RATIONALITY AND HYPERBOLICITY SUMMER SCHOOL: RATIONALITY OF THREEFOLDS OVER NON-CLOSED FIELDS EXERCISES

SARAH FREI

1. Lecture 3: Connection to curve classes

Exercise 1. Fill in the details from lecture showing that $J^1(X) \cong \ker c_1$.

Exercise 2. Recall the definition of the dual complex abelian variety given with the lecture 2 exercises. Show that $J^n(X) \cong J^1(X)^\vee$.

Exercise 3. Show that for a complex abelian variety A, the dual abelian variety satsifies $A^\vee \cong \operatorname{Pic}^0 A$. *Hint: Use the fact that* $A \cong H^0(A, \Omega^1_A)^\vee / H_1(A, \mathbb{Z})$.

Exercise 4. Note that given an abelian variety A, there is a canonical isomorphism $(A^\vee)^\vee \cong A$. Indeed, this is true over any algebraically closed field if we use Pic^0_A as the definition of $A^\vee$.

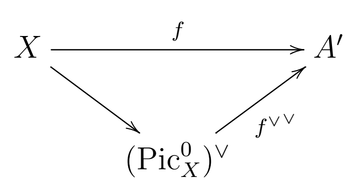
In this exercise, you'll show that $(\operatorname{Pic}^0_X)^\vee$ (that is, $\operatorname{Pic}^0_{\operatorname{Pic}^0_X}$) is Alb_X for any smooth complex projective variety X.

(1) Fix a point $x_0 \in X$. Show that there is always a morphism $X \to (\operatorname{Pic}^0_X)^\vee$ sending x_0 to $\mathcal{O}_{\operatorname{Pic}^0_X}$.

Hint: By the representability of the relative Picard functor, there is a Poincaré bundle $\mathcal{P}$ *on* $X \times \operatorname{Pic}^0_X$ *which satisfies:*
- *For all* $[L] \in \operatorname{Pic}^0_X$, $\mathcal{P}|_{X \times \{[L]\}} \cong L$, *and*
- $\mathcal{P}$ *is normalized such that* $\mathcal{P}|_{\{x_0\} \times \operatorname{Pic}^0_X} \cong \mathcal{O}_{\operatorname{Pic}^0_X}$.

(2) Let $f \colon X \to A'$ be a morphism to an abelian variety A' such that $f(x_0) = 0_{A'}$. Show that this induces a morphism of group schemes $f^{\vee\vee} \colon (\operatorname{Pic}^0_X)^\vee \to A'$.

(3) Finally, show that the morphism from (2) is the unique morphism making the following diagram commute:



Exercise 5. Show that two divisors are rationally equivalent if and only if they are linearly equivalent.

Exercise 6 (For those who like complex geometry)**.** Make clear how the Abel-Jacobi map can be given by integration over subvarieties. For example, why do rationally equivalent subvarieties give the same value in $J^m(X)$? Do you see why we must restrict to homologically trivial classes?

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Exercise 7. We say that two cycles $Z_1, Z_2 \in Z^m(X)$ are **homologically equivalent** if $\mathrm{cl}_m([Z_1]) = \mathrm{cl}_m([Z_2])$. Show that

$$\text{rationally equivalent} \implies \text{algebraically equivalent} \implies \text{homologically equivalent}.$$

(You might compare to [Har77, Exercise V.1.7] *for the case of surfaces.)*

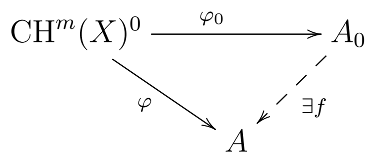
For the next exercises, here is the precise definition of a regular homomorphism: Let T be a variety and pick a point $t_0 \in T$. An **algebraic family of cycle classes** on X parametrized by T is given by $\{W_{\{t\}\times X}\}$. This family gives a map $T \to (\mathrm{CH}^m X)^0$ given by $t \mapsto W_{\{t\}\times X} - W_{\{t_0\}\times X}$.

A homomorphism $\varphi\colon (\mathrm{CH}^m X)^0 \to A$ for an abelian variety A is **regular** if for every algebraic family (T, W) as above, the composition

$$T \to (\mathrm{CH}^m X)^0 \to A$$

is a morphism of algebraic varieties.

Exercise 8. Recall that a pair (A_0, φ_0), with A_0 an abelian variety and $\varphi_0\colon \mathrm{CH}^m(X)^0 \to A_0$ a regular homomorphism, is an **algebraic representative** for $\mathrm{CH}^m(X)^0$ if it is universal among such pairs: for every (A, φ), with A an abelian variety and φ a regular homomorphism, there is a morphism $f\colon A_0 \to A$ making the following diagram commute:



(1) Show that if (A_0, φ_0) exists, then φ_0 is surjective.
(2) Show that once f exists, it must be unique.

Exercise 9. (1) Show that $\mathrm{Pic}^0 X$ is the algebraic representative for $\mathrm{CH}^1(X)^0$.
(2) Show that $\mathrm{Alb}\, X$ is the algebraic representative for $\mathrm{CH}^n(X)^0$.

References

[ABB14] Asher Auel, Marcello Bernardara, and Michele Bolognesi, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems*, J. Math. Pures Appl. (9) **102** (2014), no. 1, 249–291, DOI 10.1016/j.matpur.2013.11.009 (English, with English and French summaries). MR3212256 ↑

[GH94] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original. MR1288523 ↑

[Har92] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1992. A first course. MR1182558 ↑

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR0463157 ↑7

[Huy23] Daniel Huybrechts, *The geometry of cubic hypersurfaces*, Cambridge Studies in Advanced Mathematics, vol. 206, Cambridge University Press, Cambridge, 2023. MR4589520 ↑

[Poo17] Bjorn Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. MR3729254 ↑

Email address: sarah.frei@dartmouth.edu
URL: http://math.dartmouth.edu/~sfrei