Supplemental Material for "Incentives and Performance with Optimal Money Management Contracts"

Stefano Pegoraro*

June 26, 2022

This supplement contains variations of the model in Pegoraro (2022).

S.1 Optimal Contracts with Limited Commitment

To implement the optimal contract, the principal commits to reduce the agent's future incentives in order to reduce his ex-ante information rent. The commitment is captured by the multiplier Y_t in the dual problem. However, the optimal contract is not robust to renegotiation: At any time t, the principal would reduce her costs by renegotiating the contract and "starting over" from $Y_t = 0$. As long as the agent receives her promised continuation value V_t, he would agree to such renegotiation. In practice, a contract can be renegotiated by mutual agreement between the principal and the agent, or if the agent could transfer between principals. In this appendix, I explore optimal contracts when the principal is unable to fully commit.

S.1.1 Limited-Commitment Contracts

To understand why the optimal contract is not robust to renegotiation, suppose the principal offers the agent the contract in section 3. At time $t > 0$, the principal is committed to limit the agent's risk exposure, as indicated by a strictly positive multiplier $Y_t > 0$. At that time, the continuation contract is $\mathcal{C}^R_{(v_t,y_t,\phi_t)}$ and, by Theorem 3, its cost for the principal is $G^*(V_t, Y_t, \phi_t) - Y_t G^*_{Y+}(V_t, Y_t, \phi_t)$. If the principal could renegotiate the contract and replace it with $\mathcal{C}^R_{(v_t,0,\phi_t)}$, her cost would change to $G^*(V_t, 0, \phi_t)$. By the concavity of $G^*(V, \cdot, \phi)$, the cost for the principal would be lower because $G^*(V_t, Y_t, \phi_t) - Y_t G^*_{Y+}(V_t, Y_t, \phi_t) \geq G^*(V_t, 0, \phi_t)$, with strict inequality if $G^*_{Y+}(V_t, Y_t, \phi_t) < G^*_{Y+}(V_t, 0, \phi_t)$.

*University of Notre Dame, Mendoza College of Business – s.pegoraro@nd.edu. For the latest version please visithttps://sites.nd.edu/stefano-pegoraro/research/.

After renegotiating the contract, the agent would obtain the same continuation value V_t, but the principal would face lower cost. In this situation, the principal and the agent would mutually agree to change the terms of the contract ex post, as in the "commitment and renegotiation" model of Laffont and Tirole (1990). With the possibility of renegotiation, the optimal contract of section 3 is not credible. The terms of the contract would be continuously renegotiated, ruling out the option of an ex-ante optimal contract that requires ex-post commitment.

I therefore develop the notion of a limited-commitment contract, which defines a set of contracts that remain credible even when the principal cannot fully commit not to renegotiate. The principal may lack commitment power because she can mutually agree with the agent to alter the terms of the contract ex post, as discussed above. Alternatively, the principal may lack commitment power because the agent may transfer between two principals in a frictionless market,[1] with the second principal disregarding the first principal's commitment.

To provide a definition of a limited-commitment contract, let $\mathcal{F}_s^t := \{(R_i)_{t \leq i \leq s}\}$ be the smallest σ-algebra for which $(R_i)_{t \leq i \leq s}$ is measurable possibly augmented by the P-null sets. Hence, $(\mathcal{F}_s^t)_{s \geq t}$ is the augmented filtration generated by the history of returns starting from time t.

DEFINITION S.1. *A contract $\mathcal{C}$ is a limited-commitment contract if, conditional on V_t and ϕ_t, the continuation contract at time t is $\mathcal{F}_s^t$-adapted for all $t \geq 0$.*

Intuitively, in a limited-commitment contract, the principal "disregards" part of the history of returns leading up to any time t. From time t onward, the principal offers a contract that may depend on the agent's continuation value V_t, the posterior beliefs ϕ_t, and the agent's future performance, but not on other functions of the agent's past performance.

The optimal contract derived in section 3 is not a limited-commitment contract. In fact, conditional on V_t and ϕ_t, the continuation contract at time t depends on the principal's commitment measured by the multiplier Y_t, which is a function of the agent's history of returns up to time t. In the primal problem (14), the principal's full commitment is captured by the promise-keeping constraint on the information rent, (12). In a limited-commitment contract, the principal still commits to a continuation value for the agent, but she is unable to commit to the agent's information rent.

Similar to the main model, I define an optimal contract and a relaxed optimal contract within the class of limited-commitment contracts. Fix initial beliefs ϕ_0 and the agent's

[1] Payne (2018) studies optimal contracts when principals and agents meet in a matching market with search frictions.

initial outside option V_0. An *optimal limited-commitment contract* is a contract that minimizes the principal's cost (1) within the class of limited-commitment contracts, while delivering the agent an expected lifetime utility (2) at least as large as V_0. A *relaxed optimal contract with limited commitment* is a contract that minimizes, within the class of limited-commitment contracts, the cost for the principal (1) subject to the necessary incentive-compatibility condition (9) and subject to delivering expected promised utility V_0 to the agent if he does not shirk.

Even with limited commitment, enforcing full effort remains optimal. In other words, Lemma 1 remains valid because it does not rely on any particular assumption about the principal's commitment. Moreover, Theorem (1) also remains valid because it characterizes incentive compatibility in a generic contract with learning. Finally, the incentive-compatibility constraint (9) always binds in any optimal contract with limited commitment because expected excess returns are non-negative if the agent does not shirk.

To solve for an optimal limited-commitment contract, I proceed as in section 3. First, I focus on a relaxed optimal contract with limited commitment and characterize it as a solution of a HJB equation. Then, I provide a condition to verify whether such a candidate contract is incentive compatible with no shirking and, hence, whether it is fully optimal in the class of limited-commitment contracts. As always, I assume parameters are such that the principal's cost is positive, and I thus rule out infinite profits. Condition (O.11) in the online appendix remains a sufficient condition to rule out negative costs.

Consider the following HJB equation:

$$rj^L(\phi) = \min_{c,\hat{\beta}\geq \eta z^L(\phi)} \Bigg\{ c - \eta\phi\frac{\hat{\beta}-\eta z^L(\phi)}{\lambda}c^\rho + j^L(\phi)\left(\frac{\delta}{1-\rho} - \frac{c^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}^2\right) + \eta\hat{\beta}\phi(1-\phi){j^L}_\phi(\phi) + \frac{1}{2}\eta^2\phi^2(1-\phi)^2{j^L}_{\phi\phi}(\phi)\Bigg\}, \quad \text{(S.1)}$$

where the function z^L solves the differential equation

$$z^L(\phi)c^L(\phi)^{1-\rho} - \hat{\beta}^L(\phi)\eta\phi(1-\phi) - (1-\rho)\hat{\beta}^L(\phi)\eta\phi(1-\phi)z^L_\phi(\phi) = \frac{1}{2}\eta^2\phi^2(1-\phi)^2 z^L_{\phi\phi}(\phi), \quad \text{(S.2)}$$

and where $c^L(\phi)$ and $\hat{\beta}^L(\phi)$ are the minimizers in (S.1).

As in section 3, I associate a contract with the minimizers of (S.1). In particular, let contract $\mathcal{C}^L_{(v_0,\phi_0)} = (C^L_t, K^L_t)_{t\geq 0}$ be such that $C^L_t := v_t c^L(\phi_t)$, $K^L_t := v_t k^L(\phi_t)$ with $k^L(\phi) := (\hat{\beta}^L(\phi) - \eta z^L(\phi))\frac{c^L(\phi)^\rho}{\lambda\sigma}$, and where v_t and ϕ_t evolve as in (25) and (3) for some initial v_0 and ϕ_0, and with $c_t = c^L(\phi_t)$, $\hat{\beta}_t = \hat{\beta}^L(\phi_t)$, and $m_t = 0$ for all $t \geq 0$. By definition, $\mathcal{C}^L_{(v_0,\phi_0)}$ is a limited-commitment contract. The following proposition verifies that $\mathcal{C}^L_{(v_0,\phi_0)}$ is optimal in the class of limited-commitment contracts.

PROPOSITION S.1. *Let $j^L\colon [0,1]\to\mathbb{R}$ be a twice-differentiable solution of* (S.1) *satisfying $l_1 \leq j^L \leq l_0$ for two positive constants l_1 and l_0. Let $z^L\colon [0,1]\to\mathbb{R}$ be a positive, bounded, and twice-differentiable solution of* (S.2)*. Define $\mathcal{C}^L_{(v_0,\phi_0)}$ as the contract generated by the policy functions of* (S.1) *with $v_0 = ((1-\rho)V_0)^{\frac{1}{1-\rho}}$. Assume $M^L > 0$ exists such that $|c^L(\phi)| + |\hat{\beta}^L(\phi)| < M^L$ for all $\phi \in [0,1]$. If $\mathcal{C}^L_{(v_0,\phi_0)}$ is admissible, the following holds:*

(I) *At any time t, the agent's continuation value is $\frac{v_t^{1-\rho}}{1-\rho}$ if he does not shirk, and his information rent is $v_t^{1-\rho} z^L(\phi_t)$.*

(II) *$\mathcal{C}^L_{(v_0,\phi_0)}$ is a relaxed optimal contract in the class of limited-commitment contracts. The cost of contract $\mathcal{C}^L_{(v_0,\phi_0)}$ for the principal is $((1-\rho)V_0)^{\frac{1}{1-\rho}} j^L(\phi_0)$.*

(III) *Suppose that, for all $\phi \in [0,1]$,*

$$(1-\rho) z^L(\phi)\hat{\beta}^L(\phi) + z^L_\phi(\phi)\eta\phi(1-\phi) - \eta(1-2\phi) z^L_\phi(\phi) \geq 0. \tag{S.3}$$

Then $\mathcal{C}^L_{(v_0,\phi_0)}$ is incentive compatible with no shirking, and hence, it is an optimal limited-commitment contract.

All proofs are in the appendix of this supplement.

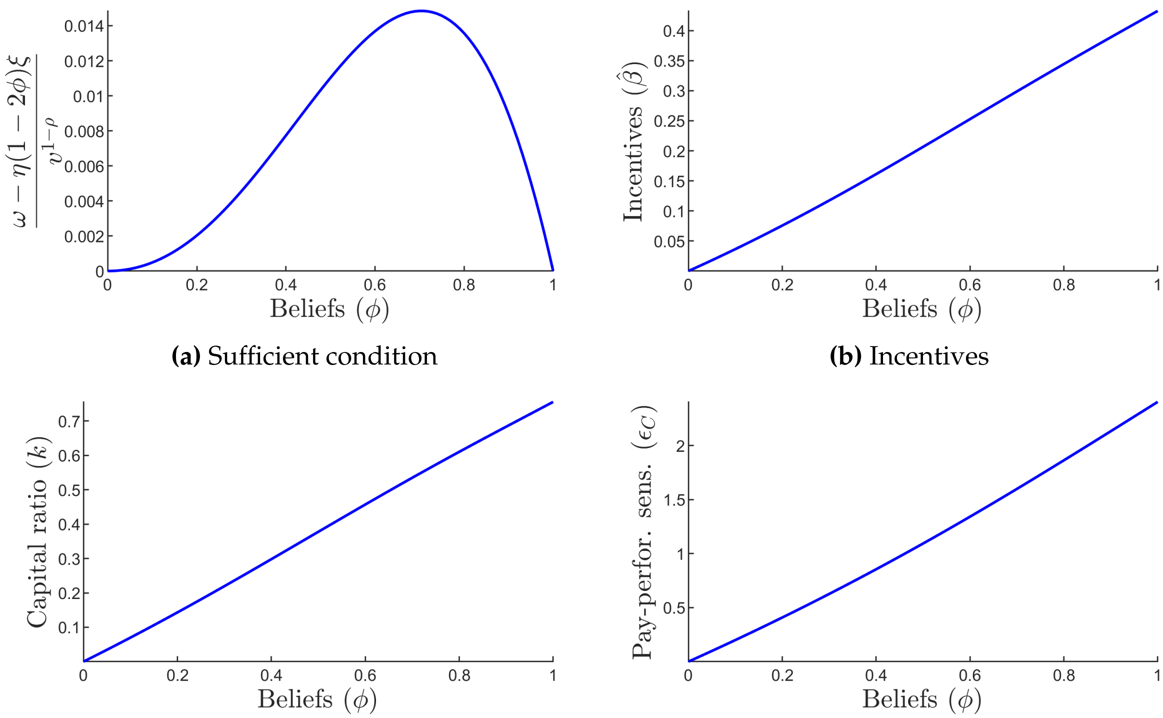
Condition (S.3) is equivalent to (13), after considering that $v_t^{1-\rho} z^L(\phi_t)$ is the agent's information rent at time t with contract $\mathcal{C}^L_{(v_0,\phi_0)}$.

Contract $\mathcal{C}^L_{(v_0,\phi_0)}$ is also internally consistent (Bernheim and Ray, 1989) and weakly renegotiation proof (Di Tella and Sannikov, 2021; Farrell and Maskin, 1989; Strulovici, 2020). In particular, consider $t \geq 0$ and $t' \geq 0$ such that $V_t = V_{t'}$ and $\phi_t = \phi_{t'}$. Then, the cost for the principal to offer the continuation contracts at t and t' is the same.

In a limited-commitment contract, learning and moral hazard still play a key role in shaping the contract's dynamics. However, the principal's commitment does not. Next, I illustrate the properties of the optimal limited-commitment contract obtained in Proposition S.1, and I highlight the differences and similarities with the (full-commitment) optimal contract of section 4.

S.1.2 INCENTIVES WITH LIMITED COMMITMENT

I numerically solve for the optimal contract using the recursive characterization in Proposition S.1. As for the optimal contract with full commitment, I first verify the contract I



(a) Sufficient condition

(b) Incentives

(c) Capital ratio

(d) Pay-performance sensitivity

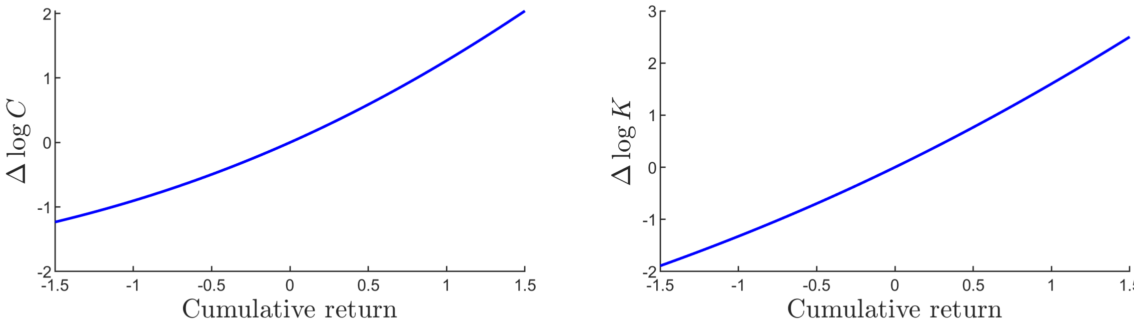
Figure S.1: Verification of condition (S.3), incentives, $\hat{\beta}^L(\phi)$, capital ratio, $k^L(\phi)$, and pay-performance sensitivity, $\varepsilon_C^L(\phi)$, in the optimal limited-commitment contract as functions of beliefs ϕ_t. Figure (a) plots the left-hand side of equation (S.3); if non-negative for all $\phi \in [0, 1]$, the contract is incentive compatible with no shirking. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

obtain satisfies condition (13), which, by Proposition S.1(III), is equivalent to (S.3). Figure S.1(a), plots the left-hand side of (S.3). Because the left-hand side of (S.3) is non-negative, the contract is incentive compatible with no shirking.

Figure S.1(b) shows the agent's incentives as a function of beliefs. As in the full-commitment contract, the principal increases the agent's incentives when beliefs increase. Whereas in the full-commitment contract incentives also depend on the principal's commitment, now beliefs are the only state variable determining incentives. Because beliefs increase with past performance by Bayesian learning, incentives also increase with past performance.

The intuition for this result is the same as in section 4 and relies on the interaction of learning and moral hazard. Because of learning, beliefs increase with the agent's past performance. Because of moral hazard, the principal must increase incentives in order to allocate more capital to the agent. In fact, (9) can be written as

$$k^L(\phi) = \frac{\hat{\beta}^L(\phi) - \eta z^L(\phi)}{\lambda\sigma}$$



(a) Performance and compensation

(b) Performance and capital

Figure S.2: Relation between cumulative performance, change in compensation, and change in capital under management in the optimal limited-commitment contract. The curves represent the change in log-compensation (Figure (a)) and log-capital (Figure (b)) as functions of cumulative performance. Curves are shifted to represent changes relative to an agent that has a zero cumulative performance. Performance, change in compensation, and change in capital are computed while assuming that returns are realized uniformly over time during the course of one year. Figures are drawn for initial beliefs $\phi_0 = 0.5$. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

for contract $\mathcal{C}^L_{(v_0,\phi_0)}$.

After good (bad) performance, the expected productivity of the agent increases (declines). The principal wants then to allocate more (less) capital to the agent and increase (reduce) the capital ratio $k^L(\phi)$. Because of the incentive-compatibility condition above, the principal must therefore increase (reduce) the agent's incentives as well.

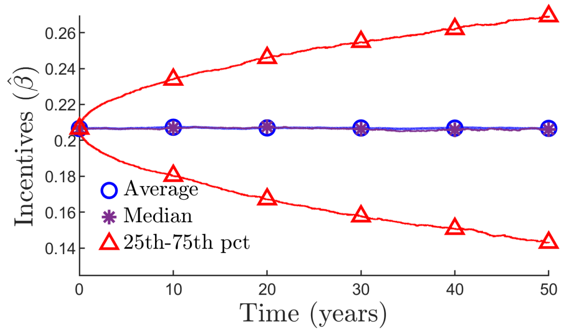
As in section 4, I define the agent's pay-performance sensitivity as the percentage change in the agent's compensation for a 1% return. Because compensation in the optimal limited-commitment contract is $C^L_t = v_t c^L(\phi_t)$, the pay-performance sensitivity can be expressed as

$$\varepsilon^L_C(\phi_t) := \frac{dC^L_t / C^L_t}{dR_t} = \frac{1}{\sigma}\left(\hat{\beta}^L(\phi_t) + \frac{\sigma^L_c(\phi_t)}{c^L(\phi_t)}\right),$$

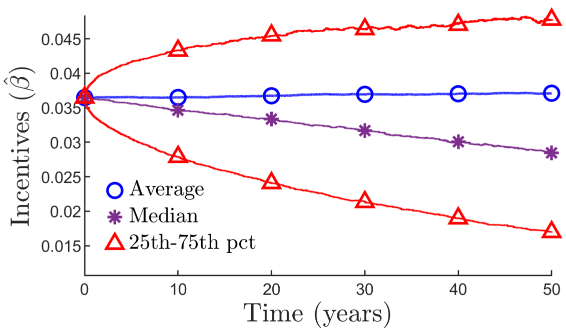
where $\sigma^L_c(\phi_t)$ is the volatility of $c^L(\phi_t)$.

Figure S.1(d) shows that, similar to the optimal full-commitment contract, the pay-performance sensitivity of the agent increases with beliefs which, in turn, increase with past performance. As a result, even with limited commitment, the model generates a positive and convex relation between changes in log-compensation and performance. Figure S.2(a) provides an illustration.

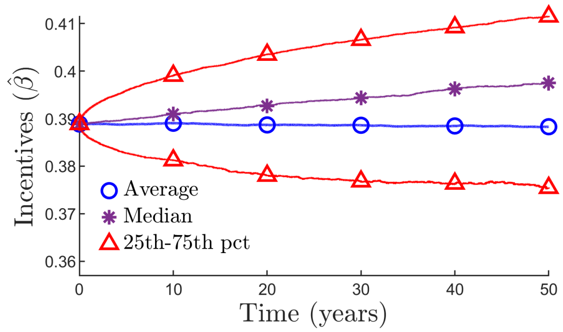
In contrast to the optimal full-commitment contract, with limited commitment incentives do not tend to decline over time, as Figure S.3 shows. In the optimal full-commitment contract, the principal optimally commits to reduce future incentives in order to lower the ex-ante information rent of the agent. With a limited-commitment contract, the principal is unable to honor such a commitment. Absent the principal's



(a) Incentives over time for $\phi_0 = 0.5$



(b) Incentives over time for $\phi_0 = 0.1$



(c) Incentives over time for $\phi_0 = 0.9$

Figure S.3: Distribution of incentives over time in the optimal limited-commitment contract for different initial values of ϕ_0. The distributions are obtained from a sample of 10,000 independent simulations in which the fraction of skilled agents is equal to the prior ϕ_0. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

commitment, expected future incentives depend solely on the expected path of beliefs and the shape of the $\hat{\beta}^L(\phi)$ function.[2] In this model, because beliefs are a martingale and because $\hat{\beta}^L(\phi)$ has minimal curvature, expected future incentives are virtually identical to the agent's current incentives

In conclusion, the optimal limited-commitment contract highlights that changes in incentives are positively correlated with performance even in the absence of commitment. The key mechanism behind this property is the interaction of learning and moral hazard. The commitment of the principal, however, is crucial to obtain a declining path of expected future incentives. With no commitment power, the principal is unable to reduce the agent's ex-ante information rent by promising low incentives in the future.

[2]Expected future incentives would tend to increase (decline) if beliefs had a positive (negative) drift, as might be the case if the agent's productivity were a stochastic process, rather than a constant. Expected incentives would also tend to increase (decline) if $\hat{\beta}^L(\phi)$ were a convex (concave) function, as Ito's lemma suggests.

S.2 Optimal Contracts with Perfect Information

To further disentangle the mechanism behind the main model's prediction, I now consider a model where the agent's type is common knowledge. Instead of considering a binary type $h \in \{0, 1\}$ as in the main model, I now assume the agent's type π takes values in the interval $[0, 1]$. An agent of type π generated returns

$$dR = (\sigma\eta\pi - m_t)dt + \sigma dW_t,$$

where m_t denotes shirking at time t.

In this setting, an optimal contract is defined exactly as in Definition 2, and Lemma 1 remains valid: Any optimal contract is incentive-compatible with no shirking.

Condition (7) is a necessary and sufficient condition for incentive compatibility for any type π. To formally see why, it is sufficient to specialize Theorems 1 and 2 to the perfect-information case, where $\xi_t = \omega_t = 0$ for all t. Moreover, (7) holds as an equality in the optimal contract because capital produces positive expected returns, and hence, the principal optimally increases capital under management until the incentive-compatibility condition binds.

I proceed as before and look for a candidate optimal contract by solving a HJB equation. I maintain parametric assumptions to ensure the principal's cost is positive. Condition (O.11) in the online appendix is sufficient to rule out negative costs. Consider the following HJB equation:

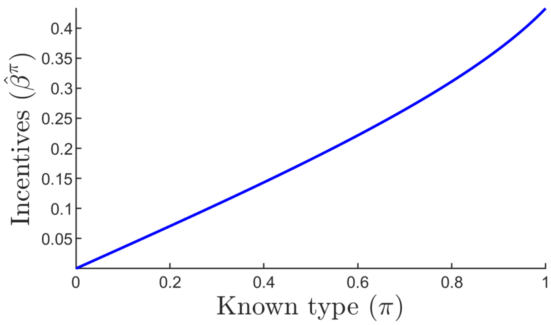
$$rj^\pi = \min_{c\geq 0, \hat{\beta}\geq 0} \left\{ c - \eta\pi\frac{\hat{\beta}c^\rho}{\lambda} + j^\pi\left(\frac{\delta}{1-\rho} - \frac{c^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}^2\right)\right\}. \tag{S.4}$$

Let c^π and $\hat{\beta}^\pi$ be the minimizers in (S.4). I associate a contract, $\mathcal{C}^\pi_{v_0} = (C^\pi_t, K^\pi_t)_{t\geq 0}$, with the policy functions of the HJB equation. In this contract, $C^\pi_t := v_t c^\pi$, $K^\pi_t := v_t k^\pi$ with $k^\pi := \hat{\beta}^\pi\frac{(c^\pi)^\rho}{\lambda\sigma}$, and v_t evolves as in (25) for some initial v_0 with $c_t = c^\pi$ and $\hat{\beta}_t = \hat{\beta}^\pi$.

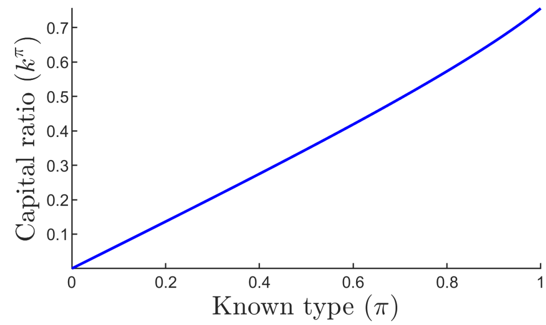
Proposition S.2 (Verification). *Let j^π be a positive and finite number satisfying* (S.4). *Then, the contract $\mathcal{C}^\pi_{v_0}$ with $v_0 = ((1-\rho)V_0)^{\frac{1}{\rho}}$ is an optimal contract for an agent of type π with outside option V_0. The cost for the principal is $v_0 j^\pi$.*

Next, I use this proposition to solve for an optimal contract for each type $\pi \in [0, 1]$. I then discuss the results and compare them to the results in section 4.

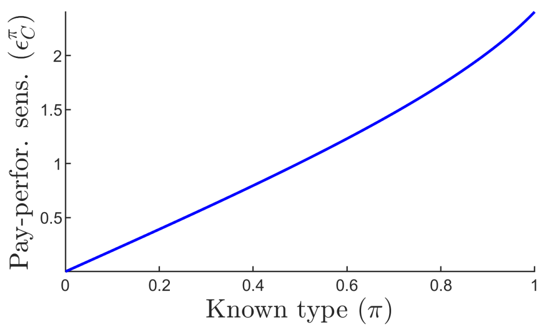
Results and Discussion. For an agent of type π, incentives $\hat{\beta}^\pi$, consumption ratio c^π, and capital ratio k^π are constant over time. Therefore, the prediction that incentives



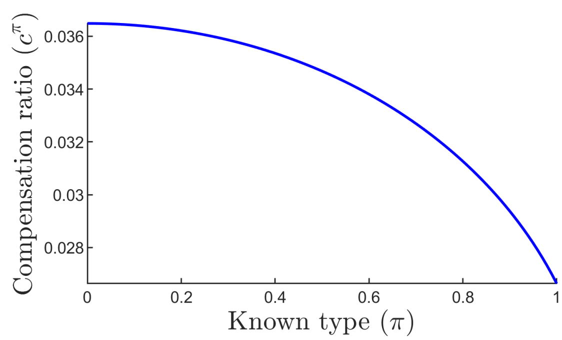
(a) Incentives by type π



(b) Capital ratio by type π



(c) Pay-performance sensitivity by type π



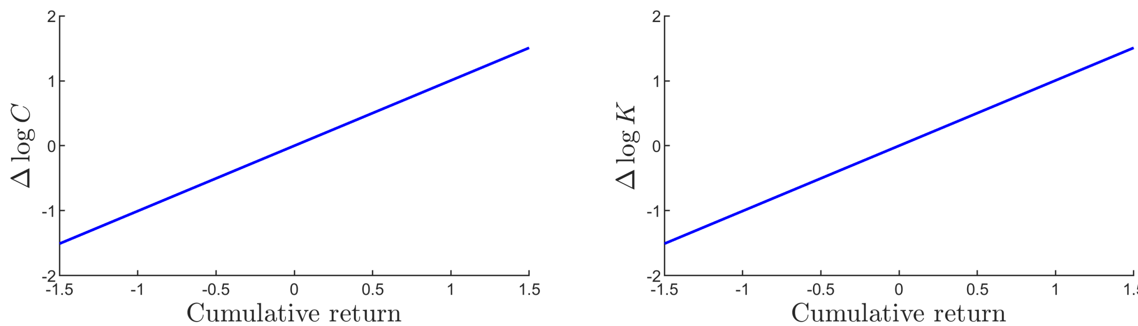
(d) Compensation ratio by type π

Figure S.4: Comparative statics with respect to the agent type. The figures show incentives, $\hat{\beta}^{\pi}$, capital ratio, k^{π}, pay-performance sensitivity, ε_C^{π}, and compensation ratio, c^{π}, in the optimal contract as functions of the agent's known type π. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

change based on past performance does not hold when the agent's type is known: Incentives are constant after any history of performance. Figure S.4(a) shows the optimal level of incentives as a function of the agent's type.

However, the comparative statics in Figure S.4, provide an intuition analogous to Figure 2 for the model with learning. The principal gives stronger incentives to a more productive agent in order to invest more capital but still deter shirking. In fact, Figure S.4(a) shows incentives increase with the agent's productivity, similar to how incentives increase with the agent's estimated productivity in Figures 2(a) and O.1(d) for the full-commitment model, and Figure S.1(b) for the limited-commitment model.

As incentives increase with the agent's type, so does the sensitivity of pay to performance, ε_C^{π}, which, without learning, is simply proportional to $\hat{\beta}^{\pi}$; that is, $\varepsilon_C^{\pi} = \hat{\beta}^{\pi}/\sigma$. Because the sensitivity of pay to performance is constant when the agent's type is known, log-compensation increases linearly with cumulative performance, as Figure S.5(a) illustrates. This is in contrast with Figure 3(a) in the paper, where log-compensation increases in a convex way with performance.



(a) Performance and compensation with known type

(b) Performance and capital with known type

Figure S.5: Relation between cumulative performance, change in compensation, and change in capital under management for an agent of type $\pi = 0.5$. The curves represent the change in log-compensation (Figure (a)) and log-capital (Figure (b)) as functions of cumulative performance. Curves are shifted to represent changes relative to an agent of type $\pi = 0.5$ that has a zero cumulative performance. Performance, change in compensation, and change in capital are computed while assuming that returns are realized uniformly over time during the course of one year. The parameters are $r = 0.02$, $\delta = 0.05$, $\rho = 1/3$, $\lambda = 0.95$, $\sigma\eta = 0.02$, and $\sigma = 0.18$.

Total compensation, C_t^π, and capital under management, K_t^π, still increase in performance for an agent with known type, as Figure S.5 shows. In fact, compensation and capital are given, respectively, by $C_t^\pi = v_t c^\pi$ and $K_t^\pi = v_t k^\pi$. Although c^π and k^π are constant, the agent's promised value, v_t, is not. The agent's promised value increases in performance in order to deter the agent from shirking, and the sensitivity of the promised value to performance is $\hat{\beta}^\pi$. As the agent's promised value increases with performance, so do compensation and capital. For a discussion of why compensation and capital scale with the agent's promised value, see section 3.4.

In conclusion, incentives are constant in a model with moral hazard but no learning. When learning is added, incentives are a function of beliefs and of the principal's commitment. Therefore, learning is crucial for incentives to change in response to performance. With learning, the agent's expected productivity changes in response to past performance (as does the principal's commitment), prompting the principal to change the level of the agent's incentives along with the amount of delegated capital.

S.A Proofs

S.A.A Proof of Proposition S.1

Proof of Proposition S.1(i). I first show $c^L(\phi)$ is uniformly bounded away from zero. Consider the first-order condition for $c^L(\phi)$ in (S.1):

$$1 - A^L(\phi)c(\phi_t)^{\rho-1} - j^L(\phi)c(\phi_t)^{-\rho} \geq 0,$$

where $A^L(\phi) := \eta\phi\frac{\hat{\beta}^L(\phi)-\eta z^L(\phi)}{\lambda} \geq 0$. Note $c^L(\phi)$ is interior because, as $c^L(\phi) \to 0$ the left-hand side of the inequality above tends to $-\infty$. Also, as $c^L(\phi) \to \infty$, the left-hand side tends to 1. Hence, $0 < c^L(\phi) < \infty$ and the above inequality holds as an equality. In turn, this also implies $1 - A^L(\phi)c(\phi_t)^{\rho-1} > 0$.

From this first-order condition, we have

$$c^L(\phi) = (1 - A^L(\phi)c(\phi_t)^{\rho-1})^{-\frac{1}{\rho}} j^L(\phi)^{\frac{1}{\rho}} \geq l_1^{\frac{1}{\rho}}.$$

Therefore, $c^L(\phi)$ is uniformly bounded from below by $l_1^{\frac{1}{\rho}} > 0$.

To show the agent's continuation value at time t is $\frac{v_t^{1-\rho}}{1-\rho}$ if he does not shirk, the proof proceeds exactly as in Theorem 4(I). I therefore omit this part of the proof.

It remains to show that $v_t^{1-\rho} z^L(\phi_t)$ is the agent's information rent. Consider a localizing sequence of increasing stopping times $(\tau_n)_{n=0}^{\infty}$ such that $\tau_0 \geq t$ and $\tau_n \to \infty$ as $n \to \infty$.

Let

$$\mathcal{A}^z[z^L; \phi; c, \hat{\beta}] = -z^L(\phi)c^{1-\rho} + (1-\rho)\hat{\beta}\eta\phi(1-\phi)z_\phi^L(\phi) + \frac{1}{2}\eta^2\phi^2(1-\phi)^2 z_{\phi\phi}^L(\phi)$$

Given contract $\mathcal{C}_{(v_0,\phi_0)}^L$, by the Dynkin's formula and equation (S.2) we have

$$\begin{aligned} v_t^{1-\rho} z^L(\phi_t) &= \mathrm{E}\left[\int_t^{\tau_n} e^{-\delta(s-t)} \left\{-v_s^{1-\rho}\mathcal{A}^z[z^L; \phi_s; c^L(\phi_s), \hat{\beta}^L(\phi_s)]\right\} ds \Big| \mathcal{F}_t\right] + \mathrm{E}\left[e^{-\delta(\tau_n - t)} v_{\tau_n}^{1-\rho} z^L(\phi_{\tau_n}) | \mathcal{F}_t\right] \\ &= \mathrm{E}\left[\int_t^{\tau_n} e^{-\delta(s-t)} v_s^{1-\rho}\hat{\beta}^L(\phi_s)\phi_s(1-\phi_s)\, ds \Big| \mathcal{F}_t\right] + \mathrm{E}\left[e^{-\delta(\tau_n - t)} v_{\tau_n}^{1-\rho} z^L(\phi_{\tau_n}) | \mathcal{F}_t\right]. \end{aligned}$$

The first term on the right-hand side converges to $\mathrm{E}\left[\int_t^\infty e^{-\delta(s-t)} v_s^{1-\rho}\hat{\beta}^L(\phi_s)\phi_s(1-\phi_s)\, ds \big| \mathcal{F}_t\right]$ as $n \to \infty$ by the monotone convergence theorem. Moreover, because $z^L(\phi)$ is positive and bounded, $\bar{z} \geq 0$ exists such that $0 \leq z^L(\phi) \leq \bar{z}$. Hence,

$$0 \leq \mathrm{E}\left[e^{-\delta(\tau_n - t)} v_{\tau_n}^{1-\rho} z^L(\phi_{\tau_n}) | \mathcal{F}_t\right] \leq \bar{z}\mathrm{E}\left[e^{-\delta(\tau_n - t)} v_{\tau_n}^{1-\rho} | \mathcal{F}_t\right].$$

As in the proof of Theorem 4(I), we have that $\mathrm{E}\left[e^{-\delta(\tau_n - t)} v_{\tau_n}^{1-\rho} | \mathcal{F}_0\right] \to 0$ as $n \to \infty$.

Therefore,

$$v_t^{1-\rho} z^L(\phi_t) = \mathrm{E}\left[\int_t^\infty e^{-\delta(s-t)} v_s^{1-\rho}\hat{\beta}^L(\phi_s)\phi_s(1-\phi_s)\, ds \Big| \mathcal{F}_t\right],$$

With contract $\mathcal{C}_{(v_0,\phi_0)}^L$, $\beta_s^L := v_s^{1-\rho}\hat{\beta}^L(\phi_s)$, is the agent's risk exposure at time s. Hence, $v_t^{1-\rho} z^L(\phi_t)$ is the information rent delivered at time t by the contract $\mathcal{C}_{(v_0,\phi_0)}^L$.

PROOF OF PROPOSITION S.1(II). Define

$$\mathcal{A}^L[j^L; \phi; c, \hat{\beta}] := j^L(\phi)\left(\tfrac{\delta}{1-\rho} - r - \tfrac{c^{1-\rho}}{1-\rho} + \tfrac{1}{2}\rho\hat{\beta}^2\right) + \eta\hat{\beta}\phi(1-\phi){j^L}_\phi(\phi) + \tfrac{1}{2}\eta^2\phi^2(1-\phi)^2{j^L}_{\phi\phi}(\phi),$$

and consider a localizing sequence of stopping times $(\tau_n)_n$ such that $\tau_n \to \infty$ as $n \to \infty$. By the Dynkin's formula and equation (S.1),

$$\begin{aligned} v_0 j^L(\phi_0) &= \mathrm{E}\left[\int_0^{\tau_n} e^{-rs}\left\{-v_s \mathcal{A}^L[j^L;\phi_s;c^L(\phi_s),\hat{\beta}^L(\phi_s)]\right\} ds \Big| \mathcal{F}_0\right] + \mathrm{E}\left[e^{-r\tau_n} v_{\tau_n} j^L(\phi_{\tau_n})|\mathcal{F}_0\right] \\ &= \mathrm{E}\left[\int_0^{\tau_n} e^{-rs} v_s \left\{c^L(\phi_s) - \eta\phi_s \frac{\hat{\beta}^L(\phi_s) - \eta z^L(\phi_s)}{\lambda} c^L(\phi_s)^\rho\right\} ds \Big| \mathcal{F}_0\right] + \mathrm{E}\left[e^{-r\tau_n} v_{\tau_n} j^L(\phi_{\tau_n})|\mathcal{F}_0\right] \\ &= \mathrm{E}\left[\int_0^{\tau_n} e^{-rs}\left(C_s^L - \sigma\eta\phi_s K_s^L\right) ds \Big| \mathcal{F}_0\right] + \mathrm{E}\left[e^{-r\tau_n} v_{\tau_n} j^L(\phi_{\tau_n})|\mathcal{F}_0\right]. \end{aligned}$$

Because $\mathcal{C}^L_{(v_0,\phi_0)}$ is admissible, by the dominated convergence theorem, the first expectation on the right-hand side converges to $\mathrm{E}\left[\int_0^\infty e^{-rs}\left(C_s^L - \sigma\eta\phi_s K_s^L\right) ds \Big| \mathcal{F}_0\right]$ as $n \to \infty$. Moreover, as in the proof of Theorem 4(II), we also have $\lim_{t\to 0} \mathrm{E}[v_t j(\phi_t)|\mathcal{F}_0] = 0$.

Therefore,

$$v_0 j^L(\phi_0) = \mathrm{E}\left[\int_0^\infty e^{-rs}\left(C_s^L - \sigma\eta\phi_s K_s^L\right) ds \Big| \mathcal{F}_0\right] \tag{S.5}$$

is the cost of contract $\mathcal{C}^L_{(v_0,\phi_0)}$.

To conclude the proof, consider an admissible limited-commitment contract $\mathcal{C} = (C_s, K_s)_{s\geq 0}$ satisfying (9) with equality. Consider another limited-commitment contract $\bar{\mathcal{C}} = (\bar{C}_s, \bar{K}_s)_{s\geq 0}$, also satisfying (9) with equality, and such that $\bar{C}_s = C_s$ and $\bar{K}_s = K_s$ for $s \leq t$, whereas $\bar{C}_s = v_s c^L(\phi_s)$ and $\bar{K}_s = v_s k^L(\phi_s)$ for $s > t$. Therefore, the continuation contract at time t of $\bar{\mathcal{C}}$ is $\bar{\mathcal{C}}_t = C^L_{(v_t,\phi_t)}$, and the implied information rent is $\tilde{\xi}(\bar{\mathcal{C}}_t, \phi_t) = v_t^{1-\rho} z^L(\phi_t)$. Let $\bar{\beta}_s = \lambda\sigma C^{-\rho}\bar{K}_s + \eta\tilde{\xi}(\bar{\mathcal{C}}_s, \phi_s)$ be the agent's risk exposure at time s under contract $\bar{C}$

Define the process

$$\hat{J}_t := \int_0^t e^{-rs}\left\{\bar{C}_s - \eta\phi_s \frac{\bar{\beta}_s - \eta\tilde{\xi}(\bar{\mathcal{C}}_s, \phi_s)}{\lambda}\bar{C}_s^\rho\right\} ds + e^{-rt} v_t j(\phi_t).$$

The drift of J_t is

$$e^{-rt} v_t \left\{\bar{c}_t - \eta\phi_t \frac{\bar{b}_t - \eta\tilde{\xi}(\bar{\mathcal{C}}_t, \phi_t)/v_t^{1-\rho}}{\lambda}\bar{c}_t^\rho + v_t \mathcal{A}^L[j^L;\phi_t;\bar{c}_t,\bar{b}_t]\right\},$$

where $\bar{c}_t := \bar{C}_t/v_t$ and $\bar{b}_t := \bar{\beta}_t/v_t^{1-\rho}$.

Because $\tilde{\xi}(\bar{\mathcal{C}}_t, \phi_t) = v_t^{1-\rho} z^L(\phi_t)$ and because of (S.1), we have

$$\bar{c}_t - \eta\phi_t \frac{\bar{b}_t - \eta\tilde{\xi}(\bar{\mathcal{C}}_t, \phi_t)/v_t^{1-\rho}}{\lambda}\bar{c}_t^\rho + v_t \mathcal{A}^L[j^L;\phi_t;\bar{c}_t,\bar{b}_t] \geq 0.$$

Hence, the drift of $\hat{J}_t$ is positive and $\hat{J}_t$ is a submartingale. In particular, $\hat{J}_0 \leq \mathrm{E}[\hat{J}_t|\mathcal{F}_0]$ for all t. Taking limits,

$$v_0 j(\phi_0) = \hat{J}_0 \leq \lim_{t\to 0} \mathrm{E}\left[\int_0^t e^{-rs}\left(C_s - \sigma\eta\phi_s K_s\right) ds + e^{-rt} v_t j^L(\phi_t)\Big| \mathcal{F}_0\right].$$

Because $(C_s, K_s)_{s\geq 0}$ is admissible, by the dominated convergence theorem,

$$\lim_{t\to 0} \mathrm{E}\left[\int_0^t e^{-rs}\left(C_s - \sigma\eta\phi_s K_s\right)\, ds \middle| \mathcal{F}_0\right] \to \mathrm{E}\left[\int_0^\infty e^{-rs}\left(C_s - \sigma\eta\phi_s K_s\right)\, ds \middle| \mathcal{F}_0\right],$$

and, as above, we also have $\lim_{t\to 0}\mathrm{E}[e^{-rt}v_t j^L(\phi_t)|\mathcal{F}_0] = 0$. Hence,

$$v_0 j(\phi_0) \leq \mathrm{E}\left[\int_0^\infty e^{-rs}\left(C_s - \sigma\eta\phi_s K_s\right)\, ds \middle| \mathcal{F}_0\right].$$

By equation (S.5), the above inequality holds as an equality when $\mathcal{C} = \mathcal{C}^L_{(v_0,\phi_0)}$. Because the contract $\mathcal{C}$ is arbitrary in the set of limited-commitment contracts satisfying (9), we conclude that $\mathcal{C}^L_{(v_0,\phi_0)}$ is a relaxed optimal contract with limited commitment, and $v_0 j^L(\phi_0)$ is the cost of contract $\mathcal{C}^L_{(v_0,\phi_0)}$ for the principal.

Proof of Proposition S.1(iii). Because $\mathcal{C}^L_{(v_0,\phi_0)}$ is a relaxed optimal contract satisfying (9), it remains to verify whether $\mathcal{C}^L_{(v_0,\phi_0)}$ is incentive compatible with no shirking. By Theorem 2, it is sufficient that (13) is also satisfied. Because $\xi_t = v_t^{1-\rho} z^L(\phi_t)$ in contract $\mathcal{C}^L_{(v_0,\phi_0)}$,

$$\omega_t = (1-\rho)v_t^{1-\rho} z^L(\phi_t)\hat{\beta}^L(\phi_t) + v_t^{1-\rho} z^L_\phi(\phi_t)\eta\phi_t(1-\phi_t).$$

Substituting the expressions for ξ_t and ω_t in (13) and dividing by $v_t^{1-\rho}$, we obtain (S.3). If this condition is satisfied, $\mathcal{C}^L_{(v_0,\phi_0)}$ is incentive compatible with full effort, and it is therefore an optimal contract in the class of limited-commitment contracts.

S.A.B Proof of Proposition S.2

Proof of Theorem S.2 I first show c^π is uniformly bounded away from zero. Consider the first-order condition for c^π in (S.4):

$$1 - A^\pi (c^\pi)^{\rho-1} - j^\pi (c^\pi)^{-\rho} \geq 0,$$

where $A^\pi := \eta\phi\frac{\hat{\beta}^\pi}{\lambda} \geq 0$. Note c^π is interior because, as $c^\pi \to 0$ the left-hand side of the inequality above tends to $-\infty$. Also, as $c^\pi \to \infty$, the left-hand side tends to 1. Hence, $0 < c^\pi < \infty$ and the above inequality holds as an equality. In turn, this also implies $1 - A^\pi(c^\pi)^{\rho-1} > 0$.

From this first-order condition, we have

$$c^\pi = (1 - A^\pi(c^\pi)^{\rho-1})^{-\frac{1}{\rho}} j^{\pi\frac{1}{\rho}} > 0.$$

Therefore, c^π is uniformly bounded from below.

To show the agent's continuation value at time t is $\frac{v_t^{1-\rho}}{1-\rho}$ if he does not shirk, the proof proceeds exactly as in Theorem 4(i). I therefore omit this part of the proof.

Define

$$\mathcal{A}^\pi[j^\pi; c, \hat{\beta}] := j^\pi\left(\frac{\delta}{1-\rho} - r - \frac{c^{1-\rho}}{1-\rho} + \frac{1}{2}\rho\hat{\beta}^2\right),$$

and consider an admissible contract $(C_t, K_t)_{t\geq 0}$ satisfying (7). Let $c_t := C_t/v_t$, $k_t := K_t/v_t$, and $\hat{\beta}_t := \beta_t/v_t^{1-\rho}$.

Consider a localizing sequence of stopping times $(\tau_n)_n$ such that $\tau_n \to \infty$ as $n \to \infty$. By the Dynkin's formula and equation (S.4),

$$\begin{aligned} v_0 j^\pi &= \mathrm{E}\left[\int_0^{\tau_n} e^{-rs}\left\{-v_s \mathcal{A}^\pi[j^\pi; c_s, \hat{\beta}_s]\right\} ds \middle| \mathcal{F}_0\right] + \mathrm{E}\left[e^{-r\tau_n} v_{\tau_n} j^\pi | \mathcal{F}_0\right] \\ &\leq \mathrm{E}\left[\int_0^{\tau_n} e^{-rs} v_s \left\{c_s - \eta\pi \frac{\hat{\beta}_s}{\lambda} c_s^\rho\right\} ds \middle| \mathcal{F}_0\right] + \mathrm{E}\left[e^{-r\tau_n} v_{\tau_n} j^\pi | \mathcal{F}_0\right] \\ &= \mathrm{E}\left[\int_0^{\tau_n} e^{-rs}\left(C_s - \eta\phi_s K_s\right) ds \middle| \mathcal{F}_0\right] + \mathrm{E}\left[e^{-r\tau_n} v_{\tau_n} j^\pi | \mathcal{F}_0\right]. \end{aligned}$$

with equality when $c_s = c^\pi$ and $\hat{\beta}_s = \hat{\beta}^\pi$ for all $s \geq 0$.

Because $(C_t, K_t)_{t\geq 0}$ is admissible, by the dominated convergence theorem, the first expectation on the right-hand side converges to $\mathrm{E}\left[\int_0^\infty e^{-rs}\left(C_s - \eta\pi K_s\right) ds \middle| \mathcal{F}_0\right]$ as $n \to \infty$. Moreover, as in the proof of Theorem 4(II), we also have $\lim_{t\to 0} \mathrm{E}[v_t j^\pi | \mathcal{F}_0] = 0$.

Therefore,

$$v_0 j^L(\phi_0) \leq \mathrm{E}\left[\int_0^\infty e^{-rs}\left(C_s - \eta\pi K_s\right) ds \middle| \mathcal{F}_0\right]$$

with equality when $C_t = C_t^\pi := c^\pi v_t$ and $K_t = K_t^\pi := k^\pi v_t$. Hence, $\mathcal{C}_{v_0}^\pi$ is an optimal contract for an agent of known type π.

References

Bernheim, B. D. and Ray, D. (1989). Collective dynamic consistency in repeated games. *Games and Economic Behavior*, 1(4):295–326.

Di Tella, S. and Sannikov, Y. (2021). Optimal asset management contracts with hidden savings. *Econometrica*, 89(3):1099–1139.

Farrell, J. and Maskin, E. (1989). Renegotiation in repeated games. *Games and Economic Behavior*, 1(4):327–360.

Laffont, J.-J. and Tirole, J. (1990). Adverse Selection and Renegotiation in Procurement. *Review of Economic Studies*, 57(4):597–625.

Payne, J. (2018). The Disruption of Long Term Bank Credit.

Pegoraro, S. (2021). Incentives and Performance with Optimal Money Management Contracts.

Strulovici, B. (2020). Renegotiation-Proof Contracts with Persistent Private Information.