RATIONALITY AND HYPERBOLICITY SUMMER SCHOOL: RATIONALITY OF THREEFOLDS OVER NON-CLOSED FIELDS EXERCISES

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(There are fewer exercises today, so use the opportunity to revisit exercises from previous days that you didn't get to!)

Lecture 5: The refined intermediate Jacobian obstruction

Exercise 1. Let X be a smooth threefold complete intersection of two quadrics.

(1) Show that all lines in X are algebraically equivalent.
(2) Show that any two lines in X are not rationally equivalent.

Exercise 2. Let X be a smooth threefold complete intersection of two quadrics. Show that $\mathrm{NS}^2 X_{\bar{k}} \cong (\mathrm{NS}^2 X_{\bar{k}})^{G_k} \cong \mathbb{Z}$, thus showing that the codimension 2 Chow scheme $\mathbf{CH}^2_X$ has a $\mathbb{Z}$ grading.

In the next exercise, we'll introduce the Albanese torsor (see [Poo17, Example 5.12.11]) and the Albanese variety for a variety with (possibly) no k-points.

Exercise 3. Let X be a geometrically integral variety over k, and $\mathcal{C}_X$ the category of triples (A, T, f) where A is an abelian variety, T is an A-torsor, and $f\colon X \to T$ is a morphism. A morphism (A, T, f) to (A', T', f') is a homomorphism $\alpha\colon A \to A'$ and a morphism $\tau\colon T \to T'$ such that the following diagrams commute:

$$\begin{array}{ccc} T \times A & \longrightarrow & T \\ \downarrow{\scriptstyle (\tau,\alpha)} & & \downarrow{\scriptstyle \tau} \\ T' \times A' & \longrightarrow & T' \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & T \\ & {\scriptstyle f'}\searrow & \downarrow{\scriptstyle \tau} \\ & & T' \end{array}$$

It is a theorem that this category has an initial object $(\mathrm{Alb}_X, \mathrm{Alb}^1_X, \iota)$; Alb_X is the **Albanese variety** of X, and Alb^1_X is the **Albanese torsor** of X.

(1) Let X be a smooth projective (geometrically integral) genus 1 curve. Show that $\mathrm{Alb}_X \cong \mathrm{Pic}^0_X$, and $\mathrm{Alb}^1_X \cong X$.
(2) Show that, if X has a k-point $x \in X(k)$, this definition of the Albanese variety agrees with the one discussed in Lecture 3.
(3) Let C be a smooth projective (geometrically integral) curve. Show that $\mathrm{Alb}_{\mathrm{Pic}^d_C} \cong \mathrm{Pic}^0_C$.

Exercise 4. Let $Y \to \mathbb{P}^1 \times \mathbb{P}^2$ be a double cover branched along a $(2, 2)$-divisor.

(1) Show that Y has the structure of a conic bundle over $\mathbb{P}^2$ as the structure of a quadric surface bundle over $\mathbb{P}^1$.
(2) Show that the discriminant curve of the conic bundle $Y \to \mathbb{P}^2$ has degree 4.

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(The conic bundle examples in [FJS+] with interesting IJT behavior are constructed as these double covers, and the quadric surface fibration is a key ingredient in our understanding of the behavior of the codimension 2 Chow torsors.)

References

[FJS+] Sarah Frei, Lena Ji, Soumya Sankar, Bianca Viray, and Isabel Vogt, *Curve classes on conic bundle threefolds and applications to rationality*, arXiv preprint arXiv:2207.07093. ↑(document)

[Poo17] Bjorn Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. MR3729254 ↑(document)