BIRATIONAL GEOMETRY EXERCISES

Unless otherwise stated all varieties are defined over $\mathbb{C}$.

Exercise 0.

(1) Let $X = \mathbb{P}^n \times \mathbb{P}^m$. Define $\mathcal{O}(a,b) = p^*\mathcal{O}(a) \otimes q^*\mathcal{O}(b)$ where p and q are the projections onto the first and second coordinates, respectively. For what values of a and b is $\mathcal{O}(a,b)$... (1) ample, (2) big, (3) nef, (4) pseudo-effective?

(2) Find an example of (1) a nef divisor which is not ample, (2) a big divisor which is not nef, (3) a pseudo-effective divisor which is not big.

Let X be a projective variety and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor.

(1) If D is semi-ample it is nef (the converse is false).

(2) If D is effective, nef, big, semi-ample or ample then it is pseudo-effective.

(3) If $f\colon Y \to X$ is any morphism from a projective variety and D is nef, then f^*D is nef.

(4) If $f\colon Y \to X$ is a birational morphism from a projective variety and D is big, then f^*D is big.

(5) If $f\colon Y \to X$ is a finite morphism and D is ample, then f^*D is ample.

Exercise 1.

(1) Let X be a smooth variety and let $f\colon X \dashrightarrow Y$ be a rational map which is not a morphism. Show that Y contains a rational curve.

(2) Let $f\colon X \dashrightarrow Y$ be rational map between smooth proper curves. Show that it is in fact a morphism. Deduce that smooth proper curves which are birational to one another are in fact isomorphic.

(3) Let C be a smooth projective curve defined over an algebraically closed field. Show that if ω_C^* is ample then $C \cong \mathbb{P}^1$.

Exercise 2. Let X be a normal projective variety.

(1) Let A be an ample divisor. Show that $A \cdot C > 0$ for any projective curve $C \subset X$.

(2) Show that semi-ample divisors are nef.

(3) Let A be an ample line bundle and let L be a nef line bundle. Then $A \otimes L$ is an ample line bundle. Deduce that the nef cone is the closure of the ample cone.
(4) We say a divisor is big if $kD \sim A + E$ for some $k \in \mathbb{N}$ where A is ample and $E \geq 0$. Show that $H^0(X, \mathcal{O}(mD)) \geq Cm^{\dim X}$ for some $C > 0$. If X is projective, show that the converse holds, i.e., if $H^0(X, \mathcal{O}(mD)) \geq Cm^{\dim X}$ for some $C > 0$ then D is big.
(5) Show that $\overline{NE}(X)$ does not contain a line.
(6) ($*$) Find an example of a non-projective variety with a big divisor. Find an example of a variety where $\overline{NE}(X)$ contains a line.

Exercise 3. Let X be a smooth projective surface such that K_X is nef. Show that if X' is smooth and birational to X' then there is a morphism $X' \to X$. Deduce the existence of minimal resolutions for singular surfaces, i.e., if Y is a singular surface then there exists a resolution of singularities $X \to Y$ such that if $X' \to Y$ is any other resolution of singularities then there exists a morphsim $X' \to X$. Do minimal resolutions exist in higher dimensions?

Exercise 4. If D is big and nef and $D \sim_{\mathbb{Q}} K_X + \Delta$ where $\Delta \geq 0$ and (X, Δ) is klt then show that D is semi-ample. ($*$) Find an example of a big and nef divisor which is not semi-ample.

Exercise 5.

(1) Let X be a smooth projective variety and let Z be a normal projective variety. Suppose that there exists a dominant rational map $\mathbb{P}^1 \times Z \dashrightarrow X$. Show that K_X is not pseudo-effective.
(2) Deduce that if X is rationally connected then X is not pseudo-effective.
(3) Find an example of a rationally connected Calabi-Yau variety. Can such a variety be smooth?

Exercise 6.

(1) Let X be a hypersurface of degree d in $\mathbb{P}^n$. When is X Fano? Calabi-Yau? General type? What if we assume that X is only a complete intersection?
(2) Let $Y \to \mathbb{P}^n$ be a double cover of $\mathbb{P}^n$ ramified along a hypersurface of degree d. Compute K_Y. When is it Fano? Calabi-Yau? General type?

Exercise 7. Let X be a smooth surface such that K_X is pseudo-effective. Suppose that $K_X \cdot C < 0$ where C is a smooth curve. Show that $C \cong \mathbb{P}^1$. Find an example of a smooth surface X, a pseudo-effective divisor D and a curve C of genus $g > 0$ such that $D \cdot C < 0$.

Exercise 8. Let $n \geq 1$. Show that $X = \{xy + z^2 + w^{2n} = 0\} \subset \mathbb{A}^4$ is not $\mathbb{Q}$-factorial, i.e., there exists an effective divisor $D \subset X$ which is not $\mathbb{Q}$-Cartier.

Exercise 9. Let X be a normal projective variety and let A be an ample line bundle. Show that $A|_Z$ is ample for any subvariety $Z \subset X$. If A is big, is it the case that $A|_Z$ is big for any subvariety $Z \subset X$?

Exercise 10.

(1) Show that $X := \{xy - zw = 0\} \subset \mathbb{A}^4$ is not $\mathbb{Q}$-factorial and that blowing up a non-$\mathbb{Q}$-Cartier Weil divisor X gives a resolution of singularities of X.
(2) Show that there are two possible ways of resolving X, and that they are connected by a flop (this is called the Atiyah flop).
(3) $(*)$ Find an example of a flip (hint: it might be easier to find an example of a fourfold flip, or try using toric geometry).
(4) Find an example of a flop which is not the Atiyah flop.

Exercise 11. Let X be the blow up $\mathbb{P}^3$ at 4 general points. Let $L_1, \dots, L_6$ be the strict transform of the lines between any two points. Show that L_i can be flopped. Show that this flop is locally isomorphic to the Atiyah flop.

Let $X \to W$ be the morphism which contracts all six lines and let $X \dashrightarrow X'/W$ be the rational map given by flopping all six lines. Show that $X' \cong X$ as varieties over Spec$k$, but are not isomorphic as varieties over W.

Exercise 12. Let $F\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the Cremona involution $[x : y : z] \mapsto [x^{-1} : y^{-1} : z^{-1}]$. Show that F can be realised as the blow up of $\mathbb{P}^2$ at 3 points, followed by the blow down of three curves. Provide a similar description of the Cremona involuation on $\mathbb{P}^3$.

Exercise 13. Compute the minimal resolution of $\{xy - z^m = 0\} \subset \mathbb{A}^3$. Find a resolution of singularities of Spec$k[x^2, y^2, z^2, xy, xz, yz]$. Can you find one with only one exceptional divisor?

Exercise 14. Let S_1 and S_2 be two smooth surfaces such that there exists a birational map $f\colon S_1 \dashrightarrow S_2$. Show that $H^0(S_1, \mathcal{O}(mK_{S_1})) = H^0(S_2, \mathcal{O}(mK_{S_2}))$. Deduce that $\mathbb{P}^2$ and a K3 surface are not birational.

Exercise 15. Let S be a smooth surface and $f\colon S \to C$ be a morphism to a curve such that at least one fibre of f is isomorphic to $\mathbb{P}^1$. Show that S is birational to $C \times \mathbb{P}^1$. Show by example that if

$X \to S$ is a morphism from a threefold to a surface such that at least one fibre is $\mathbb{P}^1$ then X is not necessarily birational to $S \times \mathbb{P}^1$ (Hint: find a curve C over the field $K := \mathbb{C}(x, y)$ which is not isomorphic to $\mathbb{P}^1_K$, but $C \times_K \overline{K} \cong \mathbb{P}^1_{\overline{K}}$ where $\overline{K}$ is an algebraic closure of K.)

Exercise 16. Let E be an elliptic curve. Show that there exists an exact sequence of vector bundles $0 \to \mathcal{O}_E \to \mathcal{F} \to \mathcal{O}_E \to 0$ such that $F \neq \mathcal{O}_E \oplus \mathcal{O}_E$. Let $X = \mathbb{P}_E(\mathcal{F})$. Show that X contains a curve C such that $C^2 = 0$, but $h^0(X, \mathcal{O}(C)) = 1$, in particular, C is not a fibre in a fibration.

Exercise 17. Let S be a smooth projective surface. Show that if S contains no rational curves then K_S is nef. Show that if S contains no rational curves and if K_S is big, then K_S is ample. Find an example of a surface where K_S is not ample, and does not contain any rational curves.

Exercise 18.

(1) Show that if A is an abelian variety, then A contains no rational curves.
(2) Deduce that if X is a smooth projective variety and $a\colon X \to \mathrm{Alb}(X)$ is albanese morphism then a maps rational curves in X to points.
(3) If $X \to \mathrm{Alb}(X)$ is an embedding deduce that K_X is nef.

Exercise 19. Find examples of projective surfaces X which contain rational curves where

(1) $-K_X$ is ample;
(2) $\mathcal{O}(K_X) \cong \mathcal{O}_X$;
(3) K_X is ample.

Exercise 20. Show that if the flip exists, then it is unique.

Exercise 21. $(*)$ Show by example that the output of the MMP is not necessarily unique. Prove the following theorem of Kawamata: if X is smooth and $f_1\colon X \dashrightarrow X_1$ and $f_2\colon X \dashrightarrow X_2$ are two MMPs, then $\alpha\colon X_1 \dashrightarrow X_2$ is a sequence of flops.

Here is the skeleton of Kawamata's proof. You can fill in the details, or try to discover a new approach.

(1) First, show that X_1 and X_2 are isomorphic in codimension one (this can be shown as a consequence of the negativity lemma).
(2) Let H_2 be an ample divisor X_2 and let $H_1 := \alpha^{1}_{*}H_2$. Show that we can run a $K_{X_1} + tH_1$-MMP for $0 < t \ll 1$.

(3) Show that the output of this MMP is X_2 and deduce that each step of this MMP is a K_{X_1}-flop.

Definition (Discrepancy) Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier. Given a birational morphism $p\colon X' \to X$ we may write $K_{X'} = p^*K_X + \sum_E a(E,X)E$ where E runs over all p-exceptional divisors. The number $a(E,X)$ is called the **discrepancy**.

We say that X is terminal (resp. canonical, log terminal, log canonical) provided $a(E,X) > 0$ (resp. ≥ 0, > -1, ≥ -1) for all divisors E on all birational models $X' \to X$.

Negativity Lemma If $f\colon X \to Y$ is a proper birational morphism and D is a divisor such that $f_*D \geq 0$ and $-D$ if f-nef then $D \geq 0$.

Exercise 22. $(*)$ Use the negativity lemma to deduce that flips preserve terminal (resp. canonical, log terminal, log canonical) singularities.

Exercise 23. Show that a cubic surface is isomorphic to $\mathbb{P}^2$ blown up in 6 points. There are several ways of doing this, but try to prove it using ideas from the MMP.

Exercise 24. Let $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be the rational map associated to a pencil of degree d curves on $\mathbb{P}^2$. Deduce that f defines a foliation $\mathcal{F}$, and compute $K_{\mathcal{F}}$. When is $\mathcal{F}$ a Fano foliation?

Exercise 25. We say a variety X is of Fano type if there exists a $\mathbb{Q}$-divisor Δ such that (X,Δ) is klt and $-(K_X+\Delta)$ is ample. Find an example of a variety which is not Fano, but is of Fano type. Is $\mathbb{P}^2$ blown up at 9 points of Fano type?

Exercise 26. Let $f\colon X \to B$ be a smooth fibration. Show that $-K_{X/B}$ cannot be ample. Deduce that there are no smooth Fano foliations.

Exercise 27. Let X be a normal variety and let $D_1,\ldots,D_k$ be big $\mathbb{Q}$-divisors such that $\bigoplus_{m_1,\ldots,m_k\geq 0} H^0(X, m_1D_1+\cdots+m_kD_k)$ is finitely generated and let V be the subcone of the big cone generated by $D_1,\ldots,D_k$.

For a $\mathbb{Q}$-diviosr $D \in V$ we define $R(D) = \bigoplus_{m\geq 0} H^0(X, mD)$. Show that $R(D)$ is finitely generated.

Fix a $\mathbb{Q}$-divisor $D \in V$ and show that $\{D' : \mathrm{Proj}R(D') \cong \mathrm{Proj}R(D)\}$ is a convex subcone of V.

Definition (MMP with scaling) Let X_1 be a normal projective variety and let $\Delta_1 \geq 0$ be a $\mathbb{Q}$-divisor such that (X_1,Δ_1) is klt and let A_1 be a divisor such $K_{X_1}+\Delta_1+A_1$ is nef.

We define $\lambda_1 := \inf\{t : K_{X_1} + \Delta_1 + tA_1 \text{ is nef.}\}$ and by the Cone theorem there exists an extremal ray R_1 such that $K_{X_1} + \Delta_1 + \lambda_1 A_1$ is zero on R_1.

The first step of our MMP with scaling $\phi\colon X_1 \dashrightarrow X_2$ is either the divisorial contraction or flip associated to R_1. Let X_2, Δ_2 and A_2 be the strict transforms of X_1, Δ_1 and A_1, respectively. We can again define $\lambda_2 := \inf\{t : K_{X_2} + \Delta_2 + tA_2 \text{ is nef.}\}$ and find an extremal ray R_2 on which $K_{X_2} + \Delta + \lambda_2 A_2$ is zero.

Continuing this process we get a sequence of divisorial contractions and flips, and rational numbers $\lambda_1 \geq \lambda_2 \geq \dots$ called the MMP with scaling of A_1.

Remark: As remarked in the lectures, if Δ is big, then an MMP with scaling of an ample divisor always terminates. There is a key distinction here with a general MMP: we are no longer allowed to choose an arbitrary $K_{X_i} + \Delta_i$-negative extremal ray, rather it is chosen for us by A_i.

Exercise 28. Show that the MMP with scaling can be run...

(1) for uniruled varieties, and terminates in a Mori fibre space; and
(2) for varieties of general type and terminates in a minimal model.

Exercise 29 Show the following fundamental properties of the discrepancy.

(1) The discrepancy is independent of the choice of birational model $X' \to X$.
(2) Smooth varieties have terminal singularities.
(3) Propose a definition of terminal, etc. for pairs (X, Δ). Show that if $p\colon X' \to X$ is birational and we write $K_{X'} + \Delta' = p^*(K_X + \Delta)$ where $p_*\Delta' = \Delta$ (note that here Δ' is not necessarily effective) then (X, Δ) is terminal (resp. canonical, ...) if and only if (X', Δ') is terminal (resp. canonical, ...).
(4) Let X be a normal variety with K_X $\mathbb{Q}$-Cartier. X is terminal (resp. canonical, log terminal, log canonical) if and only if there exists a log resolution of X which extracts divisors of discrepancy > 0 (resp. ≥ 0, > -1, ≥ -1).