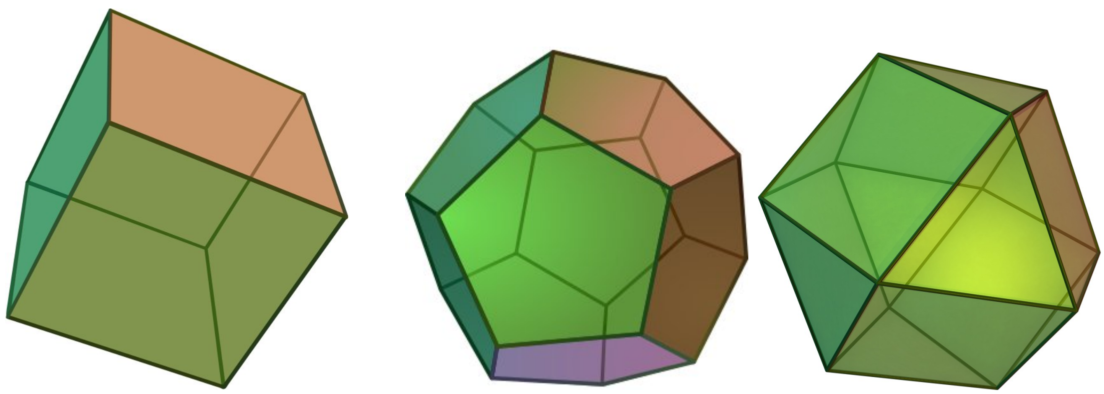
Lecture 4: Euler characteristic of the sphere

One of the key takeaways from lecture is that the sphere S^2 has Euler characteristic 2. Furthermore, this is still true for any shape S that is "topologically equivalent" to the sphere in the sense that the two can be continuously deformed into each other. In this set of exercises we will study this circle of ideas from several different perspectives.

1 Euler's formula

Euler's formula is a version of the computation $\chi(S^2) = 2$. It address a polyhedron $S \subset \mathbb{R}^3$ – that is, a bounded three-dimensional shape which has flat surfaces and straight edges. We will furthermore require our polyhedron to be convex: if p, q are two points in S then the line segment between p and q is also in S.



We are now ready to state Euler's formula for polyhedra.

Theorem 1.1 (Euler's polyhedron formula)**.** *Suppose S is a convex polyhedron in $\mathbb{R}^3$. Let V denote the number of vertices of S, E denote the number of edges of S, and F denote the number of faces of F. Then*

$$V - E + F = 2.$$

1) Why can we see Euler's formula is a special case of the computation $\chi(S^2) = 2$?

2) Verify Euler's formula for the three shapes pictured above (the cube, the dodecahedron, and the cuboctahedron).

3) Prove that it is impossible to construct a polyhedron whose faces are all hexagons.

4) A polyhedron S is said be a "Platonic solid" if all faces are congruent, every face is a regular polygon (e.g. equilateral triangle, square, regular pentagon, etc.), and whose vertices are congruent. Here is a way to classify all Platonic solids:

a) Given a regular n-sided polygon in $\mathbb{R}^2$, compute the interior angle a_n at each vertex.

b) Suppose S is a Platonic solid whose faces are regular n-sided polygons. Let b denote the number of faces which meet at each vertex. Explain why $b \cdot a_n \leq 2\pi$ and use this to determine an upper bound on b as a function of n. (Note that we also have a lower bound $b \geq 3$; only then is S actually a 3-dimensional shape!)

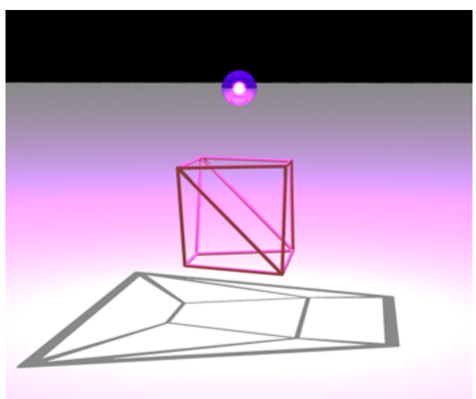
c) Using Euler's formula, prove that there are exactly 5 Platonic solids: three made using triangles, one made using squares, and one made using regular pentagons. How many faces does each Platonic solid have?

5) If you are feeling ambitious, you can try to classify the "Archimedean solids" where every face is a regular polygon and the vertices are all congruent (but we do not require all faces to be congruent). This will take some serious work.

2 Planar graphs

Our proof of Euler's formula will go through another topic that is interesting in its own right: planar graphs.

Suppose S is a convex polyhedron in $\mathbb{R}^3$. Imagine taking a light source very close to the middle of one of the faces of S and projecting the shape onto a plane. The "shadow" of the vertices and edges of S will form a 1-dimenensional cell complex inside of the plane.



The shadow of a cube

From now on we will call the resulting object a "planar graph": a 1-dimensional cell complex that can be placed in $\mathbb{R}^2$ without forcing any edges to overlap. We will denote the planar graph obtained from S by G.

Warning 2.1. We can think of G in two different ways. First, we can think of it as an abstract 1-dimensional cell complex. Second, we can "fill in" the holes of G with 2-cells to think of G as a "portion" of the polyhedron S. In this second perspective, it is important to remember that G has one fewer faces than S – often mathematicians think of the exterior of G as the "missing face" of S.
When working with Euler's formula, it is important to distinguish between these two different ways of thinking about G. In particular, if we think of G without the 2-cells the Euler characteristic will be very different from thinking about G with the 2-cells inserted!

Suppose we fill in the holes in G with 2-cells. Accounting for the "missing face" on the exterior of G, Euler's polyhedron formula is equivalent to:

Theorem 2.2 (Euler's planar graph formula)**.** *Suppose $G \subset \mathbb{R}^2$ is a planar graph. Let H denote the number of regions interior to G. Then*

$$V - E + H = 1.$$

6) Explain carefully how Euler's polyhedron formula follows from Euler's planar graph formula.

7) Prove Euler's planar graph formula in the following way. Choose any edge E inside of G. Then the two ends of E either attach to different vertices or they attach to the same vertex.

    - If the ends of E meet different vertices, collapse E down to a point and identify the two endpoints with a single vertex in the new graph.
    - If the ends of E meet the same vertex, then E is a loop. Remove E from the graph.

    In both cases, we made a new "smaller" graph G' from our original graph G. Analyze how $V - E + H$ changes under this operation. Then use your observation to prove Euler's planar graph formula by induction.

8) Suppose that G is a planar graph whose vertices are on lattice points and whose edges are straight. (It turns out that by choosing a projection correctly we can always ensure these conditions are true when G is obtained by projecting a polyhedron.) Use Pick's formula (Lecture 2: Area and lattice points) to give a different proof of $V - E + H = 1$.

Although we started our analysis by imagining that G was obtained from a polyhedron by projection, the proof did not use this structure at all. Euler's formula works for any planar graph! In turn, we can apply the planar graph formula to prove our original assertion: if we give the sphere S^2 the structure of a cellular complex then $\chi(S^2) = 2$.

9) Repeating the earlier arguments, show that $\chi(S^2) = 2$ no matter which cellular complex structure we impose on the sphere.

10) Explain why the Euler characteristic $\chi(G)$ of a planar graph agrees with the number "1 minus the number of holes in G" as suggested during lecture.

Remark 2.3. Another fun aspect of Euler's formula is that it can be used to prove the existence of non-planar graphs – graphs which cannot possibly be placed inside of $\mathbb{R}^2$ unless we allow some edges to intersect. You might enjoy watching the 3Blue1Brown video: "Why this puzzle is impossible":
`https://www.youtube.com/watch?v=VvCytJvd4H0`