expression of the thought, and this contribution of the individual [component] is its *sense*. If a name is part of the name of a truth-value, then the sense of the former name is part of the thought expressed by the latter name.[36]

In short, the sentences of *Grundgesetze* express unproblematic thoughts, ones that have determinate truth-values. This despite the fact that the functions referred to via their predicative

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[35] *Ibid.*
[36] *Grundgesetze* I §32

parts are all very partial, defined only over an extremely limited collection of objects. As Frege explains it, a first-level function-term of one argument "succeeds in referring" as long as each singular term that results from filling in its argument-place with referring terms of the formalism itself has a reference.[37] Similarly for function-terms generally. All such expressions "succeed in referring" as long as the requirement of linguistic completeness is met: the requirement of totality plays no part in Frege's *Grundgesetze* system.

Frege's means of resolving the initial linguistic incompleteness in *Grundgesetze* also enables us to make sense of a similar issue at *Grundlagen* §56. At *Grundlagen* §55, Frege raises the possibility of defining the numbers contextually by saying (a) that 0 belongs to a concept F iff $\forall x \sim Fx$; (b) that 1 belongs to a concept F iff $[(\exists x)Fx \ \& \ \forall y \forall z(Fy \ \& \ Fz \rightarrow y=z)]$; and (c) that (n+1) belongs to F iff $(\exists x)(Fx \ \& \ n$ belongs to $(Fy \ \& \ y \neq x))$. He then explains why such a definition is unsatisfactory, as follows:

> We can, of course, by using the last two definitions together, say what is meant by
>   'the number 1 + 1 belongs to the concept F'
> and then, using this, give the sense of the expression
>   'the number 1+1+1 belongs to the concept F'
> and so on; but we can never – to take a crude example – decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or not. Moreover, we cannot by the aid of our suggested definitions prove that, if the number *a* belongs to the concept F and the number *b* belongs to the same concept, then necessarily *a=b*. Thus we should be unable to justify the expression "*the* number which belongs to the concept F," and therefore should find it impossible in general to prove a numerical identity, since we should be quite unable to achieve a determinate number. It is only an illusion that we have defined 0 and 1; in reality we have only fixed the sense of the phrases "the number 0 belongs to," "the number 1 belongs to"; but we have no authority to pick out the 0 and 1 here as self-subsistent objects that can be recognized as the same again.[38]

It is easy to read the Caesar passage, taken in isolation, as complaining that the proposed definition fails to assign to e.g. "… = 0" a total function, one defined over such objects as Roman generals. Similarly, the complaint would be that the concept-term "… is a number" is not assigned a total function. This reading of the passage faces an immediate difficulty when we note that the definitions Frege eventually gives in §§68ff of *Grundlagen* "fail" in exactly the same way: the functions assigned to "… = 0" and to "… is a number" are defined only over extensions of second-level functions. More importantly, as we've seen, Frege doesn't take it to be a general condition of adequacy of function-terms that they are defined over objects outside of the intended domain. So the idea that Frege requires a good definition of these mathematical concept-phrases to assign them functions defined over historical figures is difficult to square with the texts.

In fact, the proposed *Grundlagen* definitions fail in a considerably more significant way than this: it's not that they fail to assign to the function-terms *total* functions; they fail to assign them functions at all. It's crucial for Frege's *Grundlagen* reconstruction of arithmetical discourse

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[37] *Grundgesetze* I §29
[38] Gl §56.

that "… = n," for each numeral n, and "… is a number" are assigned functions, which is to say in part that the individual numerals must be assigned objects. The proposed definitions under discussion here don't do this; as Frege says, "It is only an illusion that we have defined 0 and 1; in reality we have only fixed the sense of the phrases 'the number 0 belongs to,' 'the number 1 belongs to'." A straightforward way to dramatize this point is to note that the definitions cannot be employed to give any sense whatsoever to statements of the form "*a* = 0" or "*a* is a number," for *a* any singular term. Frege's choice of "Julius Caesar" for the term *a* doesn't illustrate a point specific to "outlying" singular terms (i.e. to non-arithmetical ones); it's a point that applies to all singular terms. The straightforward solution, and the one that Frege adopts, is to give explicit definitions of the numerals as singular terms and of the term "… is a number" as a concept-denoting phrase – but not, of course, as one that denotes a total function.

At *Grundlagen* §106, Frege gives a summary of his discussion of the nature of numbers and of the requirements on definitions of numerical terms, including a rehearsal of the lesson to be learned from the "Caesar" passage. The relevant part for our purposes is as follows:

> Let us cast a final brief glance back over the course of our enquiry. After establishing that number is neither a collection of things nor a property of such, yet at the same time is not a subjective product of mental processes either, we concluded that a statement of number asserts something objective of a concept. We attempted next to define the individual numbers 0, 1, etc., and the step from one number to the next in the number series. Our first attempt broke down, because we had defined only the predicate which we said was asserted of the concept, but had not given separate definitions of 0 or 1, which are only elements in such predicates. This resulted in our being unable to prove the identity of numbers. It became clear that the number studied by arithmetic must be conceived not as a dependent attribute, but substantivally.

In short, the difficulty with the original proposal has nothing to do with the totality of the concept of number. The difficulty illustrated by the Caesar example is that this proposal fails to treat numbers as objects. In so doing, it also fails to treat the relevant predicates, e.g. "… = 0" and "… is a number" as function-denoting phrases of any kind.

5. Conclusion

Despite some often-quoted passages that seem to indicate that Frege takes all functions to be total, there is in the end no coherent way to read him as endorsing this requirement. In the notorious passages in question, Frege insists that argumentative rigor demands (a) the rejection of conditional and of piecemeal definitions and (b) the satisfaction of the requirement of linguistic completeness. Hence in any context in which argumentative rigor is at a premium, it is important that every function be defined over every argument of the domain under discussion. That even the strictest Fregean demands of rigor fail to imply a requirement of totality on the part of functions is made most clear by the fact that the function-expressions of Frege's own extraordinarily-careful and rigorous language of *Grundgesetze* fail to refer to total functions.

Furthermore, the fact that the function-expressions of ordinary language and of ordinary scientific discourse are not defined over outlying objects and concepts – e.g. the fact that "… + …" as ordinarily used is not defined over material objects – tells us nothing, from the Fregean point of view, about the capacity of those expressions to appear in sentences expressing truth-evaluable thoughts. And finally, the fact that the function-expressions of ordinary discourse fail to have sharp boundaries even within the intended domain of discourse – e.g. that "heap" is vague, that "the eldest son of…" often delivers no value when attached to the name of a person – tells us just that these expressions are not ideally suited to the rigors of argument, not that they cannot be used, according to Frege's semantic theory, in the expression of truths and falsehoods.

This does not remove all of the difficulties for Frege's general picture of agreement and disagreement, and of the continuity of science. Indeed, it arguably does little more than clear away some distractions in order to make space for the real difficulties. Two of those difficulties in particular, which we can do little more than indicate here, are as follows.

First of all, while it is clear that Frege doesn't take the senses that we grasp when we do arithmetic to determine the values of arithmetical functions as applied to non-arithmetical objects, or even to arithmetical objects outside the domain of current discussion, it is also clear that he takes those senses to determine the values of those functions as applied to objects about whose nature we, the competent speakers of the arithmetical language, may be quite unclear. Just as we manage to refer to material objects about which we're quite ignorant, e.g. by description or by ostension, so too on Frege's picture we refer to mathematical relations and functions of whose behavior we have, on occasion, only a quite hazy grasp. For example, prior to the good hard mathematical work of the 19th century, as Frege sees it, we had only a foggy grasp of the notion of continuity, but we nevertheless managed on his view to say something determinate when we uttered sentences of the form ".. is a continuous function." As Frege puts it in *Grundlagen*,

> Often it is only after immense intellectual effort, which may have continued over centuries, that humanity at last succeeds in achieving knowledge of a concept in its pure form, in stripping off the irrelevant accretions which veil it from the eyes of the mind.[39]

The question for Frege is that of how a confused and hazy understanding, one which would not seem sufficiently rich to determine a particular reference, can constitute the grasp of a sense which does determine such reference.

Tyler Burge has argued persuasively that in cases like this, it is (as Frege understands it) the underlying mathematical facts which together with our hazy grasp suffice to determine a particular function.[40] As Burge puts it, it can take hard mathematical work, i.e. work on uncovering those mathematical facts, to determine which function it is we've been referring to all along. As long as our mathematical practice has been sufficiently rich in the surrounding area, so that there are (albeit undiscovered) mathematical facts that explain the mathematical phenomena we are

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[39] *Grundlagen* p. vii.
[40] See Burge [1984]

already aware of and seeking to explain, then there is no mystery that the explanantia can be both the things we refer to and the objects of not-yet-completed discovery. But that there are limits, some of them very nearby, to this kind of account is evident. Whenever our hazy grasp is compatible with two or more equally-good accounts of the mathematical phenomena in question, i.e. with two or more precisifications of the hazily-grasped function under discussion – or more accurately still, with two or more ways of disambiguating the terms we use in this hazy stage, the mathematical facts will be insufficient to provide the disambiguation. Armed with paradigm examples of continuous functions, and with a rich trove of results about continuity, perhaps it really is the case that mathematicians just pre-Weierstrass were referring to that concept that's picked out by the ε-δ definition.[41] But this is the case only if there is no other concept that fits those paradigm cases and those results (and perhaps our "intuitive feel" for continuity) just as well. The first difficulty for the Fregean, then, is to give a clear account of the conditions under which a hazily-understood mathematical notion is connected sufficiently clearly with a surrounding mathematical practice that, despite the haziness, the sense grasped through this understanding suffices to fix reference. If it's not possible to make sense in some such way of the increasingly-clear understanding we have historically had of central mathematical notions, then the Fregean must re-cast such conceptual progress not as the progressive clarification of objects and functions determinately referred to if only hazily understood, but rather as the process of deciding between different alternative specifications of reference after a period of ambiguous reference. That the latter option sits unhappily within the Fregean framework is clear from the fact that such ambiguity, from the Fregean perspective, is incompatible with the expression of true or false thoughts via the use of the ambiguous terms.

The second difficulty has to do not with haziness but with expansions of mathematical domains. Recall that as Frege understands it, the right way to proceed when recognizing new mathematical objects (or functions) is not to think of oneself as extending the old functions to new arguments, but instead as recognizing new functions, ones defined on the whole of the expanded domain, and for which we ought to introduce new terminology. In such cases, the senses of the new function-terms will not be the same as the senses of any of the old function-terms. Hence no thoughts expressed via the old terminology will be expressible via the new. No thought expressible via an addition-sign defined just over the positive integers, for example, will be expressible via a (later) sentence whose addition-sign is defined over the complex numbers as well.

But this means that Frege's nice, neat account of our "common intellectual life," one that proceeds in terms of the identity of the thoughts grasped across generations, will not work.

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[41] Perhaps, but it's difficult to see how. For it's difficult to see what it is about that early mathematical and applied-mathematical practice that would underwrite reference to continuity in this sense as opposed e.g. to uniform continuity, or to differentiability, and so on.

Because of the expanded domain of most of his function-signs, very few of the mathematical thoughts expressible by Newton's sentences can have been expressed by Euclid's.

The difficulty here is not as severe as was the difficulty we seemed to face earlier, when it seemed that Frege was committed to the view that future domain-expansion undermined the claim of a given mathematical sentence to express a true or false thought. There is, we now see, no difficulty in holding from the Fregean perspective that both Newton and Euclid expressed truths via their arithmetical sentences. The difficulty is that in most cases these cannot have been, strictly speaking, the same truths. There is a good deal the Fregean can say about the similarity between the thoughts expressible and graspable in each era, and it is presumably on this basis that the careful Fregean can make sense of the idea of "shared" belief, and of scientific continuity. Nothing about the current difficulty undermines the fruitful Fregean idea that belief is a matter of the affirmation of a thought, the kind of thing expressible via different sentences and the ultimate bearer of truth and of the logical relations. But the nice, neat picture on which the thoughts grasped in scientific inquiry can for the most part remain constant through developments in that science is, it seems, an over-simplification.

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