RATIONALITY AND HYPERBOLICITY SUMMER SCHOOL:
RATIONALITY OF THREEFOLDS OVER NON-CLOSED FIELDS
EXERCISES

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1. Lecture 3: Connection to curve classes

Exercise 1. Fill in the details from lecture showing that $J^1(X) \cong \ker c_1$.

Exercise 2. Recall the definition of the dual complex abelian variety given with the lecture 2 exercises. Show that $J^n(X) \cong J^1(X)^\vee$.

Exercise 3. Show that for a complex abelian variety $A$, the dual abelian variety satisfies $A^\vee \cong \text{Pic}^0 A$. Hint: Use the fact that $A \cong H^0(A, \Omega^1_A)^\vee / H_1(A, \mathbb{Z})$.

Exercise 4. Note that given an abelian variety $A$, there is a canonical isomorphism $(A^\vee)^\vee \cong A$. Indeed, this is true over any algebraically closed field if we use $\text{Pic}_A^0$ as the definition of $A^\vee$.

In this exercise, you’ll show that $(\text{Pic}_X^0)^\vee$ (that is, $\text{Pic}_{\text{Pic}_X^0}^0$) is $\text{Alb}_X$ for any smooth complex projective variety $X$.

1) Fix a point $x_0 \in X$. Show that there is always a morphism $X \to (\text{Pic}_X^0)^\vee$ sending $x_0$ to $\mathcal{O}_{\text{Pic}_X^0}$.

   Hint: By the representability of the relative Picard functor, there is a Poincaré bundle $\mathcal{P}$ on $X \times \text{Pic}_X^0$ which satisfies:
   - For all $[L] \in \text{Pic}_X^0$, $\mathcal{P}|_{X \times \{[L]\}} \cong L$, and
   - $\mathcal{P}$ is normalized such that $\mathcal{P}|_{\{x_0\} \times \text{Pic}_X^0} \cong \mathcal{O}_{\text{Pic}_X^0}$.

2) Let $f : X \to A'$ be a morphism to an abelian variety $A'$ such that $f(x_0) = 0_{A'}$. Show that this induces a morphism of group schemes $f^\vee : (\text{Pic}_X^0)^\vee \to A'$.

3) Finally, show that the morphism from (2) is the unique morphism making the following diagram commute:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & A' \\
\downarrow & & \downarrow \text{com} \\
(\text{Pic}_X^0)^\vee & \xrightarrow{f^\vee} & A' \\
\end{array}
$$

Exercise 5. Show that two divisors are rationally equivalent if and only if they are linearly equivalent.

Exercise 6 (For those who like complex geometry). Make clear how the Abel-Jacobi map can be given by integration over subvarieties. For example, why do rationally equivalent subvarieties give the same value in $J^m(X)$? Do you see why we must restrict to homologically trivial classes?

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Exercise 7. We say that two cycles $Z_1, Z_2 \in Z^m(X)$ are homologically equivalent if $\text{cl}_m([Z_1]) = \text{cl}_m([Z_2])$. Show that
\[
\text{rationally equivalent } \implies \text{ algebraically equivalent } \implies \text{ homologically equivalent.}
\]
(You might compare to [Har77, Exercise V.1.7] for the case of surfaces.)

For the next exercises, here is the precise definition of a regular homomorphism: Let $T$ be a variety and pick a point $t_0 \in T$. An algebraic family of cycle classes on $X$ parametrized by $T$ is given by $\{W_{\{t\}\times X}\}$. This family gives a map $T \to (\text{CH}^m X)^0$ given by $t \mapsto W_{\{t\}\times X} - W_{\{t_0\}\times X}$.

A homomorphism $\varphi: (\text{CH}^m X)^0 \to A$ for an abelian variety $A$ is regular if for every algebraic family $(T, W)$ as above, the composition $T \to (\text{CH}^m X)^0 \to A$ is a morphism of algebraic varieties.

Exercise 8. Recall that a pair $(A_0, \varphi_0)$, with $A_0$ an abelian variety and $\varphi_0: \text{CH}^m(X)^0 \to A_0$ a regular homomorphism, is an algebraic representative for $\text{CH}^m(X)^0$ if it is universal among such pairs: for every $(A, \varphi)$, with $A$ an abelian variety and $\varphi$ a regular homomorphism, there is a morphism $f: A_0 \to A$ making the following diagram commute:

\[
\begin{tikzcd}
\text{CH}^m(X)^0 \arrow{r}{\varphi_0} \arrow{dr}[swap]{\varphi} & A_0 \\
& A \arrow{ur}[swap]{f}
\end{tikzcd}
\]

(1) Show that if $(A_0, \varphi_0)$ exists, then $\varphi_0$ is surjective.
(2) Show that once $f$ exists, it must be unique.

Exercise 9. (1) Show that $\text{Pic}^0 X$ is the algebraic representative for $\text{CH}^1(X)^0$.
(2) Show that $\text{Alb} X$ is the algebraic representative for $\text{CH}^n(X)^0$.

References


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