Online Appendix for Risk Aversion with Nothing to Lose

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September 5, 2023

This online appendix contains proofs and additional results for Pegoraro (2023).

O.1 Additional Applications to Existing Literature

In this appendix, I establish the connection between my model and additional models of corporate finance, dynamic contracting, and delegation.

O.1.1 Models of Dynamic Corporate Finance

Other models of dynamic corporate finance share similarities with Bolton et al. (2011). Specifically, a firm manages cash holdings over time. When cash holdings are depleted, the firm may either liquidate or raise capital at a cost. Finally, the firm pays dividends above a payout boundary. Next, I consider models by Bolton et al. (2013), Décamps et al. (2011), and Hugonnier et al. (2015) who provide different applications within this common framework. As discussed in section 4.1, in these models forward-looking rents originate from the opportunity to grow cash reserves and pay dividends in the future.

O.1.1.1 Bolton et al. (2013)

Bolton et al. (2013) provide an extension of Bolton et al. (2011) by letting the firm operate under two regimes: a regime with low financing costs (regime G) and a regime with high financing costs (regime B). Regimes change according to a

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Markov chain. The HJB equation that chacterizes the firm's value function in state $S \in \{G, B\}$ is

$$rp_S(w) = \max_{i \in \mathbb{R}} \left\{ (i-\delta)p_S(w) + \hat{\zeta}_S(p_{S^-}(w) - p_S(w)) \right. \\ \left. + \left[(r-\lambda)w + m - i - g(i) - (i-\delta)w\right] p_S'(w) + \frac{\sigma^2}{2} p_S''(w) \right\}$$

in the interval $[\underline{w}_S, \bar{w}_S]$. At $\underline{w}_S$, the firm chooses to recapitalize or terminate, and $\bar{w}_S$ is a payout boundary. The parameters $\hat{\zeta}_G > 0$ and $\hat{\zeta}_B > 0$ is the hazard rate of a transition from state S to state S^-. The model is, therefore, analogous to Bolton et al. (2011), which I discussed in section 4. However, now the flow payoff includes the expected value of state transition $\hat{\zeta}_S(p_{S^-}(w) - p_S(w))$.

Using numerical methods, the authors find that, near the recapitalization thresholds, the value function in regime B, $p_B(w)$, is concave. One can show regime B is also a direct extension of the model of Bolton et al. (2011), with the only difference being the term $p_G(w) - p_B(w)$ added to the flow payoff. One only needs to note $p_G(w) - p_B(w)$ is positive and decreasing in w.[1] Therefore, link in Bolton et al. (2011), forward-looking rents emerge when the expected cash flow generated by a unit of capital, m, is sufficiently large and cash holdings are expected to grow sufficiently quickly over time.

It is important to note that, in regime G, the firm's value function may be convex near the recapitalization threshold. In fact, in this case, the firm may choose to recapitalize before cash holdings are depleted to hedge against the risk of raising financing in regime B, which involves higher costs of external financing. When $p_B(w) - p_G(w)$ is sufficiently negative, (C2') fails because the flow payoff cannot be guaranteed to be positive near the recapitalization threshold $\underline{w}_G$. Economically, recapitalization is optimal for the firm and, therefore, the firm has no forward-looking rents to forego near the recapitalization threshold. In their numerical results, Bolton et al. (2013) consider a case where the function p_G is indeed convex near recapitalization.

[1] Because the two states differ in terms of financing costs, the further w is from zero, the smaller the difference between $p_G(w)$ and $p_B(w)$.

O.1.1.2 DÉCAMPS ET AL. (2011)

In Décamps et al. (2011), the value of a firm with cash holdings m_t is given by $V(m_t)$, where V solves

$$rV(m) = [(r - \lambda)m + g]V'(m) + \frac{\sigma^2}{2}V''(m)$$

for $m \in [0, m_1]$, with $V(0) = \max\left\{0, \max_{i\in[0,\infty)} V(i/p - f) - i\right\}$, $V'(m_1) = 1$, and $V''(m_1) = 0$. The value m_1 is a payout boundary such that $V(m) = V(m_1) + m - m_1$ for all $m \geq m_1$. Parameters satisfy $r \geq \lambda > 0$, $\sigma^2 > 0$, and $f > 0$. The firm always has the option to distribute the existing cash and, hence, $V(m) > V(0) + m$ for $m > 0$.[2]

Décamps et al. (2011) show V is concave. I want to highlight this model is a particular case of the framework of section 3.1. To establish the connection, I define $q(m) := V(m) - V(0) - m$, which solves

$$rq(m) = \pi(m) + \mu(m)q'(m) + \frac{\sigma^2}{2}q''(m),$$

with $q(0) = 0$, $q'(m_1) = 0$, and $q''(m_1) = 0$ and where $\pi(m) := -\lambda m + g - rV(0)$ and $\mu(m) := (r - \lambda)m + g$.

I want to show this model satisfies the sufficient conditions for Theorem 1. Because $V(m) > V(0) + m$ for $m > 0$, $q(m) > 0$ and condition (C1) is satisfied. Moreover, $r \geq \lambda$ and $g > 0$. Hence, $\mu(m)$ satisfies condition (C2). It remains to show $\pi(0) \geq 0$. To this end, I consider the HJB equation at m_1, which, together with $q(m_1) > 0$, implies

$$q(m_1) = \frac{g - \lambda m_1}{r} - V(0) > 0.$$

From this inequality, we thus obtain $\pi(0) = g - rV(0) > \lambda m_1 > 0$. The parameter g represents the expected cash flow of the firm. Therefore, like in Bolton et al. (2011), the firm possesses forward-looking rents and becomes risk averse when expected cash flows are sufficiently large.

[2]Décamps et al. (2011) use the notation μ instead of g. I changed notation to avoid confusing this parameter with the drift of the state variable.

O.1.1.3 HUGONNIER ET AL. (2015)

In Hugonnier et al. (2015), the value of a firm with cash holdings c_t is given by $v(c_t)$ and a $\bar{c} > 0$ exists such that v solves

$$\rho v(c) = \lambda[V_1(C_1^*) - C_1^* - K + c - v(c)] + v'(c)(rc + \mu_0) + \frac{\sigma^2}{2} v''(c)$$

for $c \in [0, \bar{c}]$ with $v(0) = l_0 \in [0, \mu_0/\rho)$. Parameters satisfy $\rho > r$, $K > 0$, $\mu_0 \geq 0$, and $\sigma > 0$. The parameter λ represents the arrival rate of new investors. When new investors arrive, the firm pays a fixed cost K to transition to a new state with value function $V_1(\cdot)$. The firm is also recapitalized to the optimal level of cash holdings C_1^*.

Hugonnier et al. (2015) show a firm may implement two types of strategies, depending on how large K is and each strategy is associated with a different condition at the upper boundary of the domain. However, in both cases, a $B_0 > 0$ and $B_1 > B_0 + K$ exist such that $v(c) = V_1(B_0) + c - B_0 - K$ for any $c > B_1$. Moreover, for $c > 0$, Hugonnier et al. (2015) observe $v(c) > l_0 + c$ and $V_1(c) > l_0 + c$.

Hugonnier et al. (2015) also show the value function is not globally concave, in general. However, I want to establish that, if $\mu_0 \geq 0$ like in the numerical results of Hugonnier et al. (2015), this model satisfies the sufficient conditions for Theorem 1 and, hence, the value function is locally concave near termination, as the authors show numerically. I define $q(c) \coloneqq v(c) - l_0 - c$ and $Q_1(c) = V_1(c) - l_0 - c$. By the previous observation, $q(c) > 0$ and $Q_1(c) > 0$ for $c > 0$, thus satisfying condition (C1). The function $q(c)$ solves

$$\rho q(c) = \pi(c) + v'(c)\mu(c) + \frac{\sigma^2}{2} v''(c),$$

where $\pi(c) \coloneqq \lambda[Q_1(C_1^*) - q(c) - K] + \mu_0 - \rho l_0 - (\rho - r)c$ and $\mu(c) = (rc + \mu_0)$. Note that, for $c \geq B_1$, we have $q'(c) = q''(c) = 0$. Hence, for $c = B_1$, the HJB equation implies

$$q(B_1) = \frac{\lambda[Q_1(C_1^*) - q(c) - K] + \mu_0 - (\rho - r)B_1}{\rho} - l_0 > 0.$$

Hence, $\pi(0) = \lambda[Q_1(C_1^*) - q(c) - K] - \rho l_0 + \mu_0 > (\rho - r)B_1 > 0$. Moreover, $\mu(0) \geq 0$ when $\mu_0 \geq 0$. Therefore, the model satisfies the sufficient conditions for Theorem 1 and $q''(c) = v''(c) < 0$ in a neighborhood of 0. Similar to other models in this literature, the firm possesses forward-looking rents when the the cash flows it expects from capital injections, $\lambda[Q_1(C_1^*) - q(c) - K]$, and investment profits, μ_0, are sufficiently large.

O.1.2 Models of Dynamic Contracting

Several models in the continuous-time contracting literature extended the framework of DeMarzo and Sannikov (2006). Here, I focus on two strands of literature. First, I study the model in DeMarzo et al. (2012), which combines elements of DeMarzo and Sannikov (2006) and Bolton et al. (2011) by including investments in a contracting model. I then consider contracting models by Feng and Westerfield (2021), Piskorski and Westerfield (2016), and Szydlowski (2019), who incorporate risk choices by the principal.[3] Similar to section 4.2, the principal earns forward-looking rents and becomes endogenously risk averse near termination when the project managed by the agent is sufficiently profitable.

O.1.2.1 DeMarzo et al. (2012)

In DeMarzo et al. (2012), K_t is the stock of capital and $w_t K_t$ is the agent's continuation value at time t. The principal's value function is given by $K_t p(w_t)$, where p solves

$$rp(w) = \max_{i \in \mathbb{R}, \eta \geq \lambda} \left\{ m - c(i) + (i - \delta)p(w) + (\gamma - (i - \delta))wp'(w) + \frac{1}{2}\eta^2\sigma^2 p''(w)\lambda^2 \right\}$$

for $w \in (0, \bar{w})$, $p(0) = l \geq 0$, $p'(\bar{w}) = -1$, and $p''(\bar{w}) = 0$. The value $\bar{w}$ represents a payout boundary and $p(w) = p(\bar{w}) - (w - \bar{w})$ for all $w > \bar{w}$. The authors assume $p(0) = l < p(w^*)$, where $w^* := \arg\max_w p(w)$.[4]

Using a linear transformation like in (5), I define $q(w) := p(w) - l + w$, which solves

$$rq(w) = \max_{i \in \mathbb{R}, \eta \geq \lambda} \left\{ \pi(w, \eta, i) + (i - \delta)q(w) + \mu(w, \eta, i)q'(w) + \frac{1}{2}\eta^2\sigma^2 q''(w) \right\}$$

[3] Also Rivera (2020) models risk choices and his model satisfies conditions (C1) and (C2) implying risk aversion near termination. However, in Rivera (2020), risk-shifting takes the form of exposure to jump risk and it therefore differs from the volatility of a state variable.

[4] DeMarzo et al. (2012) use the notation μ instead of m. I changed notation to distinguish this parameter from the drift of the state variable.

where $\pi(w,\eta,i) \coloneqq m - c(i) + (i-\delta-r)l - (\gamma-r)w$ and $\mu(w,\eta,i) \coloneqq (\gamma-(i-\delta))w$. I want to show $\pi(\cdot,\cdot,\cdot)$ and $\mu(\cdot,\cdot,\cdot)$ satisfy conditions (C1′), (C2′), and (C3′). The principal can always pay $w_t K_t$ to the agent and earn the outside option lK_t. Hence, $p(w) > l - w$ for all $w \in (0,\bar{w})$ and condition (C1′) is satisfied. Moreover, $\mu(0,\eta,i) = 0$. To fully establish the conditions, it is sufficient to prove an $i \in \mathbb{R}$ exists such that $\pi(0,\eta,i) > 0$.

Because $q'(\bar{w}) = 0$ and $q''(\bar{w}) = 0$, the HJB equation for $w = \bar{w}$ implies

$$q(\bar{w}) = \frac{m - c(i(\bar{w})) - (\gamma-r)\bar{w}}{r+\delta-i(\bar{w})} - l$$

As discussed in section 4.1 for Bolton et al. (2011), parameters must satisfy $m - c(i(\bar{w})) - (\gamma-r)\bar{w} \geq 0$ and $r+\delta-i(\bar{w}) > 0$. Therefore,

$$\pi(0,\eta,i(\bar{w})) = m - c(i(\bar{w})) - (r+\delta-i(\bar{w}))l > (\gamma-r)\bar{w} \geq 0.$$

Hence, the model satisfies (C1′), (C2′), and (C3′) and Theorems 1′ and 2′ apply. For these results to hold, the project profitability, m, must be sufficiently large.

O.1.2.2 FENG AND WESTERFIELD (2021)

In Feng and Westerfield (2021), the principal's value function solves

$$rF(W) = \max_{K\geq 0, a\in(\underline{a},\bar{a})} \Bigg\{ f(K)m(a) - rK + F'(W)\gamma W + \frac{1}{2}\left[\frac{\lambda}{f'(K)} \cdot \frac{1}{m(a)-m'(a)a}\right]^2 f(K)^2 a^2 F''(W) \Bigg\}$$

for $W \in [0,\bar{W})$, with $F(0) = L \geq 0$, $F'(\bar{W}) = -1$, $F''(\bar{W}) = 0$. The value $\bar{W}$ represents a payout threshold such that $F(W) = F(\bar{W}) + (W - \bar{W})$ for all $W \geq \bar{W}$. The function f satisfies $f(0) = 0$, $f'(K) > 0$, and $f''(K) < 0$ for all K and the Inada conditions; that is, $\lim_{K\to 0} f'(K) = +\infty$ and $\lim_{K\to+\infty} f'(K) = 0$. The function m satisfies $m''(a) < 0$ and $m(a) > 0$ for all $a \in (\underline{a},\bar{a})$, with $\underline{a} > 0$.[5]

I change variables and define $\eta \coloneqq f(K)$ and, hence, $K = f^{-1}(\eta)$. I also apply a

[5] Compared to the original paper, I changed notation to remain consistent with the notation of the general model of the present paper. In particular, Feng and Westerfield (2021) use σ instead of a and $\mu(\cdot)$ instead of $m(\cdot)$.

linear transformation $q(W) \coloneqq F(W) - L + W$ so that the HJB equation becomes

$$rq(W) = \max_{\eta \geq 0, a \in (\underline{a}, \bar{a})} \left\{ \pi(W, \eta, a) + q'(W)\mu(W, \eta, a) + \frac{1}{2}\sigma(W, \eta, a)^2 \eta^2 q''(W) \right\},$$

where $\pi(W, \eta, a) \coloneqq \eta m(a) - rf^{-1}(\eta) - rL - (\gamma - r)W$, $\mu(W, \eta, a) \coloneqq \gamma W$, and $\sigma(W, \eta, a) \coloneqq \left[\frac{\lambda}{f'(f^{-1}(\eta))} \cdot \frac{a}{m(a) - m'(a)a} \right]$. I want to show this model can be mapped into the framework of Section 3.2 with $\omega = \nu = 0$ and with $\underline{\eta} = 0$. Note the function $\sigma(W, \eta, a)$ satisfies the assumption of section 3.2. In particular $\sigma(W, \eta, a) > 0$ if $\eta > 0$ and $\sigma_\eta(W, \eta, a) > 0$.

By the Inada conditions, an $\eta >$ exists such that $\eta m(a) - rf^{-1}(\eta) > 0$. Let $\pi^* \coloneqq \max_{\eta \geq 0, a \in [\underline{a}, \bar{a}]} \{\eta m(a) - rf^{-1}(\eta)\}$ be the first-best cash flow. Because liquidation is inefficient, $L < \pi^*/r$. Therefore, η and a exist such that $\pi(0, a, \eta) > 0$. Moreover, $\mu(0, a, \eta) = 0$.

Hence, the model satisfies the conditions (C1′) and (C2′). In particular, the project is sufficiently profitable that the principal possesses forward-looking rents near termination and thus becomes risk averse endogenously by the first part of Lemma 2′. In particular, either she sets $\eta(0) = 0$ or, if she does not, her value function is concave.

In Corollary 10, the authors show that, if $L > 0$, then the principal does not minimize risk and sets $\eta(0) > 0$ at the termination threshold. However, as they note in Section 6, the principal may minimize risk η at termination if $L = 0$. I want to show that, for the function $f(K) = 2K^{\frac{1}{2}}$ Feng and Westerfield (2021) use in their numerical results, setting $L = 0$ implies condition (C3′) hold. In this case, $\mu_\eta(0, a, \eta) = 0 = 2\mu(0, a, \eta)$ and

$$2\pi(0, a, \eta) = 2\eta m(a) - 2r(\eta/2)^2 > -2r(\eta/2)^2 = \eta \pi_\eta(0, a, \eta)$$

and the second part of Lemma 1′ applies to establish $\eta(0) = 0$.

O.1.2.3 PISKORSKI AND WESTERFIELD (2016)

In Piskorski and Westerfield (2016), given the agent's continuation value V, the principal's value function $F(V)$ solves

$$rF(V) = \max_{\eta\in[0,1]} \left\{ m - \frac{\theta}{V - V_F}(1-\eta) + \gamma V F'(V) + \frac{1}{2}\eta^2\sigma^2 F''(V) \right\}$$

for $V \in (V_R, V_C)$, where V_C is a payout boundary such that $F(V) = F(V_C) - (V - V_C)$ for all $V \geq V_C$ and where $F(V) = L > 0$ if $\eta(V_R) > 0$ and $rF(V_R) = m - \frac{\theta}{V_R - V_F} + \gamma V_R F'(V_R)$ if $\eta(V_R) = 0$. Parameters satisfy $\gamma > r > 0$, $m > 0$, $V_R > V_F \geq 0$, $\theta > 0$, and $\sigma > 0$.

Property 5 and Theorem 4 in Piskorski and Westerfield (2016) provide a sufficient condition for the principal to set $\eta(V_R) = 0$. In particular, they imply that, if $L \leq \frac{1}{r}\left(m - \frac{\theta}{V_R - V_F} - \gamma V_R\right)$, then $\eta(V_R) = 0$.[6] I want to show this parameter restriction is equivalent to condition (C3), which is required for Lemma 1 to hold in this model.

Like in the models I discussed above, I apply a linear transformation and define $q(V) := F(V) - L + (V - V_R)$ and the HJB equation becomes

$$rq(V) = \max_{\eta\in[0,1]} \left\{ \pi(V,\eta) + \mu(V,\eta)F'(V) + \frac{1}{2}\eta^2\sigma^2 F''(V) \right\},$$

where $\pi(V,\eta) := m - \frac{\theta}{V - V_F}(1-\eta) + rV - r(L + V_R) - \gamma V$ and $\mu(V,\eta) := \gamma V$.

For the contract to be optimal, $F(V) > L - (V - V_R)$ for $V > V_R$ and, hence, $q(V) > 0$ for $V > V_R$, thus satisfying condition (C1). Moreover, because $\mu(V_R,\eta) = \gamma V_R > 0$, the function $\mu(\cdot,\cdot)$ satisfies the conditions in (C2) and (C3) with strict inequalities.

For $\pi(\cdot,\cdot)$ to satisfy condition (C2) and (C3), we must have

$$rL \leq \left(m - \frac{\theta}{V_R - V_F} - \gamma V_R\right) + \frac{\theta}{V_R - V_F}\frac{\eta}{2} \quad \forall \eta \in [0,1],$$

which thus implies $L \leq \frac{1}{r}\left(m - \frac{\theta}{V_R - V_F} - \gamma V_R\right)$ is sufficient for $\eta(V_R) = 0$, as also shown in Piskorski and Westerfield (2016). Note this condition requires the flow

[6] Piskorski and Westerfield (2016) also obtain a necessary and sufficient condition for the principal to set $\eta(V_R) = 0$, but the condition is not derived in closed form.

payoff generated by the project, m, to be sufficiently large to generate forward-looking rents and risk aversion, like in previous models of dynamic contracting.

O.1.2.4 SZYDLOWSKI (2019)

In Szydlowski (2019), the principal's value function solves the HJB equation

$$rJ(W) = \max_{(\eta_1,\ldots,\eta_N)\in\{0,1\}^N} \left\{ \sum_{i=1}^{N} \eta_i m_i + \left(\gamma W + h \sum_{i=1}^{N} \eta_i \right) J'(W) + \frac{1}{2} J''(W) \sum_{i=1}^{N} \eta_i \psi_i \right\},$$

for $W \in [0, \bar{W}]$, with $J(0) = l$ and $J'(\bar{W}) = -1$ and $J''(\bar{W}) = 0$. The parameters satisfy $\gamma > r > 0$, $h > 0$, and $l \in [0, \bar{l})$ where, similar to previous models, $\bar{l} < \max_{W \geq 0} J(W)$, ensuring termination is inefficient.

One can immediately verify that (C1) and (C2) hold using steps similar to those I used in the contracting models discussed above. However, unlike previous models, in Szydlowski (2019) the principal chooses from discrete projects with different volatility, rather than from a continuum of values for risk exposure.

Although the risk-choice problem in Szydlowski (2019) cannot be directly mapped into my framework, Proposition 3 in Szydlowski (2019) can be seen as a special case of my Theorem 2. In Proposition 3 of Szydlowski (2019), the author assumes $l \to 0$ and shows a project with volatility $\psi = \varepsilon$, with $\varepsilon > 0$ sufficiently small, becomes optimal near the termination threshold. Under $l \to 0$, condition (C3) holds for $\pi^i(W, \eta) := \eta m_i$ and $\mu^i(W, \eta_i) := \gamma W/N + h\eta_i$ for $\eta \in [\varepsilon, \bar{\eta}]$.[7] Therefore, as W approaches the termination threshold, a project with minimal volatility is optimal for the principal. In this case, $l \to 0$, the principal earns forward-looking rents and becomes risk averse near termination if projects have a positive profitability m_i.

O.1.3 MODELS OF DELEGATION AND LEARNING

Finally, I consider models of hedge-fund delegation by Lan et al. (2013) and Panageas and Westerfield (2009) and models of delegation with learning about the manager's skill by Moreira (2019) and Kuvalekar and Lipnowski (2020). I also

[7] Note that, in Szydlowski (2019)'s HJB equation, the flow payoff and the drift are given by $\sum_{i=1}^{N} \pi^i(W, \eta_i)$ and $\sum_{i=1}^{N} \mu^i(W, \eta_i)$, respectively.

discuss the model by Panageas and Westerfield (2009), which is a special case of Drechsler (2014). As discussed in section 4.3, in these models forward-looking rents originate from the present value of the fees collected by the manager for as long as she is not terminated.

O.1.3.1 LAN ET AL. (2013)

In a numerical calibration of their model, Lan et al. (2013) find a hedge fund manager becomes endogenously risk averse and reduces risk near termination. I show their model satisfies the assumption of Theorem 1′. The value function of a fund manager with high-water mark H_t and assets under management $w_t H_t$ is $H_t f(w_t)$ where f solves

$$(\beta - g + \delta + \lambda) f(w) = \max_{\eta \in [0, \bar{\eta}]} \left\{ cw + (\eta\alpha + r - g - c) w f'(w) + \frac{1}{2}\eta^2\sigma^2 w^2 f''(w) \right\}$$

for $w \in (b, 1)$ with $b > 0$, $f(w) = 0$ for $w < b$, and $f(1) = (k+1)f'(1) - k$. In the calibration, β, g, λ, r, c, σ, and $\bar{\eta}$ are strictly positive, $\alpha \geq 0$, $\delta \geq 0$, $\beta + \delta + \lambda > g$, $\bar{\eta}\alpha + r \geq g + c$ and $\hat{w} > 0$. Without loss of generality, I restricted $\eta \geq 0$ because the present value of expected compensation increases with w and, hence, $\alpha f'(w) \geq 0$.

Thus, we can immediately verify that $\pi(b, \eta) := cb > 0$ and $\mu(b, \bar{\eta}) := (\bar{\eta}\alpha + r - g - c)b > 0$, thus satisfying conditions (C1′) and (C2′). The first part of Lemma 1 thus implies the manager is risk averse near termination. However, condition (C3′) does not necessarily hold. If $(r - g - c) < 0$, as in Figure 1 in Lan et al. (2013), the condition $\eta\mu_\eta(b, \eta) > 2\mu(b, \eta)$ fails for some η. In particular, it fails for sufficiently small and positive η. In this case, the manager reduces risk near termination, but she does not *minimize*. Lan et al. (2013) illustrate this result in Figure 1. If, instead, parameters satisfied $(r - g - c) > 0$, one could use the second part of Lemma 1 to show the manager minimizes risk near termination.

O.1.3.2 MOREIRA (2019)

Moreira (2019) studies a model of delegation to an intermediary whose skill is private information. Investors learn by observing the performance of the intermediary and, in a version of the model, the intermediary's portfolio as well. The author finds a skilled intermediary becomes risk averse near termination. I show his model can be mapped into the framework of section 3.1. Although Moreira (2019) incorporates tail risk, to streamline the presentation, I consider a model with no tail risk.

The continuation value of a skilled intermediary with log-likelihood ratio p_t is $V(p_t; S)$, where V solves

$$\rho V(p;S) = \max_{X\in\Omega} \Big\{ mX'\mu_S + C(p) + V_p(p;S)X^I(p;S)'\mu_S \left(X - \frac{1}{2}X^I(p;S)\right)' \mu_S + \frac{1}{2}V_{pp}(p;S)(X^I(p;S)'\mu_S)^2 \Big\}$$

for $p > 0$, with $V(0,S) = 0$. The quantity $C(p)$ is non-negative and represents the equilibrium compensation of the manager. The vector $X^I(p;S) \in \mathbb{R}^M$, with $M \geq 1$, represents the equilibrium strategy of the manager, and $\mu_S \in R^M$. Moreover, an $X_\mu \in \mathbb{R}^M$ exists such that $X_\mu'\mu_S = \max_X X'\mu_S > 0$. Finally, $\Omega := \{X \in \mathbb{R}^M := X'\Sigma X = 1\}$, where Σ is a positive definite matrix and $M \geq 1$.

First, Moreira (2019) considers the case in which holdings are observable and, hence, X replaces $X^I(p;S)$ in the HJB equation. In this case, $\pi(p,X;S) := mX'\mu_S + C(p)$ and $\mu(p,x;S) := \frac{1}{2}(X'\mu_S)^2$. Thus, we immediately obtain $\pi(0,X_\mu;p) > 0$ and $\mu(0,X_\mu;p) > 0$, showing this version of the model satisfies the conditions for Theorem 1.

Then, Moreira (2019) assumes holdings cannot be observed. In this case, $\pi(p,X;S) := mX'\mu_S + C(p)$ and $\mu(p,x;S) := X^I(p;S)'\mu_S \left(X - \frac{1}{2}X^I(p;S)\right)'\mu_S$. In equilibrium, we must have $X^I(p;S)'\mu_S \geq 0$ for $p > 0$, otherwise investors would terminate the fund. Given this observation, we obtain once again that $\pi(0,X_\mu;p) > 0$ and $\mu(0,X_\mu;p) > 0$. Hence, also this version of the model satisfies the conditions for Theorem 1.

O.1.3.3 Kuvalekar and Lipnowski (2020)

In Kuvalekar and Lipnowski (2020), a worker with reputation p has continuation value $v(p)$ which is a solution of

$$rv(p) = \max_{\eta\in[\underline{\eta},\bar{\eta}]} \left\{ 1 + \frac{1}{2}v''(p)\eta^2 p^2 (1-p)^2 \right\}.$$

in $p \in (\hat{p}, 1)$, where $\hat{p}$ represents the upper boundary of a probation region. The parameter r is a positive constant. At $\hat{p}$, the continuation value is $v(\hat{p}) < 1/r$. To map this model to the framework of section 3.1, I define $q(p) := v(p) - v(\hat{p})$, which solves

$$rq(p) = \max_{\eta \in [\underline{\eta}, \bar{\eta}]} \left\{ 1 - rv(\hat{p}) + \frac{1}{2} v''(p) \eta^2 p^2 (1-p)^2 \right\}.$$

One can then immediately verify that $\pi(p, \eta) := 1 - rv(\hat{p})$ and $\mu(p, \eta) := 0$ satisfy condition (C1), (C2), and (C3) and, hence, Theorems 1 and 2 apply in a right-neighborhood the probation threshold $\hat{p}$.

O.1.3.4 Panageas and Westerfield (2009)

As noted by Drechsler (2014), the model in Panageas and Westerfield (2009) is a special case with $C = g = 0$. However, condition (C3′) does not hold in Panageas and Westerfield (2009) because, with $C = 0$, we have $\eta \pi_\eta(0, \eta) = 2\pi(0, \eta) = 0$ and $\eta \mu_\eta(0, \eta) = 2\mu(0, \eta) = 0$ for any $\eta \in \mathbb{R}$. In fact, Panageas and Westerfield (2009) find the manager does not fully de-risk near termination. In particular, the manager maintains a constant risk exposure. In this case, (10) and $\eta(0)^2 > 0$ are satisfied and, according to the first part of Lemma 1, the manager is risk averse near termination with a concave value function. By contrast, Drechsler (2014) assumes $C > 0$ so that, when (10) or, equivalently, (11) holds, $0 = \eta \pi_\eta(C, \eta) < 2\pi(C, \eta)$ and $\eta \mu_\eta(C, \eta) \leq 2\mu(C, \eta)$, thus satisfying also condition (C3).

O.2 Application: Risk Aversion with Early Regulatory Intervention

I now consider a variation of the default model of section 5. I let equity holders be decision-makers, and I thus eliminate the distinction between ownership and control that has characterized the model so far. However, a regulator imposes early termination in a model otherwise analogous to the one in section 5. I then study how such regulatory intervention generates endogenous risk aversion in levered financial institutions.

In particular, a regulator imposes a threshold R such that, if assets fall below that threshold, she takes control of the firm and shareholders receive an outside option that I normalize to zero.[8] This threshold may be thought of as a minimum capital requirement when the face value of liabilities is constant.

With this variation of the model, I provide a tractable framework to study capital requirements and PCA for US financial institutions (Mehran and Mollineaux, 2012). When capital requirements are violated, US regulators may impose restrictions on financial institutions. If the institution remains severely undercapitalized, regulators would then take steps to orderly liquidate it. This regulation effectively imposes an early termination threshold on the institution. In this paper, I do not microfound why the regulator intervenes.[9] Instead, I take regulatory intervention as given and consider the investment behavior of the financial institution.

Absent any regulatory intervention, equity holders optimally default when assets fall below some level, because a version of Lemma O.3 in Online Appendix O.4.1 still applies. I denote this level by V^*. If the regulatory threshold R is lower than V^*, equity holders would still default at V^*, and the regulatory intervention is irrelevant. I, therefore, focus on the case in which $R \geq V^*$.

In this case, the value of shareholders' equity $E_R(V)$ is the unique solution of the following HJB equation:

$$rE_R(V) = \max_{\eta \in [\underline{\eta}, \bar{\eta}]} \left\{ \delta V - c - \beta E_R(V) + E_R'(V)\mu\eta V + \frac{1}{2} E_R''(V)\eta^2 V^2 \right\},$$

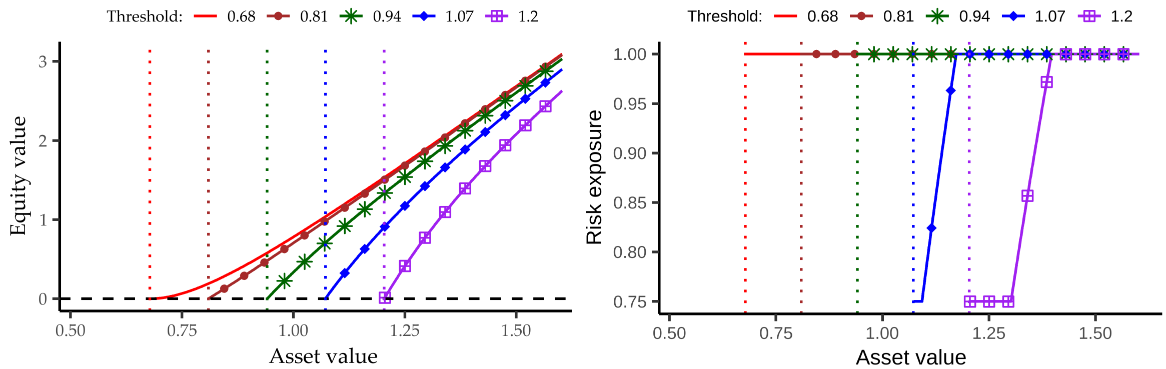
subject to the boundary condition $E_R(R) = 0$ and satisfying linear growth.

If $\delta R - c > 0$, the conditions of Theorems 1 and 2 are met and, hence, equity holders become endogenously risk averse and minimize risk near termination. However, as discussed in section 3.1.1, the conditions in those theorems are sufficient, but not necessary. In Figure O.1, I show results for different values of the termination threshold, starting from the optimal default threshold V^*. For all the thresholds I consider, the cash flow near termination is strictly negative; that is, $\delta R - c < 0$.

As we see in Figure O.1(a), the concavity of equity holders' value function and their optimal risk exposure near termination change as the regulatory threshold

[8] According to the discussion in footnote 8, the solution of a model with positive outside option O can be characterized as a solution of a model with zero outside option, but where the cost c is increased to $c + O/r$.

[9] Capital requirements and PCA are typically justified by the regulator's desire to avoid costly bankruptcy and market disruption (Mehran and Mollineaux, 2012).



(a) Equity value $E_R(V)$

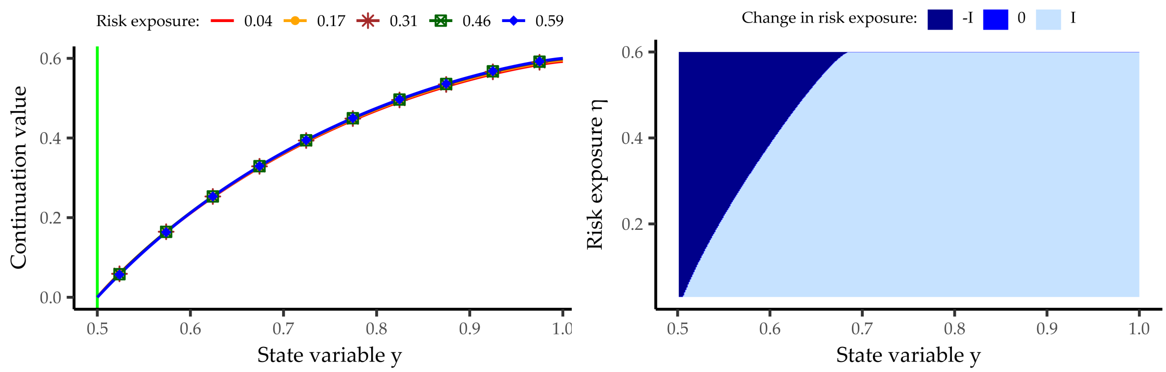
(b) Risk exposure $\eta_R(V)$

Figure O.1: Equity value and risk exposure when equity holders are decision-makers but face early termination because of regulatory intervention. The figures show results for different regulatory thresholds (dotted vertical lines), and the lowest one coincides with the optimal default threshold V^*. The parameter values are $\mu = 2\%$, $\delta = 4\%$, $c = 10\%$, $\sigma = 10\%$, $\bar{\eta} = 100\%$, $\underline{\eta} = 75\%$, $r = \rho = 3\%$.

increases. If the threshold is sufficiently close to the optimal one, the value function remains convex near termination. When the threshold is sufficiently above the optimal one, equity holders become risk averse. Moreover, as Figure O.1(b) indicates, for a low enough threshold, equity holders still maximize risk near termination. However, for higher thresholds, equity holders minimize risk exposure near termination.

O.3 Application: Overconfidence and Slow Risk Adjustment

I provide an example of a model with slow risk adjustment in which $T(\eta)$ is constant. In this case, Theorem A.1 provides condition for the decision maker to become risk averse near termination. However, the decision-maker's optimal policy near termination depends on parameters because $T'(\cdot) = 0$. I consider a simple model, unrelated to a specific economic application, just to illustrate how results



(a) Decision-maker's continuation value $u(y, \eta)$ **(b)** Change in risk exposure $i(y, \eta)$

Figure O.2: Continuation value of a Decision-maker and change in risk exposure as functions of the the state variable y and current risk exposure η. The continuation value is provided as a function of y for five distinct levels η. The change in risk exposure is represented by different colors over the entire state space. The parameter values are $r = \rho = 5\%$, $\sigma = 10\%$, $\hat{y} = 0.5$, $\underline{\eta} = 0.03$, $\bar{\eta} = 0.6$, $\beta = 1$, and $I = 1$.

depend on parameters. In particular, the decision-maker solves

$$\begin{aligned} \max_{(i_t)_{t \geq s}} \quad & \mathrm{E}^{P^m}\left[\int_s^\tau e^{-\rho(t-s)}\beta\eta_t\sigma(y-\hat{y})\,dt\right] \quad \forall s \geq 0, \\ \text{s.t.} \quad & \text{(A.4) and } i_t \in [-I, I] \quad \forall t \geq s \\ & dy_t = \eta_t^2 y_t^2(1-y_t)^2 dt + \eta_t y_t(1-y_t)dZ_t \end{aligned} \tag{O.1}$$

where $\tau = \inf_{t \geq s}\{t : y_t \leq \hat{y}\}$.

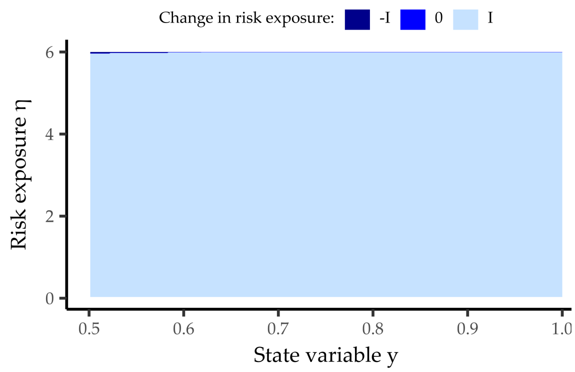
The value function is characterized by the following HJB equation:

$$\rho u(y,\eta) = \beta\sigma(1-\hat{y}) + u_y(y,\eta)\eta^2 y^2(1-y)^2 + \frac{1}{2}u_{yy}(y,\eta)\eta^2 y^2(1-y)^2 + \max_{i \in [-I,I]}\{i u_\eta(y,\eta)\}, \tag{O.2}$$

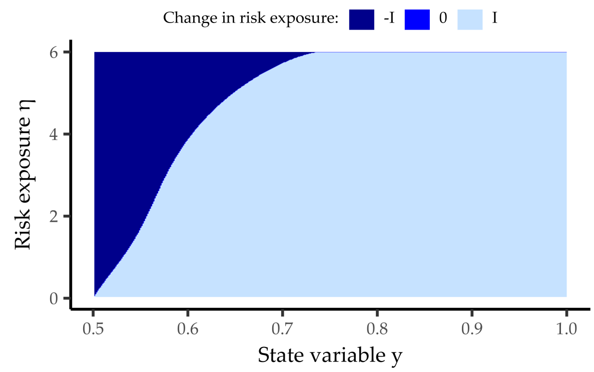
with boundary condition $u(T(\eta), \eta) = 0$ and state constraints $\underline{\eta} \leq \eta \leq \bar{\eta}$.

If a solution to this equation exists that is twice differentiable in y and once differentiable in η, arguments similar to those in Lemma O.1 imply a control $(i_t)_{t \geq 0}$ such that $i_t = i(y_t, \eta_t) = \arg\max_{i \in [-I,I]}\{i u_\eta(y_t, \eta_t)\}$ is an optimal control for the decision-maker.

Instead of plotting the three-dimensional function $u(y, \eta)$, in Figure O.2(a), I show the decision-maker's continuation value as a function of the state variable y for five values of risk exposure η. Figure O.2(b) shows the associated optimal



(a) Change in risk exposure $i(y, \eta)$ with $I = 1$



(b) Change in risk exposure $i(y, \eta)$ with $I = 100$

Figure O.3: Change in risk exposure as function of the state variable y and current risk exposure when $\bar{\eta}$ is large. The change in risk exposure is represented by different colors over the entire state space. The parameter values are $r = \rho = 5\%$, $\sigma = 10\%$, $\hat{y} = 0.5$, $\underline{\eta} = 0.03$, $\bar{\eta} = 6$, and $\beta = 1$. The parameter I changes between the two figures.

control over the entire state space. As expected from part (I) of Proposition A.1, the value function is increasing and concave in the state variable y near the termination threshold. Moreover, the numerical results also show the decision-maker reduces risk exposure in the vicinity of the termination threshold. Interestingly, the decision-maker begins reducing risk exposure at larger values of y when the current risk exposure η is higher.

However, the result in Figure O.2(b) is not always guaranteed. Figures O.3(a) and O.3(b) show optimal controls when $\bar{\eta} = 6$, instead of 0.6. In Figure O.3(a), the parameter I is set equal to 1, like in Figure O.2. All other parameters are unchanged. Although the value function continues to be increasing and concave near termination by Theorem A.1(I), the decision-maker increases risk exposure near the termination threshold. In Figure O.3(a), I increase the parameter I to 100 and keep $\bar{\eta} = 6$. In this case, we observe again a reduction in risk exposure as the state variable y approaches the termination threshold even if $\bar{\eta}$ is large.

O.4 Technical Appendix

O.4.1 Auxiliary Results for Section 3

Lemma O.1. *Let $(\eta(y_t))_{t\geq 0}$ be admissible and let $u(\cdot)$ be a twice-differentiable solution of* (4)*. If $\lim_{t\to\infty} \mathrm{E}[u(y_t)e^{-\rho t}|\mathcal{F}_0] = 0$, $u(\cdot)$ is the decision-maker's value function and*

$(\eta(y_t))_{t\geq 0}$ *is optimal for the decision-maker.*

Proof. Consider a localizing sequence of stopping times $(\tau_n)_{n=0}^{\infty}$ such that $\tau_n \to \infty$ as $n \to \infty$. Then, for any arbitrary admissible strategy $(\eta_t)_{t\geq 0}$ such that $\eta_t \in [\underline{\eta}, \bar{\eta}]$, by the Dynkin's formula (Øksendal, 2003, Chapter 7.4),

$$E[e^{-\rho\tau_n}u(y_{\tau_n})|\mathcal{F}_0] - u(y_0) = \mathrm{E}\left[\int_0^{\tau_n} e^{-\rho t}\left\{u'(y_t)\mu(y_t,\eta_t) + \frac{1}{2}u''(y_t)\eta_t^2\sigma(y_t)^2 - \rho u(y_t)\right\} dt\Big|\mathcal{F}_0\right]$$
$$\leq -\mathrm{E}\left[\int_0^{\tau_n} e^{-\rho t}\pi(y_t,\eta_t)\, dt\Big|\mathcal{F}_0\right],$$

with equality if $\eta_t = \eta(y_t)$.

By assumption $E[e^{-\rho\tau_n}u(y_{\tau_n})|\mathcal{F}_0] \to 0$ as $n \to 0$. Taking the limit and using the dominated convergence theorem, I obtain $u(y_0) \geq \mathrm{E}\left[\int_0^{\infty} e^{-\rho t}\pi(y_t,\eta_t)\, dt\big|\mathcal{F}_0\right]$, with equality if $\eta_t = \eta(y_t)$. Hence, $(\eta(y_t))_{t\geq 0}$ must be an optimal control and

$$u(y_0) = \mathrm{E}\left[\int_0^{\infty} e^{-\rho t}\pi(y_t,\eta(y_t))\, dt\Big|\mathcal{F}_0\right].$$

□

REMARK O.1. If a function $u(\cdot)$ satisfies linear growth, it also satisfies $\lim_{t\to\infty}\mathrm{E}[u(y_t)e^{-\rho t}|\mathcal{F}_0] = 0$. In fact,

$$0 \leq \left|\lim_{t\to\infty}\mathrm{E}[u(y_t)e^{-\rho t}|\mathcal{F}_0]\right| \leq \lim_{t\to\infty}\mathrm{E}[|u(y_t)|e^{-\rho t}|\mathcal{F}_0] \leq \lim_{t\to\infty}\mathrm{E}[e^{-\rho t}(C_0\mu + C_1^{\mu}|y_t|)|\mathcal{F}_0] = 0,$$

where the last equality follows from C_0^{μ} and C_1^{μ} being constants and from Lemma 1 in Strulovici and Szydlowski (2015), which applies because of Assumption (R2).

REMARK O.2. When $\eta(\hat{y}) = 0$, strict concavity does not hold in general because the decision-maker can avoid termination and can thus maintain her rents. Moreover, $u(y)$ may also be decreasing in a neighborhood of $\hat{y}$. Consider, for example, a problem with $\pi(y,\eta) = 1-y$, $\mu(\cdot,\cdot) = 0$, $\sigma(\cdot) = 1$, $\bar{\eta} = -\underline{\eta} = 1$, and $\hat{y} = 0$. The model thus satisfies conditions (C1), (C2), (C3) and the corresponding HJB equation is

$$\rho u(y) = \max_{\eta\in[-1,1]}\left\{(1-y) + \eta^2 u''(y)\right\}.$$

However, one can immediately verify that $\eta(\hat{y}) = 0$ and that the decreasing linear function $u(y) = \frac{1-y}{\rho}$ is a solution.

LEMMA O.2. *The value function in problem* (8) *takes the form* $V(y,K) = Kv(y)$ *for a continuous function* v *which is the unique solution of the HJB equation* (9).

Proof. Solving the FSDE (7), I obtain

$$K_t = K_0 \exp\left\{\int_0^t \left[g(y_s,\eta_s,a_s) - \frac{1}{2}\nu(y_s,\eta_s,a_s)^2(\rho^2+\rho'^2)\right] ds + \int_0^t v(y_s,\eta_s,a_s)(\rho dZ_s^1 + \rho' dZ_s^2)\right\}.$$

Using this expression, I write the objective function in (8) as

$$K_0 \mathrm{E}\left[\int_0^\tau B_t e^{-\int_0^t(\rho - g(y_s,\eta_s,a_s))ds}\pi(y_t,\eta_t,a_t)\,dt\right], \tag{O.3}$$

where $(B_t)_{t\geq 0}$ is a density process:

$$B_t := \exp\left\{\int_0^t \nu(y_s,\eta_s,a_s)(\rho dZ_s^1 + \rho' dZ_s^2) - \int_0^t \frac{1}{2}\nu(y_s,\eta_s,a_s)^2(\rho^2+\rho'^2)ds\right\}.$$

The maximization problem (8) is equivalent to maximizing (O.3) under the same constraints. Hence, $V(y_0,K_0) = K_0 v(y_0)$, where

$$\begin{aligned} v(y_0) = \max_{(\eta_t)_{t\geq 0},(a_t)_{t\geq 0}} \ &\mathrm{E}\left[\int_0^\tau B_t e^{-\int_0^t(\rho - g(y_s,\eta_s,a_s))ds}\pi(y_t,\eta_t,a_t)\,dt\right] \\ \text{s.t. } &\eta_t \in [\underline{\eta},\bar{\eta}] \text{ and } a_t \in A \quad \forall t \geq 0, \\ &(6). \end{aligned} \tag{O.4}$$

I next show (9) is the HJB equation associated to problem (O.4). I define a probability measure Q on $(\Omega, \mathcal{F}^*)$ so that

$$E^Q[X_t] = E[B_t X_t]$$

for any $\mathcal{F}_t$-measurable random variable X_t. By Girsanov's theorem,

$$dZ_t^Q := dZ_t^1 - v(y_t,\eta_t,a_t)\rho dt$$

is an increment of a Brownian motion under the measure Q. Therefore, under the measure Q, the state variable y_t evolves as

$$dy_t = [\mu(y_t,\eta_t,a_t) + \eta_t\sigma(y_t)\nu(y_t,\eta_t,a_t)\rho]dt + \eta_t\sigma(y_t)dZ_t^Q, \quad y_0 = Y_0. \tag{O.5}$$

Problem (O.4) can thus be written as

$$v(y_0) = \max_{(\eta_t)_{t\geq 0},(a_t)_{t\geq 0}} \mathrm{E}^Q \left[\int_0^\tau e^{-\int_0^t (\rho - g(y_s,\eta_s,a_s))ds} \pi(y_t, \eta_t, a_t)\, dt \right]$$
$$\text{s.t. } \eta_t \in [\underline{\eta}, \bar{\eta}] \text{ and } a_t \in A \quad \forall t \geq 0,$$
$$\text{(O.5)}.$$

The HJB equation associated to this problem is (9).

□

REGULARITY CONDITIONS FOR SECTION 3.2. Similar to section 3.1, one could impose regularity conditions to ensure that (8) has a solution $K_0 v(y_0)$ where v is the unique classical solution of (9) satisfying $v(\hat{y}) = 0$ and linear growth. The following regularity conditions are the counterparts of (R1), (R2), and (R3).

CONDITIONS. *(Regularity Conditions with a Scaling Variable)*

(R1′) *The functions $\mu(y,\eta,a)$, $\sigma(y,\eta,a)$, $\pi(y,\eta,a)$, $\omega(y,\eta,a)$, and $\nu(y,\eta,a)$ are Lipschitz-continuous in y and continuous in a for all $y \in \mathcal{Y}$, $\eta \in [\underline{\eta}, \bar{\eta}]$, and $a \in A$.*

(R2′) *The discount rate ρ is large enough that $\mathrm{E}\left[\int_0^\tau e^{-\rho t} K_t \pi(y_t, \eta_t, a_t)\, dt\right] < \infty$ for any $y_0 \in \mathcal{Y}$, $K_0 \in \mathbb{R}^+$, and for all admissible $(\eta_t)_{t\geq 0}$ and $(a_t)_{t\geq 0}$ such that $\eta_t \in [\underline{\eta}, \bar{\eta}]$ and $a_t \in A$ for all $t \geq 0$.*

(R3′) *$\underline{\sigma} > 0$ exists such that $\sigma(y,\eta,a) \geq \underline{\sigma}$ for all $\eta \in [\underline{\eta}, \bar{\eta}]$, $a \in A$, and $y \in \mathcal{Y}$ such that $y \geq \hat{y}$. Moreover, $\underline{\eta} > 0$.*

O.4.2 AUXILIARY RESULTS FOR SECTION 5

Here, I present technical results for the model of section 5. In particular, I show the optimal stopping rule is a *threshold strategy* and I characterize the Markov perfect equilibrium using a system of differential inequalities. I start by showing equity holders default after assets fall below a threshold, which I denote by $\hat{V}$.

LEMMA O.3. *Consider the model of section 5. In any Markov-perfect equilibrium, a $\hat{V} > 0$ exists such that $\mathcal{D} = \{V \in \mathbb{R}^+ \colon V > \hat{V}\}$ and, hence, $\tau^* = \inf\{t \geq 0 \colon V_t \leq \hat{V}\}$.*

Proof. It is sufficient to prove the complement of $\mathcal{D}$, which I denote as $\mathcal{D}^C$, is a non-empty, bounded, and closed interval in the form $[0, \hat{V}]$.

First, I prove $\mathcal{D}^C$ is non-empty. I proceed by contradiction. Suppose $\mathcal{D}^C$ is empty; then, E coincides with the equity value when $\tau = \infty$, which I denote by $\tilde{E}(V)$. Note

$$\tilde{E}(V) \leq \max_{(\eta_t)_{t \geq 0}} E\left[\int_0^\infty e^{-rt}[(1-\theta)\delta V_t - c]\, dt \Big| \mathcal{F}_0\right] = \frac{(1-\theta)\delta}{r - \bar{\eta}\mu} V - \frac{c}{r},$$

and therefore, there exists a $\tilde{V} > 0$ such that $\tilde{E}(V) < 0$ for all $V \in [0, \tilde{V})$. But this finding contradicts that $\tau = \infty$ is optimal for the equity holders. Therefore, $\mathcal{D}^C$ is not empty.

Next, I show $\mathcal{D}^C$ is bounded. In particular, I show $\mathcal{D}^C$ is a subset of the bounded interval $\mathcal{N} := \{V \in \mathbb{R}^+ \colon (1-\theta)\delta V - c \leq 0\} = \left[0, \frac{c}{(1-\theta)\delta}\right]$. To show $\mathcal{D}^C \subseteq \mathcal{N}$, I proceed by contradiction. In particular, suppose there exists $V \in \mathcal{D}^C$ such that $(1-\theta)\delta V - c > 0$. Because $V \in \mathcal{D}^C$, $E(V) = 0$. Given $V_0 = V$, consider a stopping time $\bar{\tau} = \inf\{t \geq 0 \colon (1-\theta)\delta V_t - c \leq 0\}$. Then, $\mathrm{E}\left[\int_s^{\bar{\tau}} e^{-rt-s}[(1-\theta)\delta V_t - c]\, dt\right] > 0 = E(V)$. But this result contradicts $\tau = 0$ being optimal when $V_0 = V$.[10] Therefore, $\mathcal{D}^C$ must be a subset of $\mathcal{N}$.

Then, I show $\mathcal{D}^C$ is an interval. By way of contradiction, assume there exist two sets, D_1 and D_2, subsets of $\mathcal{D}^C$, with $D_1 \cap D_2 = \emptyset$, $V^1 = \sup D_1 < \inf D_2 = V^2$ and such that $[V^1, V^2] \cap \mathcal{D} \neq \emptyset$. Then, for any $V \in [V^1, V^2] \cap \mathcal{D}$, $E(V) > 0$. With $V_0 = V$, define $\tau' := \inf\{t \geq 0 \colon V_t \in D_1 \cup D_2\}$. By the dynamic programming principle,

$$E(V) = \mathrm{E}\left[\int_0^{\tau'} e^{-rt}((1-\theta)\delta V_t - c)\, dt + e^{-r\tilde{\tau}} E(V_{\tilde{\tau}}) \Big| \mathcal{F}_0\right].$$

By the definition of τ', $E(V_{\tau'}) = 0$. Moreover, because $\mathcal{D}^C \subseteq \mathcal{N}$, $(1-\theta)\delta V - c \leq 0$ for all $V \leq V^2$, and the integral in the previous expression is (weakly) negative. This result would imply $E(V) \leq 0$, thus contradicting that $V \in \mathcal{D}$. Therefore, $\mathcal{D}^C$ is an interval.

Finally, I show $\mathcal{D}^C$ is closed. First, notice $0 \in \mathcal{D}^C$. I then need to show $\hat{V} := \sup \mathcal{D}^C \in \mathcal{D}^C$. Let $\tau_V^0 := \inf\{t \geq 0 : V_t \in \mathcal{D}^C, V_0 = V\}$. By the Blumenthal zero-

[10] Note that, by definition of a Markov equilibrium, $\tau = 0$ is optimal for equity holders when $V_0 \in \mathcal{D}^C$.

one law (Karatzas and Shreve, 1998, Chapter 2.7.C), either $P(\tau^0_{\tilde{V}} = 0|V_0 = \tilde{V}) = 0$ or $P(\tau^0_{\tilde{V}} = 0|V_0 = \tilde{V}) = 1$. By symmetry, the first case is impossible. Therefore, $P(\tau^0_{\tilde{V}} = 0|V_0 = \tilde{V}) = 1$ and $E(\tilde{V}) = 0$. □

After establishing default is determined by a threshold strategy, I characterize the Markov perfect equilibrium recursively. Consider the HJB equation associated with the manager's decision problem, (16) for $V > \hat{V}$, with $u(\hat{V}) = 0$. For any given $\hat{V}$, results in Pham (2009) and Strulovici and Szydlowski (2015) imply the manager's value function is the unique twice-differentiable solution of (16) for $V \geq \hat{V}$. Next, I show the policy function $\eta(V)$ is continuous in V.

LEMMA O.4. *Consider the model of section 5. Let $\eta(V)$ be the maximizer in* (16). *Then, $\eta(\cdot)$ is a continuous function for $V > \hat{V}$.*

Proof. By the maximum theorem, the set of maximizers of (16) is an upper-hemicontinuous correspondence in V. I, therefore, need to show it is single-valued. For $V > \hat{V}$, there are two cases in which (16) has multiple maximizers: (i) $u''(V) = 0$ and $u'(V)\mu = 0$, or (ii) $u''(V) > 0$ and $u'(V)\mu(\bar{\eta} - \underline{\eta}) + \frac{1}{2}u''(V)s^2V(\bar{\eta}^2 - \underline{\eta}^2) = 0$.

I rule out these two cases by showing u is strictly increasing for $V \geq \hat{V}$ and that, if $\mu = 0$, u is strictly concave for $V \geq \hat{V}$. If u is strictly increasing, then one can rule out both cases when $\mu > 0$. When $\mu = 0$, the strict concavity of u rules out both cases as well.

To show that u is strictly increasing, consider $V_0^1 > V_0^0 \geq \hat{V}$. For $i \in \{0, 1\}$, let

$$\ln V_t^{i,\eta} = \ln V_0^i + \int_0^t \left(\mu\eta_t - \frac{1}{2}\eta_t^2\right) dt + \int_0^t \eta_t \, dZ_t, \tag{O.6}$$

where I set $\eta_t = \eta\left(e^{\ln V_t^{0,\eta}}\right) = \eta\left(e^{\ln V_t^{1,\eta} + \ln V_0^0 - \ln V_0^1}\right) = \eta\left(V_t^{1,\eta}\frac{V_0^0}{V_0^1}\right)$, which is a Markovian control also for $V_0 = V_0^1$. In particular, it coincides with the manager's optimal control when $V_0 = V_0^0$ for any given realized path $(Z_u)_{0 \leq u \leq t}$ of the Brownian motion.

Let $\tau^0 = \inf\{t \geq 0 : V_t^{0,\eta} \leq \hat{V}\}$. Note $V_{\tau^0}^{1,\eta} > \hat{V}$. Then,

$$u(V_0^1) \geq \mathrm{E}\left[\int_0^{\tau_0} e^{-\rho t}\theta\delta V_t^{1,\eta} \, dt + e^{-\rho\tau^0}u(V_{\tau^0}^{1,\eta})\right] \geq u(V_0^0) + \mathrm{E}\left[e^{-\rho\tau^0}u(V_{\tau^0}^1)\right] > u(V_0^0). \tag{O.7}$$

Hence, u is strictly increasing for $V \geq \hat{V}$.

Next, I prove strict concavity when $\mu = 0$. First, note that, from (16), $\rho u(V) - \theta\delta V \geq \frac{1}{2}u''(V)V^2\eta^2$ for any $\eta \in [\underline{\eta}, \bar{\eta}]$. Next, note

$$u(V) < \tilde{u}(V) = \max_{(\eta_t)_{t\geq s}} \mathrm{E}\left[\int_s^\infty e^{-\rho(t-s)}\theta\delta V_t\,dt\right] = \frac{\theta\delta V}{\rho},$$

where the first inequality follows because $\theta\delta V_t > 0$ and $\tau := \inf\{t \geq 0\colon V_t \leq \hat{V}\} < \infty$. The equality follows because $\theta\delta V/\rho$ is a solution of (16) with $\hat{V} = 0$ and, by Pham (2009) and Strulovici and Szydlowski (2015), it is the unique solution of (16). Therefore, $\rho u(V) - \theta\delta V < 0$, and $u''(V) < 0$ for any $V \geq \hat{V}$.

In conclusion, the maximizer of (16) is a continuous function of V. □

I then define the default threshold $\hat{V} := \sup\{V \geq 0\colon E(V) = 0\}$, where E solves the variational inequality associated with the shareholders' problem:

$$\min\{rE(V) - H(V, E'(V), E''(V)), E(V)\} = 0, \tag{O.8}$$

with

$$H(V, E'(V), E''(V)) = (1-\theta)\delta V - c + E'(V)\mu\eta(V)V + \frac{1}{2}E''(V)\eta(V)^2V^2.$$

By Lemma O.4, the manager's optimal policy $\eta(V)$ is continuous in V for $V \geq \hat{V}$. I impose $\eta(V) = \lim_{V'\to\hat{V}}\eta(V')$ for $V \leq \hat{V}$, so that $\eta(V)$ is continuous for all $V > 0$. Hence, the equity value is the unique continuous solution of (O.8) satisfying linear growth (Pham, 2009, Theorem 5.2.1 and Remark 5.2.1).

A Markov-perfect equilibrium, therefore, solves a fixed-point problem. Given the manager's policy function $\eta(\cdot)$, the stopping time $\tau = \inf\{t \geq 0\colon V \leq \hat{V}\}$ must be optimal for the equity holders. At the same time, given the stopping time τ and its associated default threshold $\hat{V}$, the policy function $\eta(\cdot)$ must be optimal for the manager.

In particular, let the functions u and E solve the system given by (16) and (O.8) with $\hat{V} := \sup\{V \geq 0\colon E(V) = 0\}$ and where $\eta(V)$ is the maximizer in (16). Let $\eta_t^* = \eta(V_t)$ for all $t \geq 0$ and $\tau^* := \inf\{t \geq\colon V_t \leq \hat{V}\}$. If $(\eta_t^*)_{t\geq 0}$ is admissible, $(\eta_t^*)_{t\geq 0}$ and τ^* constitute a Markov-perfect equilibrium. By Lemma O.1 and Remark O.1, $(\eta_t^*)_{t\geq 0}$

is optimal for the manager. Moreover, τ^* is optimal for equity holders because

$$\mathrm{E}\left[\int_0^{\tau^*} e^{-\rho t}\{(1-\theta)\delta V_t - c\}\,dt\middle|\mathcal{F}_0\right] = E(V_0) = \max_\tau \mathrm{E}\left[\int_0^{\tau} e^{-\rho t}\{(1-\theta)\delta V_t - c\}\,dt\middle|\mathcal{F}_0\right],$$

where the first equality follows from the dynamic programming principle and $E(V_{\tau^*}) = 0$, and the second follows because $E(V_0)$ is the equity holders' value function. Therefore, τ^* is optimal for equity holders.

Finally, the following result establishes equity holders are risk-loving in a neighborhood of $\hat{V}$.

Lemma O.5. *Consider the model of section 5. An $\varepsilon > 0$ exists such that E is twice differentiable in $(\hat{V}, \hat{V}+\varepsilon)$ with $E''(V) < 0$.*

Proof. I begin by showing E is twice differentiable in a right neighborhood of $\hat{V}$. By corollary 1 and by the restriction that $\eta(V') = \lim_{V\to V^+}\eta(V)$ for $V' \le \hat{V}$, there exists $\varepsilon' > 0$ such that $\eta(V) = \underline{\eta}$ for $V < \hat{V} + \varepsilon'$. For a $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon'$, consider the following variational inequality for $V \in (\hat{V}+\varepsilon_0, \hat{V}+\varepsilon_1)$:

$$re(V) - H(V, e'(V), e''(V)) = 0,$$

with boundary conditions $e(\hat{V}+\varepsilon_0) = E(\hat{V}+\varepsilon_0)$ and $e(\hat{V}+\varepsilon_1) = E(\hat{V}+\varepsilon_1)$. It can be immediately verified that $e = E$ is a solution of this variational inequality and that, by the usual arguments in Pham (2009), it is the unique continuous solution.

Because $\eta(V) = \underline{\eta}$ for $V \le V_1$, the functions $(1-\theta)\delta V - c$, $\mu\eta(V)V$, and $\eta(V)\sigma V$ are all twice differentiable in V and bounded in the interval $(\hat{V}+\varepsilon_0, \hat{V}+\varepsilon_1)$. Classical results (Fleming and Soner, 2006, Ch. IV.4) imply e is twice differentiable in $(\hat{V}+\varepsilon_0, \hat{V}+\varepsilon_1)$. Because $\varepsilon_0 > 0$ can be arbitrarily small, both e and E are twice differentiable in $(\hat{V}, \hat{V}+\varepsilon_1)$.

Next, I show $E''(V) > 0$ in a right neighborhood of $\hat{V}$. By the smooth-fit principle (Pham, 2009), $E(V) = E'(\tilde{V}) = 0$ and $E(V) \to 0$ and $E'(V) \to 0$ as $V \to \tilde{V}^+$. By (O.8),

$$rE(V) - [(1-\theta)\delta V - c] - E'(V)\mu\eta(V)V = +\frac{1}{2}E''(V)\eta(V)^2V^2,$$

and hence,

$$\lim_{V\to\hat{V}^+}\frac{1}{2}E''(V)\eta(V)^2V^2 = \lim_{V\to\hat{V}^+} -[(1-\theta)\delta V - c].$$

It therefore suffices to show $[(1-\theta)\delta\hat{V}-c]<0$. To show this, note

$$E(V) > \min_{(\eta_t)_{t\geq 0}} \mathrm{E}\left[\int_0^\infty e^{-rt}[(1-\theta)\delta V_t - c]\,dt\right] \quad \text{s.t. } \eta_t \in [\underline{\eta},\bar{\eta}],\ \forall t \geq 0,$$

where the strict inequality follows because $\mathcal{D}^C$ is non-empty and, hence, $\tau^* < \infty$, and where the right-hand side of this expression is equal to $\frac{(1-\theta)\delta}{r-\mu\underline{\eta}}V - \frac{c}{r}$. For $V = \hat{V}$, it follows that $0 > \frac{(1-\theta)\delta}{r-\mu\underline{\eta}}\hat{V} - \frac{c}{r}$ and $(1-\theta)\delta\tilde{V} - c < -c\mu\underline{\eta}/r \leq 0$. Therefore, $E''(V) > 0$ in a right neighborhood of $\hat{V}$. □

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