Problems on jet bundles

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1 Day 5

1. Let Φ_0 and Φ be k-planes in $\mathbb{P}^n$. Show that there exists a sequence of k-planes Φ_0, $\Phi_1 \ldots, \Phi_\ell = \Phi$ such that $\Phi_i \cap \Phi_{i+1}$ is a $k-1$-plane.

2. Using the Grassmannian technique, give an alternate proof of the fact that a very general hypersurface of degree $d \geq 2n$ is algebraically hyperbolic. (Hint: Use the problem from yesterday about sweeping families of curves on general type varieties.)

3. Given a variety Y, we say two closed points p and q are rationally Chow-0 equivalent if for some $N > 0$, $Np \sim Nq$ in CH_0. A theorem of Roitman shows that if $X \subset \mathbb{P}^n$ is a very general hypersurface of degree $d \geq n+1$, then a general point of X is rationally Chow-0 equivalent to at most countably many other points of X.

   (a) Show that a general point of a very general hypersurface X of degree $d \geq n+2$ will be rationally Chow-0 equivalent to no other points of X.

   (b) Show that a very general hypersurface X of degree $d \geq 2n$ will have no points Chow-0 equivalent to any others. (first proved by Chen, Lewis, Sheng)

   (c) For X in $\mathbb{P}^n$ of degree d, bound the dimension of the space of points in X that are rationally Chow-0 equivalent to some other point.

   (d) Find a large space of points equivalent to some other point by considering the space of lines meeting X set-theoretically in 2 points.

4. The following problem lays the foundation for a generalization of the Grassmannian technique first proved by Coskun and Riedl.

   (a) Let $B \subset \mathbb{G}(k-1, n)$. We defined the covering family of B as the set $C \subset \mathbb{G}(k, n)$ of k planes containing some element of B. We say B is ℓ-clustered if a general member of C contains a $k - \ell$ dimensional family of B. Show that if B has dimension $k(n-k+1) - \epsilon$, then C has dimension $(k+1)(n-k) - \epsilon + \ell$.

(b) Let Z be some subvariety of $\mathbb{P}^n$. Show that the family B of $k-1$ planes meeting Z is 1-clustered provided is not all of $\mathbb{G}(k-1,n)$.

(c) We now try to show that all 1-clustered families B of codimension at least 2 have this form. We start with the case $n = k+1$. Consider what C must be in $(\mathbb{P}^{k+1})^*$. Show that the space of k-planes containing an element $b \in B$ must be a line in $(\mathbb{P}^{k+1})^*$.

(d) Show by considering the elements $B' \subset B$ contained in a single $c \in C$ that C must be swept out by a $k-1$-dimensional family of lines, all passing through the single point corresponding to c. Conclude that C has dimension precisely k.

(e) Since this works for any general $c \in C$, show that C must be a hyperplane in $(\mathbb{P}^{k+1})^*$.

(f) Now suppose n is arbitrary. Take some $k+1$-plane Λ containing a general element of C. Show by the above that it follows that the sets of planes of B and C that lie in Λ must all contain the same point p.

(g) Show by varying Λ through a particular $c \in C$ that all $k-1$ planes through p lie in B, and all k-planes through p lie in C.

(h) Conclude that there is some set of points $Z \subset \mathbb{P}^n$ such that B and C are the set of planes meeting Z.

(i) It follows (with a lot of work) that you can show that the codimension increases by 2 each time unless the special locus $S_{n,d}$ is the locus swept out by a particular configuration of lines.