Problems on jet bundles

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1 Day 1

Basic practice

1. Let $D$ be a big divisor and $E$ be an effective divisor. Show that $D + E$ is big.

2. Recall that a divisor $D$ is nef if its restriction to every curve $C$ in $X$ has non-negative degree. Find an example of a divisor that is nef but not ample. (Hint: there’s one on $\mathbb{P}^1 \times \mathbb{P}^1$.)

3. Find an example of a divisor that is big but not nef. (Hint: there’s one on the blowup of $\mathbb{P}^2$.

4. Suppose $X$ is a complete intersection of hypersurfaces in $\mathbb{P}^n$ of type $(d_1, d_2, \ldots, d_k)$. Find the canonical sheaf $K_X$. (Hint: use the adjunction formula.)

5. Let $L$ and $L'$ be line bundles on a smooth curve $C$, and suppose there exists a nonzero map $L \to L'$. Show that $\deg L \geq \deg L'$, with equality if and only if $L$ is isomorphic to $L'$.

6. Let $C \subset X$ be a smooth curve in $X$. Show that the degree of the normal bundle $N_{C/X}$ is $2g - 2 - K_X \cdot C$.

7. Let $h : C \to X$ be a birational map from a smooth curve to $X$. Define $N_{h/X}$ as the quotient of $h^*T_X$ by $T_C$. Show that $\deg N_{h/X}$ is $2g - 2 - K_X \cdot C$.

8. For all positive integers $d$, define the syzygy bundle $M_d$ on $\mathbb{P}^n$ via the following sequence:

$$0 \to M_d \to H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \otimes \mathcal{O} \to \mathcal{O}(d) \to 0.$$ 

Show that $H^0(M_d) = 0 = H^1(M_d)$.

9. Let $f : C \to \mathbb{P}^n$ be a smooth, degree $e$ curve mapping birationally. Show that every quotient of $f^*M_1$ has degree at least $-e$. Show further that every quotient of a direct sum of $s$ copies of $f^*M_1$ has degree at least $-se$. 
Algebraic hyperbolicity of hypersurfaces

The following series of problems takes you a proof of the algebraic hyperbolicity of general hypersurfaces in \( \mathbb{P}^n \) of degree \( d \geq 2n - 1 \). This problem has a long history, with the original idea dates back to work from Clemens and Ein, which was elaborated on by Voisin, Xu, Pacienza, Clemens-Ran, and Riedl-Coskun. This version most closely resembles the more streamlined presentation from Wern Yeong’s thesis.

1. (Setup) Suppose that a general hypersurface \( X \) in \( \mathbb{P}^n \) contains a curve of degree \( e \) and geometric genus \( g \). Let \( X \subset \mathbb{P}^n \times H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \) be the space of pairs \((f, p)\) such that \( p \) in a point in the hypersurface \( V(f) \). Consider the space \( M' \) of maps from smooth genus \( g \) curves to \( X \) such that the image is a degree \( e \) curve in the fiber of \( X \to H^0(\mathbb{P}^n(d)) \). By hypothesis, \( M' \) dominates \( H^0(\mathbb{P}^n(d)) \).
   
   (a) Show that there exists a subvariety \( B \) of \( M' \) such that \( M \) parameterizes only finitely curves for a general hypersurface.
   
   (b) Show that we can select \( B \) to be a \( PGL \)-invariant family.
   
   (c) Show that by possibly restricting to an open set, we can assume the map \( B \to H^0(\mathbb{P}^n(d)) \) is etale.
   
   (d) Base-change \( X \) to \( B \), so that we have a family of curves \( Y \to B \) and a map \( h : Y \to X_B \). From now one, we write \( X \) for \( X_B \).

2. Let \( Y_b \) be a general fiber of \( Y \to B \) and \( X_b \) the fiber of \( X \) over \( b \), with map \( h_b : Y_b \to X_b \). Show that \( N_{h/X|Y_b} = N_{h_b/X_b} \). (Hint: write down a commutative diagram with the relevant pieces and use the eight lemma.)

3. Recall we have the relative tangent sheaves \( T_{X/\mathbb{P}^n} \) and \( T_{Y/\mathbb{P}^n} \), given by the kernels of \( T_{X-\mathbb{P}^n} \) and \( T_{Y-\mathbb{P}^n} \). Show that \( N_{h/X} \) is the quotient of \( h^*T_{X/\mathbb{P}^n} \) by \( T_{Y/\mathbb{P}^n} \). (Hint: this is another diagram chase.)

4. Show that \( T_{X/\mathbb{P}^n} \) is isomorphic to the pullback of \( M_d \) from \( \mathbb{P}^n \). (Hint: this is another diagram chase, using the fact that \( T_B|_{X_b} = H^0(\mathbb{P}^n(d)) \otimes \mathcal{O} \).)

5. Show that given any degree \( d - 1 \) polynomial \( P \), we have a natural map \( M_1 \to M_d \) given by multiplication by \( P \).

6. Show that there is a surjection from a direct sum of many copies of \( M_1 \) to \( M_d \).

7. Show that this implies that there is a surjective map from a direct sum of copies of \( h_b^*M_1 \) to \( N_{h_b/X_b} \).

8. Show that in fact there is a generically surjective map from a direct sum of at most \( n - 2 \) copies of \( h_b^*M_1 \) onto \( N_{h_b/X_b} \), and that the degree of \( N_{h_b/X_b} \) is at least \( -(n-2)e \).

9. Conclude that \( X \) must be algebraically hyperbolic for \( d \geq 2n \).

10. Show that for \( d \leq 2n - 3 \), \( X \) must contain a line, leaving open only the cases \( d = 2n - 1, 2n - 2 \).