Frege on Caesar and Hume's Principle

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1 Introduction

Frege's reaction to Russell's Paradox was, in the end, to abandon his logicist project. That he did so without seriously considering a now widely-discussed alternative, the alternative of grounding arithmetic in pure logic together with Hume's Principle, is significant. Clearly there is something in such a grounding that Frege took to be incompatible with either his view of the nature of logic or of arithmetic, or his view of acceptable ways to pursue a reduction of one to the other. The purpose of this essay is to investigate what, exactly, Frege took the difficulty to be, and what we can learn from this about his view of the nature of mathematics and of logic. I will be concerned to argue that there are some important difficulties with what might be called the 'standard' answer to these questions, an answer according to which the central difficulty with the Hume's-Principle strategy is to be found in the so-called "Caesar passages" of *Grundlagen.* The alternative answer offered here involves a different account of the importance of the Caesar passages, and a revisionary, though I think better-supported, account of Frege's view of the role of identity-statements in the reduction. Finally, I'll suggest that the alternative reading attributes to Frege a more plausible view of mathematical discourse than does the standard reading.

2 Background

Frege's attempt to prove the fundamental truths of arithmetic from purely logical principles involved crucial appeal to the *extensions* of concepts. Both the pre-formal account in *Grundlagen* and the formal account in *Grundgesetze* employ a term-forming operator whose output is a collection of singular terms intended to refer to extensions, and a principle of extensionality for those extensions.[1] The most important role of extensions in the project

[1] The extensions of *Grundgesetze* are known as 'value ranges' (*Wertverläufe*). The principle of extensionality (that the extension of concept F is identical to the extension of concept G iff exactly the same

is Frege's identification of numbers as the extensions of concepts.

What Frege learned from Russell in 1902 was that his conception of extensions was incoherent: there cannot be objects which both answer to the singular terms just mentioned and obey the principle of extensionality. Frege's reaction to the discovery was that it entirely undermined his attempt to provide a logical foundation for arithmetic. He abandoned the project for this reason, holding in the end that logic is insufficient to furnish us with the objects needed for arithmetic.[2]

The neo-logicist project, following on from Crispin Wright's [?], seeks to establish that Frege's abandonment of logicism was hasty, in the sense that a logicism very much in the spirit of Frege's project can be successfully pursued by making a straightforward adjustment to the foundation Frege attempted to give. That adjustment is to discard the notion of 'extension' altogether, and to provide reference to numerical singular terms not by stipulating that those terms refer to extensions, but by characterizing the nature of those references directly by appeal to an arguably analytic principle governing numbers. That principle, now known as "Hume's Principle," is the principle that the number that belongs to a concept F is the same as the number that belongs to a concept G iff the items falling under those concepts can be correlated one to one:[3]

(HP) NxFx = NxGx iff there is a bijection from the F's to the G's.

The crucial fact about HP is that, as noted by Charles Parsons and demonstrated in detail by Crispin Wright, the addition of HP to an unproblematic system of second-order logic provides resources sufficient for the proof of those fundamental principles of natural-number arithmetic that Frege himself sought to prove.[4] Indeed, Frege's own strategy turns on just this point: Frege's proof-sketches in *Grundlagen* and rigorous proofs in *Grundgesetze* can be understood as comprising two separable steps: first, the proof of HP from principles (including extensionality) that Frege took to be purely logical, and second, the proof from HP of the fundamental truths of arithmetic. As Richard Kimberly Heck has demonstrated, not only is it the case that Frege's proof of those fundamental truths relies on no further essential appeal to extensions once HP has been established, but importantly that Frege's construction shows that he himself knew this. In short, as Heck has put it, "there was no *formal* obstacle to the logicist program, even after Russell's discovery of the contradiction, and Frege knew it."[5]

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things fall under those concepts, or more generally that the extension of F = the extension of G iff F and G give the same values for every argument) is entailed by Basic Law V of *Grundgesetze*; it is not stated explicitly in *Grundlagen*, but is assumed throughout the proof-sketches given there.

[2] See the unpublished fragment entitled "A new attempt at a foundation for arithmetic," [?] 278-281.

[3] The name, introduced by George Boolos, is prompted by Frege's mention in *Grundlagen* §63 of Hume's recognition that equinumerosity can be characterized in terms of one-one correlation. See [?]. On the infelicity of that name, see [?].

[4] [?], [?]. For helpful discussion of the history here, see [?], [?], and [?]

[5] [?] p 92

This raises the following important question: Why, in the aftermath of the paradox, didn't Frege do as just suggested: why didn't he excise appeal to the problematic extensions, adopt HP as a fundamental principle, and proceed as before? Without an understanding of this, we certainly lack an understanding of Frege's conception of his own project and of his view of the constraints on its successful completion.

The standard answer to this question is essentially that Frege felt that such a strategy would fail to determine the references of numerical terms. That it would fail to do so is due, according to this account, to the fact that the HP strategy provides truth-conditions for insufficiently-many kinds of identity sentences involving terms of the form "the number of F's."[6] Appeal to HP will determine that each sentence of the form

"The number of F's = the number of G's"

shares the truth-value of the corresponding sentence

"There is a bijection mapping the F's to the G's,"

but will not provide truth-conditions for sentences of the form

"The number of F's = q,"

for q not of the form "the number of G's." In failing to do this, it fails, on this account, to determine the reference of terms of the form "the number of F's." Finally, because numeral-reference is stipulated by appeal to terms of the form "the number of F's," the HP strategy would, on this account, fail to determine the reference of the numerals. As Dummett puts it, the difficulty Frege saw with grounding arithmetic in HP was that "it affords us no possibility of determining the truth or falsity of a sentence like 'the number of planets is Julius Caesar,' and thereby fails to determine the references of numerical terms."[7]

In what follows, it will be argued (i) that the texts usually thought to support this understanding of Frege's failure to adopt the HP strategy do not in fact do so, and (ii) that by the time he learned of Russell's paradox, Frege did not hold that a successful account of arithmetical discourse must provide a semantics or truth-conditions for such non-arithmetical sentences as "the number of planets is Julius Caesar." An alternative account of Frege's failure to adopt the HP strategy will be proposed.

3 *Grundlagen* §§55-56: The Recursive Analysis

Frege sets the groundwork for his eventual analysis of numerical discourse by first presenting two inadequate analyses, and explaining their inadequacies. All of the analyses,

[6] This, as I understand it, is the view of Michael Dummett (see [?] esp Ch 13), Richard Kimberly Heck (see [?] and [?]), Crispin Wright ([?] esp ch. 3), and Bob Hale ([?])

[7] [?] 156-7

both rejected and successful, focus on what Frege calls "assignments of number," i.e. on statements like "There are nine planets," which he understands as involving the claim that a particular object (here, the number nine) bears a particular relationship (the relation of numbering) to a given concept (in this case, the concept *planet*). In his canonical mode of expression, each such statement is equivalent to an instance of the form

(An) "The number n belongs to the concept F."

It is crucial to Frege's overall strategy to treat the place occupied by "n" in this schema as open to quantification, i.e. to treat the position marked by the ellipsis in

(A) The number ... belongs to the concept F

as one into which it is possible to quantify with an object-level quantifier. Accordingly, his analyses of e.g.

($A0$) The number 0 belongs to the concept F

and

($A1$) The number 1 belongs to the concept F

must treat these sentences as predicating the content of the open sentence (A) of the content of '0' and of '1,' respectively. This is not a trivial requirement: the analysis must provide an account of the content of ($A0$), ($A1$) etc. that preserves the function-argument structure of the originals: each of the analysans-sentences must predicate a first-level concept of an object, the number in question.

The first rejected analysis, which we'll call "the recursive analysis," appears in §55 of *Grundlagen*, and the discussion of its failings in §56. Its core is the proposal to define each (finite) instance of (An) by stipulating that:

- The number 0 belongs to F iff for all x $\neg Fx$
- The number 1 belongs to F iff $\exists x Fx$ & $\forall y \forall z(Fy$ & $Fz \rightarrow y = z)$
- The number n+1 belongs to F iff $\exists y(Fy$ & the number n belongs to $(Fx$ & $y \neq x))$

Frege's objection to the recursive analysis is as follows:

> §56 These definitions suggest themselves so spontaneously in the light of our previous results, that we shall have to go into the reasons why they cannot be reckoned satisfactory.
>
> The most likely to cause misgivings is the last; for strictly speaking we do not know the sense of the expression "the number n belongs to the concept G" any more than we do that of the expression "the number $(n + 1)$ belongs to

> the concept F." We can, of course, by using the last two definitions together, say what is meant by "the number 1+1 belongs to the concept F" and then, using this, give the sense of the expression "the number 1+1+1 belongs to the concept F" and so on; but we can never - to take a crude example - decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or is not. Moreover we cannot by the aid of our suggested definitions prove that, if the number a belongs to the concept F and the number b belongs to the same concept, then necessarily $a = b$. Thus we should be unable to justify the expression "*the* number which belongs to the concept F," and therefore should find it impossible in general to prove a numerical identity, since we should be quite unable to achieve a determinate number. It is only an illusion that we have defined 0 and 1; in reality we have only fixed the sense of the phrases
>
> "the number 0 belongs to"
>
> "the number 1 belongs to";
>
> but we have no authority to pick out the 0 and 1 here as self-subsistent objects that can be recognized as the same again.

The fundamental failing of the recursive analysis is that it fails to treat numerical statements like ($A1$) as each affirming of an object (e.g. the content of "1") that it falls under a concept (the content of (A)). In providing meaning only to a series of semantically-simple second-level predicates ("the number 0 belongs to ....," "the number 1 belongs to...," etc.) and not to the two parts that Frege needs to treat as semantically-significant units (the numerical singular terms "0," "1," etc. and the relation-term "The number ... belongs to ..."), the recursive analysis gives, as Frege says, "only an illusion" of having defined the numbers..[8]

The failure of the recursive analysis to treat the numerals in instances of (An) as singular terms blocks Frege's route to an account of the content of the concept-phrase

(Num) "... is a number."

That route (§72) is to stipulate that

(Num$_n$) "n is a number"

is to be understood as meaning

(Num$^*_n$) "there exists a concept such that n is the number which belongs to it,"

and again requires that the syntactic position marked by "n" is of the kind that can be filled by arbitrary singular terms, and hence of the kind into which one can place a first-level

[8] For a clear discussion of this point, see [**?**]

bound variable. That the recursive analysis fails to do this is made plain by noting that neither (Num^*_n) nor, consequently, (Num_n), is treated by the recursive analysis in a way that yields a syntactically well-formed sentence on the insertion of an arbitrary singular term at the position marked by "n."

If numerical discourse is understood in the way the recursive analysis proposes, then we cannot form the claim that a concept F has the number Julius Caesar belonging to it, and we cannot form the claim that Julius Caesar, or any arbitrary object, is a number. It will be impossible, that is, to understand assignments of number as affirming a relationship between an object and a concept, and impossible to understand (Num) as expressing a concept under which objects fall.

At least part of the role of the example "Julius Caesar," then, is to stand as an uncontroversial singular term, used to illustrate the failure of the recursive analysis to treat numerals as singular terms. It is sometimes supposed that the example is used to point out a further failing of the recursive analysis, i.e. a failure to provide criteria by means of which to determine that Julius Caesar is not a number. There is a sense in which the recursive analysis is guilty of this failing: in failing to treat "Julius Caesar is a number" as well-formed, it stands in the way even of asking the question whether Caesar is a number. It is important, though, to note that the failure here is no indication that an adequate account of the semantics of (Num) must come along with criteria distinguishing numbers from e.g. people. Keeping in mind Frege's central goal, an accurate analysis of the syntax and semantics of arithmetical discourse - which itself has nothing to say about the distinction between numbers and people - we should note that such a requirement is considerably stronger than is required by the solution of the more general problem on display in §§55-56.

4 *Grundlagen* §62: Identity Sentences

In the next five sections (§§57-61), Frege clarifies what he means by the term "*object*," and notes again the importance of identity-sentences, like "$1 + 1 = 2$," in understanding the role of the numerals.[9] The crucial points here regarding objects are (i) that it makes sense to affirm identity-statements about them, and (ii) that some of these identity-statements are true even when the terms flanking the identity-sign present their objects in different ways.[10] Finally, in §62, Frege makes these remarks about identity and objects:

[9] §57

[10] This passage is independently interesting for its hint of what will soon become Frege's idea that significant pieces of language carry a sense as well as a reference:

> ... what we have here is an identity, stating that the expression "the number of Jupiter's moons" signifies the same object as the word "four." And identities are, of all forms of proposition, the most typical of arithmetic. It is no objection to this account that the word "four" contains nothing about Jupiter or moons. No more is there in the name "Columbus" anything about discovery or about America, yet for all that it is the same man that we call Columbus and the discoverer of America. [§57]

> [W]e have already settled that number words are to be understood as standing for self-subsistent objects. And that is enough to give us a class of propositions which must have a sense, namely those which express our recognition of a number as the same again. If we are to use the symbol a to signify an object, we must have a criterion for deciding in all cases whether b is the same as a, even if it is not always in our power to apply this criterion. In our present case, we have to define the sense of the proposition
>
> "the number which belongs to the concept F is the same as that which belongs to the concept G";
>
> that is to say, we must reproduce the content of this proposition in other terms, avoiding the use of the expression
>
> "the Number which belongs to the concept F".
>
> In doing this, we shall be giving a general criterion for the identity of numbers. When we have thus acquired a means of arriving at a determinate number and of recognizing it again as the same, we can assign it a number word as its proper name."[11]

Parts of this passage are exactly as one would expect, given what has gone before. The requirement that we treat numerical terms as genuine singular terms, i.e. as referring to "self-subsistent objects," involves the requirement that these terms can stand in the appropriate place in sentences. And a particularly important such place is the whole of the position to either side of the identity sign. The requirement is not merely that the analysis must assign a sense to sentences that appear superficially to be identity-sentences linking singular terms, since this weak criterion is satisfiable by an analysis that fails to preserve the singular-term status of the numerals. What's required, if Frege is to be able to put the analysis to work, is that the resulting analysans-sentences themselves have the structure of identity-sentences. Just as the recursive analysis provides only "an illusion" of defining the singular terms "0" and "1" etc., an analysis of

> "The number which belongs to the concept F is the same as that which belongs to the concept G"

that does not contain two places into which uncontroversial singular terms can be meaningfully substituted, linking these places by the identity-sign, will fail to satisfy Frege's crucial requirement that arithmetical sentences treat numbers as objects.

All of this is familiar, given the discussion in §§55 - 61. But there is one odd line in this §62 passage, a line which has received a lot of attention, and which I will argue below has been the source of much confusion. This is Frege's claim that the use of a term a as a

[11] *Grundlagen* §62

genuine singular term - as he puts it, its use “to signify an object” - requires that we “must have a criterion for deciding in all cases whether b is the same as a ...”[12] At first glance, this might seem uncontroversial: after all, if a and b are singular terms, then either they refer to the same object or they don’t, and a failure to make sense of the apparent identity-claim “$a = b$” might be thought an indication that we do not, after all, have a genuine identity claim on our hands, or that we haven’t determined what (at least) one of the apparent singular terms stands for. But while this thought is indeed natural when a and b are terms for ordinary household goods, it is not at all clear that the thought is plausible once we include under “singular term” and “object” all of the things that Frege includes. Mathematical singular terms appear in contexts: terms for real numbers appear in analysis; terms for rationals and terms for natural numbers in their own respective theories, etc. And ordinary mathematical practice counts e.g. “the natural number 2” and “the real number 2” as genuine singular terms (for the reasons Frege has already given), without providing any means for determining the truth-value of the identity-sentence linking them. If one can make any sense of there being a “fact of the matter” whether the natural number 2 is identical to the real number 2, it is certainly one that has no bearing on mathematics. So if we take Frege’s (apparent) requirement at face value, we will have to understand him as insisting on a very strong, and mathematically strange, requirement. The alternative is to understand Frege’s use of “all cases” as restricted to a contextually-relevant domain, e.g. a particular mathematical theory.[13]

The two interpretive options regarding this passage are these: *First,* take Frege to have been speaking carefully and strictly in the target sentence of §62, and to have insisted on the following strong requirement:

Strong Identity Condition (SIC) t is a genuine singular term only if for every genuine singular term t', we have a criterion for determining whether the sentence $t = t'$ is true or false.

Second, take Frege’s prose here to be loose, with the intended requirement just the one we have already discussed above, namely that

Weak Identity Condition (WIC) t is a genuine singular term only if it is of the right syntactic kind to occur as a term in identity-sentences, which is to say that it can appear in the gap in true or false instances of the form “... $= t'$,” where t' is a genuine singular term.

[12] The German here reads as follows: “Wenn uns das Zeichen a einen Gegenstand bezeichenen soll, so müssen wir ein Kennzeichen haben, welches überall entscheidet, ob b dasselbe sei wie a, wenn es auch nicht immer in unserer Macht steht, dies Kennzeichen anzuwenden.” A certain unclarity that isn’t preserved in Austin’s translation is what Frege means at the opening of the sentence: “If *for us* the sign a is to designate an object... .” An interesting question is whether this qualification is relevant to what’s meant by “all cases” in this context. Thanks to Philip Ebert for pointing out the interest of this passage.

[13] For further discussion of this remark about “all cases,” and the interpretation of similar passages throughout Frege’s work, see [**?**].

The difference between the two is illustrated by the fact that the former, but not the latter, entails that at least one of the terms "the natural number 2" and "the real number 2" fails to count as a genuine singular term under ordinary conditions - i.e. in the absence of some further stipulation that grounds the truth or the falsehood of the identity sentence linking these terms.[14]

The "standard account" of Frege's later failure to adopt the HP strategy in light of Russell's paradox follows immediately from the idea that Frege holds (both here and in 1902) the stronger condition (SIC). The argument in favor of interpreting Frege as holding (SIC) is that he seems to say just this: the passage just quoted is perhaps most straightforwardly read as endorsing it. The argument against this interpretive option is that, as argued below, (SIC) is inconsistent with Frege's favored treatment of arithmetical discourse (both in *Grundlagen* and *Grundgesetze*), and that there is a straightforward alternative account of the passage in question - i.e. the contextually restricted reading just mentioned - which is consistent with (WIC) but not with (SIC). We return to this interpretive issue after laying out Frege's rejection of the second inadequate analysis of arithmetical discourse, and his own favored analysis.

5 *Grundlagen* §§63-68: The HP Strategy

Immediately after the passage just discussed, Frege presents the second inadequate proposal for the analysis of numerical terminology. The proposal is to define terms of the form "the number that belongs to the concept F" by stipulating that each instance of

(N=) "The number that belongs to the concept F = the number that belongs to the concept G"

is to mean:

"There is a 1-1 correlation between the objects falling under F and those falling under G."

That is to say, the proposal is just what we have called the "HP strategy" above, and is essentially the proposal at the heart of neo-logicism. Frege's criticism of the strategy focuses not on the actual case of central concern, i.e. on the analysis of terms of the form "the number that belongs to the concept F," but on a simpler analogical case having to do with directions. The proposal, outlined in §§64-65, is to give an analysis of direction-talk by stipulating that statements of the form

(Dir=) "the direction of line a = the direction of line b"

[14] Both conditions involve the usual Fregean "boot-strapping" in the sense that we gain from each a significant requirement on the genuine singular-term status of a term t only against the backdrop of some collection of other terms already recognized as genuine singular terms. For more on his understanding of such boot-strapping in a formal environment, see *Grundgesetze* I §30.

are to express what their counterpart sentences of the form

(Par) "line a is parallel to line b"

express. The criticism, in §66, is as follows:

> In the proposition
>
> "the direction of a is identical with the direction of b"
>
> the direction of a plays the part of an object [*footnote*], and our definition affords us a means of recognizing this object as the same again, in case it should happen to crop up in some other guise, say as the direction of b. But this means does not provide for all cases. It will not, for instance, decide for us whether England is the same as the direction of the Earth's axis - if I may be forgiven an example which looks nonsensical. Naturally no one is going to confuse England with the direction of the Earth's axis; but that is no thanks to our definition of direction. That says nothing as to whether the proposition
>
> "the direction of a is identical with q"
>
> should be affirmed or denied, except in the one case where q is given in the form of "the direction of b". *What we lack is the concept of direction;* for if we had that, then we could lay it down that, if q is not a direction, our proposition is to be denied, while if it is a direction, our original definition will decide whether it is to be denied or affirmed. So the temptation is to give as our definition:
>
> q is a direction, if there is a line b whose direction is q.
>
> But then we have obviously come round in a circle. For in order to make use of this definition, we should have to know already in every case whether the proposition
>
> "q is identical with the direction of b"
>
> was to be affirmed or denied.[15]

If we analyze ($Dir_=$) as (Par), then, for the reasons given above in the rejection of the recursive analysis, we will have failed to treat direction-terms as genuine singular terms. We will have failed to provide a means of explaining direction-talk that makes sense of direction-terms as occupying a syntactic place suitable for substitution by arbitrary singular terms, and therefore suitable for first-level generalization. Frege's means of demonstrating this failure is familiar: one demonstrates that the analysis provides no account of the meaning of instances of

(Dir_{arb}) $\langle$ *arbitrary singular term* $\rangle$ = the direction of the Earth's axis

[15] *Grundlagen* §66; emphasis added.

or of

(Dir) $\langle$ *arbitrary singular term* $\rangle$ is a direction

by pointing out that the completion of these phrases by a token, uncontroversial singular term (here, "England") produces a sentence for which the HP analysis provides no counterpart analysans. We don't even get to the question of the semantic value of the sentences delivered by the analysis; the difficulty is that the analysis delivers nothing. In failing to provide an account that treats direction-terms as occupying a position into which uncontroversial genuine singular terms can be substituted, the "parallels analysis" fails to treat the position of these terms as susceptible to first-order generalization. And it therefore fails (i) to treat directions as objects, and (ii) to treat the phrase "... is a direction" as referring to a concept. "What we lack," says Frege, "is the concept of direction." This is not because the analysis assigns to "... is a direction" a semantic value that provides no answer to whether England is a direction; it is because the analysis stands in the way of assigning to "... is a direction" any semantic value at all.

Similarly for the HP strategy and the concept of *number*. That strategy provides no account of instances of

$(N =_{arb})$ $\langle$ *arbitrary singular term* $\rangle$ = the number of planets,

and so stands in the way of an account of

(Num_{arb}) $\langle$ *arbitrary singular term* $\rangle$ is a number.

It fails, in short, to give us a concept of number.

Just as in the discussion of the recursive analysis, Frege's way of making vivid the fact that the HP analysis fails to reconstruct the apparent concept-phrase ("... is the number that belongs to the concept F," or "... is the direction of the line a") as a genuine concept-phrase is to point out via the direction analogy that that reconstruction provides no argument-place suitable for a singular term to be inserted. There is no part of the reconstructed version of (N=) that can be swapped out for "England."

The interpretive question noted above can now be framed as follows. When using the unproblematic singular terms "Julius Caesar" and "England" to make his point, does Frege mean to convey the strong requirement that an analysis of the concept-phrases "... is the number that belongs to the concept F" and "... is the direction of line a" must come along with a criterion that distinguishes numbers from people, and directions from countries? Or does he mean to convey just the weaker requirement that the analysans provided by an analysis of these concept-phrases must be one that has an argument place suited for a first-level bound variable, and hence of the kind that takes genuine singular terms as arguments?

Focusing just on the text of *Grundlagen*, it is difficult to find a conclusive basis for preferring the strong or the weak reading of Frege's texts. The difficulties are these. In favor of the strong criterion is Frege's statement, quoted above, that a is a genuine singular term only if it comes along with a criterion for deciding "in all cases" whether b is the same as a. Arguably weakening the force of this claim is that Frege makes it in service of his immediately-preceding remark, for which he has clear reasons on either interpretive option, that genuine singular terms must be able to serve as terms in identity-sentences. Most importantly (or at least most influentially) in favor of the strong reading is Frege's choice of "Julius Caesar" and "England" as examples to illustrate the failure of the recursive and the Hume analyses to secure the singular-term status of numerical terms. Arguably weakening the force of these examples is the fact that in order to make his point, as sketched above, Frege required terms that his audience would immediately grant as genuine singular terms. Since the status of numerical terms is at issue just here, Frege's point cannot be made via the use of numerical terms. Independently, then, of the question whether he requires an adequate analysis of arithmetical discourse to settle such extra-mathematical issues as the identity of numbers and people, we should expect his rhetorical examples to be non-mathematical.

The strongest, though still inconclusive, argument for attributing to Frege merely the weaker condition is the argument that his own solution in *Grundlagen* meets the weak condition and fails to meet the strong. Frege's solution, presented in §68, is to understand "the number that belongs to the concept F" as referring to the extension of that second-level concept under which fall all and only the first-level concepts equinumerous with F. Extensions are clearly objects, according to Frege, and hence our core problem is solved: the weak requirement is satisfied. In analyzing instances of (N=) as instances of

(Ext=) "The extension of the concept '*equinumerous with the concept F*' = the extension of the concept '*equinumerous with the concept G*',"

the favored analysis treats the numerical terms as occupying uncontroversially genuine singular-term positions. But it is far from clear that the analysis satisfies the strong criterion. The identity-statements involving extension-terms that are clearly taken in *Grundlagen* to be meaningful are those of the form

(ID-ext1) The extension of $F =$ the extension of G

where F and G are first-level concepts,

(ID-ext2) The extension of $\Phi =$ the extension of Ψ

where Φ and Ψ are second-level concepts, and

(ID-mix) The extension of $\Phi =$ the number that belongs to the (first-level) concept F.

Instances of (ID-mix) are provided truth-conditions by reducing them to corresponding instances of (ID-ext2). Frege's proof-sketches indicate that he takes the truth-value of each instance of (ID-ext1) and of (ID-ext2), as he would have expected his audience to know, to be given by the principle of extensionality for concepts of that level. But Frege is silent, in *Grundlagen*, about truth-conditions for any other identity-statements involving extension-terms. As a consequence, the *Grundlagen* account does not provide truth-conditions for sentences of the form

(ID-q) The number that belongs to the concept F = q,

for q not of the form "the number that belongs to the concept G" or "the extension of the (second-level) concept Φ." This is at least odd, if it has been essential to Frege's line of reasoning all along that genuine singular terms must come along with criteria determining the truth-value of all such statements. If we take Frege to have insisted on what we have called the "strong condition," (SIC), above, then his own analysis of arithmetical discourse fails, in a very straightforward way, to meet his requirement.

One *might* hold in response to this line of thought that Frege presumes his audience to know e.g. that extensions (and hence numbers) are not identical to people or to countries; this might be part of what he intends to convey in the footnote remark: "I assume it is known what the extension of a concept is."[16] But even this is not enough to satisfy the strong condition. Frege is well aware, by the time of writing *Grundlagen*, of the difference between first-level and second-level concepts: quantification over the latter is essential to all of the interesting work in Part III of *Begriffsschrift*. *Grundlagen* treats directions as extensions of first-level concepts, and numbers as extensions of second-level concepts. Even granting Frege the assumption in *Grundlagen* of the extensionality condition for concept-extensions, we still get no truth-conditions for statements of the forms

The extension of (first-level) F = the extension of (second-level) Φ,

The extension of (second-level) Φ = the direction of a;

or

The extension of (first-level) F = the number that belongs to the (first-level) concept G.

For the extensionality principle provides truth-conditions only for identity-statements between extensions of concepts of the same level. We could of course make a further assumption on Frege's part, and add to *Grundlagen*'s machinery the stipulation that all "cross-level" extension-identity claims are false. Or we could make on his behalf some other consistent assumption, one according to which some of these are true. Nothing in what happens later in *Grundlagen* will depend on how we decide such cases. The important point here is that Frege makes no attempt to provide such completeness, despite the fact that he takes himself clearly to have established that, on the analysis provided in §68, numerical terms are genuine singular terms.

[16] *Grundlagen* §69, footnote.

The question in the air is whether Frege holds, at the time of writing *Grundlagen*, that a is a genuine singular term only if we have criteria that decide, for every genuine singular term b, the truth-value of the sentence $a = b$. The alternative is that, on his conception of singular terms (and hence of objects), there can be in the usual mathematical manner some "don't-care" cases that need not be assigned truth-conditions by an account of arithmetical discourse that is to be deemed adequate. If the latter is his view, then the point of the Caesar passages is not to establish the strong condition, but to make vivid the failure of the recursive and the Hume analyses to treat numerical terms as singular terms, i.e. as occupying positions susceptible to first-level quantification.

Recall that the reason to ask this interpretive question is in order to answer the further question: why did Frege not adopt the Hume strategy in response to Russell's Paradox? For this purpose, the most important interpretive question is not the question of Frege's attitude to the strong condition in 1884, but his attitude to it in 1902. And things become much clearer in the intervening years. In the next section, we look briefly at Frege's analysis of arithmetical discourse in *Grundgesetze*, with an eye toward his allegiance to the strong condition.

6 *Grundgesetze*

In the years leading up to the *Grundgesetze*, Frege's view of functions and objects as the referents of predicative phrases and singular terms respectively becomes more precise as it is integrated into the mature semantic theory of sense and reference. His view of the semantical requirements on the kind of formal language he presents in *Grundgesetze* sharpens up as well. One clear requirement on the formal language is that every well-formed string of symbols of that language must have a determinate reference, so that each well-formed sentence has a determinate truth-value. The rationale for this requirement in Frege's hands is, as Michael Dummett has argued, straightforward: without the satisfaction of this requirement, it is impossible to provide a syntactic criterion of axioms and of rules of inference guaranteed to sanction the inference from truths only of truths.[17] It is crucial, then, that each completion of a function-expression with argument-expressions of appropriate syntactic type has a determinate reference. Call this the "linguistic completeness requirement."

A distinct requirement, which Frege is often viewed as endorsing, is the requirement that every function referred to by a functional expression of the language be *total*, which is to say that it delivers a value for every object or function (or n-tuple thereof) of appropriate type as argument. Call this the *totality requirement*. The totality requirement implies that the function referred to by e.g. an addition-sign be defined over not just pairs of numbers, but also over pairs of people, lamps, and shoestrings. Because the expression left by removing a singular term from an identity-sentence is itself a functional expression, the

[17]For a clear discussion of this point, see [?] Ch 13, esp pp 155-159.

totality requirement includes the requirement for every singular term t that there is a fact of the matter whether t co-refers with $t*$, for every singular term $t*$. Hence the totality requirement implies the strong identity condition noted above.

The totality requirement is considerably stronger than is the linguistic completeness requirement: a formal language intended for the arithmetic of the integers will satisfy the linguistic completeness requirement with respect to its addition-sign if the addition-function it refers to is defined over all pairs of integers. In general, if a formal language is restricted to a particular subject-matter, then the linguistic completeness requirement as applied to that language is satisfiable without satisfaction of (SIC): the semantics of the language need have nothing to say about singular terms from outside the language.

The usual view of Frege's texts is that he intends the totality requirement; and there is considerable textual support for this view. There are also texts that don't fit well with this requirement, so that here again there is an interpretive difficulty. I have argued elsewhere that Frege is most plausibly understood in the relevant passages as endorsing the weaker linguistic completeness requirement, and this only for formal rather than for natural languages.[18] For current purposes, the important question concerns Frege's view as confined to the formal language of *Grundgesetze*, since this is the language whose semantics bears on the question of integrating the neo-logicist proposal into Frege's mature logicist framework. The central question is this: what does Frege require of the function-terms and singular terms of the language of *Grundgesetze*?

The answer to this question is clear. *Grundgesetze* I §29 opens as follows:

> We now answer the question: when does a name refer to something?

The answer is as follows:

- A name of a first-level function of one argument *has a reference* if every result of filling its argument-place with a *referring* proper name *has a reference.*
- A proper name *has a reference* if:
  - Whenever it fills the argument-place of a *referring* first-level function of one argument, the resulting proper name *has a reference*, and
  - Whenever it fills one or the other of the argument-places of a *referring* first-level function of two arguments, the resulting function-name *has a reference.*
- A name of a first-level function of two arguments *has a reference* if the result of filling both of its argument-places with *referring* proper names *has a reference.*
- A name of a second-level function of one argument of the second kind *has a reference* if every result of filling its argument place with a *referring* first-level function *has a*

[18] See [**?**]

reference. (By "an argument of the second kind," Frege means a first-level function of one argument.[19])

- A name of a third-level function *has a reference* if every result of filling its argument-place with a *referring* second-level function of one argument of the second kind *has a reference*.[20]

As Frege notes, these criteria cannot stand alone as an answer:

> These propositions are not to be regarded as explanations of the expressions 'to have a reference' or 'to refer to something,' since their application always presupposes that one has already recognised some names as referential; but they can serve to widen the circle of such names gradually. It follows from them that every name formed out of referential names refers to something.[21]

The names that Frege will "presuppose" to have reference are primitive names of truth-values; the §29 clauses then provide a characterization of a broader class of names as having reference on the basis of that presupposition.

The first question one might ask about these clauses concerns what might seem their fragmentary character. Why, for example, does the referential character of a singular term (proper name) turn just on its behavior when filling argument-places in names of one-place and two-place first-level functions? What about the cases in which it fills argument-places in names of, say, unequal-leveled functions, or of three-place first-level functions? The answer is simply that there are no such primitive function-names in *Grundgesetze*. The standard for "having a reference" appropriate to a proper name is not that a value is determined when that proper name appears in the (or an) argument-place of *every* appropriately-leveled referring function-name. It is that a value is determined when that proper name appears in the (or an) argument place of every appropriately-leveled referring function-name *of Grundgesetze*. The extremely limited nature of this requirement is most vivid as applied to the important case of singular terms of the form "$\acute{\epsilon}\Phi(\epsilon)$" In accordance with the second of the clauses listed above, the question whether such a value-range name has a reference is the question whether, given a referring name of a first-level function of one or of two arguments, the result of filling the/an argument-place with that value-range name is a referring name. One of those first-level functions of one argument is the result of filling in one of the argument-places of the identity function with a referring singular term n, and the important question in this sub-case is whether every sentence of the form

(ID) "$n = \acute{\epsilon}\Phi(\epsilon)$,"

for referring singular term n, itself refers. The answer to this question turns out to be very easy: the only singular terms in *Grundgesetze* are names of truth-values or names

[19]See *Grundgesetze* I §23.
[20]*Grundgesetze* I §29.
[21]*Grundgesetze* I §30.

of value-ranges. If n is a name of a truth-value, then it refers either to the true or to the false, and hence via the stipulations of Vol I §10 to a particular value-range; in this case, the instance of (ID) refers to the true if $\Phi()$ is a concept under which only the true (respectively, only the false) falls. Otherwise, it refers to the false. And if n is the name of the value-range of a function $\Psi()$, then "$n = \acute{\epsilon}\Phi(\epsilon)$" refers to the true if the value of $\Phi(x)$ is the same as the value of $\Psi(x)$ for every argument, and refers to the false otherwise. And we are done with this case.

In short, then, the criterion to be met by the well-formed names of *Grundgesetze* in order that they collectively meet the condition that each "has a reference" is the criterion of linguistic completeness. It is critical, for each singular term n, that the sentence $n = m$, for every singular term m, has a determinate truth-value. But nothing about the semantics of Grundgesetze provides any answer to the question whether any singular term of *Grundgesetze* refers to Julius Caesar or to England. And the condition that Frege requires his singular terms to meet does not include the condition that such questions have answers. There are, in short, an enormous number of "don't care" cases in Frege's *Grundgesetze* analysis of value-range terms and hence of numerical singular terms. The identity-sentences that matter to arithmetic are settled, as are those extras (like the identities between finite cardinals and real numbers) that are needed for logical hygiene. But nothing even approximating (SIC) is met.

The emphasis on linguistic completeness, and the significant distance between this requirement and the totality requirement introduced above, are clear as well in the §10 stipulations regarding value-range names, referenced as just mentioned in the §31 argument. In §10, Frege raises the question of the reference of value-range names. The stipulation that a name of the form "$\acute{\epsilon}\Phi(\epsilon)$" (for "$\Phi()$" a name of a first-level function of one argument) is to refer to the value-range of that function, together with Basic Law V, settle the reference of all sentences of the form "$\acute{\epsilon}\Phi(\epsilon) = \acute{\epsilon}\Psi(\epsilon)$" in the obvious way. But this does not account for identity-sentences of the form "$\acute{\epsilon}\Phi(\epsilon) =$ q" for q not of the form "$\acute{\epsilon}\Psi(\epsilon)$" As Frege puts it,

> By presenting the combination of signs '$\acute{\epsilon}\Phi(\epsilon) = \acute{\alpha}\Psi(\alpha)$,' as co-referential with '$\forall x(\Phi(x) = \Psi(x))$', we have admittedly by no means yet completely fixed the reference of a name such as '$\acute{\epsilon}\Phi(\epsilon)$'. We have a way always to recognise a value-range as the same if it is designated by a name such as '$\acute{\epsilon}\Phi(\epsilon)$', whereby it is already recognisable as a value-range. However, we cannot decide yet whether an object that is not given to us as a value-range is a value-range or which function it may belong to...[22]

The reader who finds this discussion reminiscent of *Grundlagen* sections 56 and 66 might expect at this point that Frege will supplement his account of value-range terms by providing criteria by means of which one might determine whether their referents are

[22] *Grundgesetze* I §10. I have substituted modern universal-quantifier notation for Frege's.

identical to or distinct from e.g. Julius Caesar or England. But he does nothing of the sort. Following a brief argument in support of the claim he has just made - i.e. that the stipulations so far introduced do not "completely fix[ ] the reference" of value-range terms, Frege continues:

> Now, how is this indeterminacy resolved? By determining for every function, when introducing it, which value it receives for value-ranges as arguments, just as for other arguments. Let us do this for the functions hitherto considered. These are the following:
>
> $\xi = \zeta, \, -\xi, \, \neg\xi$
>
> [Demonstration that the last two are reducible to the first] ... After having thus reduced everything to the consideration of the function $\xi = \zeta$, we ask which values it has when a value-range appears as argument. Since so far we have only introduced the truth-values and value-ranges as objects, the question can only be whether one of the truth-values might be a value-range.

Frege then demonstrates that a stipulation that an arbitrarily-chosen value-range is the true, and a different one the false, will introduce no contradiction into the system. And finally, he makes such a stipulation, the stipulation that the value-range of any function under which just the true falls is to be identical with the true, and the value-range of any function under which just the false falls is to be identical with the false. The section concludes:

> We have hereby determined the *value-ranges* as far as is possible here. Only when the further issue arises of introducing a function that is not completely reducible to the functions already known will we be able to stipulate what values it should have for value-ranges as arguments; and this can then be viewed as a determination of the value-ranges as well as of that function.

Two remarkable features of the §10 strategy are these. First of all, the stipulations of §10 do not solve what one might have thought to be the "Caesar problem," i.e. the (pseudo-) problem that the semantic principles laid down to this point do not determine, of any value-range terms, whether they do or do not refer to Julius Caesar. The only unresolved question of reference with which Frege is concerned is the question of the truth-conditions of identity-statements linking value-range terms with terms for truth-values. Once this question is swiftly answered via stipulation, the formal language has achieved linguistic completeness (or so Frege thinks), as argued in §31. There is still no fact of the matter about whether any value-range term refers to Caesar. Should we want to define a new language by adding the term "Julius Caesar," and hence also the function-name "$\xi =$ Julius Caesar" to the *Grundgesetze* language, we would need to add stipulations sufficient to regain linguistic completeness, including stipulations regarding the truth-value of such sentences as "$\exists\Phi(\acute{\epsilon}\Phi(\epsilon) =$ Julius Caesar)." But without the presence of such a name in the language, there is nothing incomplete, in Frege's judgment, about the referential status of the terms in the language as it is.[23]

The second remarkable feature of the §10 strategy is the fundamental idea that the truth-values of the identity-sentences in question are matters for stipulation. Regarding the identity-sentences linking value-range names and truth-value names: one might have thought that, independently of any linguistic stipulation, there is a fact of the matter about whether truth-values are, or are not, value-ranges. Or more modestly, one might have thought that *if* there are such things as value-ranges and truth-values, then either some value-ranges are truth-values, or none are, independently of the semantic stipulations for any particular language. Cows and horses are this way; why not truth-values and value-ranges? But this is clearly not how Frege thinks of the matter. The question whether "$\acute{\epsilon}\Phi(\epsilon) = \forall xx = x$" is true or false is clearly not, for Frege, a matter of the identity of the objects in question, where these can be in some sense 'located' independently of their role as references of the singular terms in a variety of sentences. The order of explanation is the other way around: that value-range is identical with that truth-value if the sentence "$\acute{\epsilon}\Phi(\epsilon) = \forall xx = x$" is true; they are distinct if the sentence is false. Some identity-sentences are provided truth-values via the laws of logic; some via empirical fact. And some, like this one, are provided truth-values via semantic *fiat*. The really interesting identity-sentences for our purposes are those that are not part of the language of *Grundgesetze*, but are part of a more-comprehensive language that we could engineer by e.g. adding the term "Julius Caesar," with its ordinary reference, to the *Grundgesetze* language. It would be a simple matter to stipulate that every identity-sentence linking this term with a value-range term is false. It would also be a simple matter to choose a value-range term and stipulate that the identity-sentence linking it with "Julius Caesar" is true. The important point, for the purposes of coming to grips with Frege's view of object-reference, is that nothing about the semantics of the *Grundgesetze* language rules out either of these alternatives for later expansion. As we might put it, just as there was no fact of the matter, prior to Frege's stipulation, whether any of the value-ranges referred to in *Grundgesetze* was identical with a truth-value, so too there is no fact of the matter whether any of the value-range terms in *Grundgesetze* refers to Caesar.

Frege's most mature and most careful presentation of the logicist project involves nothing like the totality condition, and nothing like the strong condition (SIC). The numerical

[23] Dummett notes ([**?**] Ch 13 that in *Grundlagen*, Frege does not complete the answer to the question whether Julius Caesar is a number, since he doesn't there answer the question whether Caesar is an extension. Dummett takes §10 of *Grundgesetze* Vol 1 to involve the attempt to finally answer the lingering question. But this cannot be right. §10 and its stipulations are brief and clear, and the only question answered there is very straightforward: it is the question whether any value-range terms co-refer with any terms for truth-values; and if so, which. This is the only question whose answer is required for linguistic completeness, and the only one in which Frege shows any interest. There is nothing like an attempt, anywhere in *Grundgesetze*, to answer questions regarding the identity of value-ranges, or numbers, with any of the kind of "outlying" objects that purportedly give rise to the Caesar question.

singular terms used there provably satisfy the linguistic-completeness requirement,[24] and in particular each numerical singular term n satisfies the requirement that for every singular term t of the *Grundgesetze* language, the sentence $n = t$ has a determinate truth-value. Nothing about the semantics of the language determines, for objects o not referred to via terms of that language, whether any singular term of the language refers to o.

If at the time of writing *Grundlagen* Frege held a successful account of arithmetical discourse to the strong condition (SIC), then his view changed by the time of writing *Grundgesetze*. If (as I think more plausible) his view at the time of writing *Grundlagen* involved instead merely the weak condition (WIC), then his view is essentially uniform from *Grundlagen* to *Grundgesetze*.[25] The important point for our purposes is that by the time he received the letter from Russell, he did not take the logicist project to require for its success the provision of criteria by means of which one might determine whether its numerical singular terms refer e.g. to people or to countries. It therefore cannot be correct to understand his reason for failing to adopt the neo-logicist strategy to be that that strategy would fail to meet this condition.

The appeal to Hume's Principle in order to fix the reference of numerical singular terms would have posed two remediable problems for Frege. The first is the problem he presents in *Grundlagen* §66, which is that the account given thereby would not have demonstrated that the numerical terms are in fact singular terms. The remedy for this would have been to rely on the robust apparent singular-termhood of those terms in order to anchor the claim to genuine singular-termhood, in just the way that Frege actually does for extension-terms in *Grundlagen* and for value-range terms in *Grundgesetze*. That is, this part of the strategy would be to follow the main line of the neo-logicist proposal. The second remediable difficulty is the internal Caesar problem arising from the requirement of linguistic completeness. The language of *Grundgesetze* is intended not just for the development of a theory of cardinal numbers, but also for a theory of the reals, and perhaps also for a theory of "negative, fractional, irrational [and] complex" numbers.[26] Frege's actual procedure, as far as it goes, is to define all numbers as value-ranges of first-level functions, in terms of primitive vocabulary already available in *Grundgesetze*, with the result that the argument for linguistic completeness in §31 will apply to terms for all such numbers. A resort instead to Hume's Principle for the definitions of terms for cardinal numbers will be of no help in defining terms for the remaining numbers, so some entirely different procedure will be needed for them.[27] However these new terms are defined, we will have a failure of the linguistic completeness requirement: for example, identity-statements linking finite cardinals and reals will have no truth-conditions and no truth-values prior to the provision of additional stipulations. The remedy here will be more of the kind

[24]Or would have done so if not for the difficulty revealed by Russell's paradox.

[25]The qualifier "essentially" is needed because the *Grundgesetze* view is given in terms of the mature theory of (sense and) reference, while the *Grundlagen* view is concerned with the earlier notion of content.

[26]*Grundlagen* §109

[27]For discussion of this issue as it applies to the neologicist project, see [**?**].

seen already in §10: these cases, which are of no mathematical significance, will need to be settled by arbitrary stipulations, in the choice of which Frege would have had wide latitude.

But the appeal to Hume's Principle in order to fix the reference of numerical singular terms was, as Frege notes in the letter to Russell, blocked entirely by a more fundamental difficulty. We turn now to that difficulty.

7 The Paradox, Law V, and HP

The fundamental idea of Law V is that there is a very strong equivalence between the two sides of any instance of it. Frege sometimes says that the two sides "express the same sense,"[28] and sometimes puts it in terms of a guaranteed identity of reference.[29] More generally he says simply that "we can convert" one side into another,[30] and that if the universally-quantified claim is true, we can "also say that" the two functions have the same value-range;[31] and that we engage in a "transformation of the generality of an equality into a value-range equality."[32]

Frege makes similar remarks about abstraction principles in general. In *Grundlagen*: "The judgment 'line a is parallel to line b' ... *can be taken as* an identity;" in so doing, we "carve up the content" differently.[33] Similarly for shape and orientation,[34] and for length and color.[35] In each case, the two sides of the biconditional have something very like the same content, and we can unproblematically "transform" one into the other.

The strong semantic equivalence between the two sides of abstraction principles does not fit entirely easily with Frege's mature theory of sense and reference. That it is a "fundamental law of logic" that we can "transform" one side into the other requires a much stronger relation between the two sides than mere identity of reference (i.e. of truth-value); but at least some of Frege's remarks on sense-identity conflict with the idea that the two sides have the same sense. These are the remarks on which most of the post-Fregean theory of Frege's notion of sense tend to be based, the remarks according to which the possibility of a competent speaker judging that a sentence S expresses a truth while (say) being unsure about whether S' does so suffices for a difference in sense between the two sentences. Things are less clear than they might at first seem, since (i) Frege also claims on occasion that sense-identity is not so fine-grained; and (ii) even the initial test for sense-distinction provides no clear result, since a gray area surrounds the question whether a speaker who affirms one side of an abstraction-principle while failing to do so for the other

[28] "Function and Concept" p 11
[29] *Grundgesetze* I §3
[30] ibid
[31] ibid
[32] *Grundgesetze* I §9
[33] *Grundlagen* §64
[34] ibid
[35] *Grundlagen* §65

side can in fact be said to understand the sentences.[36] The difficulty here is that Frege simply has no clear criterion of sense-identity. The crucial feature of Basic Law V is that the two sides bear to one another the strong semantic similarity relation that Frege in *Grundlagen* would have called the result of a "re-carving" of their shared content, and by the time of *Grundgesetze* seems sometimes to have viewed as sense-identity and sometimes as a somewhat weaker relation of similarity, but in any case as strong enough to justify the inference from one side to the other on purely logical grounds.

Michael Dummett has argued that Frege's claim in "Function and Concept" that the two sides of V share sense is inconsistent with other central aspects of Frege's view of sense.[37] Frege claims that the senses of sub-sentential pieces of language are in some sense "parts" of the sense of the whole sentences in which they appear. How exactly we are to understand the part-whole metaphor is never made entirely clear by Frege, but Dummett argues that one fixed point is this: that if the sense of a phrase is *part* of the sense of a sentence, then grasp of the sense of that sentence must involve a prior grasp of the sense of that phrase. If this is the case, then indeed there is a problem: for it is presumably clear that a grasp of the sense of an instance of "Every F is a G and vice-versa" does not require a grasp of the sense of "the extension of F," with the result that the two halves of Law V cannot, on Dummett's understanding of the requirements on sense-parthood, express the same sense. But it is difficult to see why we should take Frege to have intended the connection between "parthood" and grasping in this way. In keeping with the *Grundlagen* idea that on recognizing a "re-carving" of a content already grasped one can come to discover new "concepts" (as he then calls the relevant content-parts), one might (perhaps more plausibly) take Frege to have intended by his talk of "parthood" merely that when a complete sense can be broken down into parts in a particular way, then *one* way of coming to grasp that complete sense is via a grasp of those parts and of their mode of composition. That the sense of "line a is parallel to line b" can be "re-carved" to yield the concept of *direction* as "the direction of a = the direction of b" does not, on this account, entail that only a speaker already in possession of the concept *direction* can understand the original sentence. On this way of understanding Frege's talk of senses and their parts, one might plausibly claim that while a grasp of the senses of *both* sides of an instance of Law V requires a fairly-immediate recognition that they express the same (or very closely similar) senses, a grasp of the universally-quantified half is possible for someone who has not yet mastered the notion of *value-range*. In any case, the crucial feature of Frege's view of the relationship between the senses expressed by the two sides is not whether or not he took them to be strictly identical, but that he took them to be sufficiently similar to underwrite an immediate inference from one to the other. This, prior to reading Russell's letter, would seem to have been Frege's view of the relationship between the two sides of abstraction principles in general; and it was certainly his view of the two sides of Basic Law V.

[36] For a fuller discussion of these issues, [**?**] chapter 2.

[37] [**?**] Ch. 14

Frege's immediate reaction to Russell's letter is that the paradox shows that the "transformation of the generality of an identity into an identity of ranges of values ... is not always permissible..."[38] A month later, he discusses the strategy of turning to a numerical abstraction principle, i.e. to Hume's Principle, for the "possibility of placing arithmetic on a logical foundation."

> We can also try the following expedient, and I hinted at this in my *Foundations of Arithmetic*. If we have a relation $\Phi(\xi, \zeta)$ for which the following propositions hold: ... [symmetry and transitivity] ..., then this relation can be transformed into an equality (identity), and $\Phi(a, b)$ can be replaced by writing, e.g., $\S a = \S b$. If the relation is e.g. that of geometrical similarity, then 'a is similar to b' can be replaced by saying 'the shape of a is the same as the shape of b.' This is perhaps what you call 'definition by abstraction.' But the difficulties here are the same as in transforming the generality of an identity into an identity of ranges of values.[39]

If the account sketched above is correct, then it is clear why the difficulties are "the same." The paradox undermines the core principle that Frege thought grounded all such biconditionals: the principle that the two halves are merely alternative expressions of essentially the same content. It is a natural thought, and it was certainly Frege's thought, that where $\Phi(\xi, \zeta)$ is an equivalence relation, then

- $\Phi(a, b)$

makes essentially the same claim as does

- The ϕ of $a =$ the ϕ of b,

for the obvious choice of a function ϕ. For equivalence in height, we have *the height of...*, for equivalence in shape, *the shape of...*, and for equivalence in extension, we have *the extension of...*. The lesson of the paradox, as Frege sees it, is that this simply isn't true.[40] We cannot assume that each equivalence-relation comes along with a function delivering objects whose identity-conditions are given by the holding of that equivalence-relation. And if this is not a general and obvious principle, then it's not a law of logic that the pairs of claims linked in this way are equivalent in truth-value. If it's true that there are, say, shapes and numbers meeting these criteria, then their existence is grounded in principles peculiar to their cases, and not in general laws of logic. Logic, as Frege now sees it, is insufficient to give us objects.

[38] Letter to Russell 22 June 1902

[39] Letter to Russell 24 July 1902

[40] For a similar diagnosis of the difficulty posed by the paradox, see [?], ch. 18.

8 Conclusion

Frege's view in 1884 of the necessary conditions on an acceptable analysis of arithmetical discourse, especially those parts of his view expressed in the "Caesar passages" of *Grundlagen*, are difficult to pin down with any precision. It is possible, though not mandated by the texts and inconsistent with some of them, that Frege held, at that time, the condition (SIC) as a requirement on the successful analysis of arithmetical discourse. But it is entirely clear that by the time of writing *Grundgeseetze*, he had no such view. Frege's account of the semantics of the *Grundgesetze* language, and the content of the claims he takes himself to have proven about that language, make it clear that by 1893 he did not view it as a necessary condition on an adequate analysis of arithmetical singular terms that the analysis provide truth-conditions for such "outlying" sentences as "Julius Caesar is the number of planets." And this is all to the good, from the point of view of the plausibility of Frege's view of the semantics of mathematical language. The idea that an understanding of mathematical discourse requires an assignment of truth-conditions to non-mathematical sentences, e.g. to sentences that identify real numbers with particular rationals, cardinals with people, or line-directions with countries, is an idea contrary to the ordinary successful functioning of mathematical discourse. I have suggested here that we should understand Frege's treatment of the few "don't-care" cases that arise for him in *Grundgesetze* - e.g. the identification of truth-values with value-ranges - in just the way we understand the decision to identify, or not, rationals with pairs of integers: of purely housekeeping significance.

Frege's own mature analysis of arithmetical language does not provide truth-conditions to sentences like "Julius Caesar is the number of finite cardinals." So his rejection of the HP strategy as a way to resurrect the logicist project in the light of Russell's paradox cannot be understood as motivated by the failure of Hume's Principle to determine truth-conditions for such sentences. I have suggested here that the inadequacy of the Hume's Principle strategy, from Frege's 1902 perspective, is rather that the treatment of Hume's Principle as fundamental within a logicist reduction requires what Russell's paradox shows to be false: that abstraction principles can be trusted, in virtue of their form, to be essentially truths of logic.[41] [42]

[41]This raises the question whether there is a way to defend the logical or analytic status of Hume's Principle from this skeptical response. I argue in [**?**] that the answer to this question, from Frege's point of view, is "no."

[42]Parts of this material were presented at the University of Connecticut Analytic Philosophy Workshop. Many thanks to the organizers of that workshop and to its participants for helpful feedback, especially to Marcus Rossberg, Philip Ebert, Junyeol Kim, Richard Kimberly Heck, Robert May, Kai Wehmeier, Roy Cook, Jamie Tappenden, Michael Hallett, and Sanford Shieh. Thanks especially to Philip Ebert for comments on an earlier draft.