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# Frege on Consistency and Conceptual Analysis†

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Gottlob Frege famously rejects the methodology for consistency and independence proofs offered by David Hilbert in the latter's *Foundations of Geometry*. The present essay defends against recent criticism the view that this rejection turns on Frege's understanding of logical entailment, on which the entailment relation is sensitive to the contents of non-logical terminology. The goals are (a) to clarify further Frege's understanding of logic and of the role of conceptual analysis in logical investigation, and (b) to point out the extent to which his understanding of logic differs importantly from that of the model-theoretic tradition that grows out of Hilbert's work.



## 1. Introduction

Gottlob Frege is notorious for having rejected, in the first decade of the twentieth century, the then-emerging methodology for independence and consistency proofs in mathematics. The target of Frege's criticism was primarily the use of that methodology in David Hilbert's classic *Foundations of Geometry*. Because Hilbert's methods were undoubtedly successful, and differ only in detail from our own contemporary means of demonstrating consistency and independence, Frege's critical stance here has struck some as simply short-sighted or worse.[1]

I have argued elsewhere [Blanchette, 1996] that Frege had an important point to make in his criticism of Hilbert, one that seems to have been largely lost in the ensuing century. If one understands the relations of *independence* and *consistency* in the way Frege does (a way which, incidentally, has something to be said for it), then neither Hilbert's own, nor more recent model-theoretic, methods are generally successful in demonstrating independence and consistency. Or so I have argued. The central points here are (a) that Hilbert's

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1 See, *e.g.*, [Dummett, 1975], [Resnik, 1973/4], [Currie, 1982b].

 

methods demonstrate what might be called a 'formal' notion of relative consistency and independence, one that applies to sentences in virtue of their explicit form; (b) that Frege is concerned with notions of consistency and independence that hold in virtue of (sometimes) more deep-seated semantic features of the claims or sentences in question; and most importantly (c) that consistency and independence in Hilbert's sense do not imply consistency and independence in Frege's sense. The clearest cases in which a set of sentences (or the claims they express) will be counted consistent in Hilbert's sense and inconsistent in Frege's sense are cases in which the inconsistency (from the Fregean point of view) is evident only after a conceptual analysis of some of the sub-sentential components of the sentences in question. For example, the set {*Point* a *lies between points* b *and* c; *it is not the case that point* a *lies between points* c *and* b} is clearly consistent in Hilbert's sense; whether it is consistent in the Fregean sense is much less clear-cut, and turns on the question of whether further conceptual analysis, especially of the relation *between*, will reveal that the first proposition implies the negation of the second.

Recently, Wilfrid Hodges [2004] has argued that this account of Frege's understanding of consistency and independence, and the consequent account of Frege's disagreement with Hilbert, cannot be right. As Hodges sees it, Frege and Hilbert have no disagreement over the role of conceptual analysis in logical investigation, and hence their disagreement over consistency proofs cannot be explained in the way just suggested. The importance of this point is that, if correct, it shows that conceptual analysis cannot play the role just suggested in Frege's conception of logic, and that Frege's rejection of Hilbert's methodology requires an entirely different explanation.

The purpose of this note is to clarify the role played by conceptual analysis in Frege's procedure, and to argue that Frege's disagreement with Hilbert is indeed as just sketched.

## 2. Analysis in Frege: a Brief Overview

Conceptual analysis is crucial to all of Frege's logicist and proto-logicist work. The proofs he offers in *Grundgesetze*, which were meant to seal the case for logicism, are proofs of highly analyzed versions of arithmetical truths, versions which use the analyses of *cardinal number, zero, successor*, *etc*. for which Frege famously argues in *Grundlagen*. These analyses are what allow Frege to break down the relevant arithmetical truths into their primitive components, and to demonstrate their provability from propositions that explicitly employ only those primitive components. As Frege says in *Grundlagen*,

> ... the fundamental propositions of arithmetic should be proved, if in any way possible, with the utmost rigour; for only if every gap in the chain of deductions is eliminated with the greatest care can we say with certainty upon what primitive truths the proof depends ...
>
> If we now try to meet this demand, we very soon come to propositions which cannot be proved so long as we do not succeed in analyzing concepts which occur in them into simpler concepts or in reducing them to something of greater generality. Now here it is above all Number which has to be either defined or recognized as indefinable. This is the point which the present work is meant to settle. On the outcome of this task will depend the decision as to the nature of the laws of arithmetic. [Frege, 1884, p. 5][2]

The idea that the truths in question 'cannot be proved so long as we do not succeed in analyzing concepts which occur in them', but can in principle be proven once such an analysis has been given, is the motivation for the central work of the *Grundlagen*. Frege's strategy here is to show how such claims as 'every natural number has a successor' can be shown to follow from purely logical principles once such central notions as *natural number* and *successor* have been broken down into simpler components.

It is not just proof from principles of logic (as in *Grundgesetze*) that is facilitated by analysis; for Frege, the provability in general of a proposition from particular premises can often be demonstrated only after the right analysis has been performed. To have a clear example in front of us, we shall look briefly at an analysis and proof offered by Frege in his [1881]. The relevant passage begins (with our labels inserted for later reference):

> I wish to prove the theorem that
>
> (SUM) the sum of two multiples of a number is in its turn a multiple of that number.
>
> ...
>
> The numbers whose multiples are to be considered are subject to no conditions other than that the following addition theorems:
>
> (P1) $(\forall m)(\forall n)(\forall p)((m+n)+p = m+(n+p))$
>
> and

[2] On Frege's use here of the word 'definition', see §3 below.

(P2) $(\forall n)(n = n + 0)$

hold for them.[3]

As Frege notes, the result is an interesting one, since the premises (P1) and (P2) are, on the surface at least, surprisingly weak by comparison with the conclusion (SUM) to be demonstrated. Having announced the project, Frege gives his reader a careful analysis of the relation 'multiple-of', to the effect that a number $n$ is a multiple of $a$ iff $n$ follows 0 in the '$+a$' series, with 'following in the ... series' cashed out as in *Begriffsschrift*, and '0' and '+' left unanalyzed. We thereby obtain a highly analyzed version of what is expressed by (SUM), presented via a complex formula which we shall abbreviate as:

(SUM′)[4]

$$(\forall m)(\forall n)(\forall a)((\mathrm{Fol}_{+a}(m, 0) \;\&\; \mathrm{Fol}_{+a}(n, 0)) \rightarrow \mathrm{Fol}_{+a}((m + n), 0).$$

That is: if $m$ follows 0 in the '$+a$' series, and $n$ follows 0 in that series, then $(m + n)$ does too. (SUM′) is then the sentence which Frege derives from (P1) and (P2) with his usual painstaking rigor.

In this brief example we have an illustration of Frege's standard procedure, marked by the following two points which will concern us below:

(i) Frege standardly discusses the theorem he is going to prove and, after the fact, the theorem he *has* proven, in ordinary language, speaking for example in terms of (SUM), and talking about ordinary arithmetical truths, as the things established by his proofs. But the propositions he actually proves are generally the results of a non-trivial *analysis* of the propositions casually discussed. This leads to the following question: Does Frege in fact take, *e.g.*, the sentences (SUM) and (SUM′) to express the same proposition, so that he is to be taken at his word when he speaks for example of his proof of (SUM′) as constituting a proof of (SUM)? (Here it is important to keep in mind that for Frege, the objects of proof are not the sentences displayed, but rather the nonlinguistic propositions expressed by those sentences.) Or are we to take him to be speaking somewhat more loosely in such cases, so that *analysandum*-sentence and *analysans*-sentence express importantly similar, rather than identical, propositions? As we will see below, Frege is not entirely clear on this point. What *is* clear, and what will be crucial for our purposes,

[3] (P1) and (P2) are formulas (1) and (2) respectively of Frege's [1881]. (SUM) is Frege's own natural-language rendering of the claim to be proven; see [1881, p. 27]. Addition and zero are left unanalyzed in (P1) and (P2).

[4] This is Frege's (12) in modern notation. See [1881, p. 31].

is that Frege takes it that the two propositions are sufficiently similar that they share logical grounds and entailments. In order to demonstrate that a mathematical truth follows from a given collection of premises, it suffices to prove an analyzed version of that truth from (analyzed versions of) those premises. This is the assumption behind all of Frege's foundational work, in which the logical grounds of mathematical truths are established by giving excruciatingly careful proofs of their highly analyzed counterparts.

(ii) Frege's strategy always involves two strictly distinct stages. The first is analysis: the components of the mathematical truths in question are broken down into simpler constituents, thereby revealing a more complex structure for those truths. The second stage is proof: the now-analyzed truths are proven via strict step-by-step proofs that follow the rules of Frege's formal system. An important point for us is that the conceptual analyses themselves make no appearance within the proofs proper; we do not find any claimed equivalence between *analysandum* and *analysans* within the context of proof. The only version of a mathematical proposition that shows up in a proof is the *analysans*; the *analysandum* is left behind, and appears only in those natural-language discussions which precede and succeed the proof itself.

## 3. Two Kinds of Definition

One point which it will be important to mark out before looking more closely at Frege's procedure is his dual use of the term 'definition' and its connection to the Fregean distinction between sentence and thought.

Frege is, again, concerned primarily not with linguistic items like sentences, but rather with the nonlinguistic items expressed by sentences. Called *thoughts* after 1891, these latter are the items between which the logical relations obtain, and they are the objects of proof.

> When one uses the phrase 'prove a proposition' in mathematics, then by the word 'proposition' we clearly mean not a sequence of words or a group of signs, but a thought; something of which one can say that it is true.[5]

But in order to present a proof of a thought, Frege of course writes down a series of sentences. Each sentence in such a series, as Frege sees it, expresses a determinate thought; as we might put it, such sentences are always, for Frege, 'fully interpreted'. To keep terminology straight, let us call such a series of sentences a 'derivation,' and call the series of

[5] [Frege, 1906a, p. 401] ([1984, p. 332]). See also [Frege, 1914] ([1979, p. 206]): 'What we prove is not a sentence, but a thought. And it is neither here nor there which language is used in giving the proof.'

thoughts expressed a 'proof'. The final sentence of a derivation, then, expresses the thought proven.

In a Fregean derivation, each sentence is either a logical axiom of the formal system, an assumed premise (clearly marked as such), a definition, or a formal consequence of previous sentences in the derivation via a rule of inference of the system. Frege is extremely explicit and careful regarding the role of definitions: a definition that appears within a formal derivation is always a stipulation of notational convention; it merely announces an abbreviative convention, and is entirely eliminable.[6] Any thought provable from a given set of assumed premise-thoughts via the use of definitions is also provable from those premise-thoughts without the use of definitions. That definitions are in this sense 'empty' and eliminable underlies Frege's claim to have demonstrated the purely logical grounding of arithmetical truths via proofs which explicitly involve appeal not just to purely logical principles, but also to definitions.[7]

Frege also recognizes a second kind of thing sometimes called a 'definition', one that never appears within a derivation. Here is Frege's description of it:

> We have a simple sign with a long established use. We believe that we can give a logical analysis of its sense, obtaining a

[6] See Frege's comment at the introduction of the first definition of the *Begriffsschrift*:

> [W]e can do without the notation introduced by this proposition and hence without the proposition itself as its definition; nothing follows from the proposition that could not also be inferred without it. Our sole purpose in introducing such definitions is to bring about an extrinsic simplification by stipulating an abbreviation. [1879, §24]

See also [1884; §67; 1893/1903, p. 2]; the letter to Hilbert of Dec. 27, 1899 [Frege, 1980, p. 36]; also [1914] ([1979, p. 208]): 'it is not possible to prove something new from a definition alone that would be unprovable without it.' and ' . . . considered from a logical point of view, [definition] stands out as something wholly inessential and dispensable.'

[7] See, *e.g.*, [1884, §87]:

> I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgements and consequently *a priori*. Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one.

Also [1893, §0]:

> In my *Grundlagen der Arithmetik* I sought to make it plausible that arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition. In the present book this shall now be confirmed, by the derivation of the simplest laws of Numbers by logical means alone.

> complex expression which in our opinion has the same sense.
> ...
> The effect of the logical analysis of which we spoke will then be precisely this—to articulate the sense clearly. Work of this kind is very useful; it does not, however, form part of the construction of the system, but must take place beforehand. [Frege, 1914, pp. 210–211]

This is the kind of analytic work discussed above, in which Frege 'defines' for example the notion of *multiple of m* in terms of the ancestral of the $+m$ relation. It is also the work that forms the heart of *Grundlagen* and which sets the stage for *Grundgesetze.* The 'analysis' or 'definition' of cardinal number, for example, in terms of the extensions of second-level concepts, is part of the work that as Frege says 'take[s] place beforehand'. These expressions of analysis, Frege says at one point, ought strictly speaking not to be called 'definitions', since they are not mere terminological stipulations; nevertheless, he does on occasion use the term in this colloquial way, sometimes clarifying by referring to these things as 'analytic definitions.'[8] The separation between the two kinds of definition, however, is strict: only definitions of the empty, stipulative kind appear in derivations; those of the analytic kind are the ones that express the results of interesting mathematical or philosophical conceptual work, and never appear in derivations.

## 4. Content and Form in Frege and Hilbert

Because the analyses that Frege offers are typically directed at the contents of particular non-logical terms (*e.g.*, of such terms as 'natural number', 'successor', 'multiple of', *etc.*), the contents of those terms is all-important. That (SUM) follows logically from (P1) and (P2) turns critically on what the term 'multiple' means, as is brought to light by Frege's analysis. That the fundamental truths of arithmetic follow from principles of pure logic is due essentially to what is meant by such terms as 'natural number', 'successor', and so on; this is of course why the analysis of the contents of these terms forms an important part of Frege's project. In general, the question of whether a given thought follows logically from others is one that cannot be determined just from the overall 'form' of the sentences used to express the thoughts, but often turns to a large extent on the contents of the mathematical (or other non-logical) terms appearing in the sentences. Logical implication, in short, does not generally supervene on form.[9]

[8] '*zerlegende Definitionen*' [1914, p. 210].

[9] This leaves open the question of whether it *ever* supervenes on form. Is there a language the structure of whose sentences gives a maximally perspicuous representation

For Hilbert, things are quite different. In the *Foundations of Geometry*, Hilbert takes up questions having to do with the consistency of various sets of geometric axioms, and with the independence of certain axioms and theorems from others. The consistency of a set of statements for Hilbert, as for Frege, is a matter of its not logically entailing a contradiction; independence is similarly, for both of them, failure of logical implication. Hilbert's means of demonstrating relative consistency and independence is the now-familiar 'reinterpretation' method: to show that a sentence $A$ is independent of a set $S$ of sentences, Hilbert provides the geometric terms in $S$ and in $A$ with new interpretations taken from a background theory of real numbers, interpretations on the basis of which $A$ expresses a falsehood, while $S$'s members all express truths. Assuming the consistency of the background theory, this shows that $A$ is not logically entailed, in Hilbert's sense, by $S$.[10] The crucial point, for our purposes, is that this conception of logical entailment is one which holds entirely independently of the contents of the non-logical terms. As Hilbert explains,

> ... it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points I think of some system of things, *e.g.* the system: love, law, chimney-sweep ... and then assume all my axioms as relations between these things, then my propositions, *e.g.* Pythagoras' theorem, are also valid for these things. In other words: any theory can always be applied to infinitely many systems of basic elements. [Frege, 1980, pp. 40–41][11]

It is of the essence of Hilbert's procedure that the question of what is logically entailed by a set of sentences has nothing to do with the specific meanings had by its non-logical terms; they can take any of a variety of meanings, or indeed can remain empty, without affecting the logical implications of the set. Logical implication, for Hilbert (in *Foundations of Geometry)* supervenes on form.

of the conceptual connections that are relevant to logical implication? (In other words: is there a 'fully analyzed' language, one the contents of whose simple terms bear no entailment-relevant conceptual connections to one another?) This is a question which seems to receive no answer in Frege's work. The important point for us is that, as far as Frege is concerned, one cannot in general suppose that the language within which one is working is a language of this kind. See §5.3 below.

[10] For a more detailed discussion of Hilbert's method, see my [1996].

[11] Hilbert to Frege 29/12/1899, excerpt by Frege.

The upshot of this difference is that Hilbert's entailment relation is strictly stronger than Frege's, in the following sense: if a sentence *A* is entailed in Hilbert's sense by a set *S* of sentences, then the thought expressed by *A* is entailed in Frege's sense by the set of thoughts expressed by *S*. But not conversely. *A* can well fail to be entailed in Hilbert's sense by *S* despite the fact that the thought expressed by *A* is entailed in Frege's sense by the members of *S*. This will happen whenever the latter entailment is due to the meanings of the non-logical terms in *A* and/or in *S*, in the kind of way which, as Frege understands it, can be brought out by conceptual analysis. Note for example that the sentence (SUM) is independent, in the sense used by Hilbert in *Foundations of Geometry*, from the sentences (P1) and (P2), though of course the thought expressed by the first of these is, from Frege's point of view, logically entailed by the thoughts expressed by the latter pair.[12]

The fact that a sentence *A* is independent, in Hilbert's sense, from a set *S* of sentences is therefore no guarantee that the thought expressed by *A* is independent, in Frege's sense, from the set of thoughts expressed by the members of *S*. Whenever a conceptual analysis is required in order to bring to light the relevant logical complexity, Hilbert will find independence while Frege may find logical entailment. Similarly for consistency: the consistency in Hilbert's sense of a set of sentences is no guarantee of the consistency in Frege's sense of the set of thoughts expressed by those sentences. This, I have argued, is the central reason for Frege's rejection of Hilbert's independence and consistency proofs.

## 5. Objections

Wilfrid Hodges raises some reasons to be skeptical of this account of the Frege-Hilbert disagreement. First of all, as Hodges notes, Hilbert himself was not uninterested in conceptual analysis, and did not see a strict separation between conceptual analysis and logical investigation. Secondly, Frege's own proposed method for demonstrating independence, as sketched in [1906a], seems to imply an understanding of logical implication in which conceptual analysis plays no role. And finally, Frege makes no mention of conceptual analysis in his controversy with Hilbert. We shall take these points in turn, along with an investigation of the role Hodges proposes for conceptual analysis in Frege's work.

[12] I do not mean to imply here that Hilbert would have applied this notion of independence (and of entailment) to arithmetic. The question at issue here concerns just the conception of independence at play in his *Foundations of Geometry*.

### 5.1. *Hilbert on Conceptual Analysis*

As Hodges points out, Hilbert takes conceptual analysis, at various times, to be an important tool in the investigation of logical properties and relations. The central passage for Hodges is the following, from [Hilbert and Ackerman, 1928], whose purpose is to point out the superiority of the full first-order predicate calculus over its monadic fragment. (I have inserted Hodges's numbering at various points, for later reference.)

> The sentence [earlier discussed] read 'If there is a son, then there is a father'. The symbolic rendering of this statement in the predicate calculus is:
>
> [(1.8)] $$(\exists x)S(x) \rightarrow (\exists x)F(x).$$
>
> $S(x)$ stands for '$x$ is a son', and $F(x)$ for '$x$ is a father'. A proof of this statement is possible only if we analyze conceptually the meanings of the two predicates which occur. In the concept 'son' is contained the property 'male', on the one hand, and, on the other, the relation of child to parents; in the concept 'father', the relation to wife and child.
>
> Accordingly, if we introduce for '$x$ is male' the symbol $M(x)$ and render the predicate '$x$ and $y$ are the parents of $z$' (or more exactly, '$x$ and $y$ as husband and wife have $z$ as their child') by the symbol $C(x, y, z)$, then we define $S(x)$ by
>
> $$M(x)\&(\exists u)(\exists v)C(u, v, x).$$
>
> ('$x$ is a son' means '$x$ is male, and there is a $u$ and there is a $v$ such that $u$ as husband and $v$ as wife are the parents of $x$'.)
>
> Likewise $F(x)$ is defined by $(\exists y)(\exists z)C(x, y, z)$.
>
> ('$x$ is a father' means 'There is a $y$ and there is a $z$ such that $x$ and $y$ as husband and wife are the parents of $z$'.)
>
> If we introduce the expressions obtained for $S(x)$ and $F(x)$, the above assertion assumes the form:
>
> [(1.7)] $(\exists x)[M(x)\&(\exists u)(\exists v)C(u, v, x)]$
> $\rightarrow (\exists x)\ (\exists y)(\exists z)C(x, y, z).$
> [Hilbert and Ackermann [1928], pp. 62–63.]

At this point, Hilbert and Ackermann go on to point out the provability of this last formula.

Here we see Hilbert's interest in, and use of, conceptual analysis when enquiring into questions of provability. As Hodges sees it, this is one reason to take my account of the Frege-Hilbert disagreement to be mistaken. Reacting to that account, and specifically to the view that Frege's understanding of the role of conceptual analysis plays a significant role in his rejection of Hilbert's strategy, Hodges writes:

> I think this account is wrong. There clearly are differences between Frege and Hilbert-1899, and between Hilbert-1899 and Hilbert-1928. But they have nothing to do with the role of conceptual analysis; on this there is no reason to think that Hilbert's views changed. There is plenty of evidence that in 1900 he and Frege had no disagreement about the relationship between conceptual analysis and logical inference in geometry. [Hodges, 2004, p. 130]

Indeed, there is no reason to view Hilbert's *Foundations of Geometry* and the Hilbert-Ackermann passage as exemplifying significantly different views about conceptual analysis. In the 1928 passage just quoted, Hilbert and Ackermann are interested in the best way to express a given content (here, a 'statement' about fathers and sons). The content of the predicate terms is therefore of the essence, and conceptual analysis is relevant in finding the right expression. In 1899 on the other hand, Hilbert is expressly unconcerned with specific geometric contents, *i.e.*, with the meanings of such non-logical terms as 'point', 'between', *etc.* His concern is exclusively with the general axiomatic framework within which different such contents can be fitted. The consistency question of 1899 has explicitly to do with the satisfiability of such general frameworks, which is to say that it has to do entirely with the forms of the axiom sets in question, and not with their specific subject matter. No conceptual analysis can have any bearing on the questions he asks here. Rather than indicating a significant change in view regarding conceptual analysis in general, the difference between the 1899 consistency proofs and the brief 1928 passage marks a difference between two entirely different kinds of project.

In any case, the issue of the evolution of Hilbert's views on conceptual analysis is beside the point with respect to our current concern. For this much is beyond controversy: that Hilbert's 1899 consistency demonstrations establish a kind of consistency that holds independently of the meanings, if any, of the non-logical terms appearing in the sentences in question. No conceptual analysis of any specific meanings those terms might have is relevant to the consistency questions he raises (and answers) there.

## 5.2. *Analysis and Entailment*

Once again, the role of conceptual analysis in Frege's work, as I have sketched it, is to help reveal relations of logical entailment which hold in virtue of content, and hence which are not immediately evident on the basis of the surface structure of the pre-analytic sentences in question. The analysis of what is expressed by a sentence $A$ can yield a new more highly structured sentence $A'$; the subsequent derivation of $A'$ from premises shows, as Frege sees it, that the original thought expressed by $A$ follows logically from those premises. This, I take it, is the use to which Frege puts analysis in all of his foundational work.

Hodges, however, argues that we should not see conceptual analysis in Frege's work as a means of revealing logical connections that already obtain (perhaps unnoticed) between the thoughts in question. We should instead take Frege's analyses as providing *new* thoughts, new 'axioms' which express the newly discovered conceptual connections.

We can illustrate the difference as follows, taking as our example the transitivity of the relation of equinumerosity. Given these three sentences,

(E1) Equinumerous (F, G),
(E2) Equinumerous (G, H),
(E3) Equinumerous (F, H),

we can ask whether (the thought expressed by) (E3) follows logically from (the thoughts expressed by) the pair of (E1) and (E2). For Frege, the answer turns on an analysis of the sense of 'equinumerous' in terms of 1-1 mappings.[13] As I have portrayed Frege's general procedure above, the analysis will yield sentences

(E1′) $\exists R(\forall x(Fx \rightarrow \exists! y(Gy \& Rxy)) \& \forall y(Gy \rightarrow \exists! x(Fx \& Rxy)))$,
(E2′) $\exists R(\forall x(Gx \rightarrow \exists! y(Hy \& Rxy)) \& \forall y(Hy \rightarrow \exists! x(Gx \& Rxy)))$,
(E3′) $\exists R(\forall x(Fx \rightarrow \exists! y(Hy \& Rxy)) \& \forall y(Hy \rightarrow \exists! x(Fx \& Rxy)))$,

the third of which is derivable from the first two, thus demonstrating that the thought expressed by (E3) does follow logically from the pair of thoughts expressed by (E1) and (E2).

As Hodges sees it on the other hand, a Fregean analysis will deliver a sentence like

(A) $\forall X \forall Y(\text{Equinumerous } (X, Y) \text{ iff } \exists R \forall x(Xx \rightarrow \exists! y(Yy \& Rxy)) \& \forall y(Yy \rightarrow \exists! x(Xx \& Rxy)))$.

Instead of deriving (E3′) from (E1′) and (E2′), if we are interested in the transitivity of equinumerosity, we will (if we follow Hodges's Frege) derive (E3) from (E1), (E2), and (A). The significance here is as follows: if appeals to conceptual analysis bring in new content (in the form of

[13] See, *e.g.*, [Frege, 1884, §§63, 73].

new 'axioms'), rather than simply explicating the thoughts already under investigation, then we cannot view conceptual analysis as a means of drawing out logically significant features of content that are already contained in the thoughts in question. Appeals to conceptual analysis in a foundational project like logicism will be problematic, since on this reading such appeals will have the result that the proven thoughts are shown to follow logically not (as Frege has it) from purely logical principles, but instead from logical principles together with the 'new' axioms revealed by analysis.[14]

Hodges's talk of 'new axioms' is taken from Frege's discussion in [1914] of the two kinds of definition discussed above, the 'stipulative' and the 'analytic'. Hodges's account of this discussion, quoting Frege at (1.17) on stipulative definitions and (1.18) on analytic definitions, is as follows:

> [Frege] distinguishes two cases [of definition]:
>
> (1.17) (1) We construct a sense out of its constituents and introduce an entirely new sign to express this sense.
>
> ...
>
> (1.18) (2) We have a simple sign with a long established use. We believe that we can give a logical analysis of its sense. ... what we should here like to call a definition is really to be regarded as an axiom.
>
> This is the case discussed by Hilbert and Ackermann. We analyse 'son' and 'father', and we see ('by an immediate insight', as Frege puts it) that the definitions of these concepts in terms of $M$ and $C$ are true. In this case it is not at all clear to me why Frege should regard (1.8) and (1.7) as expressing the same thoughts, since the two definitions (1.11), which are needed to get from one of (1.7) and (1.8) to the other, express significant thoughts and not just a notational convenience. But the key point in this case is that Frege regards the conceptual analysis not as providing new sentences $s'$ and $\Sigma'$ to express the same thoughts as before, but as supplying a new premise in the form of an axiom expressing a relation between concepts. [Hodges, 2004, pp. 137–138][15]

[14] See footnote 7 above.

[15] The sentences (1.11) to which Hodges refers are

$$\forall x(S(x) \leftrightarrow M(x) \wedge \exists u \exists v C(u, v, x)) \text{ and } \forall x(F(x) \leftrightarrow \exists y \exists z C(x, y, z)).$$

The question whether Frege would view *e.g.* (1.8) and (1.7) as expressing the same sense is an important question in its own right, but not crucial to the debate between Frege and Hilbert. Let us digress briefly to clarify this point.

Frege often talks as if successful analysis is generally sense-preserving, so that such pairs as (SUM) and (SUM′), (1.7) and (1.8), *etc.* express the same thought. Here for example we have his often-repeated claim to have proven *truths of arithmetic* when what he has actually proven are highly analyzed versions thereof, his claim to have proven what is expressed by (SUM) via a derivation of (SUM′), and so on. In this vein we find also his discussion of analysis in [1914] as already quoted above; filling in some of the gaps in Hodges's quotation of it ((1.18) above), the passage begins:

> We have a simple sign with a long established use. We believe that we can give a logical analysis of its sense, obtaining a complex expression which in our opinion *has the same sense.* We can only allow something as a constituent of a complex expression if it has a sense we recognize. The sense of the complex expression must be yielded by the way in which it is put together. *That it agrees with the sense of the long established simple sign is not a matter for arbitrary stipulation, but can only be recognized by an immediate insight.* [Frege, 1914] ([1979, p. 210]), (emphasis added).

Here, we seem to have quite explicitly a depiction of successful analysis as sense-preserving.[16] Nevertheless, there are other Fregean texts which give a different picture. Frege's well-known discussion of senses in [1892], for example, outlines a considerably more fine-grained criterion of identity for thoughts, on which two sentences can express the same thought only if they are fairly obvious synonyms. The non-triviality of Frege's analyses mean that on this latter conception, analysis cannot in general be intended to preserve sense.[17]

Here I think it is fair to say that Frege's texts as a whole are simply not clear: he has no univocal criterion of sense identity, and consequently no clear answer to the question of whether successful analyses preserve

[16] In this vein, see also, *e.g.*, [1906b] ([1979, pp. 197–198]).

[17] Michael Dummett has raised the question of the criteria of analytic adequacy in Frege's project; see, *e.g.*, [Dummett, 1991, pp. 30–31, 143]. As Dummett sees it, Frege's definitions ought to 'come as close as possible to capturing the existing sense'. See also [Currie, 1982a] for the claim that the analyses are required to preserve sense, and for discussion of difficulties with this requirement.

sense.[18] But one can see why this point was not centrally important to Frege. What is important for Frege's purposes is that the sentence delivered at the end of analysis expresses a thought which can stand in for, with respect to logical investigation, the thought expressed by the original sentence. That is to say: analysis is clarificatory, in that the thoughts expressed by *analysandum*-sentence and *analysans*-sentence are sufficiently similar that the logical grounds and implications of one are a sure guide to those of the other.[19] This is the assumption that underlies all of Frege's appeals to conceptual analysis in the course of his foundational investigations.

To return to Hodges's worry: the passage quoted by Hodges above at (1.18) ends as follows:

> No doubt we speak of a definition in this case too. It might be called an 'analytic definition' to distinguish it from the first case. But it is better to eschew the word 'definition' altogether in this case, because what we should here like to call a definition is really to be regarded as an axiom. [Frege. 1914] ([1979, p. 210])

That we should strictly speaking not use the word 'definition' in such cases is clear: for Frege, definitions strictly so-called are stipulative. But why call these expressions of analysis 'axioms'? Frege's idea here is presumably the simple one that the equivalence between *analysans* and *analysandum* expressed in such a statement is one that is not susceptible of demonstration; it is 'axiomatic' in the sense of an unprovable truth, one that is recognizable only on the basis of, as Frege puts it, 'immediate insight'.

Might Frege *also* mean, by referring to the analysis as an 'axiom', what Hodges has suggested, namely that it expresses new content not already contained in the propositions involving the *analysanda*, so that it must be counted amongst the premises of proofs involving those propositions? The idea here would be that, *e.g.*, the biconditionals linking ordinary arithmetical notions with Frege's proposed analyses thereof



[18] Frege's project is most easily described if one keeps to the more coarse-grained of his identity criteria, and hence with the idea (as expressed in the [1914] passage) that analysis preserves sense. Then we can say that a proof of the *analysans* demonstrates the logical grounds of the *analysandum* because *analysans is analysandum*. Speaking alternatively in accordance with the more fine-grained criterion of sense identity, one must say that the proof of the *analysans* demonstrates the logical grounds of the *analysandum* because of a less strict equivalence between the two, one which is at least as strong as logical equivalence. Hodges is right to point out that my tendency in [1996] to describe the project in the first way is under-determined by the texts.

[19] See the letter to Husserl of Oct. 30/Nov. 1, 1906 [1980, p. 67].

would have to appear in any proofs that demonstrate the logical grounds of arithmetical truths.

The difficulty with this proposal as an interpretation of Frege is that no Fregean proof includes such a biconditional. Frege's analytic definitions never appear in his proofs. As he says explicitly, and as his procedure uniformly exhibits, the work of analysis is preparatory: it takes place before the proofs proper. This is a crucial point for Frege. If analytic definitions expressed new content in the way proposed by Hodges, then Frege's claim to have been engaged in demonstrating the purely logical grounds of the truths of arithmetic would be straightforwardly false: he would have been demonstrating the grounding of arithmetic in purely logical truths *together with* the claims expressed by the new axioms. His claim to have shown, *e.g.*, that (SUM) follows logically from (P1) and (P2) would be false; he would have shown rather that (SUM) follows from those premises together with a new axiom. And so on. In short, Hodges's 'new axioms' reading of Frege's analyses seems to be contradicted both by Frege's practice and by his explicit discussion of that practice.

### 5.3. *What Frege Does and Does Not Say*

Hodges notes two important aspects of Frege's texts, both of which might be taken to conflict with the idea that Frege views conceptual analysis as a means of drawing out logically relevant features of the contents of non-logical terms. The first of these is that Frege's own proposed method for demonstrating independence gives no role to conceptual analysis. The second is the fact that Frege does not discuss conceptual analysis in his replies to Hilbert. We shall take these in turn.

In [1906a], Frege briefly sketches out a proposal for demonstrating independence.[20] The central idea of the proposal turns on a mapping $\mu$ from terms of a language $L$ to terms of $L$, one which preserves syntactic type[21] and maps 'logical' terms to themselves. $L$ is fully interpreted, and is 'logically perfect' just in the sense that each well-formed sentence expresses a determinate thought. The mapping $\mu$ will induce a map $\mu'$ from sentences to sentences. Given a set $P$ of premise-sentences all expressing truths, and a conclusion-sentence $C$ expressing a truth, we then assess the independence of $C$ from $P$ by looking at the set $P'$ of sentences to which the members of $P$ are mapped by $\mu'$, and the sentence $C'$ to which $C$ is mapped by $\mu'$. If each member of $P'$ expresses a truth, while $C'$ expresses a falsehood, then our original $C$ (and the thought it

[20] [1906a, §iii], esp. pp. 426–428 ([1984, pp. 337–339]).

[21] So that, *e.g.*, terms for first-level functions of one argument are mapped to the same, *etc.*

expresses) is independent of the set $P$ (and the set of thoughts expressed by its members).

Frege follows his sketch with the warning that 'it still needs more precise formulation', and that difficult questions remain to be answered. But he never returns to the issue.

Perhaps the most striking feature of the proposed test itself is that it can be used to demonstrate 'independence' only in the sense of 'formal independence', which is to say that it can show just that $C$ cannot be derived from $P$ via the kinds of purely formal laws of logic that both Frege and Hilbert employ in the course of proof. It cannot show, generally speaking, that $C$ is independent of $P$ in the sense discussed above, which is to say that no further conceptual analysis will reveal highly analyzed equivalents $C^*$ and $P^*$, the first of which is provable from the second. In cases in which this kind of further conceptual analysis is possible, Frege's proposed test will fail in exactly the same way that Hilbert's test fails. Indeed, the two are essentially equivalent, as applied to a fully interpreted language of the kind discussed by Frege.[22] Both will successfully demonstrate independence if the meanings of the terms in question are already 'fully analyzed', and both will fail if any further analysis is possible.

If Frege really means the 1906 proposal as a fully general test of independence, then there is a conflict between this proposal and the account offered here of Frege's understanding of the logical relations. For as I have presented his views, Frege takes logical implication, and hence independence, to turn not just on form, but also on content. And the 1906 proposal presupposes a purely formal understanding of these logical relations.

There are real difficulties, though, with taking the 1906 proposal as representative of Frege's general views about logic and form. Most important is the equivalence just noted between the proposal and Hilbert's own method: if Frege is seriously proposing to test for independence in the way sketched in 1906, then he is in no position to criticize Hilbert. Similarly, Frege's own views about logical implication as these emerge throughout all of his positive work, on which logical implication has everything to do with the contents of such terms as 'successor' and 'multiple', directly contradict the purely formal account of implication that seems to lie behind the 1906 proposal.

There are two ways of making sense of Frege's proposal, given the surrounding body of work. The first is to suppose that Frege had in mind the application of his independence test in a very restricted area, specifically to languages whose non-logical terminology is, as

[22] The equivalence requires that $L$ include names for all the objects and functions to which Hilbert might appeal in constructing an interpretation.

above, 'fully analyzed.' The difficulty of spelling out this condition of complete analysis is perhaps part of what Frege has in mind when he notes that a 'more precise formulation' of his proposal is still needed. [1906a, p. 429] ([1984, p. 339]) The second is that Frege is confused here, and specifically that he has briefly confused the purely formal character of the fundamental *laws* of logic (which he acknowledges) with the considerably stronger requirement of the purely formal character of logical implication (which he generally denies).[23]

However we explain the odd 1906 proposal, it must be said that Frege himself seems not to have been happy with it. He never follows up on either the proposal or the open questions surrounding it, and by 1910 seems to have concluded that independence cannot be demonstrated. In Frege's notes to Jourdain of 1910, we find:

> The indemonstrability of the axiom of parallels cannot be proved. If we do this apparently, we use the word 'axiom' in a sense quite different from that which is handed down to us. *Cf.* my essays 'On the Foundations of Geometry' ... [24]

Frege's reader is referred here, presumably, not to the brief positive proposal at the end of the second essay, but rather to the discussion constituting the bulk of both essays referenced, *i.e.*, to the criticism of Hilbert. That the second essay referred to contains the outline of a positive solution to the question of independence demonstrations is not even mentioned. Similarly, the posthumously published 'Logic in Mathematics', written in 1914, contains a rehearsal of the early criticism of Hilbert's independence proofs, and no mention of the 1906 proposal.

In short, the brief discussion of independence in [1906a, §iii] is certainly odd. It is difficult, however, to see in this discussion a coherent reason for taking Frege to have a serious or sustained view of the logical relations on which entailment, consistency, and independence are to be understood purely formally.

[23] In this vein, see Frege's curious note about his proposal that: 'One might call it an emanation of the formal nature of logical laws' ([1906a, p. 427]([1984, p. 337]). The fundamental logical laws clearly are 'formal' for Frege, in that they apply independently of the contents of non-logical terms. But this does not imply, of course, that the logical *relations*, *e.g.* entailment, are formal in this sense. (For, as is clear in the logicist reduction, the fact that the thought expressed by a sentence $S$ is logically entailed by the thoughts expressed by a set of premise-sentences $P$ is not reducible to the formal derivability of $S$ from $P$.) While the formal nature of entailment (which formality Frege rejects) would underwrite the proposed 1906 test, the formal nature of the *laws* (which formality he accepts) would underwrite such a test only for a fully analyzed language.

[24] [Frege, 1980, p. 183n]. For an enlightening discussion of this passage and its relation to the 1906 text, including an argument that Frege is not, in 1910, expressing the blanket rejection of independence proofs suggested here, see [Tappenden, 2000].

The final textual issue of this section concerns Frege's silence about conceptual analysis in his replies to Hilbert. Hodges takes the following view:

> The fact that in the whole correspondence with Hilbert and the sixty-odd pages of [[1903]] and [[1906a]] Frege never once says that he finds Hilbert's independence proofs unacceptable because they ignore the possibilities of conceptual analysis is one strong argument against Blanchette's thesis. [Hodges, 2004, p. 139]

I do not agree. The crucial difference between Frege and Hilbert, as I have argued, is that the relation of logical entailment that concerns Frege is sensitive to the meanings of non-logical terms, while the relation of logical entailment that concerns Hilbert is not. The reason for discussing conceptual analysis, here and in [Blanchette, 1996], is that Frege's use of conceptual analysis illustrates how, exactly, these meanings and their relationships are brought to bear in Frege's positive demonstrations of logical entailment. But there is no reason for Frege himself to engage in, or to discuss, conceptual analysis in his controversy with Hilbert. Frege has, in these texts, no specific claims to make about conceptual analysis in the field of geometry. His claim is not that Hilbert has failed to notice some specific conceptual analyses which he, Frege, has discovered.[25] His claim is rather that Hilbert's general methodology must be a failure because it establishes nothing about the real subject matter of geometry, *i.e.*, about the thoughts expressed by geometric axioms and theorems when the geometric primitives are given their ordinary, geometric meaning. For, again, Hilbert's method will miss out on all logical implications that turn on those specific meanings.

But let us look at what Frege says.

Frege's correspondence with Hilbert and his two 'Foundations of Geometry' essays are largely concerned with issues which, though critical for Frege, were simply pedantic from Hilbert's point of view. Frege points out for instance that Hilbert is unclear about whether his 'axioms' and 'theorems' should be understood as sentences or as thoughts, and notes Hilbert's tendency to talk as if there is just one 'axiom' in question despite shifting back and forth between different interpretations of the relevant terms. He complains that Hilbert's individual 'definitions' fail to determine unique references for their *definienda*, and that the attempt to define the geometric terms (implicitly) via collections of axioms and so-called definitions founders on the

[25] I might add here that Hodges's characterization of my account as one on which 'Hilbert's agenda in 1899 missed a crucial point that Frege noticed' [2004, p. 130] is not correct. Frege and Hilbert are, I take it, simply interested in quite different things.

existence of multiple interpretations. He notes that if we take the sentences in question to be only partially interpreted (as we would put it), then questions of consistency and independence cannot arise, since sentences so understood express no determinate thoughts. And so on.

We can see why all of these issues would loom large for Frege. As Frege sees it, the different thoughts expressible by a given sentence under its geometric and its real-number interpretations may well have radically different logical properties, so that the set of thoughts expressed by a set of sentences under one of its interpretations may be consistent, while the set expressed under the other interpretation is inconsistent. Hence the indeterminacy of meanings involved in Hilbert's treatment entirely undermines the idea that one is asking (or answering) meaningful questions about the consistency and independence of any particular sets of thoughts. For Hilbert on the other hand, the distinctions Frege insists upon are entirely beside the point. Because Hilbert is concerned with notions of consistency and independence with respect to which the specific meanings of non-logical terms have absolutely no bearing, it makes no difference whether one has in mind a geometric or a real-number interpretation of the non-logical terms; it makes no difference whether one takes the results in question to hold of sentences or of thoughts, and there is no reason (indeed quite the contrary) to expect 'definitions' to pin down unique senses or references for their *definienda*. It is no wonder that Hilbert did not find Frege's points worth pursuing.

Considerably more interesting are those parts of the texts in which Frege takes a more constructive stance, trying to state clearly what he takes it that Hilbert has achieved, and what he has not. Instead of taking Hilbert's sentences to be misguided attempts to express specific thoughts, in these passages Frege characterizes Hilbert's approach, accurately, as that of adopting a 'higher standpoint from which Euclidean geometry appears as a special case of a more general theory.'[26] That is, in taking the words 'point', 'line', *etc.* either schematically or as bound variables,[27] a set $\Sigma$ of geometric sentences can be seen to characterize a general higher-order condition satisfiable by appropriately chosen series of entities, not necessarily geometric ones.[28] Frege's way of putting the point is that when $\Sigma$'s geometric terms are assigned no definite sense, then we

[26] See the letter to Hilbert of 27 December, 1899 ([1980, p. 38]); the letter to Hilbert of 6 January, 1900 ([1980, pp. 44, 47–48]); [1903, p. 374–375] ([1984, pp. 283–284]).

[27] *I.e.*, in Frege's terminology, one takes them either as empty symbols or as letters that 'express generality;' see, *e.g.*, [1906a, p. 389–390] ([1984, pp. 319–321]).

[28] See, *e.g.*, [1903, pp. 324, 374] ([1984, pp. 277–278, 283–284]), [1906a, p. 402] ([1984, p. 332]).

have simply what he calls 'pseudo-axioms', while each way of assigning specific senses to $\Sigma$'s geometric terms gives rise to a determinate 'special geometry', *i.e.*, to a determinate set of thoughts.[29] Assigning the ordinary geometric senses to $\Sigma$'s terms gives rise to a set of thoughts of Euclidean geometry (*i.e.*, to the set of thoughts whose consistency Frege takes to be at issue); assigning Hilbert's senses gives rise to a set of thoughts about real numbers, and so on. In the interesting cases, the first set of thoughts, the geometric ones, includes some falsehoods (since otherwise consistency would not be an issue), while the second, parallel set of thoughts about reals will consist entirely of truths.[30] Hilbert's constructed entities immediately show the satisfiability of the defined higher-order condition and the truth of the 'new' set of thoughts concerning real numbers. The central question from Frege's point of view reduces to the question of whether these results imply the consistency of the original, geometric, set of thoughts.[31]

Before looking at Frege's answer, let us fix some terms. Where $\Sigma$ is a set of sentences as above, and taking as given a particular one of Hilbert's reinterpretations involving the real numbers:

- Let $T_G$ be the set of thoughts expressed by $\Sigma$ when its terms take their ordinary geometric senses. Thus $T_G$ will be a set of what Frege calls 'axioms of geometry'. We shall need to refer below also to the conjunction of these thoughts; call this conjunction '$\wedge T_G$'.
- Let $T_R$ be the set of thoughts expressed by $\Sigma$ when its terms take Hilbert's assigned meanings. $T_R$ is therefore, as Frege would put it, a set of thoughts about the real numbers. Similarly, $\wedge T_R$.
- Let $T_\exists$ be the existential thought expressed by the result of conjoining $\Sigma$'s members and existentially quantifying over the geometric terms. This last thought essentially says that there is some series of concepts, relations, *etc.*, that will satisfy the schematically understood $\Sigma$.

As above, Hilbert's reinterpretation immediately shows (assuming the truth of the background theory) that $T_R$'s members are all true, hence that $T_R$ is consistent, and further that $T_\exists$ is true and hence consistent. Frege's question is whether we can infer the consistency of $T_G$ from

[29] See, *e.g.*, the letter to Hilbert of 6 January, 1900 ([1980, p. 48]), [1906a, pp. 382, 386–395] ([1984, pp. 313, 317–326]).

[30] Assuming the truth of the background theory of real arithmetic.

[31] See, *e.g.*, [1906a, p. 402] ([1984, pp. 332–333]), the letter to Hilbert of January 6, 1900 ([1980, pp. 43 ff.]).

these results. And his answer is a resounding 'no'.[32] In the January 1900 letter to Hilbert, referring to $T_\exists$ as a 'general proposition' and each of its instances, *e.g.* $\wedge T_R$ and $\wedge T_G$, as the 'particular propositions ... contained in it', Frege writes:

> If a general proposition contains a contradiction, then so does any particular proposition that is contained in it. Thus if the latter is free from contradiction, we can infer that the general proposition is free from contradiction, but not conversely. (Letter of 6 January, 1900 ([Frege, 1980, p. 47]))

The first point is straightforward: if $T_\exists$ is inconsistent, then every instance of it, including $T_R$ and $T_G$, will be as well. So if either of these special cases has been shown consistent (as indeed one of them has by Hilbert), we can conclude that $T_\exists$ itself is consistent. The second point (the 'not conversely') contains the disagreement with Hilbert: the consistency of $T_\exists$ does not imply the consistency of $T_G$.

Similarly in the same letter, referring to instances like $T_R$ and $T_G$ as 'special geometries' and the generalized $T_\exists$ as a 'general axiom', Frege says:

> [G]iven that the axioms in special geometries are all special cases of general axioms, one can conclude from lack of contradiction in a special geometry to lack of contradiction in the general case, but not to lack of contradiction in another special case. (Letter of 6 January, 1900 ([Frege, 1980, p. 48]))

As above: from the consistency of $T_R$, we can straightforwardly infer the consistency of $T_\exists$. But—and here again is the source of disagreement with Hilbert—we cannot infer the consistency of $T_G$ from that of $T_R$.

A similar sentiment appears in Frege's letter to Liebmann in the same year (1900), in which Frege discusses the equivalent question of independence:[33]

32

> ... Mr. Hilbert appears to transfer the independence putatively proved of his pseudo-axioms to the axioms proper ... This would seem to constitute a considerable fallacy. And all mathematicians who think that Mr. Hilbert has proved the independence of the real axioms from one another have surely fallen into the same error. [1906a, p. 402] ([1984, p. 333])

33 For Frege as for Hilbert, independence and consistency are two sides of the same coin: $\varphi$ is independent of the set $\Pi$ iff $\Pi \cup \{\sim \varphi\}$ is consistent. Note that this sense of independence is weaker than another common sense, in which $\varphi$ is independent of $\Pi$ iff both $\Pi \cup \{\sim \varphi\}$ and $\Pi \cup \{\varphi\}$ are consistent.

> I have reasons for believing that the mutual independence of the axioms of *Euclidean* geometry cannot be proved. Hilbert tries to do it by widening the area so that Euclidean geometry appears as a special case; and in this wider area he can now show lack of contradiction by examples; but only in this wider area; for from lack of contradiction in a more comprehensive area we cannot infer lack of contradiction in a narrower area ... (Letter to Liebmann of 29 July, 1900 ([Frege, 1980, p. 91]))

*Why* does Frege take it that the consistency of $T_R$ and of $T_\exists$ is insufficient to guarantee the consistency of the very similar $T_G$? It is worth noting here that from Hilbert's point of view, the inference in question—*i.e.*, from the consistency of the axioms as interpreted over the reals to their consistency as interpreted over geometric entities—is entirely unproblematic. For as far as Hilbert is concerned, the consistency question has to do with what these two 'special cases' have in common, namely the overarching structure exemplified by each, and described by the set $\Sigma$ when its terms are taken schematically. But if the above account of Frege is correct, the crucial point is that the consistency question for $T_G$ does *not* have to do just with the form shared by $T_G$ and $T_R$, and described by $T_\exists$. It has to do additionally with what distinguishes $T_G$ from $T_R$, namely the specific geometric concepts, objects, and relations with which $T_G$ is concerned.

Frege completes the sentence we have left hanging above as follows:

> ... from lack of contradiction in a more comprehensive area we cannot infer lack of contradiction in a narrower area; *for contradictions might enter in just because of the restriction* (Letter to Liebmann of 29 July 1900 ([Frege, 1980, p. 91]), emphasis added)

You cannot infer the consistency of $T_G$ from that of $T_\exists$, or of $T_R$, since the specific subject matter (the 'restriction') introduced by $T_G$ may well give rise to contradiction. A similar sentiment is expressed by Frege in his earlier 'On Formal Theories of Arithmetic'. Here Frege is discussing different 'special cases' of a set of schematically understood arithmetical rules, one of which is obtained by—as we would put it—interpreting those rules over the natural numbers, and the other by interpreting them over the complex domain. The question is whether the consistency (or the truth) of the former set of thoughts entails the consistency of the latter:

> A proof of non-contradictoriness, then, cannot be given by saying that these rules have been proved as laws for the positive whole numbers and therefore must be without

> contradiction; for after all, they might conflict with the peculiar properties of the higher numbers, *e.g.*, that of yielding $-1$ when squared. And in fact, not all rules can be retained ... It is therefore evident that in virtue of the peculiar nature of the complex higher numbers there may arise a contradiction where so far as the positive whole numbers are concerned, no contradiction obtains. [Frege, 1885, p. 102] ([1984, p. 119])

I take it that Frege's point is clear: we can introduce a contradiction in the move from $T_R$ to $T_G$ simply in virtue of the move from one subject-matter to another. For as always, for Frege, the question of whether a contradiction follows from a collection of thoughts will depend in part on the subject matter of those thoughts. It will depend, that is, on the senses had by the non-logical terms of the sentences in question.

## 6. Conclusion

When we ask whether a set of sentences (or the set of thoughts expressed by them) is consistent, there are a number of different things we might mean. Following Hilbert, we might be enquiring about the existence of an interpretation of the non-logical terms which will satisfy each of the sentences. Or we might be interested in the kind of formal consistency that follows immediately from the existence of such an interpretation. Following Frege on the other hand, we might be interested in whether a contradiction is logically entailed by the specific set of thoughts expressed. The further question we have dealt with above is the question of whether consistency in either of the first two senses implies consistency in the third: does Hilbert-consistency imply Frege-consistency? The answer to this question turns on the issue of whether logical implication, for Frege, generally supervenes on the form of the sentences in question. And the answer to this final question is a clear 'no'. The logical implications of a set of thoughts are not restricted to what is formally derivable from a specific set of sentences expressing those thoughts; they may include as well thoughts that are provable only after non-trivial conceptual analysis. This semantic robustness in Frege's conception of logical implication holds just as much when asking about consistency in general as it does when asking specifically about the grounds of arithmetical truth.

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