Problems on jet bundles

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1 Day 5

1. Let $\Phi_0$ and $\Phi$ be $k$-planes in $\mathbb{P}^n$. Show that there exists a sequence of $k$-planes $\Phi_0, \Phi_1, \ldots, \Phi_\ell = \Phi$ such that $\Phi_i \cap \Phi_{i+1}$ is a $k-1$-plane.

2. Using the Grassmannian technique, give an alternate proof of the fact that a very general hypersurface of degree $d \geq 2n$ is algebraically hyperbolic. (Hint: Use the problem from yesterday about sweeping families of curves on general type varieties.)

3. Given a variety $Y$, we say two closed points $p$ and $q$ are rationally Chow-0 equivalent if for some $N > 0$, $Np \sim Nq$ in $CH_0$. A theorem of Roitman shows that if $X \subset \mathbb{P}^n$ is a very general hypersurface of degree $d \geq n + 1$, then a general point of $X$ is rationally Chow-0 equivalent to at most countably many other points of $X$.
   (a) Show that a general point of a very general hypersurface $X$ of degree $d \geq n + 2$ will be rationally Chow-0 equivalent to no other points of $X$.
   (b) Show that a very general hypersurface $X$ of degree $d \geq 2n$ will have no points Chow-0 equivalent to any others. (first proved by Chen, Lewis, Sheng)
   (c) For $X$ in $\mathbb{P}^n$ of degree $d$, bound the dimension of the space of points in $X$ that are rationally Chow-0 equivalent to some other point.
   (d) Find a large space of points equivalent to some other point by considering the space of lines meeting $X$ set-theoretically in 2 points.

4. The following problem lays the foundation for a generalization of the Grassmannian technique first proved by Coskun and Riedl.
   (a) Let $B \subset \mathbb{G}(k-1, n)$. We defined the covering family of $B$ as the set $C \subset \mathbb{G}(k, n)$ of $k$ planes containing some element of $B$. We say $B$ is $\ell$-clustered if a general member of $C$ contains a $k-\ell$ dimensional family of $B$. Show that if $B$ has dimension $k(n-k+1) - \epsilon$, then $C$ has dimension $(k+1)(n-k) - \epsilon + \ell$. 

(b) Let $Z$ be some subvariety of $\mathbb{P}^n$. Show that the family $B$ of $k-1$ planes meeting $Z$ is 1-clustered provided is not all of $G(k-1,n)$.

(c) We now try to show that all 1-clustered families $B$ of codimension at least 2 have this form. We start with the case $n = k + 1$. Consider what $C$ must be in $(\mathbb{P}^{k+1})^\ast$. Show that the space of $k$-planes containing an element $b \in B$ must be a line in $(\mathbb{P}^{k+1})^\ast$.

(d) Show by considering the elements $B' \subset B$ contained in a single $c \in C$ that $C$ must be swept out by a $k - 1$-dimensional family of lines, all passing through the single point corresponding to $c$. Conclude that $C$ has dimension precisely $k$.

(e) Since this works for any general $c \in C$, show that $C$ must be a hyperplane in $(\mathbb{P}^{k+1})^\ast$.

(f) Now suppose $n$ is arbitrary. Take some $k + 1$-plane $\Lambda$ containing a general element of $C$. Show by the above that it follows that the sets of planes of $B$ and $C$ that lie in $\Lambda$ must all contain the same point $p$.

(g) Show by varying $\Lambda$ through a particular $c \in C$ that all $k - 1$ planes through $p$ lie in $B$, and all $k$-planes through $p$ lie in $C$.

(h) Conclude that there is some set of points $Z \subset \mathbb{P}^n$ such that $B$ and $C$ are the set of planes meeting $Z$.

(i) It follows (with a lot of work) that you can show that the codimension increases by 2 each time unless the special locus $S_{n,d}$ is the locus swept out by a particular configuration of lines.