Lecture 5: Plane curves

Suppose that $C \subset \mathbb{R}^2$ is a curve defined by a polynomial equation $P(x,y) = 0$. In lecture we mentioned that one can "compactify" C by taking its closure in $\mathbb{P}^2_{\mathbb{R}}$. In this set of exercises we will discuss this operation in more detail. First a reminder:

Definition 0.1. A polynomial is said to be homogeneous if every term has the same degree.

For example, the polynomial $x^2 + 2xy - y^2$ is homogeneous of degree 2 while $x^2 - 3y$ is not homogeneous.

1 Projective space

This section recalls some basic facts about projective space (also covered in Prof. Migliore's lectures). We will think of $\mathbb{P}^2_{\mathbb{R}}$ as the set of equivalence classes of triples $(x : y : z)$ of real numbers such that x, y, z are not all zero, where two triples $(x_1 : y_1 : z_1)$ and $(x_2 : y_2 : z_2)$ are said to be equivalent if there is a non-zero $\lambda \in \mathbb{R}$ such that

$$x_1 = \lambda x_2 \qquad y_1 = \lambda y_2 \qquad z_1 = \lambda z_2$$

or expressed more compactly, $(x_1 : y_1 : z_1) = \lambda(x_2 : y_2 : z_2)$. We will somewhat lazily think of a non-zero triple $(x : y : z)$ as an element of $\mathbb{P}^2$, with the caveat that there are other triples that also define the same element.

We can identify a copy of $\mathbb{R}^2$ inside of $\mathbb{P}^2$ in the following way. Consider the function

$$\phi : \mathbb{R}^2 \to \mathbb{P}^2$$
$$\phi(x,y) = (x : y : 1)$$

This function is injective: if $\phi(x,y) = \phi(x',y')$ then $(x : y : 1)$ and $(x' : y' : 1)$ are equivalent tuples. But by looking at the z-coordinate we see the rescaling factor for these equivalent tuples must be 1, so that $x = x'$ and $y = y'$.
The image of this function is the set $U \subset \mathbb{P}^2$ consisting of all points $(x : y : z)$ such that $z \neq 0$. Indeed, if $(x : y : z)$ is a point with $z \neq 0$, then it is equivalent to the point $(\frac{x}{z} : \frac{y}{z} : 1)$ which is clearly in the image of ϕ. In other words, on U we have an inverse function

$$\phi^{-1} : U \to \mathbb{R}^2$$
$$\phi^{-1}(x : y : z) = \left(\frac{x}{z}, \frac{y}{z}\right)$$

We will think of U as a "copy" of $\mathbb{R}^2$ inside of $\mathbb{P}^2$. We will call the complement $\mathbb{P}^2\backslash U$ the "set of points at infinity" and denote it by $\mathbb{P}^1_\infty$. To be clear: the set of points at infinity is the set of points of the form $(x : y : 0)$ inside of $\mathbb{P}^2$ (which really is a copy of $\mathbb{P}^1$). Geometrically the point $(x : y : 0)$ represents the "slope direction" $\frac{y}{x}$.

Note that a polynomial $P(x, y, z)$ usually does not define a function $\mathbb{P}^2 \to \mathbb{R}$ in any meaningful way. Of course if we choose a particular triple $(x : y : z)$ then $P(x, y, z)$ is a well-defined real number. However, if we choose a rescaling $(\lambda x : \lambda y : \lambda z)$ representing the *same* point of $\mathbb{P}^2$, usually the value of $P(\lambda x, \lambda y, \lambda z)$ is *different*. Since the value of $P(x, y, z)$ depends on the choice of representative, we can't use P to define a function $\mathbb{P}^2 \to \mathbb{R}$.
However, we can do something a little weaker if $P(x, y, z)$ is a homogeneous polynomial. In this case, the rescaling operation is easier to understand: if P has degree d then

$$P(\lambda x, \lambda y, \lambda z) = \lambda^d P(x, y, z).$$

We still can't necessarily get a value of P at a point $(x : y : z) \in \mathbb{P}^2$. However, we *can* tell if $P(x, y, z) = 0$ because this property does not change when we rescale the point!

Definition 1.1. Given a homogeneous polynomial $\widetilde{P}(x, y, z)$ of degree $d \geq 1$, the curve $\widetilde{C} \subset \mathbb{P}^2$ defined by the equation $\widetilde{P}(x, y, z) = 0$ is the set of all points $(x : y : z)$ such that $\widetilde{P}(x, y, z) = 0$. (For emphasis: to check if a point of $\mathbb{P}^2$ is in $\widetilde{C}$ it does not matter which representative tuple we pick!)

2 Affine to projective

Now suppose we have a plane curve C defined by an equation $P(x, y) = 0$ in $\mathbb{R}^2$. We will associate to it a curve $\widetilde{C} \subset \mathbb{P}^2$ in the following way.

Definition 2.1. Suppose that $P(x, y)$ is a degree d polynomial with $d \geq 1$. We define the homogenization $\widetilde{P}(x, y, z)$ to be the homogeneous polynomial of degree d obtained by adding as many factors of z as necessary to the terms of P to obtain a homogeneous polynomial of degree d.

For example,

$$\begin{aligned} P(x, y) = xy^2 - 3x^2 + 2y \qquad &\Longrightarrow \qquad \widetilde{P}(x, y, z) = xy^2 - 3x^2z + 2yz^2 \\ P(x, y) = x^4 - xy + 3 \qquad &\Longrightarrow \qquad \widetilde{P}(x, y, z) = x^4 - xyz^2 + 3z^4 \end{aligned}$$

Note that it is possible that $\widetilde{P}(x, y, z) = P(x, y)$ if $P(x, y)$ is already homogeneous.

Definition 2.2. Suppose that $C \subset \mathbb{R}^2$ is a curve defined by $P(x, y) = 0$ where $P(x, y)$ is a degree d polynomial with $d \geq 1$. We define $\widetilde{C} \subset \mathbb{P}^2$ by the equation $\widetilde{P}(x, y, z) = 0$.

Using the inclusion $\phi : \mathbb{R}^2 \to \mathbb{P}^2$ we can think of C as a subset of $\mathbb{P}^2$. Loosely speaking, $\widetilde{C}$ is the "closure" of C inside of $\mathbb{P}^2$. The following theorem gives us some indication of why this is a valid perspective.

Theorem 2.3. *For any curve $C \subset \mathbb{R}^2$ defined by a polynomial $P(x, y) = 0$ we have*

$$\widetilde{C} \cap U = \phi(C).$$

Proof. A point in U can be rescaled to have the form $(x : y : 1)$. Note that $\widetilde{P}(x, y, 1) = P(x, y)$. Thus the set of points of the form $(x : y : 1)$ such that $\widetilde{P}(x, y, 1) = 0$ is the same as the set of points (x, y) such that $P(x, y) = 0$. □

Caution 2.4. Strictly speaking it is not true that $\widetilde{C}$ is the closure of C in $\mathbb{P}^2$. When we are working over $\mathbb{C}$ this is literally true! But when we are working over $\mathbb{R}$ it is not; see Exercise (4). We will ignore this minor issue.

The following exercises give you practice with this "closure" operation. Feel free to use Desmos to get a visual intuition!

1) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $x^2 - y = 0$. What does the curve $\widetilde{C}$ look like? What are its points at infinity?

2) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $6x^2 - y^2 - 5y = 0$. What does the curve $\widetilde{C}$ look like? What are its points at infinity?

3) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $2x^2 + y^2 - 6 = 0$. What does the curve $\widetilde{C}$ look like? What are its points at infinity?

4) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $x^2 + 1 = 0$. What does the curve $\widetilde{C}$ look like? What are its points at infinity?

5) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $x^3 - 3y^3 + 3x - 5y = 0$. What does the curve $\widetilde{C}$ look like? What are its points at infinity?

6) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $x^3 - 3x^2y + 3xy^2 - y^3 - 6x^2 + 5y^2 - 3x + y + 5 = 0$. What does the curve $\widetilde{C}$ look like? What are its points at infinity?

7) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $x^3 - 4x^2y + 5xy^2 - 2y^3 - 5xy + 3x - 2 = 0$. What does the curve $\widetilde{C}$ look like? What are its points at infinity?

8) Make sense of the following claims. Suppose $P(x, y)$ has degree $d \geq 1$.

- Let $P_d(x, y)$ denote all the terms of $P(x, y)$ with degree d. Then the points at infinity of $\widetilde{C}$ is the subset of $\mathbb{P}^1_\infty$ defined by the homogeneous polynomial $P_d(x, y) = 0$.
- The points at infinity of $\widetilde{C}$ are the "limits" of the tangent directions to the points of C.

9) If $\widetilde{C} \subset \mathbb{P}^2$ is defined by a homogeneous polynomial $\widetilde{P}(x, y, z) = 0$, is it true that $\widetilde{C} \cap U$ is the same as the subset defined by the "dehomogenized" polynomial $\widetilde{P}(x, y, 1) = 0$? Prove or give a counterexample!

3 Smooth points

Let $C \subset \mathbb{R}^2$ be a curve defined by a polynomial $P(x, y) = 0$. Recall that p is said to be a singular point of C if both $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ vanish at p. Otherwise p is said to be a smooth point of C. (Technically, it is best to look for singular points in $\mathbb{C}^2$ and not just $\mathbb{R}^2$ – this will give us the best sense of the behavior of our curve.)

Caution 3.1. A singular point p of C must satisfy three conditions, not two: $P(p) = 0$, $\frac{\partial P}{\partial x}(p) = 0$, $\frac{\partial P}{\partial y}(p) = 0$

10) Show that $(0, 0)$ is a singular point of C if and only every term of P has degree ≥ 2. Show that this matches your intuition by using Desmos to graph the following curves: $y^2 - x^3 - x^2 = 0$, $(x^2 + y^2)^2 + 3x^2y - y^3 = 0$.

11) Show that the following curves are smooth:

    a) $x^2 + y^2 - 3 = 0$.
    b) $x^2 - 3y^2 + 5x - 6y = 0$.
    c) $y^2 - x^3 - 3x - 1 = 0$.

12) Find all the singular points of the following curves:

    a) $y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy = 0$.
    b) $x^4 + y^4 - x^2y^2 = 0$.
    c) $x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1 = 0$.

13) Suppose that C has the Weierstrass form $y^2 = x^3 + ax + b$. Show that C is singular if and only if $4a^3 + 27b^2 = 0$. (Here I am implicitly looking for singular points over $\mathbb{C}$.)

14) Suppose that C is defined by a degree 2 equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$. What is the condition on the coefficients that determines whether C is smooth or has a singular point? (This is a little complicated – the analogous question in $\mathbb{P}^2$ is better behaved.)

Now suppose we have $\widetilde{C} \subset \mathbb{P}^2$ defined by a polynomial equation $\widetilde{P}(x, y, z) = 0$. Letting $U = \{(x : y : z) | z \neq 0\}$ as before, recall that $\widetilde{C} \cap U$ is the same as the locus $C \subset \mathbb{R}^2$ defined by the equation $\widetilde{P}(x, y, 1) = 0$. Thus it is very natural to say that a point in $\widetilde{C} \cap U$ is singular or smooth if the corresponding point of C is singular/smooth.

However, from the viewpoint of projective space, the points at infinity are as good as any other point. So we can also define a notion of singular/smooth for these points as well! One way is simply to swap the roles of the variables and "dehomogenize" with respect to x or y:

- Letting $V = \{(x : y : z) | y \neq 0\}$, we check for singularities of $\widetilde{C} \cap V$ using the equation $\widetilde{P}(x, 1, z) = 0$.
- Letting $W = \{(x : y : z) | x \neq 0\}$, we check for singularities of $\widetilde{C} \cap W$ using the equation $\widetilde{P}(1, y, z) = 0$.

This looks a little complicated. For example, it is not at all obvious that if a point $p \in U \cap V \cap W$ is singular when considered in $\widetilde{C} \cap U$, it is also singular when considered in $\widetilde{C} \cap V$ or $\widetilde{C} \cap W$. (To be clear: it is true, just not obvious.)

An easier option is given by:

Definition 3.2 (Projective criterion)**.** Suppose $\widetilde{C} \subset \mathbb{P}^2$ is defined by a homogeneous equation $\widetilde{P}(x, y, z) = 0$. We say that $p \in \widetilde{C}$ is a singular point if

$$\frac{\partial \widetilde{P}}{\partial x}(p) = 0 \qquad \frac{\partial \widetilde{P}}{\partial y}(p) = 0 \qquad \frac{\partial \widetilde{P}}{\partial z}(p) = 0$$

Otherwise we say that p is a smooth point.

15) Find all the singular points of the following curves:

    a) $xz^2 - y^3 + xy^2 = 0$
    b) $x^2y^2 + 36xz^3 + 24yz^3 + 108z^4 = 0$.

16) What condition on k determines whether the curve

$$(x + y + z)^3 - kxyz = 0$$

has singular points?

17) Suppose that $C \subset \mathbb{R}^2$ is defined by the equation $y^2 = x^3 + ax + b$. Show that all points at infinity of the corresponding curve $\widetilde{C}$ are smooth.

18) Suppose that C is defined by a degree 2 equation $ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$. What is the condition on the coefficients that determines whether C is smooth or has a singular point? (Hint: recall that every two lines in $\mathbb{P}^2$ intersect!)

19) Suppose that $\widetilde{P}(x, y, z)$ is homogeneous of degree d. Prove Euler's formula:

$$d \cdot \widetilde{P} = x\frac{\partial \widetilde{P}}{\partial x} + y\frac{\partial \widetilde{P}}{\partial y} + z\frac{\partial \widetilde{P}}{\partial z}.$$

Using this formula, prove that our two definitions of singular/smooth – one via "dehomogenizing", one via the projective criterion – are equivalent. In particular, verify that if $p \in \widetilde{C}$ is singular then it is singular in every "dehomogenization" such that the corresponding copy of $\mathbb{R}^2$ contains p.