Online Appendix for "Incentives and Performance with Optimal Money Management Contracts"

Stefano Pegoraro*

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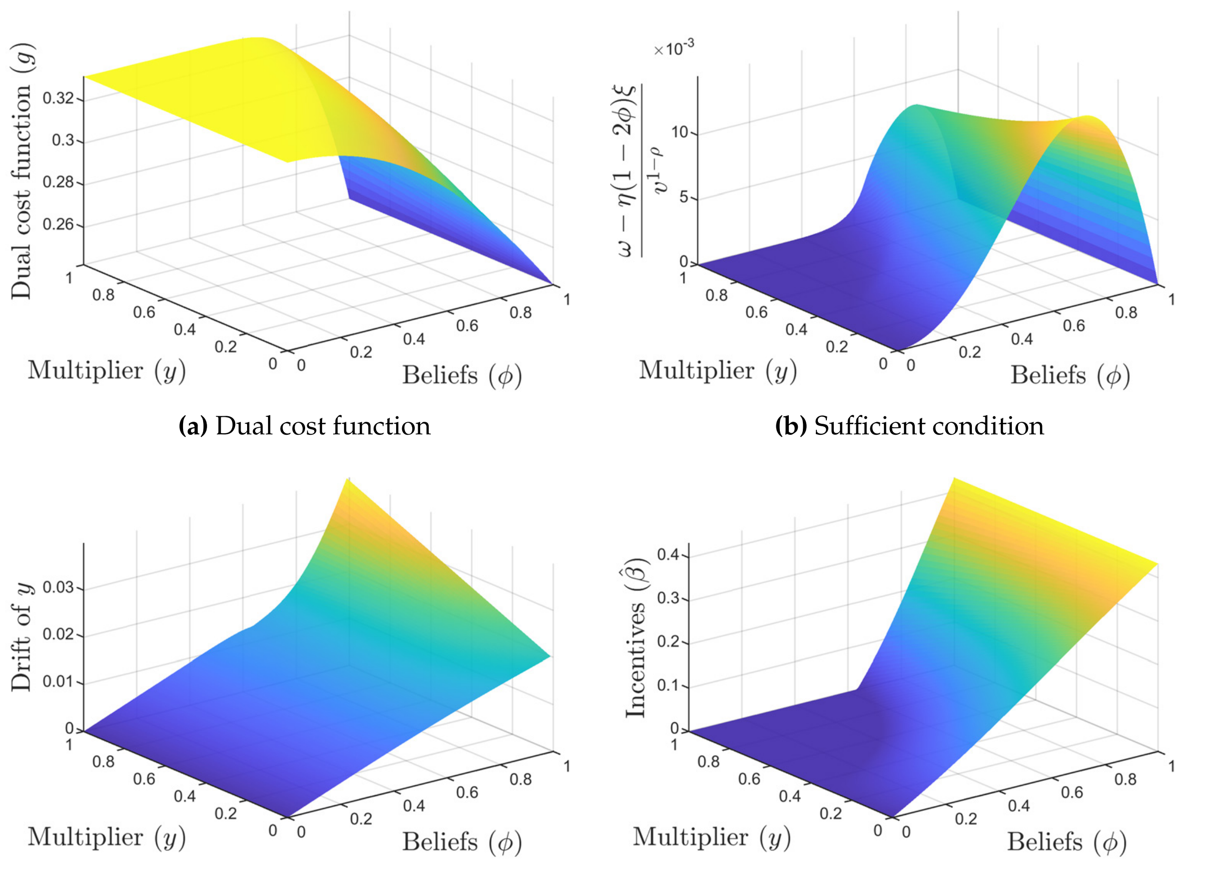
This appendix contains additional results and derivations for Pegoraro (2022).

O.1 Additional Plots



(a) Dual cost function

(b) Sufficient condition

(c) Drift of y

(d) Incentives

Figure O.1: Properties of the optimal contract as functions of the multiplier y and beliefs ϕ. Figure (a) plots the dual cost function $g(y, \phi)$; Figure (b) plots the right-hand side of (28); Figure (c) plots the drift of y; Figure (d) plots incentives $\hat{\beta}(y, \phi)$. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

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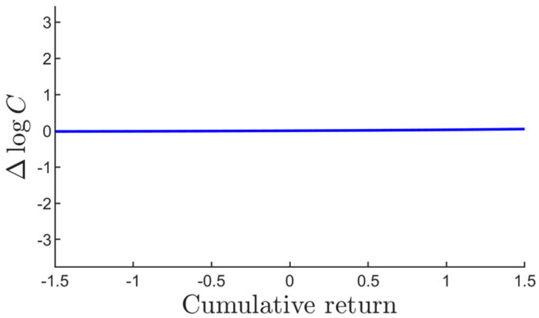
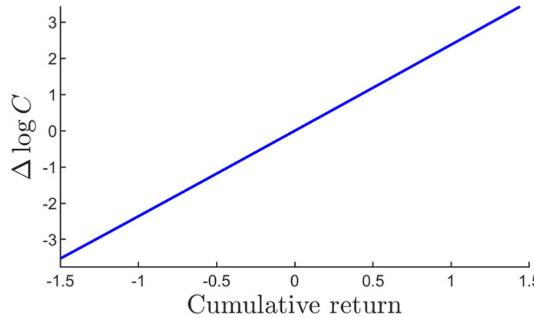
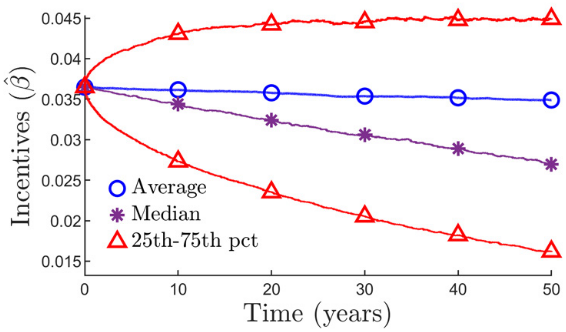


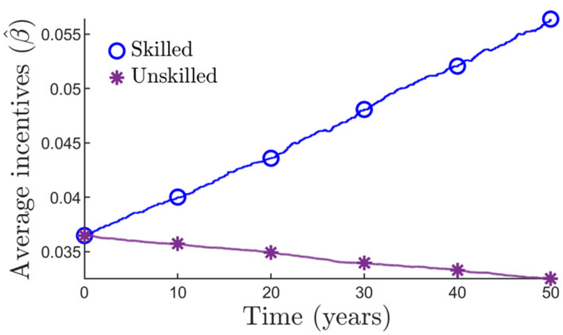


(a) Performance and compensation with $\phi_0 = 0.01$ **(b)** Performance and compensation with $\phi_0 = 0.99$

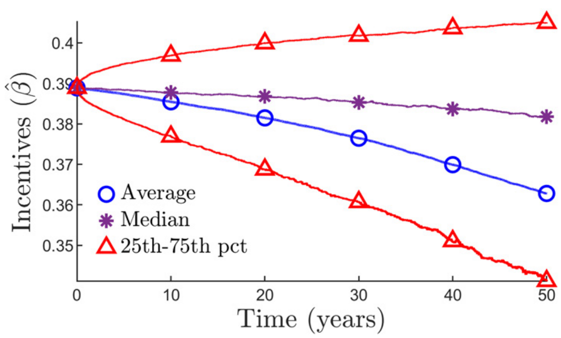
Figure O.2: Relation between cumulative performance and change in compensation when beliefs are close to zero or one. The curves represent the change in log-compensation as a function of cumulative performance. Curves are shifted to represent changes relative to an agent that has a zero cumulative performance. Performance and change in compensation are computed while assuming that returns are realized uniformly over time during the course of one year. Figures are drawn for initial multiplier $y_0 = 0$ and initial beliefs $\phi_0 = 0.01$ in (a) and $\phi_0 = 0.99$ in (b). The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.



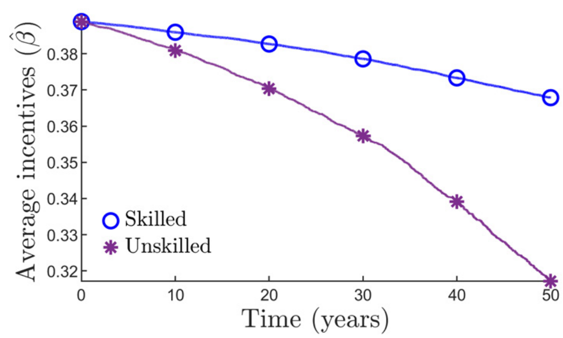
(a) Incentives for $\phi_0 = 0.1$



(b) Incentives by type for $\phi_0 = 0.1$

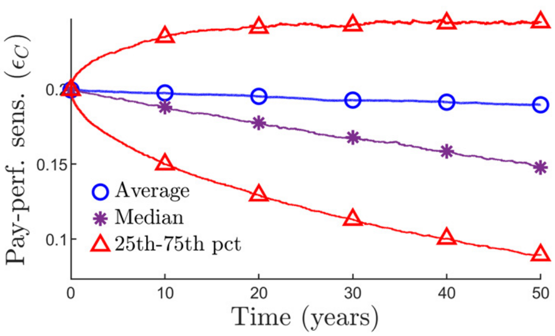


(c) Incentives for $\phi_0 = 0.9$

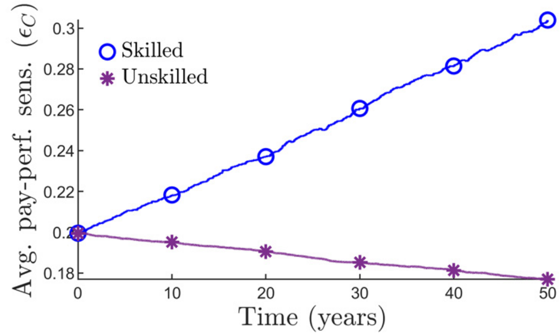


(d) Incentives by type for $\phi_0 = 0.9$

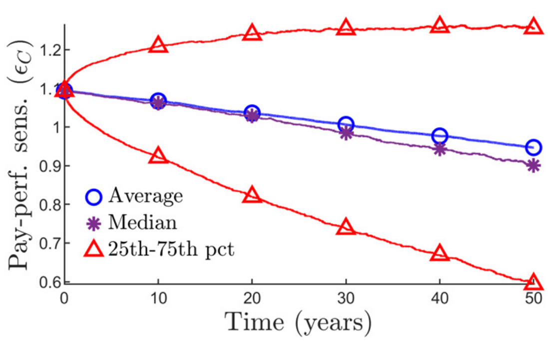
Figure O.3: Incentives, $\hat{\beta}(y_t, \phi_t)$, over time for prior ϕ_0 equal to 0.1 and 0.9. The plots on the left show the unconditional distribution of incentives at each point in time. The plots on the right show the average incentives conditional on the agent's type. The distributions are obtained from a sample of 10,000 independent simulations in which the fraction of skilled agents is equal to the prior ϕ_0. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.



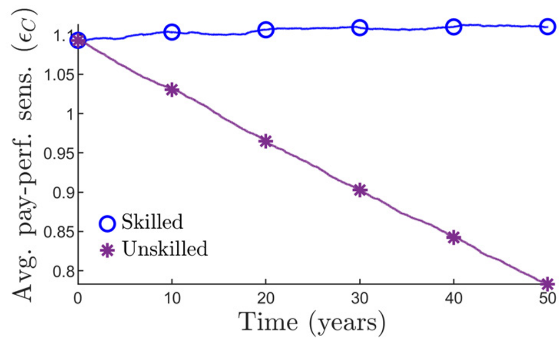
(a) Pay-performance sensitivity for $\phi_0 = 0.1$



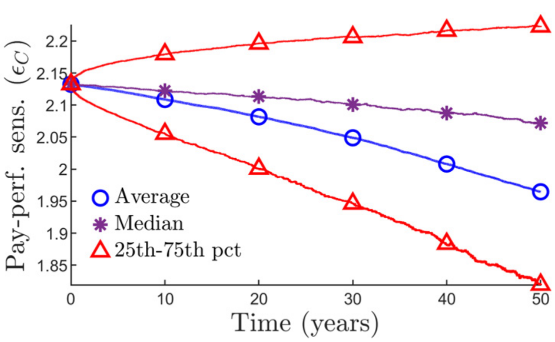
(b) Pay-performance sensitivity by type for $\phi_0 = 0.1$



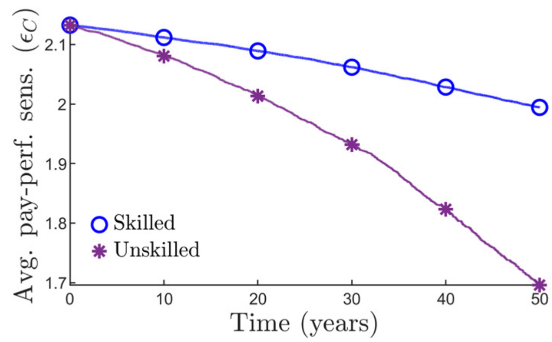
(c) Pay-performance sensitivity for $\phi_0 = 0.5$



(d) Pay-performance sensitivity by type for $\phi_0 = 0.5$

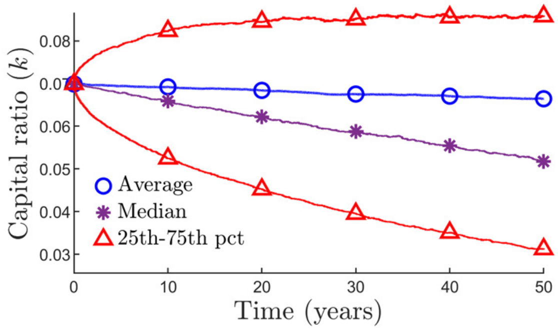


(e) Pay-performance sensitivity for $\phi_0 = 0.9$

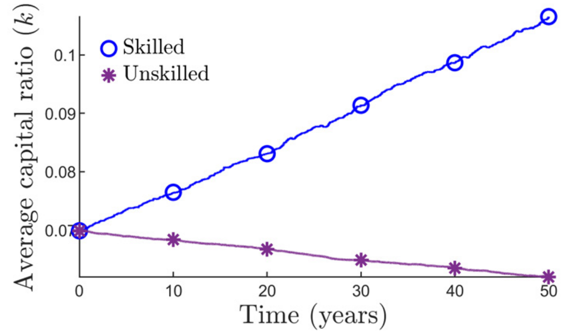


(f) Pay-performance sensitivity by type for $\phi_0 = 0.9$

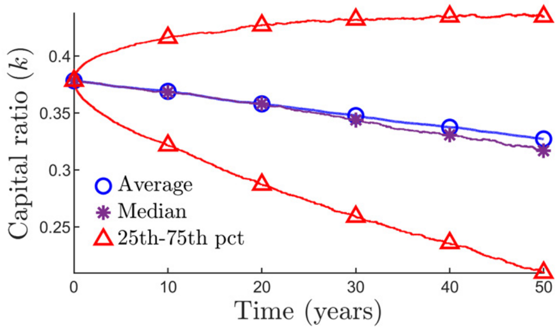
Figure O.4: Pay-performance sensitivity, $\varepsilon_C(y_t, \phi_t)$, over time for prior ϕ_0 equal to 0.1, 0.5, and 0.9. The plots on the left show the unconditional distribution of the pay-performance sensitivity at each point in time. The plots on the right show the average pay-performance sensitivity conditional on the agent's type. The distributions are obtained from a sample of 10,000 independent simulations in which the fraction of skilled agents is equal to the prior ϕ_0. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.



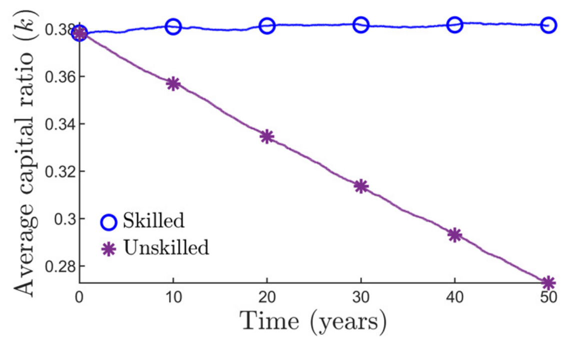
(a) Capital ratio for $\phi_0 = 0.1$



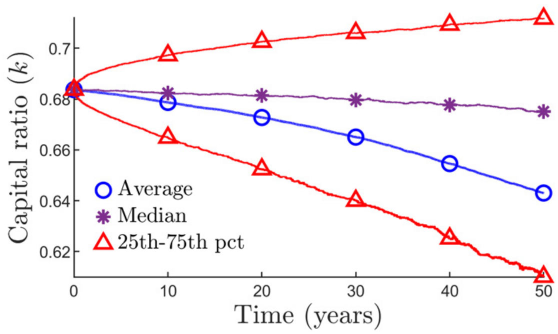
(b) Capital ratio by type for $\phi_0 = 0.1$



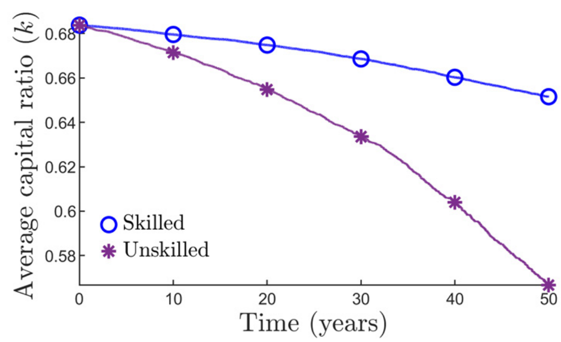
(c) Capital ratio for $\phi_0 = 0.5$



(d) Capital ratio by type for $\phi_0 = 0.5$



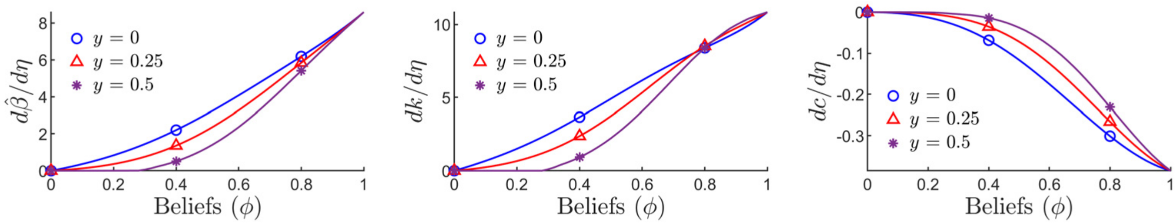
(e) Capital ratio for $\phi_0 = 0.9$



(f) Capital ratio by type for $\phi_0 = 0.9$

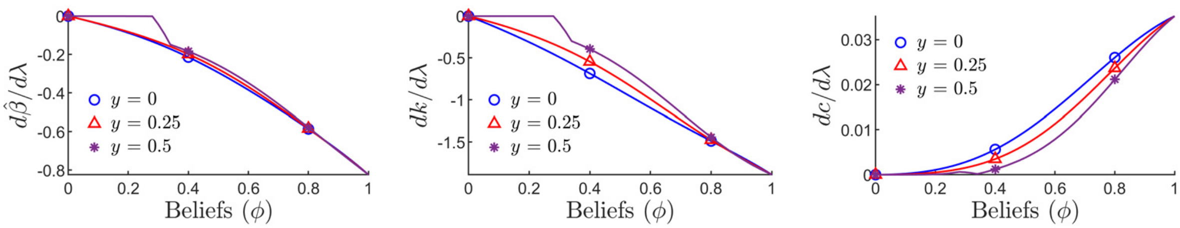
Figure O.5: Capital ratio, $k(y_t, \phi_t)$, over time for prior ϕ_0 equal to 0.1, 0.5, and 0.9. The plots on the left show the unconditional distribution of the capital ratio at each point in time. The plots on the right show the average capital ratio conditional on the agent's type. The plots on the right show the average pay-performance sensitivity conditional on the agent's type. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

O.2 Comparative Statics



(a) Incentives **(b)** Capital ratio **(c)** Compensation ratio

Figure O.6: Marginal effect of a change in the signal-to-noise ratio η on incentives $\hat{\beta}(y,\phi)$, capital ratio $k(y,\phi)$, and compensation ratio $c(y,\phi)$, shown as function of the co-state y and beliefs ϕ. Results are shown for three values of the co-state y. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

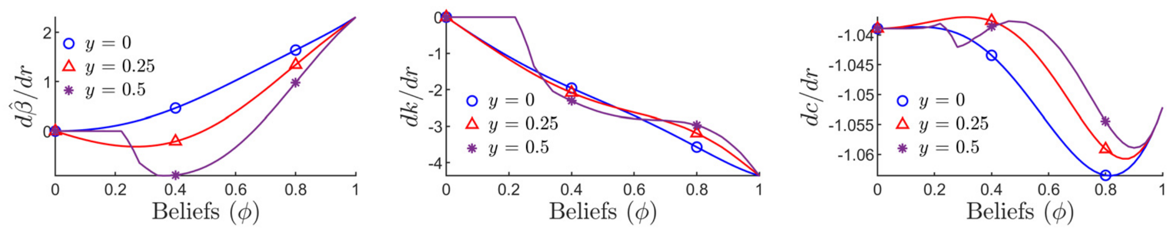


(a) Incentives **(b)** Capital ratio **(c)** Compensation ratio

Figure O.7: Marginal effect of a change in the efficiency of shirking λ on incentives $\hat{\beta}(y,\phi)$, capital ratio $k(y,\phi)$, and compensation ratio $c(y,\phi)$, shown as function of the co-state y and beliefs ϕ. Results are shown for three values of the co-state y. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

In this section, I briefly discuss the effect of a marginal change in parameters on the optimal contract. Based on the HJB equation (27), only four quantities affect the choice of the optimal controls, $\hat{\beta}(y,\phi)$ and $c(y,\phi)$, and the shape of the cost function, $g(y,\phi)$: the signal-to-noise ratio η, the efficiency of shirking λ, the relative discount rate of the two players $r - \delta(1-\rho)$, and the agent's relative risk aversion ρ.

Although the volatility of returns does not enter the HJB equation (27), it affects the pay-performance sensitivity and capital ratio through (29) and (30). In particular, when the volatility of returns increases (while keeping the signal-to-noise ratio constant), the pay-performance sensitivity declines. Moreover, less capital is allocated to the agent because, with more variability in returns, shirking becomes more difficult to detect and



(a) Incentives **(b)** Capital ratio **(c)** Compensation ratio

Figure O.8: Marginal effect of a change in the principal's discount rate r on incentives $\hat{\beta}(y, \phi)$, capital ratio $k(y, \phi)$, and compensation ratio $c(y, \phi)$, shown as function of the co-state y and beliefs ϕ. Results are shown for three values of the co-state y. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

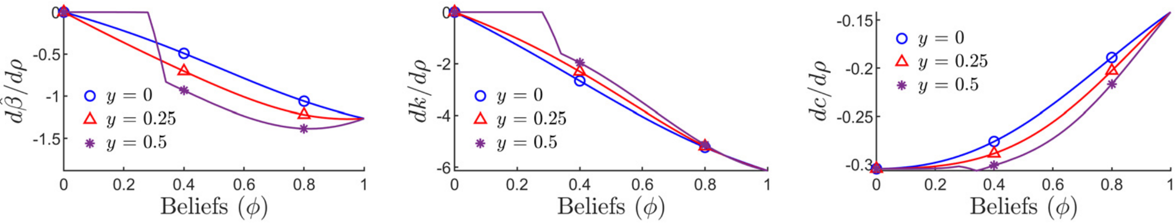
agency frictions are exacerbated.

To study how the remaining parameters affect the optimal contract, I numerically solve the model for a small change in parameters around the values I used in the paper. I then plot the effect of this marginal change on incentives, capital ratios, and compensation ratios.

Figure O.6 plots the marginal effect of an increase in the signal-to-noise ratio η. In general, incentives and capital ratio increase with η, whereas the compensation ratio declines. The mechanism for these results is analogous to the mechanism linking changes in these variables to changes in beliefs. When η and expected returns increase, the principal wants to delegate more capital to the agent and thus sets steeper incentives to prevent shirking. Furthermore, the cost of deferring compensation declines when expected returns are higher. The principal then chooses to back-load more compensation and lower the compensation ratio.

Figure O.7 shows that incentives and capital ratio decline in λ, but the compensation ratio increases. When the agent can steal more efficiently because of a higher λ, the principal faces worse agency frictions. Hence, she optimally chooses to invest less, provide more insurance to the agent, and limit the extent of compensation deferral.

In Figure O.8, I consider a change in the relative discount rate $r - \delta(1 - \rho)$ driven by a change in the principal's discount rate r. As the principal discounts the future more, she



(a) Incentives **(b)** Capital ratio **(c)** Compensation ratio

Figure O.9: Marginal effect of a change in the agent's relative risk aversion ρ on incentives $\hat{\beta}(y, \phi)$, capital ratio $k(y, \phi)$, and compensation ratio $c(y, \phi)$, shown as function of the co-state y and beliefs ϕ. Results are shown for three values of the co-state y. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

reduces the capital ratio and further back-loads consumption to the future. The effect on incentives is ambiguous, with a positive effect for a small y or a large ϕ, and a negative effect in the opposite case.

Finally, Figure O.9 shows the effect of an increase in the agent's relative risk aversion. With a more risk-averse agent, the principal faces increased costs when exposing the agent to risk. Hence, the principal reduces both incentives and capital delegation. The consumption ratio declines, and it declines more for points of the state space where $c(y, \phi)$ is larger, thus generating a smoother consumption path for the agent.

O.3 Auxiliary Lemmas

Lemma O.1. *Given a contract $\mathcal{C} = (C_t, K_t)_{t\geq 0}$ that is incentive compatible with $(m_t)_{t\geq 0}$, beliefs ϕ_t evolve as*

$$d\phi_t = \eta\phi_t(1-\phi_t)dW_t^{\mathcal{C}},$$

where

$$W_t^{\mathcal{C}} := \frac{1}{\sigma}\int_0^t [dR_t - (\sigma\eta\phi_t - m_t)dt] \tag{O.1}$$

is a standard Brownian motion under the measure of returns induced by $\mathcal{C}$.

Proof. Define

$$W_t^{\mathcal{C},0} := \frac{1}{\sigma}\int_0^t (dR_s + m_s ds),$$

which is a Brownian motion conditional on $h=0$, and define the likelihood ratio process

$$X_t := \exp\left\{\eta W_t^{\mathcal{C},0} - \frac{1}{2}\eta^2 t\right\},$$

which, by Girsanov's theorem, represents the ratio between the likelihood that the path $(R_s)_{0\leq s\leq t}$ is generated by a skilled agent ($h=1$) and the likelihood that the same path is generated by an unskilled agent ($h=0$.) Therefore, $X_t = \frac{\mathrm{E}[h|\mathcal{F}_t]}{1-\mathrm{E}[h|\mathcal{F}_t]}$.

We can then express beliefs as

$$\phi_t = \frac{\phi_0 X_t}{\phi_0 X_t + (1-\phi_0)}.$$

After applying Ito's lemma, we obtain

$$\begin{aligned}
d\phi_t &= -\frac{(1-\phi_0)\phi_0^2}{(\phi_0 X_t + (1-\phi_0))^3}(\eta X_t)^2 dt + \frac{(1-\phi_0)\phi_0}{(\phi_0 X_t + (1-\phi_0))^2}\eta X_t dW_t^{\mathcal{C},0} \\
&= -\frac{(1-\phi_0)}{\phi_0 X_t + (1-\phi_0)}\eta^2\phi_t^2 + \frac{(1-\phi_0)}{\phi_0 X_t + (1-\phi_0)}\eta\phi_t dW_t^{\mathcal{C},0} \\
&= -(1-\phi_t)\eta^2\phi_t^2 + (1-\phi_t)\eta\phi_t dW_t^{\mathcal{C},0}
\end{aligned}$$

$$= (1-\phi_t)\eta\phi_t\frac{1}{\sigma}(dR_t + m_t dt - \sigma\eta\phi_t dt).$$

□

Lemma O.2. *Let $\mathcal{C} = (C_t, K_t)_{t\geq 0}$ be a contract that is incentive compatible with the shirking process $m := (m_t)_{t\geq 0}$. Let*

$$V_t^m = \tilde{V}^m(\mathcal{C}_t, \phi_t) := \mathrm{E}\left[\int_t^\infty e^{-\delta(s-t)}u(C_s + \lambda m_s K_s)\, ds \middle| \mathcal{F}_t\right] \tag{O.2}$$

be the agent's continuation value at time $t \geq 0$. Then, there exists a progressively measurable process $(\beta_t^m)_{t\geq 0}$ such that

$$dV_t^m = (\delta V_t^m - u(C_t + \lambda m_t K_t))dt + \beta_t^m dW_t^{\mathcal{C}} \quad \textit{with} \quad \lim_{t\to\infty} \mathrm{E}\left[e^{-\delta t}V_t^m|\mathcal{F}_0\right] = 0. \tag{O.3}$$

Note equations (5) and (6) are special cases of equations (O.2) and (O.3) when the contract $\mathcal{C}$ is incentive compatible with no shirking; that is, $m_t = 0$ for all $t \geq 0$.

Proof. I use the martingale-representation approach as in Proposition 1 at p. 975 in Sannikov (2008).

Define

$$S_t^m := \int_0^t e^{-\delta s}u(C_s + \lambda m_s K_s)\, ds + e^{-\delta t}V_t^m. \tag{O.4}$$

Because $S_t^m = \mathrm{E}[S_T^m|\mathcal{F}_t]$ for all t and T such that $0 \leq t \leq T$, S_t^m is a P-martingale adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. By the martingale-representation theorem (Karatzas and Shreve, 1991), a progressively measurable process $(\beta_t^m)_{t\geq 0}$ exists such that

$$S_t^m = S_0 + \int_0^t e^{-\delta s}\beta_s^m dW_s^{\mathcal{C}}, \tag{O.5}$$

where $W_t^{\mathcal{C}}$, defined as in (O.1), is a standard Brownian motion under the measure of returns induced by $\mathcal{C}$.

By plugging (O.5) into (O.4) and differentiating, I obtain

$$e^{-\delta t}\beta_t^m dW_t^{\mathcal{C}} = e^{-\delta t}u(C_t + \lambda m_t K_t)dt - \delta e^{-\delta t}V_t^m + e^{-\delta t}dV_t^m.$$

After dividing by $e^{-\delta t}$ and rearranging terms, I obtain the stochastic differential equation in (O.3). The limit condition $\lim_{t\to\infty} \mathrm{E}\left[e^{-\delta t}V_t^m|\mathcal{F}_0\right] = 0$ ensures the solution of the backward stochastic differential equation (O.3) coincides with (O.2). □

LEMMA O.3. *Let ϕ_t be the value of the principal's beliefs at time t, and let ϕ_t^A be the value of the agent's beliefs at time t. If $m_s = 0$ for all $s > t$, then at time t, the agent's future expected utility exceeds the continuation value promised by the principal by an amount equal to*

$$\mathrm{E}\left[\int_t^\infty e^{-\delta(s-t)}\Gamma_{t,s}^A\beta_s\eta(\phi_s^A - \phi_s)\,ds\middle|\mathcal{F}_t\right],$$

where $(\Gamma_s^A)_{s\geq t}$ is a density process such that

$$\Gamma_{t,s}^A := \exp\left\{\int_t^s \eta(\phi_u^A - \phi_u)\,dW_u^{\mathcal{C}} - \frac{1}{2}\int_t^s \eta^2(\phi_u^A - \phi_u)^2\,du\right\}. \tag{O.6}$$

Proof. Let $V_t = \tilde{V}(\mathcal{C}_t, \phi_t)$ be the agent's continuation value promised by the principal, and let $V_t^A = \tilde{V}(\mathcal{C}_t, \phi_t^A)$ be the agent's future expected utility based on the agent's information.

For $s \geq t$, let

$$W_{t,s}^A := \frac{1}{\sigma}\int_t^s \left[dR_u - \sigma\eta\phi_u^A du\right]$$

be a standard Brownian motion under the measure of returns induced by contract $\mathcal{C}$ and time-t beliefs equal to ϕ_t^A.

I define a density process $(\Gamma_{t,s}^A)_{s\geq t}$, where $\Gamma_{t,s}^A$ is given by (O.6). By Girsanov's theorem, $\Gamma_{t,s}^A$ represents the change of measure for the path of returns from t to s induced by the agent's private beliefs ϕ_t^A. In particular, if $P_{t,s}^{\mathcal{C}}$ is the probability measure for which $(W_u^{\mathcal{C}} - W_t^{\mathcal{C}})_{t\leq u\leq s}$ is a standard Brownian motion, and if $P_{t,s}^A$ is the probability measure for which

$(W^A_{t,u})_{t\le u\le s}$ is standard Brownian motion, then $\Gamma^A_{t,s} = \frac{dP^A_{t,s}}{dP^{\mathcal{C}}_{t,s}}$. Therefore,

$$V^A_t = \mathrm{E}\left[\int_t^\infty e^{-\delta(s-t)}\Gamma^A_{t,s}u(C_s)\,ds\middle|\mathcal{F}_t\right].$$

Applying Girsanov's theorem to (6), we have

$$dV_s = (\delta V_s - u(C_s) + \beta_s\eta(\phi^A_s - \phi_s))ds + \beta_s dW^A_s$$

and, solving forward,

$$V_t = \mathrm{E}\left[\int_t^\infty e^{-\delta(s-t)}\Gamma^A_{t,s}\left(u(C_s) - \beta_s\eta(\phi^A_s - \phi_s)\right)\,ds\middle|\mathcal{F}_t\right].$$

I thus obtain

$$V^A_t - V_t = \mathrm{E}\left[\int_t^\infty e^{-\delta(s-t)}\Gamma^A_{t,s}\beta_s\eta(\phi^A_s - \phi_s)\,ds\middle|\mathcal{F}_t\right].$$

□

LEMMA O.4. *The information rent ξ_t is related to the marginal value of beliefs $\partial_\phi V(\mathcal{C}_t, \phi_t)$ by*

$$\xi_t = \phi_t(1-\phi_t)\partial_\phi \tilde{V}(\mathcal{C}_t, \phi_t).$$

Moreover, in any incentive-compatible contract, $\xi_t \ge 0$.

Proof. To show $\xi_t = \phi_t(1-\phi_t)\partial_\phi \tilde{V}(\mathcal{C}_t, \phi_t)$, note

$$V_t = \tilde{V}(\mathcal{C}_t, \phi_t) = (1-\phi_t)\mathrm{E}\left[\int_t^\infty e^{-\delta(s-t)}u(C_s)\,ds\middle|\mathcal{F}_t, h=0\right] + \phi_t\mathrm{E}\left[\int_t^\infty e^{-\delta(s-t)}u(C_s)\,ds\middle|\mathcal{F}_t, h=1\right].$$

The conditional expected values in the last term are functions only of the contract $\mathcal{C}$ and not of beliefs. Therefore, letting $V^h_t := \mathrm{E}\left[\int_t^\infty e^{-\delta s}u(c_s)\,ds\middle|\mathcal{F}_t, h\right]$ for $h \in \{0,1\}$, and taking partial derivatives with respect to ϕ_t in the previous expression, we obtain

$$\phi_t(1-\phi_t)\partial_\phi \tilde{V}(\mathcal{C}_t,\phi_t) = \phi_t(1-\phi_t)(V_t^1 - V_t^0) = \phi_t(V_t^1 - V_t).$$

The continuation value V_t evolves as in (6). By the martingale representation theorem and Girsanov's theorem, V_t^1 evolves as

$$dV_t^1 = (\delta V_t^1 - u(C_t) - \eta(1-\phi_t)\beta_t^1)dt + \beta_t^1 dW_t^{\mathcal{C}},$$

for some $\mathcal{F}_t$-adapted process $(\beta_t^1)_{t\geq 0}$. Using Ito's lemma, I thus derive the law of motion of $\phi_t(V_t^1 - V_t)$,

$$d[\phi_t(V_t^1 - V_t)] = (\delta\phi_t(V_t^1 - V_t) - \eta\beta_t\phi_t(1-\phi_t))dt + \omega_t' dW_t^{\mathcal{C}},$$

where $\omega_t' := \eta\phi_t(1-\phi_t)(V_t^1 - V_t) + \phi_t(\beta_t^1 - \beta_t)$. Because $\lim_{t\to\infty} \mathrm{E}[e^{-\delta t}V_t|\mathcal{F}_0] = 0$ by (6), and because $0 \leq \phi_t V_t^1 \leq V_t$, a terminal condition $\lim_{t\to\infty} \mathrm{E}[e^{-\delta t}\phi_t(V_t^1 - V_t)] = 0$ holds. Solving the BSDE for $\phi_t(V_t^1 - V_t)$ yields

$$\phi_t(V_t^1 - V_t) = \mathrm{E}\left[\int_t^\infty e^{-\delta s}\eta\beta_s\phi_s(1-\phi_s)\, ds \middle| \mathcal{F}_t\right].$$

Comparing this expression with (10), we conclude $\xi_t = \phi_t(1-\phi_t)\partial_\phi \tilde{V}(\mathcal{C}_t,\phi_t)$.

It remains to prove that in an incentive-compatible contract, $\xi_t \geq 0$. Define $o_t := \beta_t - u'(C_t)\sigma\lambda K_t - \eta\xi_t \geq 0$. Then, (12) can be written as

$$d\xi_t = [\delta\xi_t - (u'(C_t)\sigma\lambda K_t + \eta\xi_t + o_t)\eta\phi_t(1-\phi_t)]dt + \omega_t dW_t^{\mathcal{C}} \quad \text{with} \quad \lim_{t\to\infty} \mathrm{E}[e^{-\delta t}\xi_t|\mathcal{F}_0] = 0.$$

Solving this BSDE, we obtain the following expression for ξ_t

$$\xi_t = \mathrm{E}\left[\int_t^\infty e^{\int_t^s \left(\eta^2\phi_i(1-\phi_i)-\delta\right) di}\eta\phi_s(1-\phi_s)[u'(C_s)\lambda\sigma K_s + o_s]\, ds \middle| \mathcal{F}_t\right].$$

Because $u'(C_s)\lambda\sigma K_s \geq 0$ and $o_s \geq 0$ in an incentive-compatible contract, we conclude that $\xi_t \geq 0$. □

LEMMA O.5. *The dual cost function takes the form $G^*(V_0, Y_0, \phi_0) = v_0 g^*(y_0, \phi_0)$ for some function g^*, where $v_t := ((1-\rho)V_t)^{\frac{1}{1-\rho}}$ and $y_t := v_t^{-\rho} Y_t$.*

Proof. I start by deriving the laws of motion of v_t and y_t. To do so, define the scaled control variables $c_t := \frac{C_t}{v_t}$ and $\hat{\beta}_t := \frac{\beta_t}{v_t^{1-\rho}}$.

Starting from the stochastic differential equation for V_t in (6) and from $v_t := ((1-\rho)V_t)^{\frac{1}{1-\rho}}$, I apply Ito's lemma to obtain

$$dv_t = ((1-\rho)V_t)^{\frac{\rho}{1-\rho}} \left(\delta V_t - \frac{C_t^{1-\rho}}{1-\rho}\right) dt + \frac{1}{2}\rho\,((1-\rho)V_t)^{\frac{2\rho-1}{1-\rho}}\,\beta_t^2\,dt + ((1-\rho)V_t)^{\frac{\rho}{1-\rho}}\,\beta_t\,dW_t^{\mathfrak{C}}$$

$$dv_t = v_t^{\rho}\left(\delta\frac{v_t^{1-\rho}}{1-\rho} - \frac{C_t^{1-\rho}}{1-\rho}\right) dt + \frac{1}{2}\rho v_t^{2\rho-1}\beta_t^2\,dt + v_t^{\rho}\beta_t\,dW_t^{\mathfrak{C}}$$

$$dv_t = \left(\frac{\delta}{1-\rho} - \frac{c_t^{\rho}}{1-\rho}\right) v_t\,dt + \frac{1}{2}\rho v_t\hat{\beta}_t^2\,dt + v_t\hat{\beta}_t\,dW_t^{\mathfrak{C}}, \tag{O.7}$$

which coincides with (25) after dividing both sides of the equality by v_t.

Next, I combine (19), (25), and $y_t := v_t^{-\rho} Y_t$ and I use Ito's lemma to obtain

$$\begin{aligned} dy_t = v_t^{-\rho}\left((r-\delta)Y_t dt + \phi_t\frac{\eta^2}{\lambda}C_t^{\rho}dt + \eta dU_t\right) - \rho v_t^{-\rho}Y_t\left(\frac{\delta}{1-\rho} - \frac{c_t^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}_t^2\right)dt + \\ + \frac{1}{2}\rho(\rho+1)v_t^{-\rho}Y_t\hat{\beta}_t^2\,dt - \rho v_t^{-\rho}Y_t\,dW_t^{\mathfrak{C}} \end{aligned}$$

$$dy_t = \phi_t\frac{\eta^2}{\lambda}c_t^{\rho}\,dt + y_t\left(r - \frac{\delta}{1-\rho} + \rho\frac{c_t^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}_t^2\right)dt - \rho y_t\,dW_t^{\mathfrak{C}} + \eta d\hat{u}_t, \tag{O.8}$$

where $(\hat{u}_t)_{t\geq 0} \in \mathfrak{I}$ is defined as $\hat{u}_t := \int_0^t v_s^{-\rho} dU_s$. The law of motion in (O.8) coincides with (26) after observing that, in an optimal dual contract solving (20), $U_t = 0$ for all $t \geq 0$.

From equation (O.7), we obtain

$$v_t = v_0 \exp\left\{\int_0^t \left(\frac{\delta}{1-\rho} - \frac{c_s^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}_s^2 - \frac{1}{2}\hat{\beta}_s^2\right)ds + \int_0^t \hat{\beta}_s dW_s^{\mathfrak{C}}\right\}.$$

Using this expression for v_t together with $c_t = \frac{C_t}{v_t}$, $\hat{\beta}_t = \frac{\beta_t}{v_t^{1-\rho}}$ and $d\hat{u}_t = v_t^{-\rho} dU_t$, I write the objective function of the dual problem as

$$v_0 \sup_{(\hat{u}_t)_{t\geq 0}\in \mathcal{I}} \mathrm{E}\left[\int_0^\tau e^{-\int_0^t r - \left(\frac{\delta}{1-\rho} - \frac{c_s^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}_s^2\right) ds} B_t \left\{ \left(c_t - \frac{\eta}{\lambda}\phi_t\hat{\beta}_t c_t^\rho + y_t\eta\phi_t(1-\phi_t)\beta_t\right) dt - \hat{\beta}_t d\hat{u}_t \right\} \middle| \mathcal{F}_0 \right], \tag{O.9}$$

where B_t is a density process,

$$B_t := \exp\left\{\int_0^t \hat{\beta}_s \, dW_s^{\mathcal{C}} - \frac{1}{2}\int_0^t \hat{\beta}_s^2 \, ds\right\}.$$

The minimization problem in (20) is equivalent to minimizing (O.9) with respect to $(c_t, \hat{\beta}_t)_{t\geq 0}$ and subject to (O.8) and (3) with $m_t = 0$ for all $t \geq 0$. Hence, $G^*(V_0, Y_0, \phi_0) = v_0 g^*(y_0^{\hat{u}}, \phi_0)$, where

$$g^*(y_0^{\hat{u}}, \phi_0) =$$
$$\inf_{(c_t,\hat{\beta}_t)_{t\geq 0}} \sup_{(\hat{u}_t)_{t\geq 0}\in\mathcal{I}} \mathrm{E}\left[\int_0^\tau e^{-\int_0^t r - \left(\frac{\delta}{1-\rho} - \frac{c_s^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}_s^2\right) ds} B_t \left\{ \left(c_t - \frac{\eta}{\lambda}\phi_t\hat{\beta}_t c_t^\rho + y_t\eta\phi_t(1-\phi_t)\beta_t\right) dt - \hat{\beta}_t d\hat{u}_t \right\} \middle| \mathcal{F}_0 \right]$$
$$\text{s.t.} \quad \text{(O.8) and (3) with } m_t = 0 \ \forall t \geq 0,$$

thus concluding the proof. □

LEMMA O.6. *Consider a point (y, ϕ) with $y \in \mathbb{R}$ and $\phi \in [0, 1]$. Let N be a neighborhood of (y, ϕ). Consider a twice-differentiable function ψ that is concave in y and such that*

$$0 = g^{*usc}(y, \phi) - \psi(y, \phi) = \max_N \{g^{*usc} - \psi\},$$

where $g^{*usc}(y, \phi)$ is the upper-semicontinuous envelope of g^* at (y, ϕ).[1]

[1]The upper-semicontinuous envelope of g^* at (y, ϕ) is defined as

$$g^{*\text{usc}}(y, \phi) := \limsup_{(y',\phi')\to(y,\phi)} g^*(y', \phi').$$

Then,

$$\inf_{c\geq 0,\hat\beta\geq \eta\psi_y(y,\phi)}\left\{\left(c-\frac{\eta}{\lambda}\phi\hat\beta c^\rho+y\hat\beta\eta\phi(1-\phi)\right)+\mathcal{A}[\psi;(y,\phi);c,\hat\beta]\right\}\geq 0,$$

where

$$\begin{aligned}\mathcal{A}[\psi;(y,\phi);c,\hat\beta]&\coloneqq \psi(y,\phi)\left(\frac{\delta}{1-\rho}-r-\frac{c^{1-\rho}}{1-\rho}+\frac{1}{2}\rho\hat\beta^2\right)\\&\quad+\psi_y(y,\phi)\left[\frac{\eta^2}{\lambda}\phi c^\rho+y\left(r-\frac{\delta}{1-\rho}+\rho\frac{c^{1-\rho}}{1-\rho}+\frac{1}{2}\rho\hat\beta^2\right)\right]\\&\quad-\psi_y(y,\phi)y\rho\hat\beta^2+\psi_\phi(y,\phi)\eta\phi(1-\phi)\hat\beta\\&\quad+\frac{1}{2}\psi_{yy}(y,\phi)y^2\rho^2\hat\beta^2+\frac{1}{2}\psi_{\phi\phi}(y,\phi)\eta^2\phi^2(1-\phi)^2\\&\quad-\psi_{y\phi}(y,\phi)y\rho\hat\beta\eta\phi(1-\phi).\end{aligned}$$

Proof. Consider a sequence $(y^n,\phi^n)_n$ such that $(y^n,\phi^n)\to(y,\phi)$ as $n\to\infty$. Then, $g^*(y^n,\phi^n)\to g^{*\text{usc}}(y,\phi)$. Let $\delta_n\coloneqq g^*(y_n,\phi_n)-\psi(y_n,\phi_n)\to 0$ and let $(h_n)_n$ be a sequence of strictly positive numbers such that $h_n\to 0$ and $\frac{\delta_n}{h_n}\to 0$ as $n\to\infty$. Consider an arbitrary $c\geq 0$ and $\hat\beta\geq\eta\psi_y(y,\phi)$. Define $\hat\beta^n_t\coloneqq\max\{\hat\beta,\eta\psi_y(y^{n,\hat u}_t,\phi^n_t)\}$, where $(v^n_t,y^{n,\hat u}_t,\phi^n_t)_{t\geq 0}$ denote the state process with starting point $(v_0,y^n_0,\phi^n_0)=(v_0,y^n,\phi^n)$ and with $c_t=c$ and $\hat\beta_t=\hat\beta^n_t$.

Define the stopping time $\tau_n\coloneqq\inf\{t\geq 0\colon(y^{n,\hat u}_t,\phi^n_t)\notin N\}$, and let $\theta_n\coloneqq\min\{\tau_n,h_n\}$. Using the dynamic programming principle,

$$\begin{aligned}v_0g^*(y^n,\phi^n)\leq\sup_{\hat u\in\mathcal{J}}\mathrm{E}\bigg[\int_0^{\theta_n}e^{-rt}v^n_t\left\{\left(c-\frac{\eta}{\lambda}\phi_t\hat\beta^n_{t^-}c^\rho+y^{n,\hat u}_{t^-}\hat\beta^n_{t^-}\eta\phi_t(1-\phi_t)\right)dt-\hat\beta^n_{t^-}d\hat u_t\right\}\\+e^{-r\theta_n}v_{\theta_n}g^*(y^{n,\hat u}_{\theta_n},\phi^n_{\theta_n})\bigg].\end{aligned}$$

Because in N we have that $g^*\leq\psi$,

$$\begin{aligned}v_0[\psi(y^n,\phi^n)+\delta_n]\leq\sup_{\hat u\in\mathcal{J}}\mathrm{E}\bigg[\int_0^{\theta_n}e^{-rt}v^n_t\left\{\left(c-\frac{\eta}{\lambda}\phi_t\hat\beta^n_{t^-}c^\rho+y^{n,\hat u}_{t^-}\hat\beta^n_{t^-}\eta\phi_t(1-\phi_t)\right)dt-\hat\beta^n_{t^-}d\hat u_t\right\}\\+e^{-r\theta_n}v^n_{\theta_n}\psi(y^{n,\hat u}_{\theta_n},\phi^n_{\theta_n})\bigg],\end{aligned}$$

and using Ito's lemma, we obtain

$$\delta_n \le \sup_{\hat{u}\in\mathcal{I}} \mathrm{E}\bigg[\int_0^{\theta_n} e^{-rt} v_t^n \Big\{ \Big(f(y_{t^-}^{n,\hat{u}}, \phi_t^n; c, \hat{\beta}_{t^-}^n) + \mathcal{A}[\psi; y_{t^-}^{n,\hat{u}}, \phi_t^n; c, \hat{\beta}_{t^-}^n]\Big)\, dt \\ - \hat{\beta}_{t^-}^n d\hat{u}_t + \psi(y_{t^-}^{n,\hat{u}} + \eta d\hat{u}_t, \phi_t^n) - \psi(y_{t^-}^{n,\hat{u}}, \phi_t^n)\Big\}\bigg],$$

where

$$f(y, \phi; c, \hat{\beta}) \coloneqq c - \frac{\eta}{\lambda}\phi\hat{\beta}c^{\rho} + y\hat{\beta}\eta\phi(1-\phi).$$

Because ψ is concave in y and because $\hat{\beta}_{t^-}^n \ge \eta\psi_y(y_{t^-}^{n,\hat{u}}, \phi_t^n)$, the right-hand side of the previous expression is maximized by $d\hat{u}_t = 0$ for $t \in [0, \theta_n]$, and therefore,

$$\frac{\delta_n}{h_n} \le \mathrm{E}\bigg[\frac{1}{h_n}\int_0^{\theta_n} e^{-rt} v_t \Big(f(y_t^{n,0}, \phi_t^n; c, \hat{\beta}_t^n) + \mathcal{A}[\psi; (y_t^{n,0}, \phi_t^n); c, \hat{\beta}_t^n]\Big)\, dt\bigg],$$

Where I used the fact that, with $\hat{u}_t = 0$ for $t \in [0, \theta_n]$, the trajectories of $(y_t^{n,0}, \phi_t^n)_{t\in[0,\theta_n]}$ and $(\hat{\beta}_t^n)_{t\in[0,\theta_n]}$ are continuous in time.

Therefore, for n large enough, $\theta_n = h_n$ almost surely. By the mean value theorem,

$$\frac{1}{h_n}\int_0^{\theta_n} e^{-rt} v_t \Big(f(y_t^{n,0}, \phi_t^n; c, \hat{\beta}_t^n) + \mathcal{A}[\psi(y_t^{n,0}, \phi_t^n); c, \hat{\beta}_t^n]\Big)\, dt \overset{\text{a.s.}}{\to} v_0 \Big(f(y, \phi; c, \hat{\beta}) + \mathcal{A}[\psi; (y, \phi); c, \hat{\beta}]\Big).$$

Moreover, the random variable $\frac{1}{h_n}\int_0^{\theta_n} e^{-rt} v_t \Big(f(y_t^{n,0}, \phi_t^n; c, \hat{\beta}_t^n) + \mathcal{A}[\psi; (y_t^{n,0}, \phi_t^n); c, \hat{\beta}_t^n]\Big)\, dt$ is bounded almost surely in N. Hence, by the dominated convergence theorem,

$$v_0 f(y, \phi; c, \hat{\beta}) + v_0\mathcal{A}[\psi; (y, \phi); c, \hat{\beta}] \ge 0.$$

Because $v_0 > 0$, and because $c \ge 0$ and $\hat{\beta} \ge \eta\psi_y(y, \phi)$ are arbitrary, we obtain the result. □

LEMMA O.7. *$G^*(V, Y, \phi)$ is differentiable with respect to Y for $Y \neq 0$.*

Proof. First note that by Lemma O.5, $G^*(V, Y, \phi)$ is constant and equal to zero when $V = 0$. Hence, $G^*(V, \cdot, \phi)$ is differentiable for $V = 0$. Second, note $g^*(V, Y, 0)$ and $G^*(V, Y, 1)$ are constant in Y, and hence, $G^*(V,\cdot,\phi)$ is differentiable when $\phi\in\{0,1\}$.

To prove the result when $V>0$ and $\phi\in(0,1)$, I use Lemma O.5 to write $G^*(V,Y,\phi)=vg^*(y,\phi)$ for $y\coloneqq v^{-\rho}Y$. Because G^* is concave and increasing in Y, g^* is also concave and increasing in y. Therefore, the right and left derivatives of vg^* with respect to y always exist in the interior of the domain. Moreover, $G^*_Y(V,Y,\phi)$ exists if and only if $vg^*_y(y,\phi)$ exists.

To prove $G^*(V,Y,\phi)$ is differentiable in Y for $V>0$, $Y\neq 0$, and $\phi\in(0,1)$, I proceed by contradiction. Suppose a point (V',Y',ϕ') exists where $G^*(V',\cdot,\phi')$ is not differentiable. Then, $g^*(\cdot,\phi')$ is not differentiable at $y'\coloneqq((1-\rho)V')^{\frac{-\rho}{1-\rho}}Y'$. Because $g^*(\cdot,\phi')$ is increasing and concave, we must have that $0\leq g^*_{y^+}(y',\phi')<g^*_{y^-}(y',\phi')$. Consider a twice differentiable function F such that $g^{*\text{usc}}(y,\phi)-F(y,\phi)$ has a local maximum at (y',ϕ') and such that $g^{*\text{usc}}(y',\phi')-F(y',\phi')=0$.

Next, consider a $p\in(g^*_{y^+}(y',\phi'),g^*_{y^-}(y',\phi'))$ and take another function,

$$\psi_\varepsilon(y,\phi)\coloneqq F(y',\phi)+p(y-y')-\frac{1}{2\varepsilon}(y-y')^2.$$

For any arbitrary $\varepsilon>0$, $g^*(y,\phi)\leq\psi_\varepsilon(y,\phi)$ in a small enough neighborhood of (y',ϕ'), and (y',ϕ') is a local maximizer of $g^{*\text{usc}}(y,\phi)-\psi_\varepsilon(y,\phi)$ with $g^{*\text{usc}}(y',\phi')-\psi_\varepsilon(y',\phi')=0$. Moreover, ψ_ε is concave in y. Hence, by Lemma O.6,

$$\begin{aligned}\inf_{c,\hat\beta\geq\eta p}\Bigg\{&c-\frac{\eta}{\lambda}\phi'\hat\beta c^\rho+y'\hat\beta\eta\phi'(1-\phi')+\psi_\varepsilon(y',\phi')(y',\phi')\left(\frac{\delta}{1-\rho}-r-\frac{c^{1-\rho}}{1-\rho}+\frac{1}{2}\rho\hat\beta^2\right)\\&+p\left[\frac{\eta^2}{\lambda}\phi'c^\rho+y'\left(r-\frac{\delta}{1-\rho}+\rho\frac{c^{1-\rho}}{1-\rho}+\frac{1}{2}\rho\hat\beta^2\right)\right]\\&-py'\rho\hat\beta^2+F_\phi(y',\phi')\eta\phi'(1-\phi')\hat\beta+\frac{1}{2}F_{\phi\phi}(y',\phi')\eta^2\phi'^2(1-\phi')^2\\&-\frac{1}{\varepsilon}(\rho y'\hat\beta)^2\Bigg\}\geq 0.\end{aligned}\tag{O.10}$$

For $y'\neq 0$ and $\phi'\in(0,1)$, $(\rho y'\hat\beta)^2>0$ because $\hat\beta\geq\eta p>0$. Because $(\rho y'\hat\beta)^2$ is strictly positive and ε can be arbitrarily small, the inequality in (O.10) is a contradiction. Hence, we must have $g^*_{y^+}(y,\phi) = g^*_{y^-}(y,\phi)$ for any $y \neq 0$ and $\phi \in (0,1)$, from which it follows that $G^*(V,\cdot,\phi)$ is differentiable also for $V > 0$, $Y \neq 0$, and $\phi \in (0,1)$. □

LEMMA O.8. *If*

$$\frac{\eta}{\lambda\rho} \leq \sqrt{\frac{\delta - r(1-\rho)}{\rho}}\sqrt{2(1-\rho)}, \tag{O.11}$$

then $g^*(0,\phi) > 0$ *for all* $\phi \in [0,1]$.

Proof. Note $g^*(0,\phi) \geq g^*(0,1)$ for all ϕ. Hence, it suffices to prove $g^*(0,1) > 0$. With perfect information, $g^*(0,1) = j^1$, where $v_0 j^h$, for $h = i \in \{0,1\}$, is the principal's cost of delivering continuation value $\frac{{v_0}^{1-\rho}}{1-\rho}$ to an agent of known type h. The quantity j^h is a solution of the HJB equation

$$rj^h = \min_{c\geq 0,\hat{\beta}\geq 0}\left\{c - \frac{\eta h}{\lambda}\hat{\beta}c^\rho + j^h\left(\frac{\delta - c^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}^2\right)\right\}. \tag{O.12}$$

Let c_i and $\hat{\beta}_i$ be the minimizers of (O.12) for $h = i \in \{0,1\}$. For $h = 0$, we have

$$c_0 = (j^0)^{\frac{1}{\rho}} = \left(\frac{\delta - r(1-\rho)}{\rho}\right)^{\frac{1}{1-\rho}} > 0,$$

and $\hat{\beta}_0 = 0$.

Define $\bar{c} := (2\rho)^{\frac{1}{1-\rho}} c_0 > 0$. To show that $j^1 = g^*(0,1) > 0$, I proceed as in Di Tella and Sannikov (2021) and show that $j^1 \geq (1-\rho)\bar{c}^\rho$.

Standard viscosity solution arguments (Fleming and Soner, 2006; Pham, 2009) imply that j^1 is a viscosity solution of (O.12) for $h = 1$ and that a comparison principle applies. It is therefore sufficient to show that $\bar{j} := (1-\rho)\bar{c}^\rho > 0$ is a subsolution of (O.12) for $h = 1$. By the comparison principle, then $j^1 \geq (1-\rho)\bar{c}^\rho$.

To show $\bar{j}$ is a subsolution of (O.12), consider the following:

$$r\bar{j} - \min_{c,\hat{\beta}}\left\{c - \frac{\eta}{\lambda}\hat{\beta}c^\rho + \bar{j}\left(\frac{\delta - c^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}^2\right)\right\}$$

$$
\begin{aligned}
&= \max_c \left\{ r\bar{j} - c - \bar{j}\left(\frac{\delta - c^{1-\rho}}{1-\rho}\right) + \frac{1}{2}\rho\frac{1}{\bar{\bar{j}}}\left(\frac{\eta}{\rho\lambda}\right)^2 c^{2\rho} \right\} \\
&\leq \max_c \left\{ r\bar{j} - c - \bar{j}\left(\frac{\delta - c^{1-\rho}}{1-\rho}\right) + \frac{1}{2}\rho\frac{1}{\bar{\bar{j}}}\left(\frac{\delta - r(1-\rho)}{\rho}\right) 2(1-\rho)c^{2\rho} \right\} \\
&= \max_c \left\{ -c - \bar{j}\left(\frac{\rho(c^0)^{1-\rho} - c^{1-\rho}}{1-\rho}\right) + \frac{1}{2}\rho\frac{1}{\bar{j}}(c^0)^{1-\rho} 2(1-\rho)c^{2\rho} \right\} \\
&= \max_c \left\{ -c - \bar{c}^{\rho}\left(\rho c_0^{1-\rho} - c^{1-\rho}\right) + \frac{1}{2}\rho\bar{c}^{-\rho}c_0^{1-\rho}2c^{2\rho} \right\} \\
&= \max_c \left\{ -c - \bar{c}^{\rho}\left(\bar{c}^{1-\rho}/2 - c^{1-\rho}\right) + \frac{1}{2}\bar{c}^{1-2\rho}c^{2\rho} \right\} \\
&= \max_c \left\{ -c - \bar{c}/2 + c^{1-\rho}\bar{c}^{\rho} + \frac{1}{2}\bar{c}^{1-2\rho}c^{2\rho} \right\} \\
&= \max_c S(c, \bar{c})
\end{aligned}
$$

where the inequality follows from (O.11), and where $S(c, \bar{c}) := -c - \bar{c}/2 + c^{1-\rho}\bar{c}^{\rho} + \frac{1}{2}\bar{c}^{1-2\rho}c^{2\rho}$.

Note $S_c(\bar{c}, \bar{c}) = 0$ and

$$
S_{cc}(c, \bar{c}) = -\rho(1-\rho)c^{-\rho-1}\bar{c} - \rho(1-2\rho)\bar{c}^{1-2\rho}c^{2\rho-2} < 0.
$$

Hence, $S(c, \bar{c})$ is globally concave in c and it is maximized by $c = \bar{c}$. Thus,

$$
r\bar{j} - \min_{c,\hat{\beta}} \left\{ c - \frac{\eta h}{\lambda}\hat{\beta}c^{\rho} + \bar{j}\left(\frac{\delta - c^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}^2\right) \right\} \leq S(\bar{c}, \bar{c}) = 0.
$$

Therefore, $\bar{j} := (1-\rho)\bar{c}^{\rho} > 0$ is a subsolution of (O.12) for $h = 1$ and, by the comparison principle, $g^*(0, 1) = j^1 \geq \bar{j} = (1-\rho)\bar{c}^{\rho} > 0$. □

O.4 Additional Derivations

O.4.1 Derivation of Equation (31)

Consider all the terms containing $\hat{\beta}$ in (27):

$$\underbrace{-\eta\phi\hat{\beta}\frac{c^{\rho}}{\lambda} + y\hat{\beta}\eta\phi(1-\phi)}_{\text{flow cost}} + g(y,\phi)\underbrace{\left(\frac{1}{2}\rho\hat{\beta}^2\right)}_{\text{drift of } dv/v} + g_y(y,\phi)\underbrace{\left(\frac{1}{2}y\rho\hat{\beta}^2\right)}_{\text{drift of } dy} + g_y(y,\phi)\underbrace{\left(-\rho y\hat{\beta}^2\right)}_{\text{Cov}(dv/v,dy)}$$
$$+ g_\phi(y,\phi)\underbrace{\hat{\beta}\eta\phi(1-\phi)}_{\text{Cov}(dv/v,d\phi)} + \frac{1}{2}g_{yy}(y,\phi)\underbrace{(y\rho\hat{\beta})^2}_{\text{Var}(dy)} + g_{y\phi}(y,\phi)\underbrace{\left(-\rho y\hat{\beta}\eta\phi(1-\phi)\right)}_{\text{Cov}(dy,d\phi)}.$$

Assuming an interior solution $\hat{\beta}(y,\phi) > \eta g_{y^+}(y,\phi)$, the first-order condition for $\hat{\beta}(y,\phi)$ is

$$-\eta\phi\frac{c(y,\phi)^{\rho}}{\lambda} + y\eta\phi(1-\phi) + g(y,\phi)\left(\rho\hat{\beta}(y,\phi)\right) + g_y(y,\phi) + g_y(y,\phi)(\rho y\hat{\beta}(y,\phi))\left(-2\rho y\hat{\beta}(y,\phi)\right)$$
$$+ g_\phi(y,\phi)\eta\phi(1-\phi) + g_{yy}(y,\phi)(y\rho)^2\hat{\beta}(y,\phi) + g_{y\phi}(y,\phi)\left(-\rho y\eta\phi(1-\phi)\right) = 0.$$

After rearranging terms, I obtain

$$\eta\phi\frac{c(y,\phi)^{\rho}}{\lambda} - g_\phi(y,\phi)\eta\phi(1-\phi) = y\eta\phi(1-\phi) +$$
$$\rho\hat{\beta}(y,\phi)\left[g(y,\phi) - yg_y(y,\phi) + y^2\rho g_{yy}(y,\phi)\right] - g_{y\phi}(y,\phi)\rho y\eta\phi(1-\phi),$$

which coincides with (31) after I define

$$R(y,\phi) := \hat{\beta}(y,\phi)\left[g(y,\phi) - yg_y(y,\phi) + y^2\rho g_{yy}(y,\phi)\right] - g_{y\phi}(y,\phi)y\eta\phi(1-\phi). \quad \text{(O.13)}$$

O.4.2 Derivation of Equation (32)

Consider all the terms containing c in (27):

$$\underbrace{c - \eta\phi\hat{\beta}\frac{c^\rho}{\lambda}}_{\text{flow cost}} + g(y,\phi) \underbrace{\left(-\frac{c^{1-\rho}}{1-\rho}\right)}_{\text{drift of } dv/v} + g_y(y,\phi) \underbrace{\left[\phi\frac{\eta^2}{\lambda}c^\rho + y\left(\rho\frac{c^{1-\rho}}{1-\rho}\right)\right]}_{\text{drift of } dy}.$$

Rearranging terms, we thus have

$$c - \sigma\eta\phi\frac{\hat{\beta} - \eta g_y(y,\phi)}{\sigma\lambda}c^\rho + (g(y,\phi) - y g_y(y,\phi))\left(-\frac{c^{1-\rho}}{1-\rho}\right),$$

and the first-order condition for $c(y,\phi)$ is

$$1 - \sigma\eta\phi\frac{\hat{\beta}(y,\phi) - \eta g_y(y,\phi)}{\sigma\lambda}\rho c(y,\phi)^{\rho-1} + (g(y,\phi) - y g_y(y,\phi))\left(-c(y,\phi)^{-\rho}\right) = 0.$$

Note the capital ratio $k(y,\phi)$ is given by (30) and, by the differentiability of $g(\cdot,\phi)$, we have $g_y(y,\phi) = g_{y^+}(y,\phi)$ in the interior of the domain. Hence, in the optimal contract, a marginal change in $c(y,\phi)$ is associated with a marginal change in the capital ratio equal to

$$\frac{dk(y,\phi)}{dc(y,\phi)} := \frac{\hat{\beta}(y,\phi) - \eta g_y(y,\phi)}{\sigma\lambda}\rho c(y,\phi)^{\rho-1}. \tag{O.14}$$

Furthermore, we have $vg(Yv^{-\rho},\phi) = G^*(V,Y,\phi)$ with $V = v^{1-\rho}/(1-\rho)$ and $Y = v^\rho y$. Differentiating $vg(Yv^{-\rho},\phi)$ and $G^*(V,Y,\phi)$ by v, we have

$$\frac{dG^*(V,Y,\phi)}{dv} = v^{-\rho}G^*_V(V,Y,\phi) = g(y,\phi) - \rho y g_y(y,\phi). \tag{O.15}$$

Finally, let

$$\mu_v(y,\phi) := \frac{\delta}{1-\rho} - \frac{c(y,\phi)^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}(y,\phi)^2$$

be the drift of dv/v seen in (25) when the principal offers the contract obtained from the optimal controls of the HJB equation (27). In this contract, a marginal change in the consumption ratio $c(y, \phi)$ is associated with a marginal change in the drift of dv/v equal to

$$\frac{d\mu_v(y, \phi)}{dc(y, \phi)} := -c(y, \phi)^{-\rho}. \tag{O.16}$$

After substituting (O.14), (O.15), and (O.16) into the first-order condition, I obtain

$$1 - \sigma\eta\phi\frac{dk(y, \phi)}{dc(y, \phi)} + \frac{dG^*(V, Y, \phi)}{dv}\frac{d\mu_v(y, \phi)}{dc(y, \phi)} = 0,$$

which coincides with (32) after rearranging its terms.

References

Di Tella, S. and Sannikov, Y. (2021). Optimal asset management contracts with hidden savings. *Econometrica*, 89(3):1099–1139.

Fleming, W. H. and Soner, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York, 2 edition.

Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. Springer, New York, NY, second edition.

Pegoraro, S. (2022). Incentives and Performance with Optimal Money Management Contracts.

Pham, H. (2009). *Continuous-time stochastic control and optimization with financial applications*. Springer-Verlag.

Sannikov, Y. (2008). A Continuous-Time Version of the Principal–Agent Problem. *Review of Economic Studies*, 75(3):957–984.