MINI-COURSE: SYMPLECTIC AND POISSON GEOMETRY IN FIELD THEORY CMND 2024 THEMATIC PROGRAM IN FIELD THEORY AND TOPOLOGY SUGGESTED PROBLEMS AND REFERENCES

"Everything is a Lagrangian submanifold". Alan Weinstein

1. General information

Hello! These problems are suggested to be discussed/worked out during the problem sessions. Please feel free to ask any questions, share your thoughts and ideas. There are also some open-ended questions, in case you want to continue your exploration of symplectic and Poisson geometry in field theory. The last section contains some suggested references that complement the content of the lectures, and it might help to put these problems into a broader context.

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2. Problems

2.1. Lecture 1: Symplectic Geometry.

(1) Prove that every weak symplectic form on a finite dimensional vector space is actually symplectic.
(2) Prove that the symplectic vector spaces from Example 1 (standard symplectic structure) and Example 2 (cotangent space) are symplectomorphic.
(3) Let V be a weak symplectic space and W, Z subspaces of V. Prove:
  (a) $W \subset Z \Longrightarrow Z^{\perp} \subset W^{\perp}$.
  (b) $(W + Z)^{\perp} = W^{\perp} \cap Z^{\perp}$.
  (c) $W^{\perp} + Z^{\perp} \subseteq (W \cap Z)^{\perp}$.
  (d) $W \subseteq W^{\perp\perp}$.
  (e) $W^{\perp} = W^{\perp\perp\perp}$.
(4) Let $f : (V, \omega_V) \to (W, \omega_W)$ be a symplectomorphism. Prove that the graph $\mathrm{gr}(f)$ is a Lagrangian subspace of $(V \oplus W, \omega_V - \omega_W)$
(5) Prove Theorem 2 (the symplectic reduction theorem).

Theorem 2.1. *Let W be a linear subspace of a symplectic vector space V. We define*

$$\underline{W} := W/W \cap W^{\perp},$$

called the reduction of V. Prove that $\underline{W}$ is symplectic.

2.1.1. *Open Problems.*

Problem 6: Classify (up to conjugation by symplectomorphism) all the Lagrangian relations on a finite dimensional vector space V [1].

Problem 7: Determine exactly which Lagrangian relations on $V \times V \times V$ are associative [2].

[1] For more details on this problem, see [3]
[2] For more details on this problem, see [8]

2.2. **Lecture 2: Poisson Geometry.**

(8) Prove that a bivector Π satisfies the Jacobi identity if and only if the following (so called the *Schouten-Nijenhuis* equation) holds:

$$\sum_r \Pi^{sr}(x)(\partial_r)\Pi^{lk}(x) + \Pi^{kr}(x)(\partial_r)\Pi^{sl}(x) + \Pi^{lr}(x)(\partial_r)\Pi^{ks}(x) = 0,$$

(9) Prove that every bivector Π in $\mathbb{R}^2$ is Poisson.

(10) Let $M = \mathbb{R}^3$, and consider the Poisson bracket on M determined by

$$\{x, y\} = z, \{z, x\} = y, \{y, z\} = x.$$

Determine the Casimir functions and the corresponding orbits (symplectic leaves) of M.

2.2.1. *Open Problems.*

Problem 11: Let $M = \mathbb{R}^n$ (you may start with $n = 3$). A bivector Π is polynomial, if $\Pi(x)$ is a polynomial in the variables $x_1, x_2, \cdots, x_n$.

(a) Determine conditions (necessary and sufficients) for which a polynomial bivector Π is Poisson.

(b) For $n = 2, 3$, classify all polynomial Poisson bivectors up to Poisson isomorphism.

2.3. **Lecture 3: Classical Field Theory.**

(12) Use Euler–Lagrange equations to prove that on $\mathbb{R}$, the shortest path between two points is the straight segment connecting them.

(13) Prove that the form $\omega = d\alpha$ on TM is symplectic if and only if the Legendre map Φ_L is invertible.

(14) Prove that the evolution relation $L_M := \pi_M(EL_M)$ is a Lagrangian submanifold of $C_{\partial M}$.

2.3.1. *Open Problems.*

Problem 15: Under which conditions the evolution relation L_M is Lagrangian, and not just isotropic? [3]

2.4. **Lecture 4: The Poisson Sigma Model.**

(16) Prove that the tangent space $T_e(SU(2))$ is a Lie algebra isomorphic to the cross product Lie algebra.

(17) Prove that if (M, Π) is a Poisson manifold, then T^*M is a Lie algebroid with anchor map $\Pi^\sharp$

2.4.1. *Open Problems.*

Problem 18: Under which conditions a polynomial Poisson bivector admits a symplectic groupoid, i.e. the reduced phase space of the corresponding Poisson Sigma Model is a smooth manifold? [4]

3. Suggested References

3.1. **Symplectic Geometry.**

[1] Bates, S. and Weinstein, A. (1997). Lectures on the Geometry of Quantization, Vol. 8 (American Mathematical Soc.).

[2] Weinstein, A. (1977). Lectures on symplectic manifolds, 29 (American Mathematical Soc.).

[3] Lorand, J. and Weinstein, A. (2015). (Co)isotropic pairs in Poisson and presymplectic vector spaces SIGMA, Symmetry Integrability Geom. Methods Appl. 11.

3.2. **Poisson Geometry.**

[4] M. Crainic, Fernandes, R. and Marcut, I. (2021). Lectures on Poisson Geometry, 217 (Graduate Texts in) Mathematics.

[3] For more details on this problem see [5] and [6]

[4] For more details on this problem see [4] and [7]

3.3. **Classical Field Theory.**

[5] Cattaneo, A. Mnev, P. and Reshetikhin, N. (2011). Classical and Quantum Lagrangian Field Theories with Boundary. Conference: Proceedings of the Corfu Summer Institute.
[6] Cattaneo, A. and Contreras, I. (2021). Split Canonical Relations, Annales Henri Lebesgue 4.

3.4. **The Poisson Sigma Model.**

[7] Cattaneo, A. S. and Felder, G. (2001). Poisson sigma models and symplectic groupoids, in Quantization of singular symplectic quotients, Progr. Math., Vol. 198 (Birkhauser, Basel)
[8] Cattaneo, A. S. and Contreras, I. (2015). Relational symplectic groupoids, Letters in Mathematical Physics 105, 5.