Projective Planes and Beyond

Thematic Program on Rationality and Hyperbolicity

Undergraduate Workshop

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June 12-16, 2023

Slides available by emailing migliore.1@nd.edu
or from the conference website.

Outline for the week

Projective Planes and Beyond

- Lecture 1: Projective Planes
- Lecture 2: A first look at Hilbert functions
- Lecture 3: Who or what lives in projective space?
- Lecture 4: Lefschetz properties
- Lecture 5: Geproci sets

Lecture 1: Projective planes

Introduction

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- commutative algebra
- combinatorics

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Today we'll focus on the projective plane, looking at it from different points of view.

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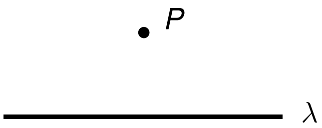

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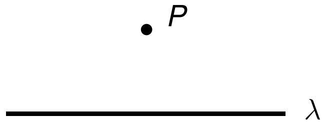

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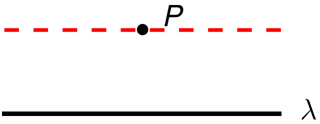

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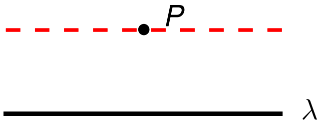

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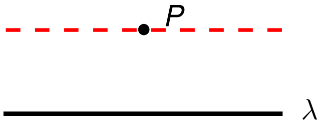

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So what’s the idea behind the projective plane? Let’s start from scratch: axioms!

There are different ways of presenting them; we’ll follow G. Eric Moorhouse, “Incidence Geometry” (available online).

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P3. There exist four points such that no three are collinear.

Remark. One defines an affine plane by just changing P2:

AP1. For any two distinct points, there is exactly one line through both.

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Remarks.

1. Answering our original question, in the affine plane there is exactly one parallel line. In the projective plane, none.

2. Notice that in either of these sets of axioms, we don't assume the real numbers $\mathbb{R}$, or even any underlying field.

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One of these was discovered by Gino Fano (1871 – 1952).

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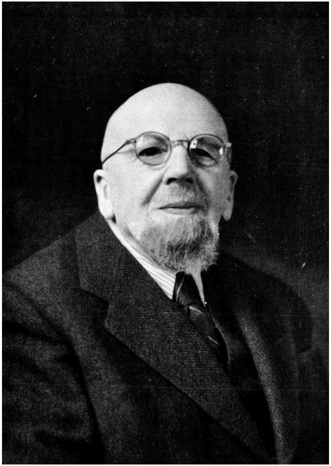

Gino Fano
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(Biblioteca Digitale Italiana di Matematica)

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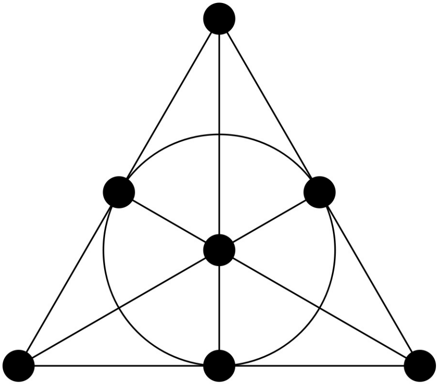

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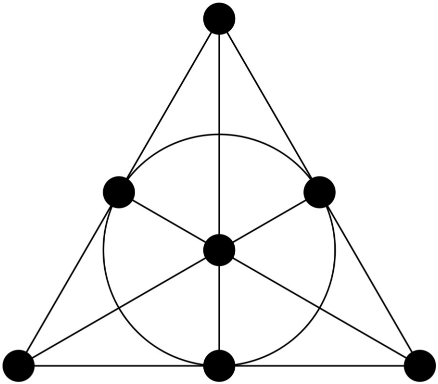

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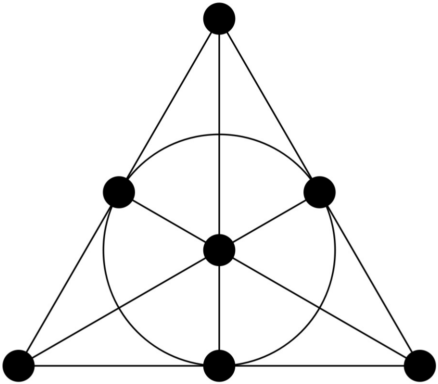

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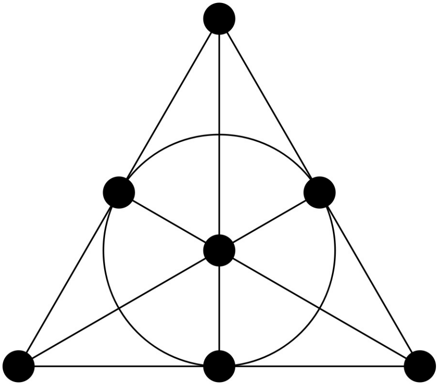

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You can check axioms P1 – P3. We'll come back to this plane.

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(Notice the duality between lines and points.)

Proof: (From Moorhouse.)

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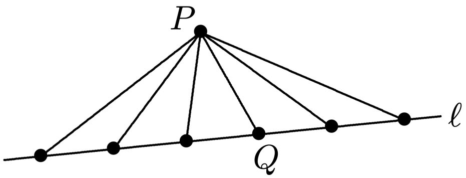
Denote by $[P]$ the set of lines through a point P, and by $[\ell]$ the set of points on a line ℓ.

If $P \notin \ell$ then an obvious bijection $[\ell] \to [P]$ consists of mapping each point $Q \in \ell$ to the line PQ.



Exercise. Use the axioms to see that it's a bijection.

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(b) There are also $d + 1$ lines through every point.

We haven't quite proven (b) yet since the P guaranteed by the exercise might conceivably depend on the choice of ℓ and m.

Let P be any point. By axiom (P3) there exists a line ℓ not passing through P; then we have seen

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(c) The plane contains $d^2 + d + 1$ points, i.e. $|\mathfrak{P}| = d^2 + d + 1$.

The total number, $d^2 + d + 1$, of points comes from looking at all lines through P, counting points on each other than P, then adding in P:

$$|\mathfrak{P}| = (d + 1) \cdot d + 1 = d^2 + d + 1.$$

(d) The plane also has $d^2 + d + 1$ lines, i.e. $|\mathfrak{L}| = d^2 + d + 1$.

Fix a line ℓ and for each of the $d + 1$ points on ℓ count lines through it other than ℓ. Then add 1 for ℓ:

$$(d + 1) \cdot d + 1 = d^2 + d + 1. \qquad \square$$

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2. Note that the statements about points and the statements about lines look the same, and even the proofs are the same. This is not a coincidence.

   We have the important notion of duality: any theorem about (points, lines) remains true about (lines, points).

Theorem (duality)

If $(\mathfrak{P}, \mathfrak{L})$ is a projective plane, then the dual structure $(\mathfrak{L}, \mathfrak{P})$ is also a projective plane (replacing "contained in" by "contains").

Proof:

(Again from Moorhouse.) Let $(\mathfrak{P}, \mathfrak{L})$ be a projective plane. Since $(\mathfrak{P}, \mathfrak{L})$ satisfies (P1) and (P2), the dual structure $(\mathfrak{L}, \mathfrak{P})$ satisfies (P2) and (P1).

Since $(\mathfrak{P}, \mathfrak{L})$ has four lines with no three concurrent (easy exercise), $(\mathfrak{L}, \mathfrak{P})$ satisfies (P3). $\square$

Some more remarks:

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6. (Bruck-Ryser theorem (1949).) *If a finite projective plane of order d exists, and d is congruent to 1 or 2 (mod 4), then d must be the sum of two squares.*
7. That rules out order 6. Order 10 was ruled out by a complicated computer check. First open case: order 12.

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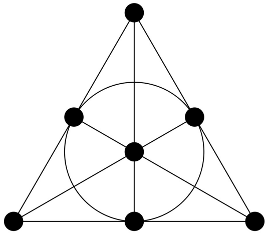

- $2 + 1 = 3$ points on each line and 3 lines through each point.
- $2^2 + 2 + 1 = 7$ points in all and 7 lines in all.

Now let's look at a different projective plane, namely the real projective plane, $\mathbb{P}^2_{\mathbb{R}}$.

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The real projective plane

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Look at this pair of parallel lines:

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And what if there are more than two parallel lines?

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This is the point of view of projective geometry (at least over the reals).

Let’s see how to get the real projective plane from the familiar Euclidean plane by adding points at infinity.

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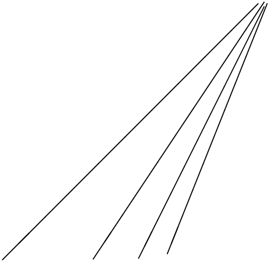

parallel lines
with slope m_1

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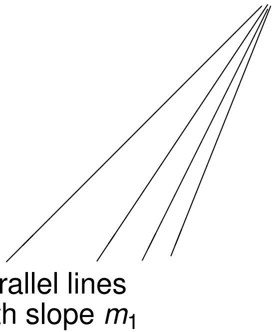

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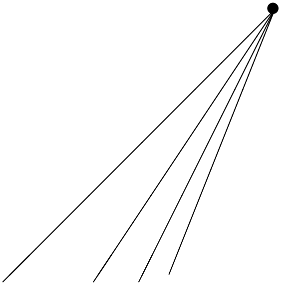

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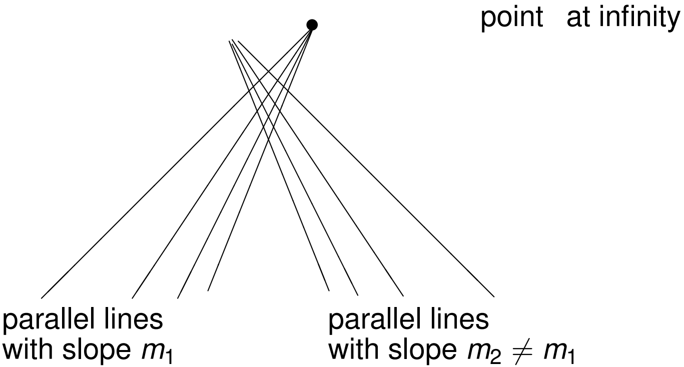

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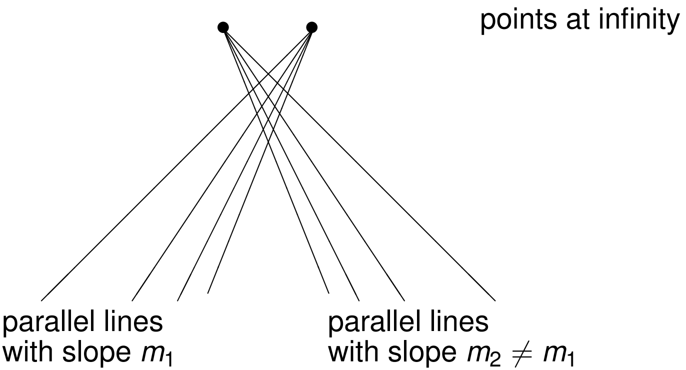

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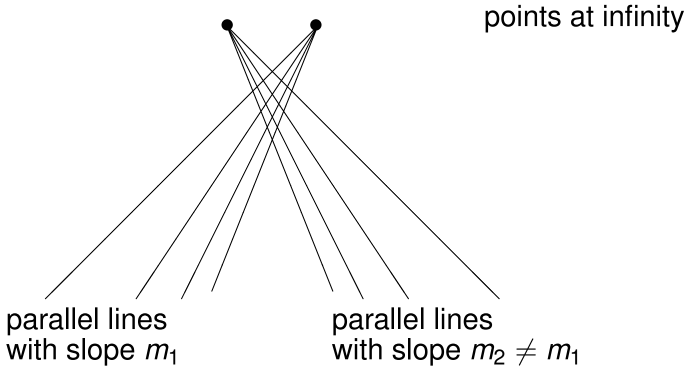

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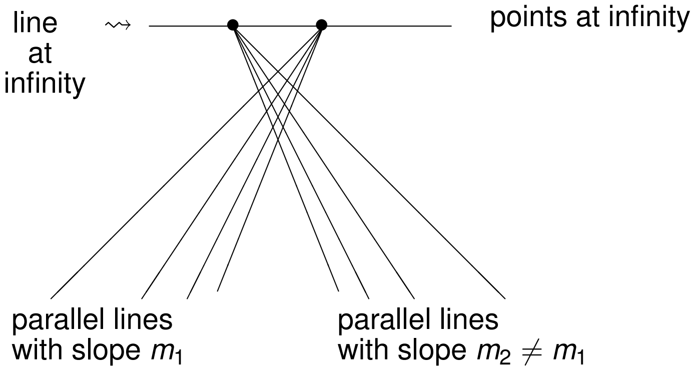

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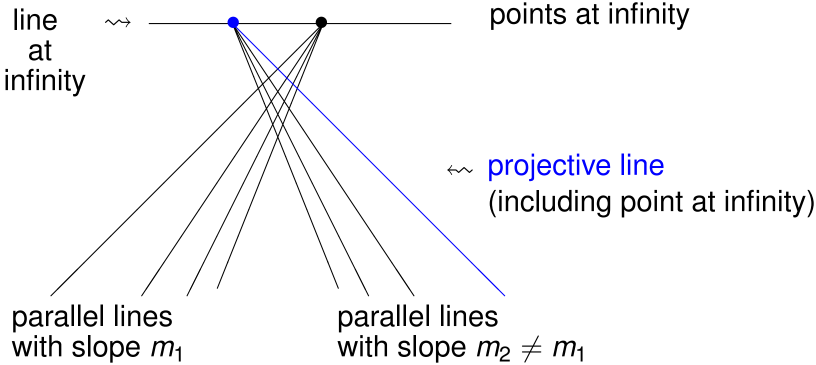

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Exercise. Verify that our real projective plane $\mathbb{P}^2_{\mathbb{R}}$ satisfies the axioms for a projective plane!

Let's look at a different approach to get $\mathbb{P}^2_{\mathbb{R}}$.

A different (but equivalent) construction of $\mathbb{P}^2_{\mathbb{R}}$

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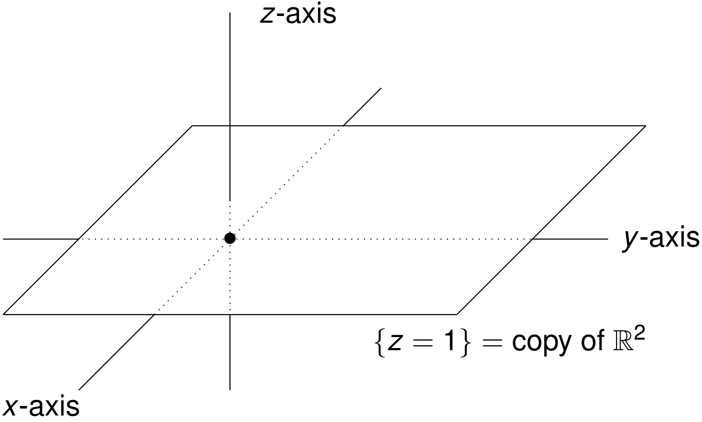

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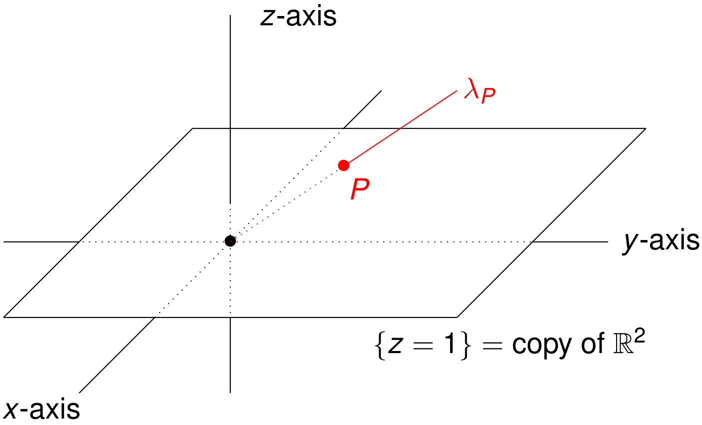
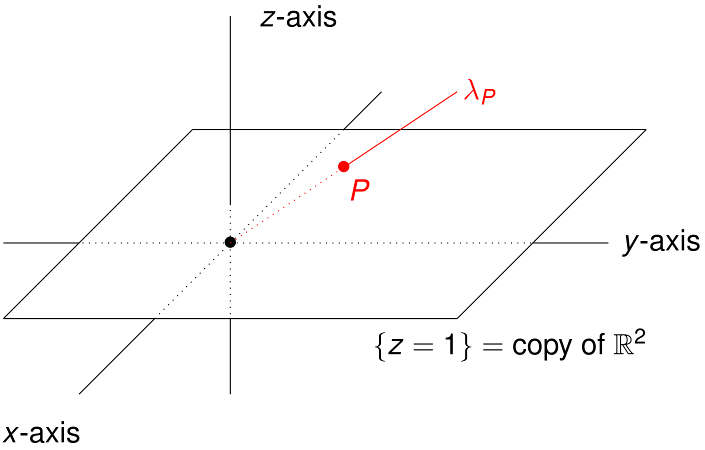
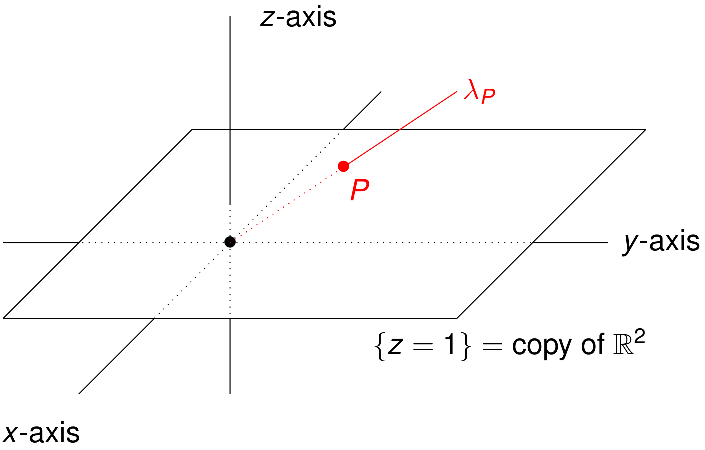


Line λ_P not on (x, y)-plane $\leftrightarrow P \in \mathbb{R}^2$.



z-axis
λ_P
P
y-axis
$\{z = 1\}$ = copy of $\mathbb{R}^2$
x-axis



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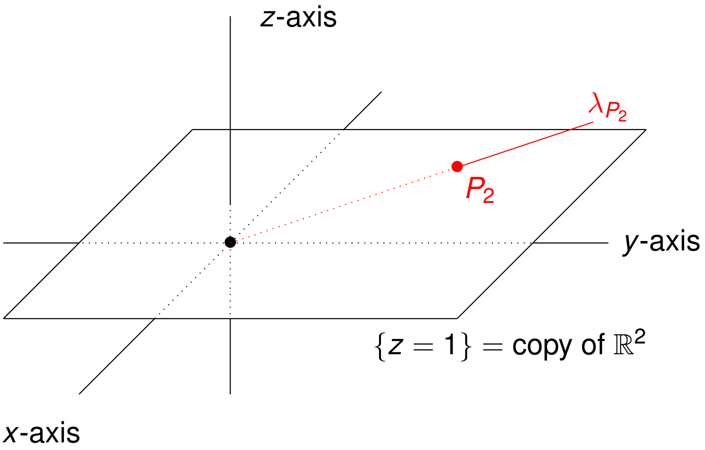

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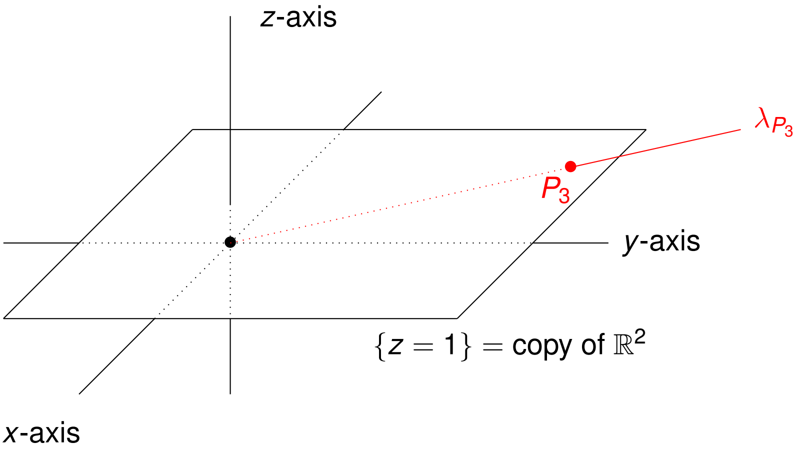

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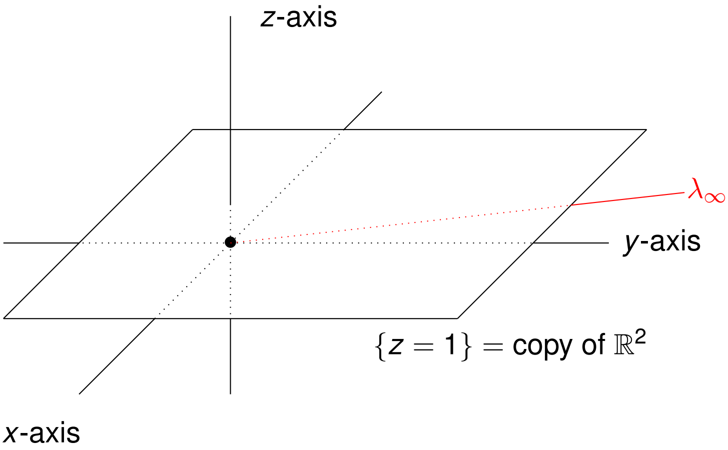

As $P \rightsquigarrow \infty$ in $\mathbb{R}^2$ in any direction, $\lambda_P \rightsquigarrow$ corresponding line through the origin in the (x, y)-plane.

We can identify

$$\{ \text{ lines through the origin in the } (x, y)\text{-plane } \}$$

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Aside: We can also think of $\mathbb{P}^2_{\mathbb{R}}$ topologically as a sphere with antipodal points identified, but we'll skip this point of view.

Next: From this point of view, $\mathbb{P}^2_{\mathbb{R}}$ comes equipped with a set of coordinates over $\mathbb{R}$.

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A line through the origin passing through a point $(a, b, c) \in \mathbb{R}^3$ $((a, b, c) \neq (0, 0, 0))$ can be described as

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So

$$\mathbb{P}^2_{\mathbb{R}} = \left\{ [a, b, c] \;\middle|\; \begin{array}{c} (a, b, c) \neq (0, 0, 0) \text{ and} \\ \\ [a, b, c] = [ta, tb, tc] \;\forall\; 0 \neq t \in \mathbb{R} \end{array} \right\}$$

(E.g. $[1, 2, 3] = [2, 4, 6]$.)

Furthermore, lines in this projective plane (union of points) are those projective points (lines through the origin) lying on the same plane.

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This leads to what Moorhouse calls the classical projective planes. (Again follow Moorhouse p. 35.)

We've already seen the main idea when we talked about $\mathbb{P}^2_{\mathbb{R}}$, and we've seen the first example when we saw the Fano plane!!

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In this case the line $\overline{PQ}$ spanned by points P and Q is the 2-dimensional subspace spanned by the 1-dimensional subspace P and the 1-dimensional subspace Q.

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Any non-zero vector (a, b, c) in V is identified with any scalar multiple (ta, tb, tc), so we get coordinates $[a, b, c]$ for $\mathbb{P}^2_F$ in the same way as we did when $F = \mathbb{R}$.

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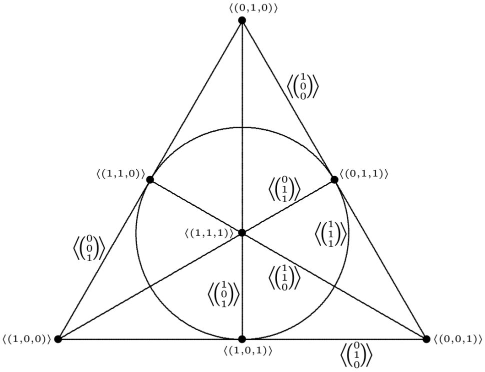
The smallest classical projective plane, constructed from the field $k = \mathbb{Z}_2$, is the Fano plane. (The following picture is from Moorhouse.)



Here points are denoted $\langle(a, b, c)\rangle$, which we are denoting $[a, b, c]$.

The line $ax + by + cz$ is denoted by $\left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle$.

In Lecture 3 we'll see what we can do with these coordinates.

Meanwhile, in the next lecture we'll introduce a special case of the important notion of Hilbert functions coming from the finite projective planes we've studied today.

In Lecture 3 we'll continue our study of Hilbert functions and bring in some more geometry.