The Borel Conjecture

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Definition. *$A \subseteq \mathbb{R}$ has strong measure 0 if for all sequences $\epsilon_i : i \in \omega$, $\epsilon_i > 0$ there exists intervals I_i with $len(I_i) < \epsilon_i$ such that $A \subseteq \bigcup I_i$.*

Remark. *Clearly if A has strong measure 0 then A has measure 0. However, e.g. the Cantor set has measure 0 but does not have strong measure 0.*

Definition. *The Borel conjecture states that every strong measure 0 set is countable.*

Theorem 0.1. *The Borel conjecture is independent of* **ZFC**.

1 Failure of the Borel Conjecture

Recall: **MA** states that for any ccc forcing $\mathbb{P}$ and family of dense subsets $\mathcal{D} = \{D : D \subseteq \mathbb{P} \text{ dense in } \mathbb{P}\}$ with $|\mathcal{D}| < 2^\omega$, there exists a filter G meeting all $D \in \mathcal{D}$. Also recall that **CH** implies **MA**.

It is known that **CH** implies that there is an uncountable strong measure zero set, i.e. the Borel Conjecture is false. In fact, more generally:

Theorem 1.1. *If* **MA** *holds then there is a strong measure 0 set of size $\mathfrak{c}$.*

Definition. *We say that $X \subseteq [0,1]$ has the Rothberger property if for all intervals $I_{xn} : n \in \omega$ all containing x with $x \in X$, there exists a sequence $x_n \in X$ s.t. $X \subseteq \bigcup I_{x_n n}$.*

Observe that if X has the Rothberger property then X has strong measure 0: fix ϵ_n, just take $len(I_{xn}) < \epsilon_n$ for all x, n.

Lemma 1.2 (Martin–Solovay)**.** *If* **MA** *holds then the union of $< \mathfrak{c}$-many meagre sets is meagre. Equivalently, any $< \mathfrak{c}$ intersection of dense open sets is dense.*

Lemma 1.3. *If* **MA** *holds then every $X \subseteq [0,1]$ with size $< \mathfrak{c}$ has the Rothberger property, hence has strong measure 0.*

Proof. For all $x \in X$ we have a sequence of intervals I_{xn}. WLOG we shrink them to have rational endpoints. So for each n, we can enumerate $\{I_{xn} : x \in X\}$ (after shrinking this is countable) as $\{I_n^0, I_n^1, \dots\}$. For each $x \in X$, define $A_x = \{f \in \omega^\omega : x \notin \bigcup_{n<\omega} I_n^{f(n)}\}$. Then A_x is nowhere dense: suppose A_x is dense in some open set given by finite stem s, then for any $g \succ s$ there is finite $t \prec g$ s.t. if $f \succ t$ then $x \notin \bigcup I_n^{f(n)}$.

But above s, we can define $g(n)$ to be some m such that $x \in I_n^m$, which exists since $x \in I_{xn}$ and I_{xn} appears somewhere in the nth enumeration. Then for any finite $t \succeq s$ with $t \prec g$, g is a counterexample.

Now since $|X| < \mathfrak{c}$, by Lemma 1.2, $\bigcup_{x \in X} A_x$ is meagre, so there is $f \in \omega^\omega - \bigcup_{x \in X} A_x$. So $X \subseteq \bigcup I_n^{f(n)}$ and we are done. □

Remark. *The above lemma already implies that the Borel conjecture fails in a lot of models, e.g. if* $\mathbf{MA} + \mathfrak{c} = \aleph_2$ *holds. But it not yet shows Theorem 1.1.*

Definition. *We say that* $X \subseteq [0,1]$ *is concentrated around a set* $C \subseteq [0,1]$ *if* $|X - \bigcup_{c \in C} I_c| < \mathfrak{c}$ *for all intervals* I_c *which contains* c.

Lemma 1.4. *If* $\mathbf{MA}$ *holds and* X *is concentrated around a countable set* C*, then* X *has strong measure 0.*

Proof. Fix ϵ_i. Enumerate $C = \{c_n\}$. Pick J_n containing c_n with length $< \epsilon_{2n}$. $X - \bigcup J_n$ has size $< \mathfrak{c}$, so by Lemma 1.3 has strong measure 0. So there are intervals K_n with $len(K_n) < \epsilon_{2n+1}$ which cover it. Then J_n and K_n together cover X. □

Definition. *A generalized Luzin set is a subset of* $\mathbb{R}$ *which is concentrated around every dense subset of* $[0,1]$.

Lemma 1.5. *If* $\mathbf{MA}$ *holds then there is a generalized Luzin set of size* $\mathfrak{c}$.

Proof. Enumerate the dense open subsets of $[0,1]$ as $D_\alpha : \alpha < \mathfrak{c}$. We construct $\{x_\alpha : \alpha < \mathfrak{c}\}$ inductively.

At stage α, choose x_α from $\bigcap_{\beta<\alpha} D_\beta$ distinct from the previous choices $\{x_\beta : \beta < \alpha\}$. This is possible since by Lemma 1.2, $\{x_\beta : \beta < \alpha\}$ is meagre and $\bigcap_{\beta<\alpha} D_\beta$ is dense, so the complement of $\{x_\beta : \beta < \alpha\}$, which is comeagre, must meet $\bigcap_{\beta<\alpha} D_\beta$.

Now we verify that $X = \{x_\alpha : \alpha < \mathfrak{c}\}$ is a generalized Luzin set. Let C be dense, fix intervals I_c for $c \in C$, then $\bigcup_{c \in C} I_c$ is dense open, which appears as D_α for some $\beta < \mathfrak{c}$. By construction, if $\alpha > \beta$ then $x_\alpha \in D_\beta$, so $X - \bigcup I_c \subseteq \{x_\alpha : \alpha \leq \beta\}$ which has size $< \mathfrak{c}$. (This follows since under $\mathbf{MA}$, $\mathfrak{c}$ is regular.) □

Proof for Theorem 1.1. Follows from Lemma 1.4 and 1.5. □

2 Consistency of the Borel Conjecture

Theorem 2.1 (Laver)**.** *Let $M \models \mathbf{ZFC} + \mathbf{CH}$, then there is a forcing extension $M[G]$ where the Borel conjecture holds and $\mathfrak{c} = \aleph_2$.*

Definition. *The Laver forcing $\mathbb{J}$ is the following forcing: the conditions are trees $T \subseteq \omega^{<\omega}$, ordered by $S \leq T$ iff $S \subseteq T$. Each condition $T \in \mathbb{J}$ has a stem s_T, i.e. for any $s \in T$ either $s_T \preceq s$ or $s \preceq s_T$. Moreover, if $s_T \preceq s$ and $s \in T$ then $|\{n : s^\frown n \in T\}| = \omega$. I.e. T is countably branching at every node including or extending the stem s_T.*

Definition. *$\mathbb{D}_{\omega_2}$ is the countable support iteration of $\mathbb{J}$ of length ω_2, which is given by the following:*

- *$\langle \mathbb{D}_\alpha : \alpha \leq \omega_2 \rangle$ and $\langle \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ s.t. $\dot{Q}_\alpha$ is a $\mathbb{D}_\alpha$-name, $\mathbb{1}_\alpha \Vdash \dot{Q}_\alpha$ to be $\mathbb{J}$ in $M[G_\alpha]$;*
- *$\mathbb{D}_1 = \{p : p \text{ is a function } 1 \to \mathbb{J}\} \simeq \mathbb{J}$;*
- *$\mathbb{D}_{\alpha+1} = \mathbb{D}_\alpha * \dot{Q}_\alpha = \{p : p|_\alpha = p \in \mathbb{D}_\alpha, p(\alpha) = \dot{q}, \text{ which is a } \mathbb{D}_\alpha\text{-name}, p \Vdash \dot{q} \in \dot{Q}_\alpha\}$;*
- *At limit stages β with $cf(\beta) = \omega$ we take the inverse limit, i.e. $\mathbb{D}_\beta = \{p : p \text{ is a sequence of length } \beta, p|_\alpha \in \mathbb{D}_\alpha \text{ for } \alpha < \beta\}$, at β with $cf(\beta) \geq \omega_1$ we take the direct limit, i.e. the subset of the inverse limit such that $p(\alpha) = \dot{\mathbb{1}}_{\dot{Q}_\alpha}$ for all sufficiently large α. Equivalently, for any limit β and $\alpha < \beta$, $p|_\alpha \in \mathbb{D}_\alpha$ and $supp(p) = \{\alpha < \beta : p(\alpha) \neq \dot{\mathbb{1}}_{\dot{Q}_\alpha}\}$ is countable.*

Remark. *Adding a Cohen real falsifies the Borel Conjecture, and a finite support iterations add Cohen reals at stages with cofinality ω, so a countable support iteration is needed.*

Definition. *$\mathbb{D}^{\alpha\beta}$ is the forcing defined in $M[G_\alpha]$, which contains the functions p with $dom(p) = [\alpha, \beta)$ s.t. $\mathbb{1}_\alpha^\frown p \in \mathbb{D}_\alpha$, ordered by $p \leq q$ iff $\exists r \in G_\alpha$ such that $r^\frown p \leq_\beta r^\frown q$. $\mathbb{D}^\alpha$ is the tail forcing $\mathbb{D}^{\alpha\omega_2}$.*

We let G be $\mathbb{D}_{\omega_2}$-generic over M. It suffices to show that in $M[G]$ there is no strong measure 0 set of reals of size $\aleph_1$, since any $\aleph_1$-subset of an uncountable strong measure 0 set has strong measure 0 as well. The theorem will follow from the following claims:

1. If X is a set of reals of size $\aleph_1$ in $M[G]$, then $X \in M[G_\alpha]$ for some $\alpha < \omega_2$;
2. If X is a set of reals in M and has strong measure 0 in $M[G]$, then it is countable;
3. $M[G_\alpha] \models (\mathbb{D}_{\omega_2})^{M[G_\alpha]} \simeq \mathbb{D}^\alpha$.

Proof outline for Theorem 2.1. Let $M[G] \models X$ has size $\aleph_1$ and has strong measure 0. By Claim 1 $X \in M[G_\alpha]$ for some $\alpha < \omega_2$. We know $M[G] = M[G_\alpha][G^\alpha]$ where G^α is $\mathbb{D}^\alpha$-generic over $M[G_\alpha]$. By Claim 3, G^α and the isomorphism induces some $H \in M[G]$ which is $(\mathbb{D}_{\omega_2})^{M[G_\alpha]}$-generic over $M[G_\alpha]$, and moreover $M[G_\alpha][H] = M[G]$. So $M[G_\alpha][H] \models X$ has strong measure 0. By Claim 2 (with $M[G_\alpha]$ as the ground model and H as the generic), X is countable. □

We need to define some orders $\leq^n$ on $\mathbb{J}$ and the iteration poset $\mathbb{D}_{\omega_2}$. For this we fix an enumeration of $\omega^{<\omega} = \{x_0, x_1, \dots\}$ such that $x_i \prec x_j \Rightarrow i < j$ and $s^\frown m, s^\frown n, m < n \Rightarrow i < j$. For each T, this induces an enumeration $T\langle 0\rangle, T\langle 1\rangle, \dots$ of its elements, with $T\langle 0\rangle = s_T$.

Definition. *For $S, T \in \mathbb{J}$, $S \leq^n T$ means $S \leq T$ and $S\langle i\rangle = T\langle i\rangle$ for all $i \leq n$. In particular, $S \leq^0 T$ means $S \leq T$ and S and T has the same stem.*

For $t \in T$, let $T_t = \{s \in T : s \preceq t \text{ or } s \succeq t\}$. $\{T\langle 0\rangle, \dots, T\langle n\rangle\}$ induces $n+1$-many subtrees of T in the following way: S_i $(i \leq n)$ is the subtree $\bigcup T_t : t \in T$ and t is an immediate successor of $T\langle i\rangle$, which is not on the list $\{T\langle 0\rangle, \dots, T\langle n\rangle\}$. Note that since $T\langle 0\rangle$ is the stem and the other points are above it, T branches at all these points, and so there must be countably many immediate successors not on the list. It follows that $S_i \in \mathbb{J}$ with $S_i\langle 0\rangle = T\langle i\rangle$ as its stem.

Definition. *$\{S_0, \dots, S_n\}$ is called the set of components of T at stage n.*

Note that the set of components forms a maximal antichain below T in $\mathbb{J}$.

Lemma 2.2. *Let $m < \omega$ and $\cdots \leq^{m+2} T_{m+2} \leq^{m+1} T_{m+1} \leq^m T_m$. Let $T_\omega = \bigcup_{i \geq m}\{T_i\langle 0\rangle, \dots, T_i\langle i\rangle\} \cup \{s : s \preceq T_m\langle 0\rangle\}$, then T_ω is the unique $T \in \mathbb{J}$ such that $T \leq^i T_i$ for all $i \geq m$.*

Proof. In fact, $T_\omega = \{s : s \preceq T_m\langle 0\rangle\} \cup \{T_m\langle 0\rangle, \dots, T_m\langle m\rangle, T_{m+1}\langle m+1\rangle, \dots\}$. Let $T_k\langle k\rangle \in T$, then $T_k\langle k\rangle$ falls in T_i for $i > k$ because in particular $T_i \leq^k T_k$. It falls in T_i for $i < k$ because in particular $T_k \leq T_i$. So $T_\omega \subseteq T_i$ for all i.

Observe that the latter part of the expression of T_ω lists $T_\omega\langle n\rangle$. Observe that the first $i+1$ components of T_i are $T_m\langle 0\rangle, \dots, T_m\langle m\rangle, \dots, T_{i-1}\langle i-1\rangle, T_i\langle i\rangle$. so clearly $T_\omega \leq^i T_i$. Any such T is unique because if $T \leq^i T_i$ then T must have $T_m\langle 0\rangle$ as stem, $\{T_m\langle 0\rangle, \dots, T_m\langle m\rangle\}$ as the first $m+1$ terms, and satisfy $T\langle k\rangle = T_k\langle k\rangle$ afterwards. □

Lemma 2.3 (Laver Condition)**.** *Let $T \in \mathbb{J}$, $\varphi_n : n \leq k$ are sentences of the forcing language, $T \Vdash \bigvee_{n \leq k} \varphi_n$. Then for all i there is T' with $T' \leq^i T$ and $I \subseteq [0, k]$ of size $\leq i+1$ such that $T' \Vdash \bigvee_{n \in I} \varphi_n$. (In particular, when $i = 0$, this says there is $T' \leq T$ with the same stem as T forcing one of the φ_n.) (Also in particular, we can decide any φ without extending the stem.)*

Proof. Proof by induction on i, starting from $i = 0$. Let

$$(*)_s : \text{ there are no } T' \in \mathbb{J}, n \text{ s.t. } T' \leq^0 T_s, T' \Vdash \varphi_n$$

Suppose the claim is false. We construct a tree S starting from the stem $T\langle 0\rangle$. $I = \{n : (*)_{T\langle 0\rangle^\frown n}\}$ is infinite: otherwise $I^C = \{n : \neg(*)_{T\langle 0\rangle^\frown n}\}$ is cofinite, so $J = I^C \cap \{n : T\langle 0\rangle^\frown n \in T\}$ is infinite, so the tree $T' = \bigcup\{T_{T\langle 0\rangle ^\frown n} : n \in J\}$ is in $\mathbb{J}$ and $T' \leq^0 T$. Now we take $s = T\langle 0\rangle ^\frown n \in T'$, since $(*)_s$ fails there are $T'' \leq T' \leq T$ and k s.t. $T'' \Vdash \varphi_k$, contradiction. We extend S to all the nodes in I. For each node we repeat the argument and complete the construction of S. Since $S \leq^0 T$, no extension of S can force any φ_k, so $S \Vdash \neg \bigvee \varphi_n$, contradiction.

For the inductive step, given T, we partition T into components at stage i, $S_0, \dots, S_i$. We apply the base case to each S_k and get $S'_k \leq^0 S_k$ s.t. $S'_k \Vdash \varphi_{n_k}$. Then $T' = \bigcup S'_k \leq^i T$ and $T' \Vdash \bigvee_{0\leq k\leq i} \varphi_{n_k}$, which is a disjunction of $i+1$ formulas. □

Lemma 2.4. *Let $T \in \mathbb{J}$, $T \Vdash \dot{a} \in M$, then for all i there is a countable $A \in M$ and a $T' \in \mathbb{J}$ with $T' \leq^i T$ such that $T' \Vdash \dot{a} \in A$ (checks omitted).*

Proof. The proof is analogous as above, by induction on i. For the base case, suppose the statement is false. Let

$$(*)_s : \text{ there are no } T' \in \mathbb{J}, A \text{ countable s.t. } T' \leq^0 T_s, T' \Vdash \dot{a} \in A$$

Start from $T\langle 0\rangle$, by the same argument as above $I = \{n : (*)_{T\langle 0\rangle^\frown n}\}$ is infinite, we construct $S \leq^0 T$ analogously, then for all countable $A \in M$, $S \nVdash \dot{a} \in A$. But since $S \leq T$, $S \Vdash \dot{a} \in M$. So there is $S' \leq S$ s.t. $S' \Vdash \dot{a} = \check{x}$ for some $x \in M$. Then $S' \leq^0 T_{S'\langle 0\rangle}$ and yet $S' \Vdash \dot{a} \in A$ for some countable A, contradicting $(*)_{S'\langle 0\rangle}$.

For the inductive step we apply the base case to components and take the finite union of the corresponding As, which is still countable. □

Lemma 2.5. *Let $T \in \mathbb{J}$, $T \Vdash \dot{A}$ is a countable subset of M, then for each n there is a countable $A \in M$ and a $T' \in \mathbb{J}$ with $T' \leq^n T$ such that $T' \Vdash \dot{A} \subseteq A$ (checks omitted).*

Proof. Let $\dot{A} = \{\dot{a}_0, \dot{a}_1, \dots\}$. Proof by induction on n. For base case $n = 0$, start with $T_0 = T$. By Lemma 2.4, there is $T_1 \leq^0 T_0$ and countable $A_0 \in M$ such that $T_1 \Vdash \dot{a}_0 \in A_0$. Use the lemma to find $\dots T_3 \leq^2 T_2 \leq^1 T_1$ and $A_1, A_2, \dots$. Use Lemma 2.2 to find $T_\omega \leq^n T_n$. Then $T_\omega \Vdash \dot{A} \subseteq \bigcup A_i$, which is countable.

For inductive case, use base case on stage n components. □

We generalize the orders $\leq^n$ to be defined on the iteration poset $\mathbb{D}_{\omega_2}$ and prove generalizations of the above lemmas.

Definition. *Let $\beta \leq \omega_2$, $F \subseteq \beta$ be finite, $p, q \in \mathbb{D}_\beta$. $p \leq^n_F q$ means $p \leq q$ and for all $\alpha \in F$, $p|_\alpha \Vdash p(\alpha) \leq^n q(\alpha)$.*

Lemma 2.6. *Let $p_n : n < \omega$, $p_n \in \mathbb{D}_\beta$, $F_n : n < \omega$ be an increasing chain of finite sets s.t. $\bigcup F_n = \bigcup supp(p_n)$, and $p_{n+1} \leq^n_{F_n} p_n$. Then there is a unique $p_\omega \in \mathbb{D}_\beta$ s.t. $p_\omega \leq^n_{F_n} p_n$ for all n.*

Proof. For $\alpha < \beta$, $p_\omega(\alpha)$ is a $\mathbb{D}_\alpha$-name forced to be the tree generated by $\{p_n(\alpha)\langle i\rangle : n < \omega\}$ where $\alpha \in F_n$ and $i \leq n$. ($p_n(\alpha)$ are $\mathbb{D}_\alpha$-names for trees in $\mathbb{J}^{M[G_\alpha]}$.) If $\alpha \notin \bigcup F_n$ then $p_\omega(\alpha)$ is forced to be $\mathbb{1}_\alpha$. p_ω has countable support.

We prove by induction on γ for $1 \leq \gamma \leq \beta$ that $p_\omega|_\gamma \in \mathbb{D}_\gamma$ and for each n, $p_\omega|_\gamma \leq^n_{F_n\cap\gamma} p_n|_\gamma$.

$\gamma = 1$: follows from Lemma 2.2.

$\gamma = \delta + 1$: By induction hypothesis, we have $p_\omega|_\delta$, a condition in $\mathbb{D}_\delta$, which satisfies the requirement. If $\delta \notin \bigcup F_n$, then $p_\omega|_{\delta+1} = p_\omega|_\delta \frown \mathbb{1}_\delta \in \mathbb{D}_{\delta+1}$. $p_\omega|_{\delta+1} \leq^n_{F_n\cap\delta} p_n|_{\delta+1}$ clearly follows. If $\delta \in \bigcup F_n$, then $p_\omega|_\delta$ forces $p_\omega(\delta)$ to be the tree generated by $\{p_n(\delta)\langle i\rangle : n < \omega, i \leq n\}$. Let N be the least such that $\delta \in F_N$, then $p_\omega(\delta)$ is enumerated by $\{p_N(\delta)\langle 0\rangle, \dots, p_N(\delta)\langle N\rangle, p_{N+1}(\delta)\langle N+1\rangle, \dots\}$. The conclusion follows from the analysis in Lemma 2.2.

γ is limit: suppose $p_\omega|_\gamma \not\leq^n_{F_n\cap\gamma} p_n|_\gamma$, then there exists $n, \delta \in F_n \cap \gamma$ s.t. $p_\omega|_\delta \not\Vdash p_\omega(\delta) \leq^n p_n(\delta)$, contradicting the definition of p_ω. □

Lemma 2.7. *Let* $1 \leq \beta \leq \omega_2$*,* $p \in \mathbb{D}_\beta$*,* $F = \{\alpha_1 < \cdots < \alpha_i\} \subseteq \beta$*,* $n < \omega$*. Then:*

i) If $p \Vdash \bigvee_{j\leq k} \varphi_j$ *then there is* $I \subseteq [0,k]$ *with* $|I| \leq (n+1)^i$ *and* $p' \leq^n_F p$ *s.t.* $p' \Vdash \bigvee_{j\in I} \varphi_j$*.*

ii) If $p \Vdash \dot a \in M$ *then there is a countable* $A \in M$ *and* $p' \leq^n_F p$ *s.t.* $p' \Vdash \dot a \in A$*.*

iii) If $p \Vdash \dot A$ *is a countable subset of* M *then there is a countable* $A \in M$ *and* $p' \leq^n_F p$ *s.t.* $p' \Vdash \dot A \subseteq A$*.*

iv) If $\beta < \delta \leq \omega_2$ *and* $p \Vdash \dot f \in \mathbb{D}^{\beta\delta}$ *then there is an* $f \in \mathbb{D}^{\beta\delta}$ *and* $p' \leq^n_F p$ *s.t.* $p' \Vdash \dot f = f$*.*

Proof. By induction on β.

Base case. $\beta = 1$. i) is Lemma 2.3. ii) is Lemma 2.4. iii) let $\dot A = \{\dot a_1, \dot a_2, \dots\}$. Start with $p_n = p$ and $\dot a_1$, use the base case of ii), get $p_{n+1} \leq^n p$ and $p_{n+1} \Vdash \dot a_1 \in A_1$, which is countable. Use ii) again over $n+1/\dot a_2$, get $p_{n+2} \leq^{n+1} p_{n+1}$ and $p_{n+2} \Vdash \dot a_2 \in A_2$, which is countable. And so on. By Lemma 2.2 we get $p_\omega \leq^i p_i$ which forces $\dot A \subseteq A = \bigcup_{i\geq 1} A_i$, which is countable. iv) by definition, $p \Vdash \dot f \in \mathbb{D}^{1\delta}$ means $p \Vdash \mathbb{1}_{\mathbb{J}} \frown \dot f \in \mathbb{D}_\delta$. In particular, $p \Vdash supp(\dot f)$ is countable. By iii) there is $p' \leq^n p$ and countable A s.t. $p' \Vdash supp(\dot f) \subseteq A$. Let $f(\gamma)$ be a $\mathbb{D}_\gamma$-name such that $\Vdash f(\gamma) = \dot f(\gamma)$ for $\gamma \in A$, otherwise $f(\gamma)$ names $\mathbb{1}$. Then f has countable support so $f \in \mathbb{D}^{\beta\delta}$.

Successor case. $\beta = \sigma + 1$. i) WLOG $\alpha_i = \sigma$. We have $p \Vdash \bigvee_{j\leq k} \varphi_j$. Let $\dot S_t$: $t \leq n$ name the tth stage n components of $p(\sigma)$. We have $p|_\sigma \frown \dot S_t \Vdash \bigvee_{j\leq k} \varphi_j$. So $p|_\sigma \Vdash (\dot S_t \Vdash \bigvee_{j\leq k} \varphi_j)$. Since Lemma 2.3 holds in $M[G_\sigma]$,

$$\mathbb{1}_\sigma \Vdash (\dot S_t \Vdash \bigvee_{j\leq k} \varphi_j \rightarrow \exists \dot S'_t \leq^0 \dot S_t, \dot S'_t \Vdash \varphi_j \text{ for some } j)$$

So for each $t \leq n$, there is some $\dot S'_t$ s.t. $p|_\sigma \Vdash \dot S'_t \leq^0 \dot S_t$ and $p|_\sigma \Vdash (\bigvee_{j\leq k} \dot S'_t \Vdash \varphi_j)$. Now $p|_\sigma \in \mathbb{D}_\sigma$, so we can apply IH over $G = F \cap \sigma = \{\alpha_1 \leq \cdots \leq \alpha_{i-1}\}$ and get $q_0 \leq^n_G p|_\sigma$ and I_0 with $|I_0| \leq (n+1)^{i-1}$ s.t. $q_0 \Vdash (\bigvee_{j\in I_0} \dot{S}'_0 \Vdash \varphi_j)$. We iterate and find $q_n \leq^n_G \cdots \leq^n_G q_0$ where $q_k \Vdash (\bigvee_{j\in I_k} \dot{S}'_k \Vdash \varphi_j)$. Let $\dot{S}'$ name the union of $\dot{S}'_k$ and $I = \bigcup I_k$, we have $|I| \leq (n+1)^i$, then we have $q_n \Vdash (\dot{S}' \Vdash \bigvee_{j\in I} \varphi_j)$, so $q_n \frown \dot{S}' \Vdash \bigvee_{j\in I} \varphi_j$, moreover $q_n \Vdash \dot{S}' \leq^n p(\sigma)$, as desired.

ii) We have $p \Vdash \dot{a} \in M$, so $p|_\sigma \Vdash (p(\sigma) \Vdash \dot{a} \in M)$. By Lemma 2.4 applied in $M[G_\sigma]$ there is some $\dot{T}$ and $\dot{A}$ s.t. $\dot{A}$ is a $\mathbb{D}_\sigma$-name for a countable set in $M[G_\sigma]$, $p|_\sigma \Vdash \dot{T} \leq^n p(\sigma)$, and $p|_\sigma \Vdash (\dot{T} \Vdash \dot{a} \in \dot{A})$. Since $p|_\sigma \Vdash \dot{A}$ is countable in $M[G_\sigma]$, we can apply iii) at stage σ. We get $p' \leq^n_G p|_\sigma$ and a countable $A \in M$ s.t. $p' \Vdash \dot{A} \subseteq A$. We have $p' \frown \dot{T} \Vdash \dot{A} \subseteq A$, while $p' \Vdash \dot{T} \leq^n p(\sigma)$, as desired.

iii) We have $\dot{A} = \{\dot{a}_1, \dot{a}_2, \dots\}$ which is a $\mathbb{D}_{\sigma+1}$-name and $p \Vdash \dot{A}$ is a countable subset of M. We are free to apply ii) at stage $\sigma+1$, starting with $F_0 = F$. So we get $p_1 \leq^n_{F_0} p$ and A_1 s.t. $p_1 \Vdash \dot{a}_1 \in A_1$. We need to get an increasing sequence of finite sets $\cdots \supseteq F_1 \supseteq F_0$ so that $\bigcup F_i = \bigcup supp(p_i)$ (since we have countable support this is possible), $p_{k+1} \leq^n_{F_k} p_k$, and $p_k \Vdash \dot{a}_k \in A_k$. Now we are in a position to apply Lemma 2.6, so we get $p_\omega \leq^{k+n}_{F_k} p_k$ which forces $\dot{A} \subseteq \bigcup A_k$. In particular $p_\omega \leq^n_F p$, as desired.

iv) Same as base case, using iii) as inductive hypothesis.

Limit case.

i) β is a limit. We have $p \in \mathbb{D}_\beta$ forcing $\bigvee \varphi_j$. We go to $\mathbb{D}_{\alpha_i+1}$ where α_i is the greatest member of F, we have $p|_{\alpha_i+1} \Vdash (p|_{[\alpha_i+1,\beta)} \Vdash \bigvee \varphi_j)$. Working in $M[G_{\alpha_i+1}]$ with $p|_{\alpha_i+1} \in G_{\alpha_i+1}$, we know there is $f \leq p|_{[\alpha_i+1,\beta)}$ s.t. $f \Vdash \varphi_j$ for some j, named by some $\mathbb{D}_{\alpha_i+1}$-name $\dot{f}$. By the Forcing Theorem, $p|_{\alpha_i+1} \Vdash \dot{f} \leq p|_{[\alpha_i+1,\beta)}, \bigvee_{j\leq k} \dot{f} \Vdash \varphi_j$. By the successor case of iv), there is $q \leq^n_F p|_{\alpha_i+1}$ and $f \in \mathbb{D}^{\alpha_i+1,\beta}$ s.t. $q \Vdash \dot{f} = f$. By IH on i), we have $q' \leq^n_F q$ and I s.t. $q' \Vdash \bigvee_{j\in I} f \Vdash \varphi_j$ (in particular $q' \Vdash f \Vdash \bigvee_{j\in I} \varphi_j$). Let $p' = q' \frown f$.

ii) Similar as above, we have $p|_{\alpha_i+1} \Vdash (p|_{[\alpha_i+1,\beta)} \Vdash \dot{a} \in M)$, use iii) and iv) at stage $\alpha_i + 1$.

Proofs for iii) and iv) in the limit case are the same as the successor case. □

Lemma 2.8. *For $\alpha \leq \omega_2$, $\mathbb{D}_\alpha$ preserves ω_1.*

Proof. Suppose in $M[G_\alpha]$ there is countable A cofinal in $(\omega_1)^M$, then some $p \in G$ forces that $\dot{A}$ is countable subset of M. By Lemma 2.7 iii), $\{p' : p' \Vdash \exists \check{A} \dot{A} \subseteq \check{A}, \check{A} \text{ is countable}\}$ is dense below p, so A is in fact covered by some countable ground model set, which can't be cofinal in $(\omega_1)^M$, contradiction. □

Lemma 2.9. *Assume M satisfies* **CH**. *For $\alpha \leq \omega_2$, $\mathbb{D}_\alpha$ has the $\aleph_2$-c.c.*

Proof. This is shown by showing that the iteration of length $\alpha < \omega_2$ has a dense subset of cardinality $\aleph_1$. See Jech *Third Millennium Edition* p. 568. □

Corollary 2.10. *Assume M satisfies* **CH**. *For $\alpha \leq \omega_2$, $\mathbb{D}_\alpha$ preserves all cofinalities hence cardinals.*

Claim 1. *If X is a set of reals of size $\aleph_1$ in $M[G]$, then $X \in M[G_\alpha]$ for some $\alpha < \omega_2$.*

Proof. Let r be any real in $M[G]$, there is a nice name for r whose members all have the form (n, p) where $p \in A_n$ for some antichain A_n. By $\aleph_2$-c.c. and countable support, $\{A_n : n < \omega\}$ are decided at some stage α with cofinality ω_1. So r appears in $M[G_\alpha]$ as the G_α interpretation of the nice name.

Now let X be a sequence of reals of size $\aleph_1$ in $M[G]$ forced by p to be so, below p there are conditions $p_\alpha \in G$ and reals r_α s.t. $p_\alpha \Vdash X(\alpha) = \dot{r}_\alpha$ for $\alpha < \omega_1$. Again we can go to some stage with cofinality ω_1 where all r_α appear and all the p_α appear in the generic up to that stage, by countable support. □

The above things can be achieved by more general iteration theorems. (See Jech *Third Millennium Edition* p. 568 and Halbeisen *Combinatorial Set Theory* p. 362.)

- A countable support iteration of proper forcings is proper;
- Any proper forcing preserves ω_1.
- A countable support iteration of proper forcing with size $\leq \aleph_1$ preserves **CH**.
- A countable support iteration of proper forcings with size $\leq \aleph_1$ has the $\aleph_2$-c.c.

In particular, Claim 1 is true for all such iterations with length ω_2.

We need to make some further definitions.

Definition. *Disjunction of conditions. Let $\alpha < \beta$, $q' \in \mathbb{D}_\alpha$, Q be a maximal antichain compatible with q' in $\mathbb{D}_\alpha$. Suppose for each $q \in Q$ there is $p_q \in \mathbb{D}_\beta$ s.t. $p_q|_\alpha = q$. Then there is $p \in \mathbb{D}_\beta$ which satisfies $p|_\alpha = q'$ and for $\gamma \in [\alpha, \beta)$, $p(\gamma)$ is a name such that $q^\frown \mathbb{1}_{[\alpha,\gamma)} \Vdash p(\gamma) = p_q(\gamma)$. This follows from the Mixing Lemma.*

Definition. *Fix $F = \{\alpha_1 < \cdots < \alpha_i\} \subseteq \omega_2$, let $(r_1, \ldots, r_i)$ with each $r_j \leq n$, $p \in \mathbb{D}_{\omega_2}$. Then $p^{r_1,\ldots,r_i}$ is the condition that takes the r_jth stage n component of $p(\alpha_j)$ at position α_j, and equal to p at other positions.*

Note that $\{p^{r_1,\ldots,r_i}\}$ forms a maximal antichain below p of size n^i.

Definition. *Amalgamation of conditions. Fix $F = \{\alpha_1 < \cdots < \alpha_i\} \subseteq \beta$, $p \in \mathbb{D}_\beta$. Let $q \in \mathbb{D}_\beta$ be s.t. $q \leq^0_F p^{r_1,\ldots,r_i}$, where each $r_j \leq n$. Then the amalgamation of p and q is a condition p' s.t. $p' \leq^n_F p$ and $p'^{r_1,\ldots,r_i} \leq^0_F q$. p' can be constructed by the following: within F, update the r_jth component of $p(\alpha_j)$ to $q(\alpha_j)$. For $\alpha \notin F$, use mixing: $p'(\alpha)$ is forced to be $q(\alpha)$ by the condition $p'|_\alpha^{r_1,\ldots,r_m}$, otherwise remains $p(\alpha)$.*

Claim 2. *If X is a set of reals in M and has strong measure 0 in $M[G]$, then it is countable in $M[G]$.*

Lemma 2.11. *Let $\dot{a}$ name a real in $M[G]$, $p \in \mathbb{D}_{\omega_2}$, $F \subseteq \omega_2$ be finite. There is $p' \leq^0_{F\cup\{0\}} p$ and a real $u \in M$ such that for all $\epsilon > 0$, there are cofinitely many immediate successors t of $p'(0)\langle 0\rangle$ in $p'(0)$ s.t.*

$$p'(0)_t \frown p'|_{[1,\omega_2)} \Vdash |\dot{a} - u| < \epsilon\,.$$

Proof. We enumerate the immediate successors of $p(0)\langle 0\rangle$ as $t_0, t_1, \ldots$. For each n we apply Lemma 2.7 i) in $M[G_1]$ and get $\dot{f}_n$ s.t.

$$p(0)_{t_n} \Vdash \dot{f}_n \leq^0_{F-\{0\}} p|_{[1,\omega_2)}, \dot{f}_n \Vdash \dot{a} \in I_n$$

Where I_n is among $[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), \ldots$. By Lemma 2.7 iv) there is $q_n \leq^0 p(0)_{t_n}$ and $f_n \in \mathbb{D}^1$ s.t. $q_n \Vdash \dot{f}_n = f_n$. Let $p_n = q_n \frown f_n$.

Now there is an infinite $A \subseteq \omega$ such that $\langle I_n : n \in A\rangle$ converges to a real u. (By Bolzano–Weierstrass.) Let p' be the disjunction of $\{p_n : n \in A\}$. We verify that p' is as desired. We have $p'(0) = \bigcup\{q_n : n \in A\}$ (since $\{q_n : n \in A\}$ is a maximal antichain below this condition, and we've extended each q_n in this antichain to p_n). Fix ϵ, go to $N \in A$ such that $len(I_N) < \epsilon$. Then for all $n > N$, $n \in A$, q_n (which is a tree above an immediate successor of the root of $p'(0)$) forces $f_n \Vdash \dot{a} \in I_n$, so $f_n \Vdash |\dot{a} - u| < \epsilon$. Also $q_n \Vdash p'|_{[1,\omega_2)}$ to be f_n, by definition of p', so $q_n \frown p'|_{[1,\omega_2)} \Vdash |\dot{a} - u| < \epsilon$, as desired. □

Lemma 2.12. *Let $\dot{a}$ name a real in $M[G]$, $p \in \mathbb{D}_{\omega_2}$, $F \subseteq [1, \omega_2)$ be finite. There is p' s.t. $p'(0) \leq^0 p(0)$, $p'|_{[1,\omega_2)} \leq^n_F p|_{[1,\omega_2)}$, and a finite set of reals $U \in M$ such that for all $\epsilon > 0$, there are cofinitely many immediate successors t of $p'(0)\langle 0\rangle$ in $p'(0)$ s.t.*

$$p'(0)_t \frown p'|_{[1,\omega_2)} \Vdash \exists u \in U |\dot{a} - u| < \epsilon\,.$$

Proof. Let $|F| = i$, $b = (n+1)^i$, we break $p(\alpha) : \alpha \in F$ into stage n components (there are $n + 1$ many at each α). Let $\vec{r}_0, \ldots, \vec{r}_{b-1}$ enumerate the sequences $(r_1, \ldots, r_i)$ with each $r_j \leq n$. Start from $p_0 = p$, we construct p' in b steps. At step $j \leq b - 1$, we apply Lemma 2.11 to $p_j^{\vec{r}_j}$, get $q_j \leq^0_{F\cup\{0\}} p_j^{\vec{r}_j}$ and a ground model real u_j s.t. for all ϵ there are cofinitely many immediate successors t of the root of $q_j(0)$ s.t.

$$q_j(0)_t \frown q_j|_{[1,\omega_2)} \Vdash |\dot{a} - u_j| < \epsilon\,.$$

Define p_{j+1} to be the amalgamation of p_j and q_j. Define p' to be p_b and $U = \{u_0, \ldots, u_{b-1}\}$. We verify that p' and U are as desired. Fix ϵ. There are infinitely many immediate successors t of the root in $p'(0)$. They are also immediate successors of the root in $q_j(0)$ for all j, so there are cofinitely many s.t.

$$q_j(0)_t \frown q_j|_{[1,\omega_2)} \Vdash |\dot{a} - u_j| < \epsilon\,.$$

It follows that

$$q_j(0)_t \frown q_j|_{[1,\omega_2)} \Vdash \exists u \in U |\dot{a} - u| < \epsilon\,.$$

Since there are finitely many js and $p'(0)_t {}^\frown p'|_{[1,\omega_2)}$ extends $q_j(0)_t {}^\frown q_j|_{[1,\omega_2)}$, it follows that

$$p'(0)_t {}^\frown p'|_{[1,\omega_2)} \Vdash \exists u \in U |\dot{a} - u| < \epsilon .$$

For cofinitely many ts, as desired. □

Lemma 2.13. *Let $\dot{a}_0, \dot{a}_1, \ldots$ be names for reals in $M[G]$, $p \in \mathbb{D}_{\omega_2}$. Then there is $p' \leq p$ with $p'(0) \leq^0 p(0)$ and some finite sets of reals U_s, one for each node $s \in p'(0)$ extending the root $p'(0)\langle 0\rangle$ with $len(s) = len(p'(0)\langle 0\rangle) + j$, satisfying: for any $\epsilon > 0$, there are cofinitely many immediate successors t of s in $p'(0)$ s.t.*

$$p'(0)_t {}^\frown p'|_{[1,\omega_2)} \Vdash \exists u \in U_s |\dot{a}_j - u| < \epsilon .$$

Proof. Start from $p(0)\langle 0\rangle$. Apply the previous lemma on p, $\dot{a}_0$, and some finite F_0, we get p_1 and a finite $U_{p(0)\langle 0\rangle}$ with the stated properties. Let t be an immediate successor of the root in $p_1(0)$, apply the previous lemma on $p_1(0)_t {}^\frown p_1|_{[1,\omega_2)}$, $\dot{a}_1$, and some finite $F_1 \supseteq F_0$, we get q^t and a finite U_t with the stated properties, in particular we can make sure $q^t|_{[1,\omega_2)} \leq^1_{F_1} p_1|_{[1,\omega_2)}$. Do this for all the immediate successors of the root. Let p_2 be the disjunction of all q^t. Then we still have $p_2|_{[1,\omega_2)} \leq^1_{F_1} p_1|_{[1,\omega_2)}$. We choose $F_2 \supseteq F_1$ and repeat the process for all the nodes 2 levels above the root in p_1 to take care of $\dot{a}_2$, etc. By countable support, we can get p_j, F_j s.t. $\bigcup F_j = \bigcup supp(p_j) - \{0\}$ and $p_{j+1} \leq^j_{F_j} p_j$. Let $p'(0) = \bigcap p_j(0)$ (note that nodes j levels above the root are determined by p_j). We can apply Lemma 2.6 in an extension which contains $p'(0)$, and get the fusion of $\{p_j|_{[1,\omega_2)}\}$, this is the tail of p'. (To be more rigorous, we can apply Lemma 2.7 iv) here.) □

Proof idea: Each node s is associated with a finite U_s with the above property. We know that the strong measure 0 set X is forced to be covered by $\bigcup \dot{I}_j$ and want to show that X is countable by showing that $X \subseteq \bigcup_s U_s$. Assume $v \notin \bigcup_s U_s$, we show that $v \notin X$. It suffices to show that some condition forces $v \notin \bigcup \dot{I}_j$, i.e. v is far from all $\dot{a}_j$ by a margin ϵ_j, which are the midpoints of the intervals. At level j, cofinitely many successors force $\dot{a}$ to be near some $u \in U_s$. Since U_s is finite, we can choose ϵ and u s.t. $|v - \dot{a}_j| \geq |v - u| - |\dot{a}_j - u| > 2\,\epsilon - |\dot{a}_j - u| > \epsilon$ for cofinitely many successors. We can let ϵ_j be inversely proportional to the value of the generic at j, so that $\epsilon_j < \epsilon$ whenever the condition makes it to the generic, by getting rid of the finitely many numerically small immediate successors. In this way the condition forces the generic to hit a large value, so $\epsilon_j < \epsilon$ at all nodes at level j, forcing v out of I_j.

Proof for Claim 2. Let $X \in M$ be a set of ground model reals, $p \in G$ forces X has strong measure 0. Let $n = len(p(0)\langle 0\rangle)$. Let $g : \omega \to \omega$ be the generic real added by G_1. For $j \geq n$, take $\epsilon_j = \frac{1}{g(j)}$. Take intervals $I_j : j \geq n$ with length $< \epsilon_j$. Let a_j be the midpoints of I_j. These are named by $\dot{I}_j, \dot{a}_j$, resp.

Apply Lemma 2.13 to $\dot{a}_j$, we get $p' \leq p$ and finite sets of ground model reals U_s for each node $s \in p'(0)$ above the root, with the stated properties. We claim that $X \subseteq \bigcup_s U_s$, so X is countable.

Let $v \notin \bigcup_s U_s$, it suffices to show that $v \notin X$. We show that there is $T \leq^0 p'(0)$ s.t.

$$T^\frown p'|_{[1,\omega_2)} \Vdash v \notin \bigcup \dot{I}_j.$$

Since $p' \Vdash X$ has strong measure 0, $p' \Vdash X \subseteq \bigcup \dot{I}_j$, so $T^\frown p'|_{[1,\omega_2)} \Vdash v \notin X$. If $v \in X$ this would be a contradiction. So the conclusion follows from here.

To finish the proof, we construct T starting from the root $p'(0)\langle 0 \rangle$. Suppose the construction at level $j \geq n$ is finished. For each t on the level j, we choose cofinitely many immediate successors of t in $p'(0)$, as the following. Since $v \notin U_t$ which is finite, there is $\epsilon < \frac{|v-u|}{2}$ for all $u \in U_t$. By construction, we can discard finitely many immediate successors of t in $p'(0)_t$ (and everything above them), and have the remaining part of $p'(0)_t$ (call it q) satisfying $q^\frown p'|_{[1,\omega_2)} \Vdash \exists u \in U_t |\dot{a}_j - u| < \epsilon$. Furthermore, we fix k s.t. $\frac{1}{k} < \epsilon$. We discard another finitely many immediate successors of t in $p'(0)_t$ whose value at j is less than k. The resulting infinitely many immediate successors are chosen to extend t in T. We do this for all t on the level j and the level $j+1$ of T is finished. We now check that T is as desired.

Suppose $T^\frown p'|_{[1,\omega_2)} \in G$. Take any $j \geq n$. In particular, $g|_j \in p'(0)$ (it has length j). So we have a finite $U_{g|_j}$ and a corresponding ϵ. Moreover $g(j) \geq k$ (by construction of T) so that $\epsilon_j = \frac{1}{g(j)} \leq \frac{1}{k} < \epsilon$. By construction, $T^\frown p'|_{[1,\omega_2)} \Vdash \exists u \in U_{g|_j} |\dot{a}_j - u| < \epsilon$, so this is true in $M[G]$, let u be the witness. In $M[G]$, we have $|v - \dot{a}_j| \geq |v-u| - |\dot{a}_j - u| > 2\,\epsilon - |\dot{a}_j - u| > \epsilon > \epsilon_j$, so $v \notin \dot{I}_j$. This is true for all j, so $M[G] \models v \notin \bigcup \dot{I}_j$, as desired. □

Claim 3. $M[G_\alpha] \models (\mathbb{D}_{\omega_2})^{M[G_\alpha]} \simeq \mathbb{D}^\alpha$.

Proof. Informally: by a standard lemma in iterated forcing, there is a name which in $M[G_\alpha]$ is forced to be the tail forcing. Moreover, $M[G_\alpha]$ thinks this name is a countable support iteration of $\mathbb{J}$ of length ω_2. □