# Lecture 4: Euler characteristic

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Today we start with a game that assigns numbers to shapes.

We start by assigning 1 to a single point.

$$\bullet \qquad = 1$$

One point

# Numbers from shapes



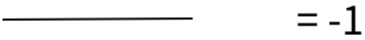
When we add on more shapes, the numbers add as well:


Several points

# Numbers from shapes

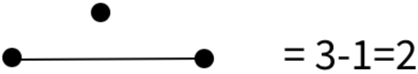


Next, we assign a line segment the number $-1$.




Line segment

Again, we can combine shapes:



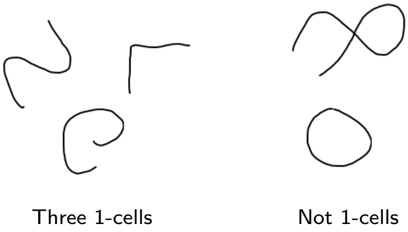

A shape

Now we start formalizing these constructions. First, some terminology:

**Definition**

A 0-cell is a point.

A 1-cell is any shape that can be "continuously deformed" to a line segment.



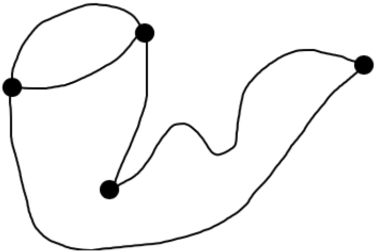
# Numbers from shapes

## Definition

A 1-dimensional cellular complex is a union of 0-cells and 1-cells that satisfies the following rules:

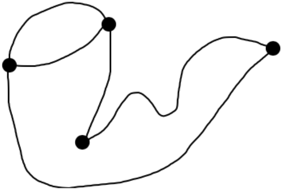
1. The ends of every 1-cell must be attached to 0-cells.
2. The interior of a 1-cell does not intersect any other cells.

(This is the same as the mathematical object called a "graph.")

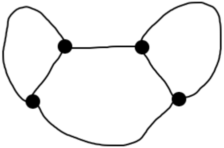


Four 0-cells, five 1-cells

Given a 1-dimensional cellular complex $S$, we define its Euler characteristic $\chi(S)$ by adding up the numbers associated to its pieces.

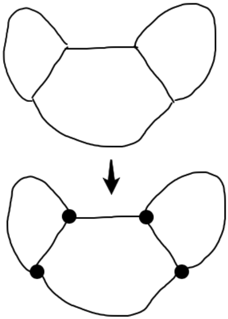


= 4-5=-1



= 4-6=-2

Let $S$ be a one-dimensional shape. Then we can give $S$ the structure of a cellular complex: we cover $S$ with 0-cells and 1-cells which satisfying the earlier rules. Once we do this procedure, we also obtain an associated Euler characteristic for $S$.



= -2

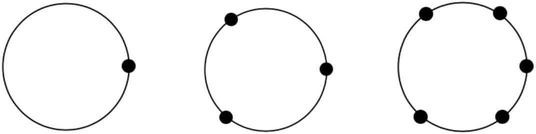
There are many different cellular structures we can give $S$. However, these different choices will end up giving us the same Euler characteristic!

**Observation 1:** The Euler characteristic of $S$ does not depend on which cellular structure we use for $S$.
**Observation 2:** If we "continuously deform" the shape $S$ the Euler characteristic does not change.

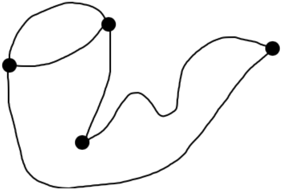
## Example

Let $S$ be the circle. Then $\chi(S) = 0$.

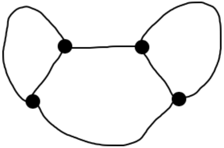
In fact, the number of 0-cells is equal to the number of 1-cells no matter what cellular structure we give to $S$.



When $S$ is connected, we can think of $\chi(S)$ as "1 - (number of holes in $S$)."



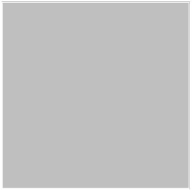
= 4-5=-1



= 4-6=-2

# Two dimensions

We now introduce a new shape: a 2-cell (with no boundary).



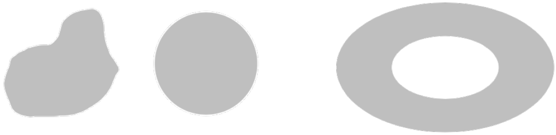
$= 1$

A 2-cell

Since a 2-cell is a product of 1-cells, we assign it the number $(-1) \times (-1) = 1$.

# Two dimensions

Just as with 1-cells, we can "deform" 2-cells freely. However, we are not allowed to tear them, glue them, etc.



More 2-cells

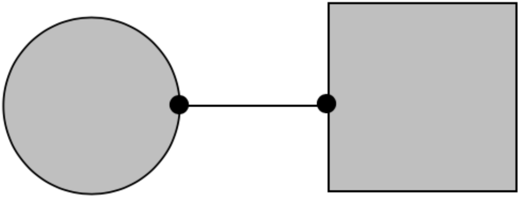
Not a 2-cell

# Two dimensions

## Definition

A 2-dimensional cellular complex is a union of 0-cells, 1-cells, and 2-cells that satisfies the following rules:

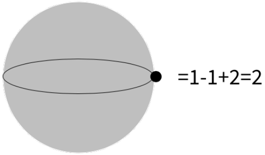
1. The entire boundary of a $k$-cell must be attached to cells of dimension $\leq (k-1)$.
2. The interior of a $k$-cell does not intersect any cells of dimension $\leq k$.



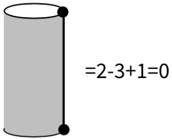
Cellular complex: two 0-cells, three 1-cells, two 2-cells

# Two dimensions

Given a 2-dimensional cellular complex $S$, we define the Euler characteristic $\chi(S)$ by adding up the numbers associated to the various cells.




Sphere: one 0-cell, one 1-cell, two 2-cells




Cylinder (no caps): two 0-cells, three 1-cells, one 2-cell

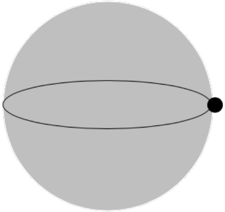
# Two dimensions

**Definition**

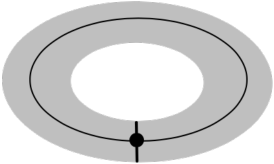
The Euler characteristic $\chi(S)$ of a 2-dimensional shape $S$ is found by giving $S$ the structure of a cellular complex and computing the associated number.

Just as in the 1-dimensional case:
**Observation 1:** The Euler characteristic of $S$ does not depend on which cellular structure we use.
**Observation 2:** If we "continuously deform" the shape $S$ the Euler characteristic does not change.

# Two dimensions



Sphere: one 0-cell, one 1-cells, two 2-cells



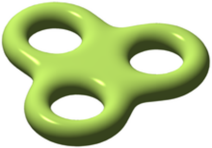
Torus: one 0-cell, two 1-cells, one 2-cell

There is no way to continuously deform a torus to a sphere! If there were such a deformation, their Euler characteristics would be the same. But the sphere has Euler characteristic 2 and the torus has Euler characteristic 0.

# Two dimensions



We will be particularly interested in the the connected compact orientable surfaces (which we will simply call “surfaces”).



A surface

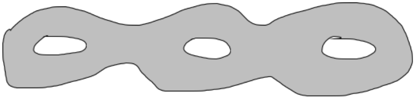
Loosely, one can think of a surface as a “doughnut with $g$ holes.” The quantity $g$ is called the “genus.”

# Two dimensions

## Theorem

*Suppose that $S$ is a surface with $g$ holes. Then*

$$\chi(S) = 2 - 2g.$$



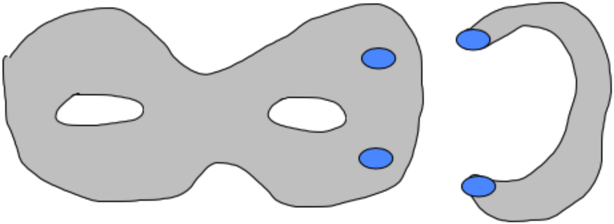
A picture of $S$

## Proof.

The proof is by induction on $g$. We have already seen the base case $g = 0$: a sphere has Euler characteristic 2.

**Proof.**

For the induction step, any surface $S$ with $g$ holes can be constructed by taking a surface $T$ with $g-1$ holes, removing two disks, and “gluing on” a shape $C$ that deforms to a cylinder.

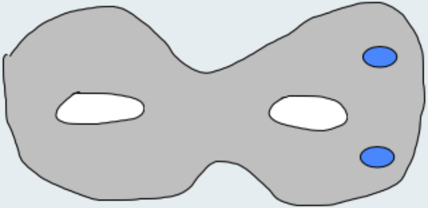


Constructing $S$ from $T$ and $C$

**Proof.**

Let's analyze how this operation changes the Euler characteristic.

First, we need to remove two disks $D$ from $T$ to create a space to glue the cylinder. If we choose our cellular complex carefully, we can simply remove two 2-cells from $T$. Thus the Euler characteristic of the new shape is $\chi(T) - 2$.



Removing two 2-cells from $T$

□

**Proof.**

Next, let's glue on the cylinder. In terms of cellular complexes, we can choose one 0-cell on the boundary of each disk in $T$, connect them with a 1-cell, and then glue on a 2-cell to form the cylinder. Altogether, we have

$$\chi(S) = (\chi(T) - 2) + (0 - 1 + 1) = \chi(T) - 2.$$

Our induction assumption is that $\chi(T) = 2 - 2(g-1) = 4 - 2g$. Altogether

$$\chi(S) = \chi(T) - 2 = 2 - 2g.$$

□

# Curvature



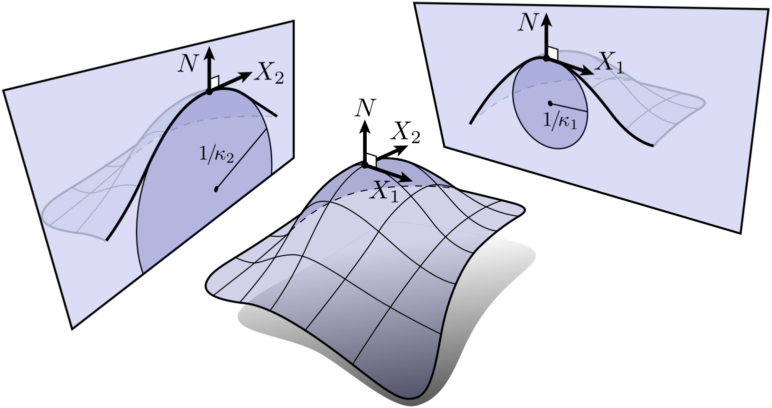
We next present a different perspective on the Euler characteristic: it measures the "total amount of curvature" of a surface. This deep result is known as the Gauss-Bonnet Theorem.

The Gauss-Bonnet theorem gives another connection between different areas of math. The Euler characteristic of a surface is a "topological" invariant: it does not change under continuous deformation. The curvature of a surface is a "metric" invariant: it depends on how our surface is embedded. The fact that these two quantities are related is quite surprising!

# Curvature

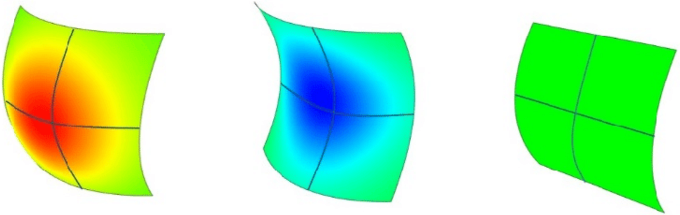


Suppose we have a compact orientable surface. The Gaussian curvature at a point is a measurement of "how much" the surface curves at a point.




Gaussian curvature

The Gaussian curvature at $p$ is a signed quantity: it can be positive, negative, or zero. The sign will depend on whether or not the curves through $p$ all "bend in the same direction."



Positive, negative, and zero curvature

# Curvature



We will only give a precise definition in one special case: a smooth parametrized surface $S \subset \mathbb{R}^3$.

## Definition

Suppose that $S \subset \mathbb{R}^3$ is a smooth surface parametrized locally by an equation $f$:

$$f(u, v) = (f_1, f_2, f_3).$$

We define the vectors:

$$f_u = \left(\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u}, \frac{\partial f_3}{\partial u}\right)$$
$$f_v = \left(\frac{\partial f_1}{\partial v}, \frac{\partial f_2}{\partial v}, \frac{\partial f_3}{\partial v}\right)$$

Similarly, we define $f_{uu}$, $f_{uv} = f_{vu}$, $f_{vv}$ as vectors of second partial derivatives.

**Definition**

We also consider the normal unit vector

$$\widehat{n} = \frac{f_u \times f_v}{\|f_u \times f_v\|}.$$

(For concreteness we choose the outward direction, although it does not matter.) Then the Gaussian curvature is

$$K = \frac{\langle f_{uu}, \widehat{n}\rangle\langle f_{vv}, \widehat{n}\rangle - \langle f_{uv}, \widehat{n}\rangle^2}{\langle f_u, f_u\rangle\langle f_v, f_v\rangle - \langle f_u, f_v\rangle^2}$$

## Example

Let's compute the Gaussian curvature of the sphere of radius $R$. A parametric equation for the sphere is

$$f(u,v) = (R\sin(u)\cos(v), R\sin(u)\sin(v), R\cos(u)).$$

We can identify the various vectors we need for computation:

$$\begin{aligned} f_u &= (R\cos(u)\cos(v), R\cos(u)\sin(v), -R\sin(u)) \\ f_v &= (-R\sin(u)\sin(v), R\sin(u)\cos(v), 0) \end{aligned}$$

and so the denominator is

$$\langle f_u, f_u\rangle\langle f_v, f_v\rangle - \langle f_u, f_v\rangle^2 = R^4\sin^2(u)$$

**Example**

Furthermore

$$\begin{aligned} f_{uu} &= (-R\sin(u)\cos(v), -R\sin(u)\sin(v), -R\cos(u)) \\ f_{uv} &= (-R\cos(u)\sin(v), R\cos(u)\cos(v), 0) \\ f_{vv} &= (-R\sin(u)\cos(v), -R\sin(u)\sin(v), 0) \\ \widehat{n} &= (\sin(u)\cos(v), \sin(u)\sin(v), \cos(u)) \end{aligned}$$

and so the numerator is

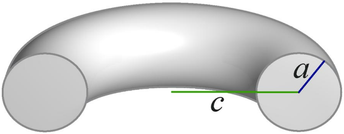
$$\langle f_{uu}, \widehat{n}\rangle\langle f_{vv}, \widehat{n}\rangle - \langle f_{uv}, \widehat{n}\rangle^2 = R^2\sin^2(u)$$

Altogether the curvature (at every point!) is the constant value $1/R^2$.

### Example

Consider the torus parametrized by

$$f(u,v) = ((c + a\cos(v))\cos(u), (c + a\cos(v))\sin(u), a\sin(v))$$




Major radius $c$, minor radius $a$

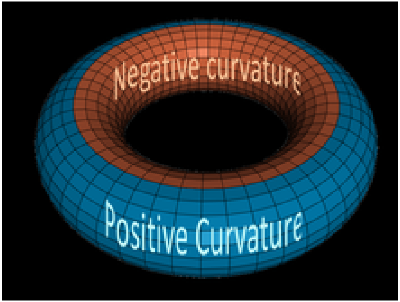
# Curvature

## Example

A similar computation shows that the curvature is

$$K(u, v) = \frac{\cos(v)}{a(c + a\cos(v))}$$

Note that this can be positive or negative depending on the sign of $\cos(v)$.



# Gauss-Bonnet

The following theorem relates the Euler characteristic to curvature.

**Theorem (Gauss-Bonnet)**

*Let $S$ be a smooth compact orientable surface with no boundary. Then*

$$\int_S K \, dA = 2\pi\chi(S)$$

*where $dA$ is the area element on $S$.*

Although the curvature can vary a lot, the "total curvature" will be the Euler characteristic! If we continuously deform $S$, the curvature can change but the net effect on the total curvature will cancel out.

## Example

Consider the sphere of radius $R$:

$$f(u,v) = (R\sin(u)\cos(v), R\sin(u)\sin(v), R\cos(u)).$$

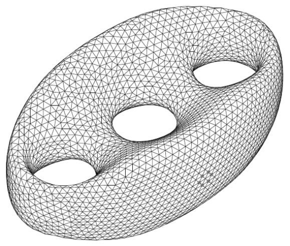
The area element is $\|f_u \times f_v\|\,du\,dv = R^2\sin(v)\,du\,dv$. Integrating:

$$\int_S K\,dA = \int_0^{\pi}\int_0^{2\pi}\frac{1}{R^2}(R^2\sin(v))\,du\,dv = 4\pi = 2\pi\chi(S)$$

verifying Gauss-Bonnet in this case.

## Sketch of proof.

Every smooth compact orientable surface admits a "triangulation": a cellular complex structure whose pieces $P$ each consist of three smooth edges connecting three vertices:



Triangulated surface

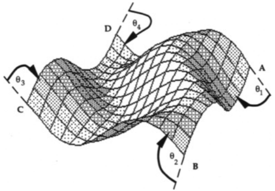
Our plan is to analyze $\int_P K\, dA$ for each piece $P$ and then add up the results.

**Sketch of proof.**

For each triangle $P$ one can prove a "local Gauss-Bonnet" theorem:

$$\int_P K\,dA + \int_{\partial P} \kappa_g + \sum_{\text{vertices } v \text{ in } P} \theta_v = 2\pi$$

where $\kappa_g$ denotes the geodesic curvature and $\theta_v$ denotes the exterior angle at the vertex $v$. (One can think of the $2\pi$ as the "total rotation" as we traverse around the boundary of $P$.)




Geodesic curvature and exterior angles

**Sketch of proof.**

We can compute $\int_S K\,dA$ by adding up all the pieces:

$$\begin{aligned}\int_S K\,dA &= \sum_P \int_P K\,dA \\ &= \sum_P \left(2\pi - \int_{\partial P} \kappa_g - \sum_{\text{vertices } v \text{ in } P} \theta_v\right)\end{aligned}$$

Let's compute the three terms in the sum one-by-one. We will let $V$ denote the total number of vertices, $E$ denote the total number of edges, and $F$ denote the total number of triangles in our triangulation.

**Sketch of proof.**

Since we have one contribution of $2\pi$ for each triangle $P$, the total contribution of this term is $2\pi F$.

Next, consider $\sum_P \int_{\partial P} \kappa_g$. Every edge in $S$ lies between exactly two triangles $P_1, P_2$. Note that this integral changes sign when we reverse the orientation of our curve. Since the orientation of our edge in $P_1$ is the opposite of its orientation in $P_2$, the total contribution of the integral of $\kappa_g$ over this edge to the integral is 0. Adding up, the entire contribution of this integral is 0.

The remaining term is $\sum_P \left(\sum_{\text{vertices } v \text{ in } P} \theta_v\right)$. To compute this sum we will reverse the summation: we first fix a vertex $v$ and then sum all the exterior angles $\theta_P$ for triangles $P$ containing $v$.

**Sketch of proof.**

For a vertex $v$ of a triangle $P$, the sum of the exterior angle $\theta_P$ and the interior angle $\alpha_P$ of $P$ at $v$ is equal to $\pi$. Moreover, the sum of all the interior angles at a given vertex is $2\pi$. Thus the sum of the exterior angles at a vertex $v$ is

$$\begin{aligned}\sum_{P \text{ meeting } v} \theta_P &= \left(\sum_{P \text{ meeting } v} \theta_P + \alpha_P\right) - 2\pi \\ &= \pi \cdot (\text{number of } P \text{ meeting at } v) - 2\pi \\ &= \pi \cdot (\text{number of edges meeting at } v) - 2\pi\end{aligned}$$

**Sketch of proof.**

Returning to our earlier sum,

$$\int_S K\,dA = \sum_P \left( 2\pi - \int_{\partial P} \kappa_g - \sum_{\text{vertices } v_i \text{ in } P} \theta_i \right)$$

$$= 2\pi F - 0 - \sum_{\text{vertices } v} (\pi(\text{number of edges meeting at } v) - 2\pi)$$

$$= 2\pi F + 2\pi V - \sum_{\text{vertices } v} \pi(\text{number of edges meeting at } v)$$

Since each edge meets two vertices, every edge appears twice in the sum above. Altogether

$$\int_S K\,dA = 2\pi V - 2\pi E + 2\pi F = 2\pi\chi(S).$$

# Higher dimensions



One can also define the Euler characteristic for a cellular complex of dimension $n$: to each cell of dimension $n$ we assign the number $(-1)^n$.

**Question (Hopf sign conjecture)**

Suppose that $X$ is a compact Riemannian manifold of dimension $2d$. If $X$ has positive curvature at every point, is $\chi(X) > 0$?

For surfaces, this is true by Gauss-Bonnet. But the general case has been open for 100 years!

# Exercises

Exercises:

1. Euler's formula: you can explore Euler's formula and its relationship with Euler characteristics.
2. Gauss-Bonnet: you can get more practice with the Gauss-Bonnet formula.

Images made using sketch.io and taken from Wikipedia, Mark Irons, Keenan Crane, Markus Wallner-Novak