# Short Introduction to Admissible Recursion Theory

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## 1 Axioms of KP and Admissible Sets

An **admissible set** is a transitive set $A$ satisfying the axioms of **Kripke-Platek Set Theory (KP):**

- The regular axioms of pairing and union. Explicitly, unordered pairing and arbitrary union:
  - $x, y \in A \to \{x, y\} \in A$
  - $x \in A \to \cup x \in A$
- $\Delta_0$ **Separation Axiom Schema / Axiom of Subsets:** For every parameter $B \in A$ and $\Delta_0$ formula $\phi$ with parameters, $\{x \in B : \phi(x)\}$ is a set. Recall briefly that in the language of set theory, $\Delta_0$ formulas are allowed to contain *bounded quantifiers* $\exists x \in y$, $\forall x \in y$.
- $\Delta_0$ **Collection Schema / Bounding Schema:** For all $B \in A$ and $\Delta_0$ formulas $\phi$ with parameters,

$$[\forall x \in B, \exists y \phi(x, y)] \to [\exists V, \forall x \in B, \exists y \in V, \phi(x, y)]$$

Now even though these axioms are called the axioms of Kripke-Platek *Set Theory*, they were NOT introduced as alternative axioms to ZFC. Some authors, such as Barwise, have certainly taken them this way. However both Kripke and Platek [cite, cite] introduced these axioms (independently) with the idea that they expressed the essential features a set needed to exhibit in order to be a well-behaved domain of *computation*. As we will see a little later, there is a natural way to interpret the natural numbers as an admissible set.

So what are these essential features? Why did Kripke and Platek care about $\Delta_0$ Separation and Collection? To address this question, we will have to introduce a few more notions first.

Let $A$ be an admissible set. A set $X \subseteq A$ is called **$A$-computably enumerable** if it is $\Sigma_1$-definable over $A$- namely, if it is definable by a $\Sigma_1$ formula containing finitely many parameters from $A$. A set $X \subseteq A$ is **$A$-computable** if both it and its complement in $A$ is $A$-c.e. In other words, a set is computable if it is $\Delta_1$-definable over $A$. The members of $A$ are the **$A$-finite** sets. A function is said to be **partial $A$-computable** if its graph is $A$-c.e.

Note briefly that $A$ generalizes $\omega$ as a domain of computation - the original computable sets are subsets of $\omega$, which is why we would like our more general computable sets to be *subsets* of $\omega$.

Now, we can return to the question of what the axioms of KP are intended to capture. The point of $\Delta_0$ separation and collection is that they can be used to prove $\Delta_1$ separation and $\Sigma_1$ collection. Since the $\Delta_1$ sets are the computable sets, $\Delta_0$ separation implies that the intersection of a computable subset of $A$ with an $A$-finite set is still $A$-finite. Together with $\Delta_0$ collection this implies that the image of a finite set under a computable map is finite. This latter property is precisely what the axioms were designed to capture; it is equivalent to $\Sigma_1$ replacement, which the axiom that Platek used in his original definition of admissible set.

To get familiar with the axioms, let's prove:

**Theorem 1.** *Every admissible set $A$ also satisfies $\Delta_1$ Separation. Equivalently, if $X$ is finite and $Y$ is $\Delta_1(A)$ then $X \cap Y$ is finite.*

*Proof.* Since $Y$ is $\Delta_1$, both it and its complement are $\Sigma_1$-definable. So let $x \in Y \iff \exists z \phi_0(x, z)$ and $x \notin Y \iff \exists z \phi_1(x, z)$ where each $\phi_i$ is $\Delta_0(A)$. Then

$$\forall x \in X \ \exists z \ (\phi_0(x, z) \vee \phi_1(x, z)$$

since every element of $X$ is either in $Y$ or not. Hence $3\exists V \in A \ \forall x \in X \ \exists z \in V \ (\phi_0(x, z) \vee \phi_1(x, z))3$ by $\Delta_0$ bounding/collection. Thus

$$\{x \in X : \exists z \in V(\phi_0(x, y))\} = X \cap Y$$

is in $A$ by $\Delta_0$ separation. □

## 2 Admissible Ordinals

The objects of study in classical recursion theory are sets of natural numbers, or sets $X \subseteq \omega$. But what if we replaced $\omega$ with $\alpha$, and considered sets of ordinals? Kripke in particular designed his axioms so that the intended models would essentially be ordinals $\alpha$ that were well-behaved domains like $\omega$. This generalization seems to be the most straightforward, since our direct intuition of operating on numbers applies. But our domains needs not be $\alpha$ strictly, just as in classical recursion theory our domain need not strictly be $\omega$. For example, the elements of $\omega$ are just symbols - we may just as well have created a notion of computability that operates only on the even numbers, and the two would be considered conceptually equivalent. What formalizes this intuition is the fact that there is a *computable bijection*, or *computable isomorphism*, between the set of even numbers and $\omega$. In the general case also, we will consider domains computably isomorphic to $\alpha$.

To be precise, an **admissible ordinal** is an ordinal $\alpha$ for which $L_\alpha$ is an admissible set. Why the switch from $\alpha$ to $L_\alpha$? It turns out that both domains are equivalent - we will show that there is a computable bijection between the two sets. One advantage of using $L_\alpha$ as our admissible set is that we can use nice properties we already know about constructible sets. However, one should always keep in mind the intuition that we are working on a domain of ordinals, just as in the classical case - this is indispensible for developing a strong intuition of how recursion on $L_\alpha$ is supposed to look.

Another reason $L_\alpha$ is the intended model of KP instead of $\alpha$ is that the structure $(\alpha, \in)$ never satisfies KP outright. For example, if $\beta, \gamma < \alpha$ the set $\{\beta, \gamma\}$ is never an ordinal unless $\beta = \emptyset$ and $\gamma = 1$. Thus ordinals as sets do not generally satisfy pairing.

**Lemma 2.** *The ordinal $\alpha$ is admissible iff $L_\alpha$ satisfies $\Delta_0$ collection.*

*Proof.* ($\rightarrow$) Trivial, by definition of admissible ordinal.
($\leftarrow$) Consider the case where $\alpha$ is a limit ordinal. Recall that each $L_\beta$ is transitive. Since $\alpha$ is a limit ordinal, it is easy to check that pairing and union hold in $L_\alpha$, by almost exactly the same proofs that they hold in $L$. We now check $\Delta_0$ separation. Fix a $B$ and a $\Delta_0$ formula $\phi(\bar{u}, x)$. Then all of $\phi$'s parameters are in $L_\beta$ for some $\beta < \alpha$ so $\{x : x \in B \wedge \phi(\bar{u}, x)\}$ is in $L_{\beta+1}$ and so in $L_\alpha$. Note briefly that $(L_\alpha, \in)$ really thinks this set is what it is because of the absoluteness of $\Delta_0$ formulas for every transitive set.
Now consider the case where $\alpha = \beta + 1$ is a successor ordinal. We will show ($\leftarrow$) holds vacuously, namely it is never the case that $L_\alpha$ satisfies $\Delta_0$ collection. For suppose it did. Recall that the ordinals in $L_\alpha$ are just those less than $\alpha$. Consider the constant $\Delta_0$ function (with parameter $\beta$) defined by $f(x) = \beta$. By collection, $\beta \in X$ for some $X \in L_{\beta+1}$, which implies $X \subseteq L_\beta$. Thus $\beta \in L_\beta$, a contradiction. □

**Theorem 3.** *The ordinal $\omega$ is the first admissible ordinal.*

*Proof.* Clearly it is the first, if it is admissible, since no successor ordinal can be admissible. We need only show $\Delta_0$ collection by the previous theorem. Recall that $L_\omega = V_\omega$ and both sets are equal to $HF$, the collection of hereditarily definable sets. Fix an HF set $B$ and suppose $\exists y \phi(x, y)$ for each $x \in B$. Pick some such $y_i$ for each $x \in B$ and consider the collection $\{y_i\}$. Since $B$ is finite and each $y_i$ is hereditarily finite, this set is hereditarily finite. This set witnesses collection and thus $\Delta_0$ collection. (In fact, while we're at it, remember that HF satisfies all axioms of ZF besides infinity.) □

Intuitively speaking, it is clear that $HF$ is in recursive bijection with $\omega$ - HF can be coded by natural numbers.

In general, recall that each $L_\alpha$ is well-ordered by some order $<_L$, which is easy to describe.

**Lemma 4.**

*The relation $<_L$ on $L_\alpha$ is computable.*

*The map taking $\beta \in \alpha$ to the $\beta$'th element of $L_\alpha$ a computable bijection.*

*Proof.* Omitted. The complete proof is given in Barwise. It is quite long and technical and detail-oriented. □

## 3 Amenability and Oracle Computability

Let $X \subseteq A$ for an admissible set $A$. We define the $X$-computable sets to be the the ones which are computable in the structure $(A, X, \in)$ as a model of KP.

Not all subsets of an admissible ordinal $\alpha$ work well as oracles. The following property identifies which ones do:

**Definition 1.** *A set $X \subseteq \alpha$ is called* ***amenable*** *for $\alpha$ or* ***regular*** *if for every $U \in L_\alpha$, $X \cap U \in L_\alpha$. In other words, the intersection of such a set with any finite set is still finite.*

By $\Delta_1$ separation, all of the $\Delta_1$ sets are amenable for $\alpha$. Next, we will show that if $\alpha$ is a successor cardinal, then all of the c.e. sets are amenable.

**Definition 2.** *Let $\alpha$ be an an admissible ordinal. The $\Sigma_1$ projectum, denoted $\alpha^*$ or $\sigma_1 p(\alpha)$ is the least $\beta$ such that there is a total $\alpha$-computable one-to-one function from $\alpha$ into $\beta$.*

Assume $A \subset \delta < \alpha^*$. If $A$ is $\alpha$-c.e. then $A$ is $\alpha$-finite.

*Proof.* Let $A$ be $\alpha$-c.e. and suppose $g$ is a computable function whose range is $A$ and domain is an initial segment of $\alpha$. The domain of $g$ can't be all of $\alpha$, since then $g$ would be a total one-to-one $\alpha$-c.e. map into $\delta$ ¡ $\alpha^*$, a contradiction. So the domain of $g$ is some $\beta \in \alpha$. But the image of a finite sets under a computable map is finite. □

**Theorem 5.** *The projectum $\alpha^*$ is the least $\beta$ such that some $\alpha$-c.e. subset of $\beta$ is not $\alpha$-finite.*

*Proof.* By the previous proposition, we only need to find a subset of $\beta$ which is not $\alpha$-finite. Let $f$ be a one-to-one total recursive map into $\beta$. The image of $\alpha$ in $\beta$ is certainly c.e. - we claim further that it is not finite. Suppose for the same of a contradiction that it is. Since $f$ is injective, there is a partial recursive map $f^{-1} : f(\alpha) \to \alpha$. But the image of a finite set under a recursive function is finite, so $f^{-1}(f(\alpha)) = dom(f)$ is finite; this is only possible if it is not all of $\alpha$, a contradiction. □

**Corollary 6.** *There exists a non-regular $\alpha$-c.e. set iff $\alpha^* < \alpha$. In other words, if $\alpha^* = \alpha$ then every $\alpha$-c.e. set is regular. In particular, if $\alpha$ is a successor cardinal then we immediately see that $\alpha = \alpha^*$.*

As a final remark, if $V = L$ then every set is amenable for $\alpha$ for every admissible $\alpha$, modulo some conditions on $\alpha$. A proof of this is found in Sacks. In modern computable structure theory using $\alpha$ recursion theory, this is a common assumption.