The Number of Models of Unstable Theories without Choice

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Abstract

This poster describes and proves some results proved in the paper [2] entitled "Model Theory without Choice: Categoricity" by S. Shelah.

Stable Theories

- Fix a theory T. A formula $\varphi(\overline{x}, \overline{y})$ has the *order property* if for any $M \models T$ there is $\overline{a}_i, \overline{b}_j \in M, i, j < \omega$ s.t. $M \models \varphi(\overline{a}_i, \overline{b}_j)$ iff $i < j$.
- T is *stable* if no formula has the order property.

A ZFC Theorem

(Shelah [1]) If T is unstable and $\kappa > |T|$ is a cardinal, then $I(\kappa, T) = 2^\kappa$, the maximal possible number.

Ehrenfeucht–Mostowski Models

Fix a some unstable theory T and a linear order I. In our context, an EM model $M(I) \models T$ is the Skolem hull of order-indiscernible elements $a_s : s \in I$ indexed by I, where the order $a_s < a_t$ is given by the formula witnessing unstablity and is such that $a_s < a_t$ iff $s <_I t$. By the Ramsey theorem and the compactness theorem, $M(I)$ exists for any infinite linear order I. I is called the *skeleton* of $M(I)$.

Partition of Stationary Sets

Solovay famously showed that any S stationary in a regular cardinal κ can be partitioned into κ-many stationary sets, assuming $\mathbf{AC}$. A variation of this is: any S stationary in κ with uncountable cofinality can be partitioned into countably many stationary sets. Both statements are independent of $\mathbf{AC}$.

Idea of Proof

We define some orders. For $A \subseteq \kappa$ and $\alpha < \kappa$, define I^A_α to be ω_1^* if $k \in A$, and I^A_α is ω^* if $\alpha \notin A$. Define $I^A = \sum_{\alpha<\kappa} I^A_\alpha$.

Lemma. Let $\kappa > \aleph_1$ be regular, $A, B \subseteq \kappa$. If $S = A \Delta B$ is stationary then $M(I^A)$ and $M(I^B)$ are not isomorphic.

Proof. Let $F : M(I^A) \to M(I^B)$ be an isomorphism. We show that $A \Delta B$ is nonstationary, i.e. it avoids some club set. Let $a(s) : s \in I^A$ and $b(t) : t \in I^B$ be the skeletons. For every $\alpha \in \kappa$, pick $s_\alpha \in I^A_\alpha$. Then $F(a(s_\alpha)) \in M(I^B)$ has the form $\tau_\alpha(b(\overline{c}_\alpha), b(\overline{d}_\alpha))$, where $\overline{c}_\alpha \in I^B_{<\alpha}$, $\overline{d}_\alpha \in I^B_{\geq\alpha}$. We observe that there is a club C of α where $I^A_{<\alpha}$ and $I^B_{<\alpha}$ both have size $|\alpha|$. Also, there is a club D of α where for all $\beta < \alpha$ $\overline{d}_\beta \in I^B_{<\alpha}$, so $S \cap C \cap D$ is stationary, we update S to this set.

Idea of Proof

Note that since T is countable and κ is regular, we can shrink S such that for $\alpha \in S$, τ_α is the same term τ, and $\overline{c}_\alpha, \overline{d}_\alpha$ have the same arity for different $\alpha \in S$. Furthermore, since $\overline{c}_\alpha \in I^B_{<\alpha}$, which has size $< |\alpha|$, the map $\alpha \mapsto c^1_\alpha$ is regressive. By Fodor's lemma we can shrink S such that c^1_α is constant on S. Apply this $|\overline{c}|$-many times, now $\overline{c}_\alpha$ is constant on S. At this point, for $\alpha \in S$, $F(a(s_\alpha)) = \tau(b(\overline{c}), b(\overline{d}_\alpha))$ for $\alpha \in S$. We fix $\delta \in S$ to be a limit point of S.

Fix $a \in I^A_\delta$, then $F(a) = \sigma(\overline{x}, \overline{y})$ where $\overline{x} \in I^B_{<\delta}$, $\overline{y} \in I^B_{\geq\delta}$. Choose $x \in I^B_{<\delta}$ above $\overline{c}$ and $\overline{x}$, and $y \in I^B_\delta$ below $\overline{y}$. By indiscernibility, if $\overline{i}, \overline{j}$ are between x and y, $t(\overline{c}, \overline{i}) < F(a)$ iff $t(\overline{c}, \overline{j}) < F(a)$. Since δ is a limit point of S, there is $\alpha < \delta, \alpha \in S$ such that $x < \overline{d}_\alpha < y$. Now $t(\overline{c}, \overline{d}_\alpha) = F(a_\alpha) < F(a)$ (a_α is chosen from I^A_α), so in fact if $x < \overline{j} < y$ then $t(\overline{c}, \overline{j}) < F(a)$.

Finally we can prove the lemma. Since $\delta \in A \Delta B$, WLOG we can take $I^A_\delta = \omega^*$ and $I^B_\delta = \omega_1^*$. If $\alpha < \delta$, then $\overline{d}_\alpha \in I^B_{<\delta}$ and $\overline{d}_\delta \notin I^B_{<\delta}$, so $\overline{d}_\alpha < \overline{d}_\delta$. Because $a_\alpha = t(\overline{c}, \overline{d}_\alpha) < a_\delta = t(\overline{c}, \overline{d}_\delta)$, by indiscernibility $a_\alpha < t(\overline{c}, \overline{j})$ for all $\overline{j} \in I^B_\delta$. On the other hand, we fix a descending sequence $z_0 > z_1 > \ldots$ cofinal in I^A_δ and use the previous conclusion to get $x_i \in I^B_{<\delta}, y_i \in I^B_\delta$ for each j_i. Since I^B_δ is ω_1^*, there is $\overline{j}$ below all y_i and above all x_i. Since z_0 is cofinal, $t(\overline{c}, \overline{j}) < F(I^A_\delta)$. This means that $t(\overline{c}, \overline{j}) \in M(I^B)$ is between the images of $I^A_{<\delta}$ and I^A_δ, but there is no such thing in $M(I^A)$, contradiction.

The theorem for regular $\kappa > \aleph_1$ follows from the lemma and Solovay's partition theorem.

Analogue in ZF

(Shelah [2]) If T is well-orderable and unstable, $|T| = \aleph_\beta < \aleph_\alpha = \kappa$, then $I(\kappa, T) \geq |\alpha - \beta|$.

Idea of Proof

For $\gamma \leq \aleph_\alpha$, we define $J_\gamma = \gamma + (\gamma)^*$. For $\xi \in [\beta, \alpha]$, we let $J^\xi = \sum_{\gamma \leq \aleph_\xi} J_\gamma + J_\infty$, where $J_\infty = (\aleph_\alpha + 1) \times \mathbb{Q}$, ordered lexicographically. Let $M^\xi = M(J^\xi)$. We show that these models are mutually not isomorphic.

Let θ be a regular cardinal in some inner model $L[T, Y]$ which contains $M = M^\xi$. $\otimes_\theta$ is the following statement: if p is a set of formulas with parameters from M of cardinality θ that has the form $\varphi(\overline{x}, \overline{a}) \wedge \neg\varphi(\overline{x}, \overline{b})$ (φ is the formula witnessing unstability) and any subset of p of cardinality $< \theta$ is realized in M then some subset of p of cardinality θ is realized in M.

Lemma. M^ξ satisfies $\otimes_\theta$ iff $\theta > \aleph_\xi$.

Proof. On the one hand, suppose $\theta \leq \aleph_\xi$. Then J_θ is in the skeleton of M, so let p be the type which says $\overline{x}$ is φ-between θ and θ^*. Since θ is regular, clearly any $q \subseteq p$ with size $< \theta$ is realized. However, any $q \subseteq p$ with size θ must have parameters that are cofinal in θ and θ^*, by indiscernibility this can't be realized.

On the other hand, suppose $\theta > \aleph_\xi$. Fix $p = \{\varphi(\overline{x}, \overline{a}_i) \wedge \neg\varphi(\overline{x}, \overline{b}_i) : i < \theta\}$, for $j < \theta$ let $p_j = \{\varphi(\overline{x}, \overline{a}_i) \wedge \neg\varphi(\overline{x}, \overline{b}_i) : i < j\}$, let p_j be realized by $\overline{c}_j$.

Idea of Proof

$\overline{a}_i, \overline{b}_i, \overline{c}_i$ can be expressed as Skolem terms. Since θ is regular, for i in some S unbounded in θ, $\overline{a}_i$ is given by the same term evaluated with different input. Similar for $\overline{b}_i$ and $\overline{c}_i$. Moreover, similar in the previous proof, S can be shrinked such that for i in S, the inputs can be separated into two parts, the first part being a constant $\overline{d} \in \sum_{\gamma \leq \aleph_\xi} J_\gamma$ and the second part being $\overline{e}^i \in J_\infty$.

For each $l < n$ where n is the arity of $\overline{e}^i_l$, we write it as (ϵ^i_l, q^i_l), where $\epsilon^i_l \in \aleph_\alpha + 1$, $q^i_l \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable, we can make q^i_l constantly q^*_l for $i \in S$. Since $\aleph_\alpha + 1$ is well-founded, we can make ϵ^i_l either strictly increases with limit ϵ^*_l or constantly ϵ^*_l as i increases in S.

We choose $t_1, \ldots, t_n$ s.t. if ϵ^i_l is constantly ϵ^*_l then $t_l = (\epsilon^*_l, q^*_l)$, otherwise $t_l = (\epsilon^*_l, \min(\{0, q^*_1, \ldots, q^*_n\}) - (n - l))$.

Now we have that if $i < j \in S$, then the position of $(\overline{e}^i, \overline{e}^j)$ and $(\overline{e}^i, \overline{t})$ in the linear order J^ξ are the same. Let $\overline{c} = \sigma(\overline{t})$, by indiscernibility, the type $\{\varphi(\overline{x}, \overline{a}_i) \wedge \neg\varphi(\overline{x}, \overline{b}_i) : i \in S\}$ is realized, so $\otimes_\theta$ is true.

Finally we prove the theorem. We fix M^{ξ_1}, M^{ξ_2}, $\xi_1 < \xi_2$, and assume $f : M^{\xi_1} \to M^{\xi_2}$ is an isomorphism. Let $Y \subseteq \mathrm{Ord}$ code $f, M^{\xi_1}, M^{\xi_2}, T$. Work in $L[T, Y]$, where we have $\mathbf{AC}$. Let θ be the successor cardinal in $L[T, Y]$ of $\aleph^V_{\xi_1}$, so $\aleph^V_{\xi_1} < \theta \leq \aleph^V_{\xi_1+1} \leq \aleph^V_{\xi_2}$. Note that in $L[T, Y]$, θ is regular. By the Lemma, M^{ξ_1} satisfies $\otimes_\theta$ but M^{ξ_2} does not, contradiction.

Questions

- The exact number of nonisomorphic models in $\mathcal{ZF}$ alone does not appear to be known.
- There are independent questions under the broad theme of "model theory without choice" in Shelah's paper. Ask me about them if you are interested!

References

[1] Saharon Shelah.
The number of non-isomorphic models of an unstable first-order theory.
Israel Journal of Mathematics, 9(4):473–487, 1971.

[2] Saharon Shelah.
Model theory without choice? categoricity.
The Journal of Symbolic Logic, 74(2):361–401, 2009.