Projective Planes and Beyond

Thematic Program on Rationality and Hyperbolicity

Undergraduate Workshop

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Slides available by emailing migliore.1@nd.edu
or from the conference website.

Lecture 3: Who or what lives in projective space?

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Or: Projective varieties and graded rings

Introduction to $\mathbb{P}_k^n$

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Questions.

- What happens if we pass to an $(n+1)$-dimensional vector space? (We'll define $\mathbb{P}^n_k$.)
- And are there other interesting subsets of $\mathbb{P}^2_k$ or $\mathbb{P}^n_k$ besides points and lines?

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So

$$\mathbb{P}^2_{\mathbb{R}} = \left\{ [a, b, c] \;\middle|\; \begin{array}{c} (a, b, c) \neq (0, 0, 0) \text{ and} \\ [a, b, c] = [ta, tb, tc] \; \forall \; 0 \neq t \in \mathbb{R} \end{array} \right\}$$

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And the same works over any field k in place of $\mathbb{R}$. More precisely:

A line through the origin in k^{n+1} passing through a point

$$(a_0, a_1, \ldots, a_n) \in k^{n+1}, \quad (a_0, a_1, \ldots, a_n) \neq (0, 0, \ldots, 0)$$

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These are homogeneous coordinates for points in $\mathbb{P}^n_k$.

In the same way that

a line in $\mathbb{P}^2_k \leftrightsquigarrow$ a 2-dimensional linear vector subspace of k^3,

we get

an s-dimensional linear variety in $\mathbb{P}^n_k$	$\leftrightsquigarrow$	$(s+1)$-dimensional (linear) vector subspace of k^{n+1}

But we can do much more!

Vanishing loci and projective varieties

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So we focus on $R = k[x_0, x_1, \ldots, x_n]$.

Question: Let $P \in \mathbb{P}^n_k$ and $f \in R$. Is the statement

f vanishes at P

well-defined?

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Example 2. $n = 2, \quad P = [1, 2, 3], \quad f_2 = x_0 + x_1 - x_2.$

$$f_2(1, 2, 3) = 1 + 2 - 3 = 0$$

and in fact

$$f_2(t, 2t, 3t) = t + 2t - 3t = 0 \quad \text{for all } t.$$

The important difference between

$$f_1 = x_0 + x_1 + x_2 - 6 \quad \text{and} \quad f_2 = x_0 + x_1 - x_2$$

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Lemma. *Let*

$$P = [a_0, a_1, \ldots, a_n] \in \mathbb{P}^n_k$$

and let

$$f \in R = k[x_0, x_1, \ldots, x_n]$$

be a homogeneous polynomial of degree d. Then

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Proof: $\forall t, \quad f(ta_0, \ldots, ta_n) = t^d f(a_0, \ldots, a_n)$ (exercise). □

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Definition. A subset $V \subseteq \mathbb{P}^n_k$ is a projective algebraic variety if there exist homogeneous polynomials

$$f_1, \ldots, f_s \in R = k[x_0, x_1, \ldots, x_n]$$

such that

$$V = \left\{ P \in \mathbb{P}^n_k \mid f_i(P) = 0 \quad \text{for all} \quad 1 \leq i \leq s \right\}.$$

We write $V = \mathbb{V}(f_1, \ldots, f_s)$. Note $\{f_1, \ldots, f_s\}$ is a finite set.

Note also

$$\emptyset \quad \text{and} \quad \mathbb{P}^n_k$$

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What are the connections with projective varieties?

- 1st connection. Let I be a homogeneous ideal. Define

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$\subseteq$: Obvious because $f_i \in I$ for $1 \leq i \leq s$.

$\supseteq$: Let $P \in \mathbb{V}(f_1, \ldots, f_s)$ and $f \in I$ (not necessarily homogeneous). Then $f(P) = \sum_{i=1}^{s} A_i(P) f_i(P) = 0$. □

- 2nd Connection. Let $S \subset \mathbb{P}^n_k$ be any set (not necessarily a variety) and assume k is infinite.

Definition.
$\mathbb{I}(S) = \{f \in R = k[x_0, x_1, \ldots, x_n] \mid f(P) = 0 \text{ for all } P \in S\}$.

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Only issue: why is it a homogeneous ideal?

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We know $f(P) = 0$, and

$$P = [ta_0, ta_1, \ldots, ta_n]$$

for any $0 \neq t \in k$ (which, again, is infinite).

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Thus $\mathbb{I}(S)$ is a homogeneous ideal. $\square$

So we have

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$$V \mapsto \mathbb{I}(V)$$

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Questions. How are these maps related? Are they inverses of each other? Are there other relations?

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$\supseteq$: Let $P \in V$. $\boxed{\text{We want to show that } P \in \mathbb{V}(\mathbb{I}(V)).}$

I.e. We want to show that if $f \in \mathbb{I}(V)$ then $f(P) = 0$. Clear.

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I.e. We want to show that if $f \in \mathbb{I}(V)$ then $f(P) = 0$. Clear.

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□

Note for $\subseteq$ it was crucial that V was a variety.

Reversing the roles does not (quite) work:

Example. $I = \langle x_0^2 \rangle$. Then

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Example. $I = \langle x_0^2 \rangle \Rightarrow \sqrt{I} = \langle x_0 \rangle$.

Proposition. *If I is homogeneous then so is $\sqrt{I}$.* (Exercise.)

Theorem. (Hilbert's Projective Strong Nullstellensatz) *Let k be algebraically closed. Let I be a homogeneous ideal. If $V = \mathbb{V}(I)$ is non-empty then*

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}.$$

Fact. (Largely Hilbert's Projective Strong Nullstellensatz.)

Assume k is algebraically closed (e.g. $k = \mathbb{C}$). We have a bijection

$$\left\{ \begin{array}{c} \text{nonempty} \\ \text{varieties in } \mathbb{P}^n_k \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{proper, radical} \\ \text{homogeneous ideals} \\ \text{in } k[x_0, \ldots, x_n] \end{array} \right\}$$

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where $\mathbb{V}$ and $\mathbb{I}$ are inverses of each other; in particular

$$\mathbb{I}(\mathbb{V}(I)) = I.$$

Hence $\mathbb{V}$ and $\mathbb{I}$ are order-reversing bijections that are inverses of each other.

Fact. Unions and intersections of projective varieties are again projective varieties:

$$\mathbb{V}(I_1) \cup \mathbb{V}(I_2) = \mathbb{V}(I_1 \cap I_2).$$

$$\mathbb{V}(I_1) \cap \mathbb{V}(I_2) = \mathbb{V}(I_1 + I_2).$$

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In fact, finite unions and arbitrary intersections of projective varieties are again projective varieties.

Projective varieties form the closed sets in the Zariski topology for $\mathbb{P}^n_k$. (Details omitted.)

Independent conditions and linear equations

We want to talk about what it means for a set of points to impose independent conditions on forms of fixed degree d.

Then we'll talk about how you check if this holds or not for a given set and given degree.

To simplify things, let's do this by a series of examples.

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To simplify things, let's do this by a series of examples.

Example 1. Let $P = [2, 3, 4] \in \mathbb{P}^2$ and fix the degree to be 3. How do we describe the set of homogeneous polynomials of degree 3 vanishing at P?

To work in $\mathbb{P}^2$ we need 3 variables, say x, y, z. A typical homogeneous polynomial of degree 3 has the form

$$a_1x^3 + a_2x^2y + a_3x^2z + a_4xy^2 + a_5xyz + a_6xz^2 +$$

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To vanish at the point $[2, 3, 4]$ means we need

$$a_1(2^3) + a_2(2^2)(3) + a_3(2^2)(4) + a_4(2)(3^2) + a_5(2)(3)(4)+$$

$$a_6(2)(4^2) + a_7(3^3) + a_8(3^2)(4) + a_9(3)(4^2) + a_{10}(4^3) = 0.$$

i.e.

$$8a_1 + 12a_2 + 16a_3 + 18a_4 + 24a_5 + 32a_6 + 27a_7 +$$

$$36a_8 + 48a_9 + 64a_{10} = 0.$$

So the condition on the polynomial

$$a_1x^3 + a_2x^2y + a_3x^2z + a_4xy^2 + a_5xyz + a_6xz^2 + \\ a_7y^3 + a_8y^2z + a_9yz^2 + a_{10}z^3.$$

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In the 10-dimensional vector space of forms of degree 3 in x, y, z, the solution space is $10 - 1 = 9$ dimensional.

Example 2. Suppose we have points $P_1, \ldots, P_7 \in \mathbb{P}^2$. How do we describe the homogeneous polynomials of degree 3 vanishing on these 7 points?

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So we get a system of seven homogeneous linear equations!

Each new equation "should" knock the dimension down by one.

How do we check if that's the case, i.e. if the points impose independent conditions?

We want to know if our equations are independent.

That's true if and only if none of the equations is a linear combination of the other six.

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It means that any solution of all the first six equations is automatically a solution of the 7th.

Translation: it means that any homogenous polynomial of degree 3 that vanishes at the first six points has to vanish at the 7th point as well.

Conclusion: The equations are independent if and only if the following statement is true:

> *Given any of the seven points, say P_i, you can find a homogeneous polynomial of degree 3 vanishing at the other six but not vanishing at P_i.*

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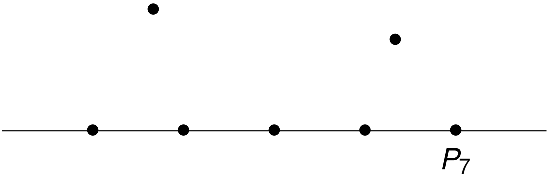
Example 3.



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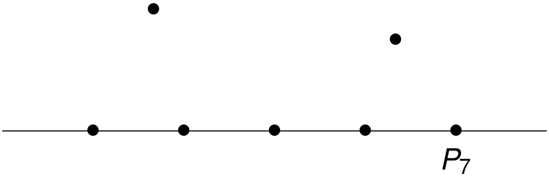

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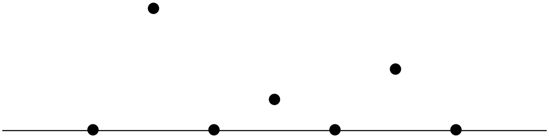

Exercise: For each of the points, P_i, find a cubic (union of 3 lines) containing the remaining 6 but not containing P_i.

Conclusion: the points do impose independent conditions on cubics.

Remark. Let Z be a set of d points in $\mathbb{P}^n$.

Assume that the Hilbert function $h_{R/\mathbb{I}(Z)}(t) = d$ for some t. (We know it's true for $t \gg 0$.)

Then Z imposes independent conditions on homogeneous polynomials of degree t.

Why?

$$h_{R/\mathbb{I}(Z)}(t) = \dim R_t - \dim \mathbb{I}(Z)_t$$

so

$$\dim R_t - \dim \mathbb{I}(Z)_t = d \quad \Rightarrow \quad \dim \mathbb{I}(Z)_t = \dim R_t - d,$$

i.e. the d points impose independent conditions.

Hilbert functions and Hilbert polynomials for varieties

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What follows are some illustrations, and are not central to this lecture.

Theorem. (Hilbert-Serre)

There is a polynomial $P_V(t) \in \mathbb{Q}[t]$ *such that*

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Takeaway: the Hilbert polynomial comes from the Hilbert function, and it gives important information about V.

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2. V is a curve if and only if $P_V(t)$ is a polynomial of degree 1.

   Then
$$P_V(t) = (\deg V)t - p_a(V) + 1.$$

2. (cont.) Examples:

2.1 Let V be the so-called twisted cubic curve in $\mathbb{P}^3_{\mathbb{R}}$,

$$V = \{[s^3, s^2t, st^2, t^3] \; [s,t] \in \mathbb{P}^1_{\mathbb{R}}\}.$$

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$$\deg(V) = ab \quad \text{and} \quad p_a(V) = \frac{1}{2}ab(a+b-4)+1.$$

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and $P_V(t) = 3t + 1$.

So V has degree 3 and arithmetic genus 0.

2.2 Let V be the complete intersection in $\mathbb{P}^3_k$ of a surface of degree a and a surface of degree b. Then

$$\deg(V) = ab \quad \text{and} \quad p_a(V) = \frac{1}{2}ab(a+b-4)+1.$$

E.g. $a = 2,\ b = 3$ gives a curve of degree 6 and arithmetic genus 4.

2. (cont.)

2.3 Let V be a curve of degree 6 and arithmetic genus 0 in $\mathbb{P}^3_k$ (e.g. a smooth rational sextic curve).

Then $P_V(t) = 6t + 1$.

In particular for $t \gg 0$ (in fact $t \geq 3$ will do)

$$\dim[R/I(V)]_t = 6t + 1,$$

so

$$\dim[I(V)]_t = \binom{t+3}{3} - (6t+1) = \frac{1}{6}\left(t^3 + 6t^2 - 25t\right).$$

This gives the dimension of the vector space of forms of degree t vanishing on V.

Since the Hilbert function determines the Hilbert polynomial, it is (at least) as important.

In fact, looking at specific (low) degrees of the Hilbert function gives info you'd never spot from the Hilbert polynomial.