(There are fewer exercises today, so use the opportunity to revisit exercises from previous days that you didn’t get to!)

**Lecture 5: The refined intermediate Jacobian obstruction**

**Exercise 1.** Let $X$ be a smooth threefold complete intersection of two quadrics.

(1) Show that all lines in $X$ are algebraically equivalent.

(2) Show that any two lines in $X$ are not rationally equivalent.

**Exercise 2.** Let $X$ be a smooth threefold complete intersection of two quadrics. Show that $\text{NS}^2 \frac{X}{k} \cong (\text{NS}^2 \frac{X}{k})^G_k \cong \mathbb{Z}$, thus showing that the codimension 2 Chow scheme $\text{CH}^2_X$ has a $\mathbb{Z}$ grading.

In the next exercise, we’ll introduce the Albanese torsor (see [Poo17, Example 5.12.11]) and the Albanese variety for a variety with (possibly) no $k$-points.

**Exercise 3.** Let $X$ be a geometrically integral variety over $k$, and $C_X$ the category of triples $(A, T, f)$ where $A$ is an abelian variety, $T$ is an $A$-torsor, and $f: X \to T$ is a morphism. A morphism $(A, T, f)$ to $(A', T', f')$ is a homomorphism $\alpha: A \to A'$ and a morphism $\tau: T \to T'$ such that the following diagrams commute:

$$
\begin{array}{ccc}
T \times A & \longrightarrow & T \\
\downarrow \scriptstyle{(\tau, \alpha)} & & \downarrow \scriptstyle{\tau} \\
T' \times A' & \longrightarrow & T'
\end{array}
\quad
\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & T \\
\downarrow \scriptstyle{\tau} & & \downarrow \scriptstyle{\tau} \\
T' & \overset{f'}{\longrightarrow} & T'
\end{array}
$$

It is a theorem that this category has an initial object $(\text{Alb}_X, \text{Alb}_X^1, \iota)$; $\text{Alb}_X$ is the **Albanese variety** of $X$, and $\text{Alb}_X^1$ is the **Albanese torsor** of $X$.

(1) Let $X$ be a smooth projective (geometrically integral) genus 1 curve. Show that $\text{Alb}_X \cong \text{Pic}_X^0$, and $\text{Alb}_X^1 \cong X$.

(2) Show that, if $X$ has a $k$-point $x \in X(k)$, this definition of the Albanese variety agrees with the one discussed in Lecture 3.

(3) Let $C$ be a smooth projective (geometrically integral) curve. Show that $\text{Alb}_{\text{Pic}_C^0} \cong \text{Pic}_C^0$.

**Exercise 4.** Let $Y \to \mathbb{P}^1 \times \mathbb{P}^2$ be a double cover branched along a $(2, 2)$-divisor.

(1) Show that $Y$ has the structure of a conic bundle over $\mathbb{P}^2$ as the structure of a quadric surface bundle over $\mathbb{P}^1$.

(2) Show that the discriminant curve of the conic bundle $Y \to \mathbb{P}^2$ has degree 4.

*Date: June 23, 2023.*
(The conic bundle examples in [FJS+] with interesting IJT behavior are constructed as these double covers, and the quadric surface fibration is a key ingredient in our understanding of the behavior of the codimension 2 Chow torsors.)

REFERENCES
